AI1103-Assignment 5

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Download all latex-tikz codes from

https://github.com/AravindShounik/AI1103/blob/main/Assignment−5/assignment−5.tex

Question GATE 2005 (ME), Q.30

Consider a single server queuing model with Poisson arrivals ($\lambda = 4/hour$) and exponential service ($\mu = 4/hour$). The number in the system is restricted to a maximum of 10. The probability that a person who comes leaves without joining the queue is

Solution

Let P_n represent the probability that there are n people in the queue
The queue is in the notation $(M/M/1) : (10/FIFO)$ in Kendall's notation.
Here, **M** stands for a Markovian or exponential distribution property of the model where the first M is for arrival and second M is for departure and the **1** stands for the number of servers in the model and **10** indicates the number of places in the queue and **FIFO** represents that the server obeys First in First out service at the server.

Consider the time interval (t,t+h), where h $\to$ 0

$$\Pr(1 \text{ arrival}) = \lambda h \tag{0.0.1}$$
$$\Pr(1 \text{ service}) = \mu h \tag{0.0.2}$$
$$\Pr(\text{no arrival}) = 1 - \lambda h \tag{0.0.3}$$
$$\Pr(\text{no service}) = 1 - \mu h \tag{0.0.4}$$
$$\tag{0.0.5}$$

Here, we can say that

$$\begin{aligned} P_n(t+h) &= P_{n-1}(t) \times \Pr(1 \text{ arrival}) \times \Pr(\text{no service}) \\ &+ P_{n+1}(t) \times \Pr(\text{no arrival}) \times \Pr(1 \text{ service}) \\ &+ P_n(t) \times \Pr(\text{no arrival}) \times \Pr(\text{no service}) \\ &+ P_n(t) \times \Pr(1 \text{ arrival}) \times \Pr(1 \text{ service}) \end{aligned} \tag{0.0.6}$$

$$\begin{aligned} P_n(t+h) &= P_{n-1}(t)(\lambda h)(1-\mu h) \\ &+ P_{n+1}(t)(1-\lambda h)(\mu h) \\ &+ P_n(t)(1-\lambda h)(1-\mu h) \\ &+ P_n(t)(\lambda h)(\mu h) \end{aligned} \tag{0.0.7}$$

Here, we can neglect higher order terms of h

$$\begin{aligned} \implies P_n(t+h) &= P_{n-1}(t)(\lambda h) + P_{n+1}(t)(\mu h) \\ &+ P_n(t)(1-\mu h-\lambda h) \end{aligned} \tag{0.0.8}$$

$$\begin{aligned} \implies \frac{P_n(t+h)-P_n(t)}{h} &= \lambda P_{n-1}(t) + \mu P_{n+1}(t) \\ &- P_n(t)(\lambda+\mu) \end{aligned} \tag{0.0.9}$$

At steady state, $P_n(t+h) = P_n(t)$

$$\implies (\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1} \tag{0.0.10}$$

Claim:
The P_n follows the pattern $P_n = \frac{\lambda}{\mu}P_{n-1}$
We will prove this theorem by induction. Checking the condition for $n = 1$
Now, substituting $n = 0$ in (0.0.6)

$$\begin{aligned} \implies P_0(t+h) &= P_1(\mu h)(1-\lambda h) \\ &+ P_0(1-\lambda h) \end{aligned} \tag{0.0.11}$$

$$\implies \frac{P_0(t+h)-P_0(t)}{h} = P_1(\mu) - P_0(\lambda) \tag{0.0.12}$$

In steady state, $P_0(t+h) = P_0(t)$

$$P_1\mu = P_0\lambda \tag{0.0.13}$$
$$P_1 = \frac{\lambda}{\mu}P_0 \tag{0.0.14}$$

So, it is proved for $n = 1$.
Inductive step:
Consider it is true for all integers less than $n + 1$.

Consider the equation (0.0.10)

$$\implies (\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1} \tag{0.0.15}$$
$$\implies (\lambda + \mu)P_n = \mu P_n + \mu P_{n+1} \tag{0.0.16}$$
$$\implies \lambda P_n = \mu P_{n+1} \tag{0.0.17}$$
$$\implies P_{n+1} = \frac{\lambda}{\mu} P_n \tag{0.0.18}$$

We can see that if it is true for n then it is true for $n + 1$. So, by mathematical induction we can say that $P_n = \frac{\lambda}{\mu} P_{n-1}$ for all n
Here, we assume $\frac{\lambda}{\mu} = \rho$ and name it as the utilisation factor. So, we can say that

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \tag{0.0.19}$$
$$P_n = \rho^n P_0 \tag{0.0.20}$$

Here, Utilisation Factor ρ in the question is

$$\rho = \frac{\lambda}{\mu} = \frac{4}{4} = 1 \tag{0.0.21}$$

Here, the maximum number of people that can stand in the line is 10.
The sum of all probabilities is 1. So, we can say that

$$\sum_{n=0}^{n=10} P_n = 1 \tag{0.0.22}$$
$$P_0 + P_1 + P_2 .... + P_{10} = 1 \tag{0.0.23}$$
$$P_0 + \rho P_0 + \rho^2 P_0 ... + \rho^{10} P_0 = 1 \tag{0.0.24}$$
$$P_0(1 + \rho + \rho^2 .. + \rho^{10}) = 1 \tag{0.0.25}$$
$$P_0(1 + 1 + ..1) = 1 \tag{0.0.26}$$
$$P_0(11) = 1 \tag{0.0.27}$$
$$P_0 = \frac{1}{11} \tag{0.0.28}$$

Here, the person doesn't stand in queue if he comes

n	0	1	2	3	4	5	6	10
P_n	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$

after the 10 people i.e, if he is the 11th person
When a person comes, if $n = 11$, then the person leaves without joining the queue.
So, P_{11} is the probability of people leaving without joining the line.

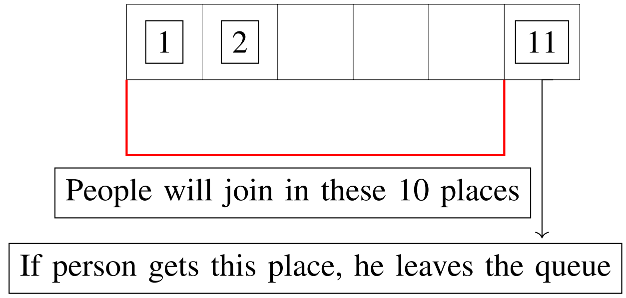

Fig. 0: Queue

$$P_{11} = \rho^{11} P_0 \tag{0.0.29}$$
$$P_{11} = 1^{11} P_0 \tag{0.0.30}$$
$$P_{11} = \frac{1}{11} \tag{0.0.31}$$

So, the probability that a person who comes and doesn't join the queue is $\frac{1}{11}$