

AI1103-Assignment 5

Name: Vanga Aravind Shounik, Roll Number: CS20BTECH11055

Download all latex-tikz codes from

<https://github.com/AravindShounik/AI1103/blob/main/Assignment-5/assignment-5.tex>

QUESTION GATE 2005 (ME), Q.30

Consider a single server queuing model with Poisson arrivals ($\lambda = 4/\text{hour}$) and exponential service ($\mu = 4/\text{hour}$). The number in the system is restricted to a maximum of 10. The probability that a person who comes leaves without joining the queue is

SOLUTION

Let P_n represent the probability that there are n people in the queue

The queue is in the notation $(M/M/1) : (10/FIFO)$ in Kendall's notation.

Here, **M** stands for a Markovian or exponential distribution property of the model where the first M is for arrival and second M is for departure and the **1** stands for the number of servers in the model and **10** indicates the number of places in the queue and **FIFO** represents that the server obeys First in First out service at the server.

Consider the time interval $(t, t+h)$, where $h \rightarrow 0$

$$\Pr(1 \text{ arrival}) = \lambda h \quad (0.0.1)$$

$$\Pr(1 \text{ service}) = \mu h \quad (0.0.2)$$

$$\Pr(\text{no arrival}) = 1 - \lambda h \quad (0.0.3)$$

$$\Pr(\text{no service}) = 1 - \mu h \quad (0.0.4)$$

$$(0.0.5)$$

Here, we can say that

$$\begin{aligned} P_n(t+h) &= P_{n-1}(t) \times \Pr(1 \text{ arrival}) \times \Pr(\text{no service}) \\ &+ P_{n+1}(t) \times \Pr(\text{no arrival}) \times \Pr(1 \text{ service}) \\ &+ P_n(t) \times \Pr(\text{no arrival}) \times \Pr(\text{no service}) \\ &+ P_n(t) \times \Pr(1 \text{ arrival}) \times \Pr(1 \text{ service}) \quad (0.0.6) \end{aligned}$$

$$\begin{aligned} P_n(t+h) &= P_{n-1}(t)(\lambda h)(1 - \mu h) \\ &+ P_{n+1}(t)(1 - \lambda h)(\mu h) \\ &+ P_n(t)(1 - \lambda h)(1 - \mu h) \\ &+ P_n(t)(\lambda h)(\mu h) \quad (0.0.7) \end{aligned}$$

Here, we can neglect higher order terms of h

$$\begin{aligned} \Rightarrow P_n(t+h) &= P_{n-1}(t)(\lambda h) + P_{n+1}(t)(\mu h) \\ &+ P_n(t)(1 - \mu h - \lambda h) \quad (0.0.8) \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{P_n(t+h) - P_n(t)}{h} &= \lambda P_{n-1}(t) + \mu P_{n+1}(t) \\ &- P_n(t)(\lambda + \mu) \quad (0.0.9) \end{aligned}$$

At steady state, $P_n(t+h) = P_n(t)$

$$\Rightarrow (\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1} \quad (0.0.10)$$

Claim:

The P_n follows the pattern $P_n = \frac{\lambda}{\mu} P_{n-1}$

We will prove this theorem by induction. Checking the condition for $n = 1$

Now, substituting $n = 0$ in (0.0.6)

$$\begin{aligned} \Rightarrow P_0(t+h) &= P_1(\mu h)(1 - \lambda h) \\ &+ P_0(1 - \lambda h) \quad (0.0.11) \end{aligned}$$

$$\Rightarrow \frac{P_0(t+h) - P_0(t)}{h} = P_1(\mu) - P_0(\lambda) \quad (0.0.12)$$

In steady state, $P_0(t+h) = P_0(t)$

$$P_1\mu = P_0\lambda \quad (0.0.13)$$

$$P_1 = \frac{\lambda}{\mu} P_0 \quad (0.0.14)$$

So, it is proved for $n = 1$.

Inductive step:

Consider it is true for all integers less than $n + 1$.

Consider the equation (0.0.10)

$$\Rightarrow (\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1} \quad (0.0.15)$$

$$\Rightarrow (\lambda + \mu)P_n = \mu P_n + \mu P_{n+1} \quad (0.0.16)$$

$$\Rightarrow \lambda P_n = \mu P_{n+1} \quad (0.0.17)$$

$$\Rightarrow P_{n+1} = \frac{\lambda}{\mu} P_n \quad (0.0.18)$$

We can see that if it is true for n then it is true for $n + 1$. So, by mathematical induction we can say that $P_n = \frac{\lambda}{\mu} P_{n-1}$ for all n

Here, we assume $\frac{\lambda}{\mu} = \rho$ and name it as the utilisation factor. So, we can say that

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \quad (0.0.19)$$

$$P_n = \rho^n P_0 \quad (0.0.20)$$

Here, Utilisation Factor ρ in the question is

$$\rho = \frac{\lambda}{\mu} = \frac{4}{4} = 1 \quad (0.0.21)$$

Here, the maximum number of people that can stand in the line is 10.

The sum of all probabilities is 1. So, we can say that

$$\sum_{n=0}^{n=10} P_n = 1 \quad (0.0.22)$$

$$P_0 + P_1 + P_2 + \dots + P_{10} = 1 \quad (0.0.23)$$

$$P_0 + \rho P_0 + \rho^2 P_0 + \dots + \rho^{10} P_0 = 1 \quad (0.0.24)$$

$$P_0(1 + \rho + \rho^2 + \dots + \rho^{10}) = 1 \quad (0.0.25)$$

$$P_0(1 + 1 + \dots + 1) = 1 \quad (0.0.26)$$

$$P_0(11) = 1 \quad (0.0.27)$$

$$P_0 = \frac{1}{11} \quad (0.0.28)$$

Here, the person doesn't stand in queue if he comes

n	0	1	2	3	4	5	6	10
P_n	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$

TABLE 0: Probability Distribution

after the 10 people i.e, if he is the 11th person

When a person comes, if $n = 11$, then the person leaves without joining the queue.

So, P_{11} is the probability of people leaving without

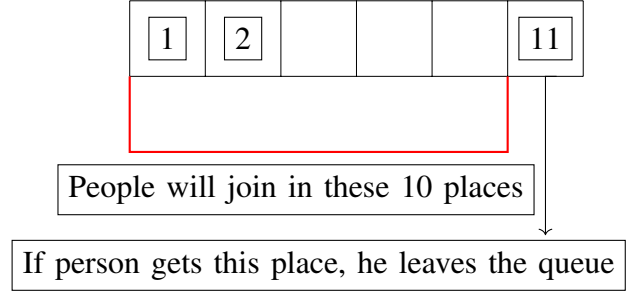


Fig. 0: Queue

joining the line.

$$P_{11} = \rho^{11} P_0 \quad (0.0.29)$$

$$P_{11} = 1^{11} P_0 \quad (0.0.30)$$

$$P_{11} = \frac{1}{11} \quad (0.0.31)$$

So, the probability that a person who comes and doesn't join the queue is $\frac{1}{11}$