2 Discrete Random Variables

2.1 Random Variables

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Random Variables

Chance experiments: Recall, we defined the sample space Ω = all possible (elementary) outcomes ω, and probability function $\Pr(E)$ for events $E \subseteq \Omega$.

Definition 2.1: Random variable (r.v.). a real-valued function $X(\omega)$ of outcomes, that is, a function

$$X : \Omega \to \Re.$$

Discrete r.v: If X takes only finintly or countably many values (often, the integers).

Example: rolling two dice, with, e.g., ω = (⚀, ⚂).
Define X = sum of the two dice. Then, $X(\omega) = 4$, etc.
Or Y = difference. Then $Y(\omega) = -2$, etc.

Events of special interest are then, e.g., $A = \{X = 4\}$, that is, the event of all possible outcomes ω that are mapped to $X(\omega) = 4$.

Example (ctd.): $\{X = 4\}$ = {(⚀⚂), (⚁⚁), (⚂⚀)}.

2.2 Independent Random Variables

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Independent R.V.'s

Use prob function $\Pr(A)$ with $A = \{X = x\}$ for a r.v. X.
It's important and common enough to give it a name :-)

Probability (mass) function (p.m.f.):

$$p_X(a) = \Pr(X = a)$$

(or just $p(a)$ when X is clear from the context).

(Cumulative) Distribution function: we often use

$$F_X(a) = \Pr(X \leq a)$$

Note, for an integer-valued r.v., $p_X(a) = F_X(a) - F_X(a-1)$.

Joint prob function: Similarly for a pair (X, Y) of (jointly distributed) r.v.'s we define

$$p_{X,Y}(a, b) = \Pr\left((X = a) \cap (Y = b)\right)$$

For short, we often write just $\Pr(X = a, Y = b)$.

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Lemma 2.0: $p_X(a) = \sum_b p_{X,Y}(a, b)$

Proof: this is just the law of total probability with $E_b = \{Y = b\}$.
The rest is just about notation:

$$p_X(a) = \Pr(X = a) = \sum_b \Pr\{(X = a) \cap (Y = b)\} = \sum_b p_{X,Y}(a, b)$$

Example: rolling two dice, X = sum of the faces,
$p_X(2)$ = Pr{(⚀, ⚀)} $= \frac{1}{36}$, $p_X(3) = \frac{2}{36}$ etc.

Let Y = difference of the faces,
$p_{X,Y}(2, 0) = p_{X,Y}(3, 1) = p_{X,Y}(3, -1) = \ldots = p_{X,Y}(12, 0) = \frac{1}{36}$.

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Def 2.2: Independent r.v.'s. using the definition of independent events we say X and Y are independent iff

$$\begin{aligned} p_{X,Y}(x, y) &= \\ &\Pr(X = x, Y = y) = \Pr(X = x) \cdot \Pr(Y = y) \\ &= p_X(x) \cdot p_Y(y) \end{aligned}$$

for all values x and y, using independence of the events $A = \{X = x\}$ and $B = \{Y = y\}$.

Mutually independent r.v's: Similarly, $X_1, \ldots, X_k$ are *mutually independent* if for any $I \subseteq \{1, \ldots, k\}$ we have

$$p_{X_1 \ldots X_k}(x_1, \ldots, x_k) = \Pr\left(\bigcap_{i \in I} X_i = x_i\right) = \prod_{i \in I} p_{X_i}(x_i)$$

Example: 2 dice with X_1 = first die, X_2 = 2nd die.

$$p_{X_1,X_2}(1, 3) = \frac{1}{36} = p_{X_1}(1) \cdot p_{X_2}(3) = \frac{1}{6} \cdot \frac{1}{6}$$

2.3 Expectations

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Expectation

The concept of r.v's is so useful because they attach numeric values to chance experiments, which can then be manipulated and summarized as real numbers.

Def 2.3: Expectation. The expectation of a discrete r.v. X is the average value, averaging over all possible values x, weighted with the corresponding probability

$$E(X) = \sum_x x \cdot p_X(x).$$

If the sum does not converge, we say the expectation is "unbounded".

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Example 1: back to 2 dice, and X = sum of the two dice.

$$E(X) = \frac{1}{36} \cdot 2 + \ldots\ldots + \frac{2}{36} \cdot 11 + \frac{1}{36} \cdot 12 = 7$$

Example 2: Consider another r.v. Y with $\Pr(Y = 2^i) = 1/2^i$, $i = 1, 2, \ldots$. Find $E(Y)$?

$$EY = \sum_{i=1}^{\infty} 2^i \cdot \frac{1}{2^i} = \infty.$$

We say Y has no expectation.

2.4 Linearity of Expectations

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Linearity of Expectation

Being defined as a sum, expectations inherit any properties of sums, e.g.

Lemma 2.2: for any constants a, b,

$$E(aX + b) = aE(X) + b.$$

This is easily proven by writing out the sum, factoring a and noting $\sum_i b\, p_X(i) = b \sum p_X(i) = b \cdot 1$.

In other words, for a linear function $g(\cdot)$

$$E[g(x)] = g(EX)$$

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Theorem 2.1: Linearity of Expectation. For r.v.'s $X_1, \ldots, X_n$, consider a (new, derived) r.v. $Y = \sum_{i=1}^n X_i$. Then

$$E(Y) = E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E(X_i)$$

Proof: for $n = 2$ variables X, Y with prob function $p(i, j)$. Recall the law of total probability for r.v.'s, $\sum_j p_{X,Y}(i, j) = p_X(i)$.

$$\begin{aligned} E(X+Y) &= \sum_i \sum_j (i + j) p(i, j) = \sum_i \sum_j i\, p(i, j) + \sum_j \sum_i j\, p(i, j) \\ &= \sum_i i \sum_j p(i, j) + \sum_j j \sum_i p(i, j) \\ &= \sum_i i\, p_X(i) + \sum_j j\, p_Y(j) = E(X) + E(Y) \end{aligned}$$

2.5 Examples

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Examples

For the following calculations we will need the identities

$$\sum_{j=1}^k j^2 = \frac{k(k+1)(2k+1)}{6} \text{ and } \sum_1^k j = \frac{k(k+1)}{2}.$$

 # 2.1: roll a fair k-sided die with the numbers 1 through k. Let X = number that appears. Find $E(X)$ =?

Solution: $E(X) = \sum_{j=1}^k j \frac{1}{k} = \frac{k+1}{2}$

 # 2.9a: rolling the k-sided die twice, let X_1 and X_2 denote the number that appears. Find $E[\max(X_1, X_2)]$?

Solution: Let $M = \max(X_1, X_2)$. First find

$$F_M(j) = \Pr(X_1 \le j, X_2 \le j) = (j/k)^2$$

and therefore $p_M(j) = F_M(j) - F_M(j-1) = \frac{j^2 - (j-1)^2}{k^2} = \frac{2j-1}{k^2}$.

$$\Longrightarrow E(M) = \sum_{j=1}^k j \frac{2j-1}{k^2} = \frac{2}{k^2} \sum j^2 - \frac{1}{k^2} \sum j = \ldots$$

2.6 Jensen's Inequality

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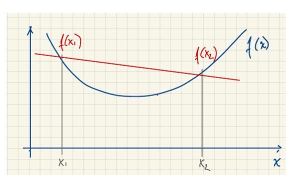
Jensen's Inequality

Definition 2.4: Convex functions. A function $f : \Re \to \Re$ is *convex* if for any x_1, x_2 and $0 \le \lambda \le 1$

$$\begin{array}{ccc} f(\lambda x_1 + (1-\lambda) x_2) & \le & \lambda f(x_1) + (1-\lambda) f(x_2) \\ \text{function betw } x_1, x_2 & & \text{line segment betw } (x_1, f(x_1)), (x_2, f(x_2)) \end{array}$$



If f is twice differentiable, f is convex $\iff$ $f''(x) \ge 0$.

Theorem 2.4: Jensen's Inequality. If $f(\cdot)$ is a *convex* function, then

$$E[f(X)] \ge f(E[X])$$

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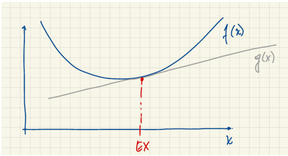
Why is this true? Let $\bar{x} = EX$.



By Lemma 2.2,

$$E[g(x)] = g(\bar{x}) = f(\bar{x})$$

for the linear function $g(x)$.

We have

$$g(x) \le f(x) \implies E[f(x)] \ge E[g(x)] = f(\bar{x}).$$

(formal) Proof of Th 2.4: $\to$ book.
In short, use a Taylor series expansion, using the mean-value form of the remainder.

2.7 Binomial R.V's

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Binomial R.V's

Bernoulli r.v.: a binary r.v. $Y \in \{0, 1\}$ with

$$p_Y(y) = \begin{cases} p & \text{for } y = 1 \text{ "success"} \\ (1-p) & \text{for } y = 0 \text{ "failure"} \end{cases}$$

We write $Y \sim \text{Bern}(p)$. Note: $E(Y) = p \cdot 1 + (1-p) \cdot 0 = p$.

Binomial experiments: Many experiments can be described as counting the number of successes ($Y_i = 1$) in a fixed number (n) of Bernoulli trials. For example,

- Flipping n coins, and counting $X = \#$ heads;
- Treating n patients, and recording $X = \#$ of patients who respond;
- Observing change in stock price over n days, and recording $X = \#$ days it rises; etc.

All these have a common structure, and we can argue for a prob function for X.

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Binomial experiments: Binom r.v. X arise when we

- repeat a basic (Bernoulli) experiment with $Y_i \in \{0, 1\}$,
- independently, with always the same $p_{Y_i}(1) = p$ (success), i.e., $Y_i \sim \text{Bern}(p)$,
- a *fixed* number of times (n),
- and $X = \sum Y_i$ counts the number of successes.

We write $X \sim \text{Bin}(n, p)$.

To find $p_X(j)$ note

- $$\Pr\{(\underbrace{1, \ldots, 1}_{[j \text{ times}]}, \underbrace{0, \ldots, 0}_{[n-j \text{ times}]})\} = p^j (1-p)^{n-j}$$
  and same for any other sequence of j successes and $n-j$ failures.
- There are $\binom{n}{j}$ such sequences.

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Prob Function & Expectation of a Bin r.v.

Definition 2.5: Binomial r.v. $X \sim \text{Bin}(n, p)$ if

$$p_X(j) = \binom{n}{j} p^j (1-p)^{n-j}$$

Result: if $X \sim \text{Bin}(n, p)$, then $E(X) = np$

Proof: By Theorem 2.1.,

$$E(X) = \sum_{i=1}^{n} E(Y_i) = n \cdot p.$$

2.8 Examples

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Example

A family has n children with probability αp^n $n \ge 1$, where $\alpha \le (1-p)/p$.

1. Let $X = \#$ of children. i.e., $p_X(n) = \alpha p^n$ for $n \ge 1$.
What proportion of families has no children? That is, find $p_X(0)$.

Recall a geometric series $S = \sum_{n \ge 0} q^n = \frac{1}{1-q}$, for $0 < q < 1$. Then use

$$\begin{aligned} p_x(0) &= 1 - \sum_{n \ge 1} \alpha p^n = \\ &= 1 - \alpha p \sum_{\ell \ge 0} p^\ell = 1 - \frac{\alpha p}{1-p}. \end{aligned}$$

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2. If each child is equally likely to be a boy or a girl (independently of each other), what proportion of families consist of k boys (and any number of girls)? That is, letting $Y = \#$ boys, find $p_Y(k)$.

Solution: We will use:

1. Law of total probability,
   $p_Y(k) = \sum_n \Pr(Y = k \mid E_n) \Pr(E_n)$, with $E_n = \{X = n\}$.

2. Recall again $S = \sum_{n\geq 0} q^n = \frac{1}{1-q}$,

$\implies \frac{dS}{dq} = \sum_{i\geq 1} i\, q^{i-1} = (1-q)^{-2}$, and in general

$$\frac{d^\ell S}{dq^\ell} = \sum_{i\geq \ell} i(i-1)\cdots(i-\ell+1)\, q^{(i-\ell)} = \ell!\, (1-q)^{-(\ell+1)}.$$

We will this with $q = (p/2)$ and $\ell = k$.

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- For the law of total prob use $\Pr(E_n) = p_X(n)$.
- Note that for given n, $X \sim \text{Bin}(n, 1/2)$. That is, $\Pr(Y = k \mid E_n) = \binom{n}{k}(1/2)^n$ for $n \geq k$ (and 0 for $n < k$).

We get for $k \geq 1$:

$$\begin{aligned}
\Pr(Y = k) &= \sum_{n\geq k} \Pr(Y = k \mid E_n) \cdot \Pr(E_n) = \sum_{n\geq k} \binom{n}{k}(1/2)^n \cdot \alpha p^n \\
&= \frac{\alpha}{k!}\left(\frac{p}{2}\right)^k \sum_{n\geq k} n(n-1)\cdots(n-k+1)\left(\frac{p}{2}\right)^{(n-k)} \\
&= \frac{\alpha}{k!}\left(\frac{p}{2}\right)^k (k!)\left(1-\frac{p}{2}\right)^{-(k+1)} = \alpha(p/2)^k\left(1-\frac{p}{2}\right)^{-(k+1)}
\end{aligned}$$

And $p_Y(0) = 1 - \sum_{k\geq 1} p_Y(k)$.

2.9 Conditional Distribution

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Conditional Distribution

Recall the definition of conditional probabilities $\Pr(A \mid B)$.

We use conditional probabilities for $A = \{Y = y\}$ and $B = \{X = x\}$ to define a conditional distribution and expectations.

Definition: Conditional distribution. we call

$$\begin{aligned}
p_{Y|X}(Y = y \mid X = x) &= \\
&= \Pr(Y = y \mid X = x) = \frac{\Pr(Y = y, X = x)}{\Pr(X = x)} = \\
&= \frac{p_{Y,X}(y, x)}{p_X(x)}
\end{aligned}$$

the conditional distribution of Y given X.
Then $p_{Y|X}$ is a probability mass function.

"Conditional prob's are *probabilities*" $\implies$ all results for prob's apply.

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Example

For r.v.'s X and Y let $p(x, y)$ denote the joint probability function, with

$$\begin{array}{ll} p(0,0) = .4 & p(0,1) = .2 \\ p(1,0) = .1 & p(1,1) = .3 \end{array}$$

Find $p_{X|Y}(x \mid Y = 1)$.

Solution: First find $p_Y(1) = .2 + .3 = .5$, giving

$$p_{X|Y}(x \mid Y = 1) = \frac{p(x, 1)}{p_Y(1)} = \begin{cases} 2/5 = 0.4 & \text{for } x = 0 \\ 3/5 = 0.6 & \text{for } x = 1. \end{cases}$$

2.10 Conditional Expectation

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Conditional Expectation

Since cond probabilities are probabilities $\implies$ can define expectation, as before

Definition 2.6: Conditional expectation.

$$E(Y \mid Z = z) = \sum_y y\, p_{Y|Z}(y \mid z)$$

Note that $E(Y \mid Z = z)$ is a number $\in \Re$.

Example: rolling two dice: Y = number on 1st die, and X = sum of the numbers on both. Then

$$E(X \mid Y = 2) = \sum_x x \Pr(X = x \mid Y = 2) = \sum_{x=3}^{8} x \cdot \frac{1}{6} = \frac{11}{2}$$

For later reference, recall that $E(X \mid Y = 2) = 5.50$ is a number. If we were not told $Y = 2$, we would just have

$$\sum_{x=Y+1}^{Y+6} \ldots = 3.50 + Y.$$

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Lemma 2.5: the average conditional expectation = expectation,

$$E(X) = \sum_y E(X \mid Y = y)\, p_Y(y).$$

Proof: Use the law of total prob with $E_y = \{Y = y\}$ to get

$$p_X(x) = \sum_y \Pr(X = x \mid Y = y)\, p_Y(y)$$

and therefore

$$\begin{aligned}
E(X) = \sum_x x\, p_X(x) &= \sum_x \left\{\sum_y x \Pr(X = x \mid Y = y)\, p_Y(y)\right\} \\
&= \sum_y \left\{p_Y(y) \sum_x x \Pr(X = x \mid Y = y)\right\} \\
&= \sum_y p_Y(y)\, E(X \mid Y = y)
\end{aligned}$$

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Conditional expectation as a r.v.

Recall that a r.v. $Y : \Omega \to \Re$ is a real-valued function, and we use $Y = y$ to indicate a specific realization $y \in \Re$.

A second (and *different*) definition of *conditional expectation* is as a function of Y:

In $E(X \mid Y = y)$, if we remove the $= y$, we are left with a function of the r.v. Y:

Example: Recall the 2 dice, Y = 1st die, and X = sum of the two.

$$
\begin{array}{lllcccc}
E(X \mid Y = 2) & = & 5.5 & \qquad & E(X \mid Y\ \ \ ) & = & Y + \frac{7}{2} \\
 & & 5.5 & & \{\text{⚀}, \text{⚁}, \cdots, \text{⚅}\} & \mapsto & \Re \\
\text{a value} \in \Re & & & & \text{a function } \Omega \mapsto \Re \text{ (a r.v.)} & &
\end{array}
$$

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Definition 2.7: Conditional expectation. $E(X \mid Y)$ is a r.v., which takes value $E(X \mid Y = y)$ when $Y = y$.

i.e. $W = E(X \mid Y)$ is a r.v., not a value! It's a mapping $\Omega \to \Re$.

Theorem 2.7: $E(X) = E[E(X \mid Y)]$

Proof: $E(X \mid Y) = f(Y)$ with $f(Y) = f(y)$ when $Y = y$

$$
\begin{aligned}
\Longrightarrow E\left[E(X \mid Y)\right] &= E[f(Y)] = \\
&= \sum_y f(y)\, p_Y(y) \\
&= \sum_y E(X \mid Y = y)\, p_Y(y) = E(X).
\end{aligned}
$$

The last equality is by Lemma 2.5.

2.11 Examples

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Example

We repeatedly roll a fair die. Let X = # of rolls until the first ⚅ (including the ⚅ itself), and Y = # of rolls until the first ⚄.

(a) Find $p_X(x)$, $p_{X|Y}(x \mid Y = 1)$ and $p_{X|Y}(x \mid Y = 2)$.

Solution: Let $\pi = 1/6$ and $q = 1 - \pi$, and $\tilde{\pi} = 1/5$. Then

$$
\begin{aligned}
p_X(j) &= (1-\pi)^{j-1}\pi, \\
p_{X|Y}(X = j \mid Y = 1) &= (1-\pi)^{j-2}\pi, \ j = 2, \ldots, \\
p_{X|Y}(X = j \mid Y = 2) &= \begin{cases} \tilde{\pi} & j = 1 \\ (1-\tilde{\pi})(1-\pi)^{j-3}\pi, & j \geq 3 \end{cases}
\end{aligned}
$$

$$\bullet, \text{⚄}, \underbrace{\bullet, \cdots \bullet}_{(j-3)\times}, \text{⚅}$$

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(b) Find $E(X)$, $E(X \mid Y = 1)$ and $E(X \mid Y = 2)$.

Solution:

$$E(X) = \pi \sum_{j=1}^{\infty} j\, q^{\,j-1} = \pi \frac{d}{dq} \sum_{j=0}^{\infty} q^{\,j} = \frac{\pi}{(1-q)^2} = \frac{1}{\pi}$$

(see also later discussion on the geometric distribution). Next

$$E(X \mid Y = 1) = \pi \sum_{j=2}^{\infty} j\, q^{\,j-2} = \pi \sum_{\ell=1}^{\infty} (\ell + 1)\, q^{\,\ell-1} =$$

$$= \pi \sum_{\ell=1}^{\infty} \ell\, q^{\,\ell-1} + \pi \sum_{\ell=1}^{\infty} q^{\,\ell-1} = EX + 1$$

For the last equality use $\pi q^{\ell-1} = p_X(\ell)$. Similarly,

$$E(X \mid Y = 2) = \tilde{\pi} \cdot 1 + (1-\tilde{\pi}) \sum_{j=3}^{\infty} j\, q^{\,j-3}\pi =$$

$$\tilde{\pi} + (1-\tilde{\pi}) \sum_{\ell=1}^{\infty} (\ell+2) q^{\,\ell-1}\pi = \tilde{\pi} + (1-\tilde{\pi})(EX + 2)$$

2.12 Example: Recursive Function Calls

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Example 4: Recursive function calls

Setup: a function includes Y_1 recursive calls to itself. If $Y_1 \sim \text{Bin}(n, p)$, find the expected total # of calls to the function?

Calls in generation i: Let Y_i = number of calls in generation i (spawned by another call in generation $i-1$).

Let X_k = # of calls spawned by the k-th call in the $(i-1)$-st generation, $k = 1, \ldots, Y_{i-1}$.
Then also $X_k \sim \text{Bin}(n, p)$, and therefore

$$E(Y_i \mid Y_{i-1} = y_{i-1}) = \sum_{k=1}^{y_{i-1}} E(X_k \mid Y_{i-1} = y_{i-1}) = \sum_k np = y_{i-1} \cdot np$$

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Example ④: Recursive function calls

Expectation $E(Y_i)$: Using Theorem 2.7 we get

$$E(Y_i) = E\{E(Y_i \mid Y_{i-1})\} = E(Y_{i-1} np) = np\, E(Y_{i-1})$$

and therefore, starting with $Y_0 = 1$, by induction $E(Y_i) = (np)^i$.

Total # calls:

$$E\left(\sum_{i=1}^{\infty} Y_i\right) = \sum_{i=1}^{\infty} E(Y_i) = \sum_i (np)^i.$$

If $np < 1$, the expected total # calls converges; otherwise it diverges (and our program crashes ...).

2.13 Geometric Distribution

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Geometric Distribution

An important detail about binomial experiments is the fixed number of repetitions (of the binary experiment). This is violated, for example, if we flip a coin until we see a head.

Definition 2.8: Goemetric r.v. A geometric r.v. with parameter p is defined by the probability distribution

$$\Pr(X = n) = (1-p)^{n-1} p.$$

It arises, for example, if we count the # coin flips until the first head.

Lemma 2.8: Memoryless property.
$\Pr(X = n + k \mid X > k) = \Pr(X = n)$

Proof: exercise. Use $\Pr(X > k) = (1-p)^k$ (need k failures for $X > k$).

For later reference, $\Pr(X \geq k) = (1-p)^{k-1}$

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Mean of Geom(p)

Lemma 2.9: If $X > 0$ is a discrete r.v., then

$$E(X) = \sum_{i=1}^{\infty} \Pr(X \geq i)$$

Proof:

$$\begin{aligned}\sum_{i=1}^{\infty} \Pr(X \geq i) &= \sum_{i=1}^{\infty}\left(\sum_{j=i}^{\infty} \Pr(X = j)\right) = \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{j} \Pr(X = j) = \sum_j j \Pr(X = j) = E(X)\end{aligned}$$

$E(X)$: expectation of a Geom(p) r.v. X:

$$E(X) = \sum_{i=1}^{\infty} \Pr(X \geq i) = \sum_{i=1}^{\infty} (1-p)^{i-1} = \frac{1}{1-(1-p)} = 1/p.$$

2.14 Example: Coupon Collector's Problem

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Example ⑤: Coupon Collector's Problem

The following setup is encountered in many problems. The coupon collector is just the traditional story around it.

Setup: Assume each box of cereal includes a coupon, randomly chosen from n possible coupons.
How many boxes to get a complete set of all n coupons?

Let X_i = number of boxes bought while you had $i - 1$ coupons. Then $X = \sum_{i=1}^{n} X_i$ is the total number of boxes you need.

Geometric r.v: $X_i \sim \text{Geom}(p_i)$, $p_i = \frac{n-(i-1)}{n}$ (since you already have $i-1$ coupons), and therefore $E(X_i) = 1/p_i = n/(n-i+1)$.

$$E(X) = E\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \underbrace{\sum_{i=1}^{n} \frac{1}{i}}_{H(n)}.$$

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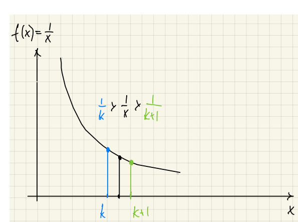
Coupon Collector (ctd.)

Lemma 2.10: $H(n) = \ln n + \Theta(1)$

Proof: noting $\frac{1}{k+1} \leq \frac{1}{x} \leq \frac{1}{k}$ for $k \leq x \leq k+1$,



$$\ln n = \int_{x=1}^{n} \frac{1}{x} dx \leq \sum_{k=1}^{n-1} \frac{1}{k} < \sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k} \leq 1 + \int_{x=1}^{n} \frac{1}{x} = 1 + \ln n$$

$\implies \ln n \leq H(n) \leq \ln n + 1.$

2.15 Example: Expected Run Time of Quicksort

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Example 6: Expected Run Time of Quicksort

Quicksort: recursively sort a list $S = \{x_1, \ldots, x_n\}$:

- If $|S| \le 1$, return S, otherwise
- Randomly select $y \in S$ ("pivot"), let $S_1 = \{x \in S : x < y\}$ and $S_2 = \{x \in S : x \ge y\}$.
- Return quicks(S_1) $\cup \{y\} \cup$ quicks(S_2).

Claim: with random y, # of comparisons is

$$2n \ln n + O(n).$$

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Proof

First some observations:

- Let $y_1, \ldots, y_n$ denote the ordered elements $x_1, \ldots, x_n$; let $X_{ij} = \begin{cases} 1 & \text{if } x_i, x_j \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$ and similarly Y_{ij}.
- Total # comparisons $X = \sum_{i<j} X_{ij} = \sum Y_{ij}$.
- Note $EX_{ij} = \Pr(X_{ij} = 1)$, and same for Y_{ij}.
- $Y_{ij} = 1 \iff y_i$ or y_j is the first pivot selected from $\underbrace{\{y_i, \ldots, y_j\}}_{j-i+1 \text{ \#'s}}$

  $\implies \Pr(Y_{ij} = 1) = \frac{2}{j-i+1}$

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We have

$$\begin{aligned} E(X) &= \sum_{i=1}^{n-1} \sum_{j>i} EX_{ij} = \sum_{i=1}^{n-1} \sum_{j>i} EY_{ij} = \\ &= \sum_{i=1}^{n-1} \sum_{j>i} \frac{2}{\underbrace{j-i+1}_{k}} = \sum_{k=2}^{n} \sum_{i=1}^{n+1-k} \frac{2}{k} = \sum_{k=2}^{n} (n+1-k)\frac{2}{k} \\ &= 2(n+1)\sum_{k=2}^{n} \frac{1}{k} - 2(n-1) = (2n+2)\underbrace{\sum_{k=1}^{n} \frac{1}{k}}_{H(n)} - 2(n-1) \end{aligned}$$

Recall $\sum_{k=1}^{n} \frac{1}{k} = H(n) = \ln n + \Theta(1) \implies EX = 2n \ln n + \Theta(n)$.

2.16 Poisson distribution

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Poisson Distribution

Note: this is §5.3 in the book.

Definition 5.1: Poisson distribution. A discrete r.v. X with

$$p_X(j) = \frac{e^{-\lambda}\lambda^j}{j!}, \quad \text{for } j = 0, 1, 2, \ldots$$

We write $X \sim \text{Poi}(\lambda)$.

General setup: Poisson probabilities are good approximations for probabilities in many problems. For example:

Poisson probabilties can be used as an approximation for Binomial probabilities with large n and small p (such that np remains moderate):

If $X \sim \text{Bin}(n, p)$ with $np = \lambda$ (or $p = \lambda/n$) and large n, then $p_X(i) \approx e^{-\lambda}\frac{\lambda^i}{i!}$, i.e., approximated by Poi(λ) probabilities.

Proof: exercise, problem 7[1]

2.17 Examples

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Examples

1. The probability of a "three of a kind" in poker is approximately $p = 1/50$.

Use the Poisson approximation to estimate the probability you will get at least one "three of a kind" if you play $n = 20$ hands.

That is, letting X = # hands with three of a kind, find $\Pr(X \ge 1)$.

Solution: use the Poi approx with $\lambda = np = 2/5$ to get

$$\Pr(X \ge 1) = 1 - p_X(0) \approx 1 - e^{-\lambda} = 1 - e^{-0.4} = 0.33.$$

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2. Let N_t = # of earthquakes in the western portion of the United States in t weeks. Assume $N_t \sim \text{Poi}(\lambda t)$, with $\lambda = 2$. That is earthquakes occur at a rate of $\lambda = 2$ per week.

(a) Find the probability of ≥ 3 earthquakes in 2 weeks.

Solution: Let $X = N_2 \sim \text{Poi}(4)$. Then

$$\Pr(X \ge 3) = 1 - p_X(0) - p_X(1) - p_X(2) = \ldots = 1 - 13e^{-4}.$$

[1] problem # reference in the video is outdated

(b) Letting Y = time until next earthquake, find $\Pr(Y \leq t)$.

Hint: Find first $\Pr(Y > t) = \Pr(N_t = 0) = \ldots$.

Solution: Use $N_t \sim \text{Poi}(\lambda t)$ to get

$$\Pr(Y > t) = p_{N_t}(0) = e^{-\lambda t} \implies \Pr(Y \leq t) = 1 - e^{-\lambda t}.$$

Note: Did you notice that the r.v. Y here is not a discrete r.v? We will talk more about this in Chapter 8.

2.18 More Properties of Poi R.V's

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More Properties of Poi R.V's

Problem 2. is an example for a (very common) application of Poisson r.v.'s to represent the number of certain events:

(1) For a *short time interval h* the probability of observing an event is proportional to the length of the interval, i.e. $\approx \lambda h$;
(2) prob of ≥ 2 events in a short interval $\rightarrow 0$ as $h \rightarrow 0$; and
(3) # of events in non-overlapping intervals are independent.

Let N_t = # events in a time interval t and assume

(1) $\Pr(N_h = 1) = \lambda h + o(h)$;

(2) $\Pr(N_h \geq 2) = o(h)$;

(3) For any n non-overlapping time intervals the numbers of events in those intervals are independent.

Under these assumptions $N_t \sim \text{Poi}(\lambda t)$.

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Sums of Poi r.v.'s

Lemma 5.2: Sums of Poi r.v.'s. If $X_i \sim \text{Poi}(\mu_i)$, $i = 1, \ldots, n$, independently, then $S = \sum_{i=1}^n X_i \sim \text{Poi}(\sum_{i=1}^n \mu_i)$.

Proof: For $n = 2 \rightarrow$ problem 9 [2]. By induction we get the desired result for $n > 2$.

[2] problem # given in the video is outdated