DEGREE PROJECT IN TECHNOLOGY,
FIRST CYCLE, 15 CREDITS
STOCKHOLM, SWEDEN 2017

Pricing Financial Derivatives with the Finite Difference Method

SARGON DANHO

KTH ROYAL INSTITUTE OF TECHNOLOGY
SCHOOL OF ENGINEERING SCIENCES

Pricing Financial Derivatives with the Finite Difference Method

SARGON DANHO

Degree Projects in Applied Mathematics and Industrial Economics
Degree Programme in Industrial Engineering and Management
KTH Royal Institute of Technology year 2017
Supervisor at University of Wollongong: Dr Xiaoping Lu
Supervisors at KTH: Henrik Hult,
Examiner at KTH: Henrik Hult

TRITA-MAT-K 2017:21
ISRN-KTH/MAT/K--17/21--SE

Royal Institute of Technology
School of Engineering Sciences
KTH SCI
SE-100 44 Stockholm, Sweden
URL: www.kth.se/sci

Pricing Financial Derivatives with the Finite Difference Method

Sargon Danho

A Thesis for Bachelor of Science in Industrial Engineering and Management with Specialization in Applied Mathematics

School of Mathematics and Applied Statistics
University of Wollongong

ABSTRACT

In this thesis, important theories in financial mathematics will be explained and derived. These theories will later be used to value financial derivatives. An analytical formula for valuing European call and put option will be derived and European call options will be valued under the Black-Scholes partial differential equation using three different finite difference methods. The Crank-Nicholson method will then be used to value American call options and solve their corresponding free boundary value problem. The optimal exercise boundary can then be plotted from the solution of the free boundary value problem.

The algorithm for valuing American call options will then be further developed to solve the stock loan problem. This will be achieved by exploiting a link that exists between American call options and stock loans. The Crank-Nicholson method will be used to value stock loans and their corresponding free boundary value problem. The optimal exit boundary can then be plotted from the solution of the free boundary value problem.

The results that are obtained from the numerical calculations will finally be used to discuss how different parameters affect the valuation of American call options and the valuation of stock loans. In the end of the thesis, conclusions about the effect of the different parameters on the optimal prices will be presented.

KEYWORDS: American Call Option, Black-Scholes Equation,European Option, Finite Difference Method, Heat Equation, Optimal Exercise Boundary, Optimal Exit Boundary, Stock Loan

Prissättning av finansiella derivat med den finita differensmetoden

I det här kandidatexamensarbetet kommer fundamentala teorier inom finansiell matematik förklaras och härledas. Dessa teorier kommer lägga grunden för värderingen av finansiella derivat i detta arbete. En analytisk formel för att värdera europeiska köp- och säljoptioner kommer att härledas. Dessutom kommer europeiska köpoptioner att värderas numeriskt med tre olika finita differensmetoder. Den finita differensmetoden Crank-Nicholson kommer sedan användas för att värdera amerikanska köpoptioner och lösa det fria gränsvärdesproblemet (free boundary value problem). Den optimala omvandlingsgränsen (Optimal Exercise Boundary) kan därefter härledas från det fria gränsvärdesproblemet.
Algoritmen för att värdera amerikanska köpoptioner utökas därefter till att värdera lån med aktier som säkerhet. Detta kan åstadkommas genom att utnyttja ett samband mellan amerikanska köpoptioner med lån där aktier används som säkerhet. Den finita differensmetoden Crank-Nicholson kommer dessutom att användas för att värdera lån med aktier som säkerhet. Den optimala avyttringsgränsen (Optimal Exit Boundary) kan därefter härledas från det fria gränsvärdesproblemet.
Resultaten från de numeriska beräkningarna kommer slutligen att användas för att diskutera hur olika parametrar påverkar värderingen av amerikanska köpoptioner, samt värdering av lån med aktier som säkerhet. Avslutningsvis kommer slutsatser om effekterna av dessa parametrar att presenteras.

Nyckelord:

Amerikanska köpoptioner, Black-Scholes ekvation, europeiska optioner, finita differensmetoden, värmeledningsekvationen, optimala omvandlingsgräns, optimala avyttringsgräns, lån med aktier som säkerhet

Acknowledgements

I would like to acknowledge Dr. Xiaoping Lu for her guidance and knowledge. This would not be possible without her.

I would also like to thank the Royal Institute of Technology and the University of Wollongong for making this exchange semester possible.

Lastly, I would like to thank anyone unmentioned that have supported through the project.

Table of Contents

List of Tables . . . iii
List of Figures/Illustrations . . . vi

ABSTRACT . . . vii
Acknowledgements . . . viii
Nomenclature . . . ix

1 Introduction **1**
Purpose and Problem Formulation . . . 2
Limitations . . . 3
Methodology . . . 3
Overview . . . 3

2 Literature Review **5**
Options . . . 5
History of Option Pricing . . . 6
Hedging with Options . . . 7
Put Options . . . 7
Call Options . . . 9
Option Sensitivities - The Greek Letters . . . 10
American Options . . . 15
Marketable collateral . . . 18
Different Types of Stock Loans . . . 18
Arbitrage . . . 21
Efficient-Market Hypothesis . . . 22
Random Walk Hypothesis . . . 24

3 Mathematical Background **26**
Brownian Motion . . . 26
Heat Equation . . . 29
The Heat Kernel . . . 30
Normal Distribution . . . 30
Black-Scholes Model Derivation . . . 32
European Call Option . . . 35
European Put Option . . . 41

Put-Call Parity . . . 45
Including a Continuous Dividend Yield . . . 47

4 Finite Difference Method **49**
Implementing the Finite Difference Method . . . 49
Applying the Finite Difference Method on Option Pricing . . . 54
Explicit Method . . . 56
Implicit Method . . . 58
Crank-Nicholson Method . . . 60
Choice of Method . . . 62

5 American Options **68**
Pricing American Call Options . . . 69
Optimal Exercise Boundary . . . 70

6 Stock Loan **71**
The Connection between American Call Options and Stock Loans . . . 72
Deriving the Stock Loan Value from American Call Options . . . 73
Loan Value using American Call Options . . . 74
Optimal Exit Price . . . 75

7 Results and Discussion **76**
American Options . . . 76
Stock Loan . . . 87

8 Conclusion and Future Work **100**

A European Options **103**

B Performance Measurement for varying δS **108**

C Performance Measurement for varying δt **110**

D American Options **112**

E Optimal Exercise Boundary **115**

F Stock Loan **117**

G Optimal Exit Price **120**

References **124**

List of Tables

4.1 A comparison between the performance of the explicit method, implicit method and the Crank-Nicholson method for a European option with $K = 100$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. The computational time is the average computational time for 100 trials. Explicit: dt = 0.0001 and ds = 1. Implicit: dt = 0.001 and ds = 0.5. Crank-Nicholson: dt = 0.01 and ds = 0.5. 64

7.1 Results for American call value using the Crank-Nicholson method. The average computational times is for 100 calculations. Parameters used: $K = 100$, $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $T = 1$, $\delta t = 0.0001$, $\delta S = 0.5$ . . . 78

7.2 Optimal exercise prices for an American call option with the parameters: $K = 100$, $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$. 80

7.3 Stock loan values for different principals. The following parameters were used: $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $T = 5$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$ 88

7.4 Optimal exit prices for a stock loan with the parameters: $T = 20$, $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$ . . 89

7.5 Optimal exit prices for a stock loan with the parameters: $T = 5$, $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$ 91

List of Figures

2.1 An illustration of how the strike price and premium affects the profit or loss of a put option. . . . 8
2.2 An illustration of how the strike price and premium affects the profit or loss of a call option. . . . 10
2.3 An illustration of the variations in delta for different stock prices. The parameters for the option with the stock as an underlying asset are the following: K=100, T=1, $\sigma = 0.2$ and r=0.1 12
2.4 An illustration of how a variation in the volatility affects the delta. The constant parameters for the option are the following: K=100, T=1 and r=0.1. The volatility is $\sigma = 0$ for the first option, $\sigma = 0.3$ for the second option and $\sigma = 0.6$ for the third option. 14
2.5 A two-step binomial tree for a European option with the parameters: $\sigma = 0.2$, $r = 0.05$, $S = 30$, $K = 30$, $T = \frac{2}{12}$ and $q = AUD\ 1$. The dividend is discrete and is distributed at $t = \frac{1}{12}$. The red number indicate the stock price movement and the blue prices are the option value for that stock price at that given time. 16
2.6 A two-step binomial tree for an American option with the parameters: $\sigma = 0.2$, $r = 0.05$, $S = 30$, $K = 30$, $T = \frac{2}{12}$ and $q = AUD\ 1$. The dividend is discrete and is distributed at $t = \frac{1}{12}$. The red number indicate the stock price movement and the blue prices are the option value for that stock price at that given time. 17
2.7 An illustration of how the market-efficiency varies from weak to strong. The level of efficiency is a scale rather than being seither weak, semi-strong or strong. The red dot shows an semi-efficient market, which in most cases is the highest level of efficiency a market will have in practice. 24
2.8 An illustration of the Random Walk Hypothesis. The random arrival ofnew information will induce a random movement in the price of the asset. 25

3.1 An illustration of a normal distribution. 31

4.1 An illustration of a graph with three different points. The distance between each point is equal to h. 51

4.2 An illustration of how the different types of finite difference methods approximate a given point. A forward approximation is approximating $f_1^{'}$ with a secant line through f_1 and f_2. A backward approximation is approximating $f_3^{'}$ with a secant line through f_2 and f_3. A central approximation is approximating $f_2^{'}$ with a secant line through f_1 and f_3. 52
4.3 An illustration of the price-time mesh used for implementing the finite difference method. The x-axis is divided into i number of steps, with the distance δt between each step. The y-axis is divided into j number of steps, with the distance δS between each step. 55
4.4 Explicit method: An illustration of how the blue, known values, are used to calculate the red, unknown option value, backward in time. . . 57
4.5 Implicit method: An illustration of which points in the mesh are used for each iteration in the implicit method. Known values are indicated by a blue colour and unknown values are indicated by a red colour. . . 59
4.6 Crank-Nicholson method: An illustration of which points in the mesh are used for each iteration in the Crank-Nicholson method. Known values are indicated by a blue colour and unknown values are indicated by a red colour. 61
4.7 A comparison between the explicit method, the implicit method and the Crank-Nicholson method. The plot illustrates the accuracy for a constant $\delta S = 1$ for an increasing number of time steps. The performance is measured on a European option with $K = 100$, $S = 80$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. 65
4.8 A comparison on the accuracy between the Crank-Nicholson method and the Implicit method. The upper plot illustrated the accuracy for a constant $\delta S = 0.5$ with an increasing number of time steps. The lower plot illustrates the accuracy for a constant $\delta t = 0.01$ with an increasing number of Stock Price steps. The performance is measured on a European option with $K = 100$, $S = 80$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. 66

7.1 An illustration of the value of an American call option for varying stock prices at $t = 0$. The blue line is the value of the American option and the red line is the pay-off function. The parameters of the option are the following: $K = 100$, $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $T = 1$, $\delta t = 0.0001$, $\delta S = 0.5$. 79
7.2 An illustration of the optimal exercise boundary of an American call option with the following parameters: $K = 100$, $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$. 81
7.3 An illustration of the optimal exercise boundary of an American call option for varying volatilities. The following parameters are used: $K = 100$, $r = 0.05$, $q = 0.1$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$. The blue line have the volatility $\sigma = 0.2$, the red line $\sigma = 0.3$ and the yellow line $\sigma = 0.4$. . 82

7.4 An illustration of the optimal exercise boundary of an American call option for varying risk free interest rates. The following parameters are used: $K = 100$, $\sigma = 0.2$, $q = 0.1$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$. The blue line have the risk free interest rate $r = 0.05$, the red line $r = 0.075$ and the yellow line $r = 0.1$. 84
7.5 An illustration of the optimal exercise boundary of an American call option for varying dividend yield rates. The following parameters are used: $K = 100$, $\sigma = 0.2$, $r = 0.05$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$. The blue line have the dividend yield rate $q = 0.05$, the red line $q = 0.1$ and the yellow line $q = 0.15$. 86
7.6 An illustration of the optimal exit boundary of a stock loan with the following parameters: $T = 20$, $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$ 90
7.7 An illustration of the optimal exit boundary of a stock loan with the following parameters: $T = 5$, $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$ 92
7.8 An illustration of the optimal exit boundary of a stock loan for varying volatilities. The following parameters are used: $T = 5$, $r = 0.06$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$. The blue line have the volatility $\sigma = 0.2$, the red line $\sigma = 0.4$ and the yellow line $\sigma = 0.6$. . . . 93
7.9 An illustration of the optimal exit boundary of a stock loan for varying risk free interest rates. The following parameters are used: $T = 5$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$. The blue line have the risk fre interest rate $r = 0.06$, the red line $r = 0.1$ and the yellow line $r = 0.15$. 94
7.10 An illustration of the optimal exit boundary of a stock loan for varying dividend yield rates. The following parameters are used: $T = 5$, $\sigma = 0.4$, $r = 0.06$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$. The blue line have the dividend yield rate $q = 0$, the red line $q = 0.03$ and the yellow line $q = 0.05$. 96
7.11 An illustration of the optimal exit boundary of a stock loan for varying loan interest rates. The following parameters are used: $T = 5$, $\sigma = 0.4$, $r = 0.06$, $Q = 0.4$, $q = 0.03$, $\delta S = 0.001$, $\delta t = 0.01$. The blue line have the loan interest rate $\gamma = 0.1$, the red line $\gamma = 0.15$ and the yellow line $\gamma = 0.2$. 98

Nomenclature

S	-	Stock price
S_0	-	Current stock price
S_f	-	Optimal Price
K	-	Strike price
σ	-	Volatility
T	-	Expiration time
r	-	The risk-free interest rate
q	-	Continuous dividend yield rate
$P(S,t)$	-	Put option value at time t and stock price S
$C(S,t)$	-	Call option value at time t and stock price S
$V(S,t)$	-	Option value at time t and stock price S
i	-	Index for the time in the price-time mesh. i = 0,1,2,...,N
j	-	Index for the stock price in the price-time mesh. j = 0,1,2,...,M
δt	-	Time step in Mesh (Finite Difference Method)
δS	-	Stock price step in Mesh (Finite Difference Method)
$f_{i,j}$	-	Mesh point value (Finite Difference Method)
S_{max}	-	Maximum stock price (Finite Difference Method)
$L(S,t)$	-	Stock loan value at time t and stock price S
Q	-	Principal
γ	-	Loan interest rate
ϕ	-	Standard normal cumulative distribution
RWH	-	Random Walk Hypothesis
EMH	-	Efficient-Market Hypothesis
LTP	-	Loan-to-portfolio ratio
LTV	-	Loan-to-value ratio

Chapter 1

Introduction

Options are financial derivatives whose value depends on an underlying asset. Options are traded as the most common financial derivatives to manage risk. Risk is managed through hedging against variations in the asset price. Hedging results in a decreased exposure to risk at the expense of a reduction in potential profits. When trading with options, one is faced with several choices which will impact the hedge:

- The underlying asset.
- The time to expiration date.
- The strike price.

It is if great significance to understanding how these option settings affect the valuation in order to efficiently hedging a portfolio.

An alternative method that can be utilized to hedge a portfolio is attaining a non-recourse loan with the portfolio as collateral. The mechanisms behind this type of hedging are different from the hedging with options. This is because there exists an additional setting that will impact the possible pay-off. The loan interest rate is an additional parameter that must be considered when hedging with stock loans. Stock loans can also be utilized for other purposes than hedging. They are used to increase the liquidity or to increase the volatility of the portfolio. Understanding the importance of the effects of the stock loan settings on the valuation is important when trading with stock loans.

Purpose and Problem Formulation

The purpose of this thesis is to immerse in important theories in financial mathematics and to use these theories to value stock loans and American call options. The valuation of non-recourse loans with stocks as collateral will be achieved by exploiting a relationship that exists between stock loans and American call options. From the valuations of the stock loans and the American call options, the optimal prices will be calculated. The optimal prices will be used to analyse how variations in parameters will affect the valuation. The main questions that will be answered in this thesis are the following:

- How do changes in the volatility, the dividend yield rate and the risk free interest rate affect the optimal exercise price for an American call option?
- When is it optimal to exercise an American option?
- How do changes in the volatility, the dividend yield rate, the risk free interest and the loan interest rate affect the optimal exit price for a non-recourse loan where stocks are used as collateral?
- When is it optimal to exit a stock loan, i.e. sell the collateral and repay the loan?
- How does an increased leverage affect the growth of the stock loan value?

Limitations

This thesis will be limited to valuing American call options under the Black-Scholes equation. For the valuation of stock loans, only non-recourse loans will be considered.

Methodology

In order to fulfill the purpose of this thesis, two different methods will be used. The methods included in this thesis are:

- A literature review
- Numerical calculations

The purpose of the literature review is to present theory that will lay the foundation on which the thesis will build on.

The purpose of the numerical calculations is to obtain results from the mathematical theory that will provide a basis for discussion from which conclusions can be drawn from.

Overview

The thesis is organised as follows.

- Chapter 2 will present a review from literature in the area with the purpose of laying the theoretical foundation on which the rest of the report will build on.

- In chapter 3, important mathematical theories will be derived and explained. This chapter will lay the mathematical foundation of this thesis.
- Chapter 4 will present how the finite difference method will be applied on the mathematical problems explained in chapter 3. Furthermore, different finite difference methods will be evaluated in this chapter.
- Chapter 5 will present the American option problem and the optimal exercise boundary.
- Chapter 6 will present the connection between American call options and stock loans. The optimal exit price will also be presented in this chapter.
- Chapter 7 will present and comment on the results attained from the numerical calculations.
- Chapter 8 will conclude the thesis and provide answers to the problem formulation. Suggestions on future work will then be presented.

Chapter 2

Literature Review

Options

An option is a contract between two parties that agrees upon either selling or buying an asset at a determined strike price in the future. The buyer of an option will pay a premium to get the right to hold the contract. The price of the option depends on the underlying asset, which most commonly is either a stock, commodity, currency or an index. From a game theory point of view, options are a zero-sum game because the sum of each party's gain or loss is exactly equal [19].

Options are commonly used to eliminate risk. The idea is to buy options that have a negative correlation with the portfolio that will be hedged. The result of this is a decreased overall volatility of the portfolio. Important properties that are part of an option contract are the following [32]:

- The time to expiration: This indicates the lifetime of the option.
- Put option or call option: This indicates whether the investor wants to short or

long the underlying asset.

- The strike price: This is the price at which the option can be exercised. For a put option, this is the price at which the underlying asset can be sold. For a call option, this is the price at which the underlying asset can be purchased to.

Furthermore, there are two different option styles: European options and American options. The fundamental difference that distinguish them from one another is that the American option includes the additional right of exercising the option at any time prior to or at the time of expiration. European options lack this right as they can only be exercised at the time of expiration [17].

History of Option Pricing

The use of options can at least be traced back to 350 B.C. when Aristotle wrote down the story of a person named Miletus who made fortunes from options on the right to use olive presses [6]. However, it was not until the 1900s before the first mathematical attempt to explain options was made. Louis Bachelier introduced the important theory of Brownian motions and stated that the true value of an options is equal to the expected value of all future pay-offs [7]. Bachelier's findings were further developed by Black, Scholes and Merton with their research 1973 [9]. They created an analytical formula for valuing European options and introduced the theory of self-financing portfolios. Since then, the popularity of options has grown tremendously and options are now considered to be one of the most common financial derivatives [17].

Hedging with Options

Put Options

Put options are financial contracts that regulate the selling of an asset. The holder of the contract aims to short the underlying asset and have the right, but not the obligation, to sell the asset to the strike price on a determined future time [17]. The writer of the put option will, in contrast to the holder of the contract, long the underlying asset. An example will be explained to illustrate how put options can be exploited to minimize the risk of a portfolio. Assuming that a portfolio is worth AUD 800 and holds ten stocks, e.g. The Commonwealth Bank of Australia, with the current stock price 80 AUD. The owner of the portfolio aims to to minimize the exposure to risk and speculates that the stocks will decrease in price. Therefore, the owner of the portfolio purchases 10 put options for 1 AUD each with the strike price AUD 80. Under the assumption that the stock price on the time of expirations is AUD 75, a profit equal to $10 \cdot (80 - 75) - 10 \cdot 1 = AUD$ 40 will be made from the options and the total loss is reduced from 100 AUD to 60 AUD. However, if the underlying asset price never decreased below the strike price, then the loss would have been equal to 10 AUD. Therefore, the total loss from the hedging with options is limited to the premium, in this example 10 AUD, but the possible theoretical profit that can be made from each put option in the strike price. The profit is limited to the strike price because the underlying asset cannot be worth less than zero. The relationship between the strike price and the premium with the profit for put options is illustrated in figure 2.1.

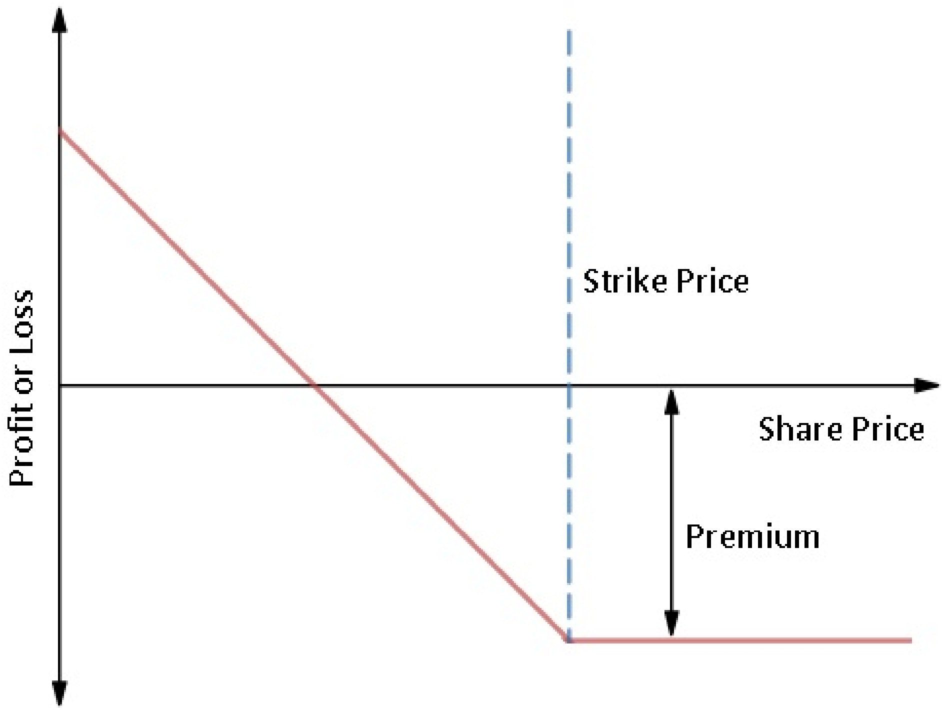

Figure 2.1: An illustration of how the strike price and premium affects the profit or loss of a put option.

The profit is equal to the pay-off minus the premium. The pay-off from a put option can be expressed algebraically as follows:

$$P(S,t) = max(K - S, 0)$$

The profit is equal to the pay-off minus the premium as follows:

$$Profit = P(S,t) - Premium$$

Call Options

Call options are financial contracts that regulate the buying of an asset. The holder of the contract aims to long the underlying asset and have the right, but not the obligation, to purchase the underlying asset to the strike price on a determined future time. Call options are used to eliminate risk, much like put options [17]. An example will be presented to illustrate how call options can be exploited to minimize the risk of a portfolio. Assuming that a portfolio is worth 50 AUD and holds 10 stocks, e.g. Qantas Airways Limited, with the current stock price 5 AUD per stock. The investor speculates that the stock will decrease in value because of increased oil prices and wants to buy a call option to hedge the portfolio. Therefore, the holder of the portfolio purchases 10 options for 1 AUD each with the commodity oil as the underlying asset to the strike price AUD 30 and the current price AUD 30. Under the assumption that the commodity price on the time of expiration had increased to AUD 32 and the stock price had decreased to AUD 3, the total value of the portfolio would have been equal to $(50 - 10 \cdot 2) + 10 \cdot (32 - 30) - 10 \cdot 1 = 40$. The loss was reduced by 10 AUD, since the portfolio would have been been worth AUD 30 if it was not hedged. However, if the underlying asset never increased above the strike price, the owner of the portfolio would have suffered a loss equal to 10 AUD. The relationship between the strike price and the premium with the profit for a call option is illustrated in figure 2.2.

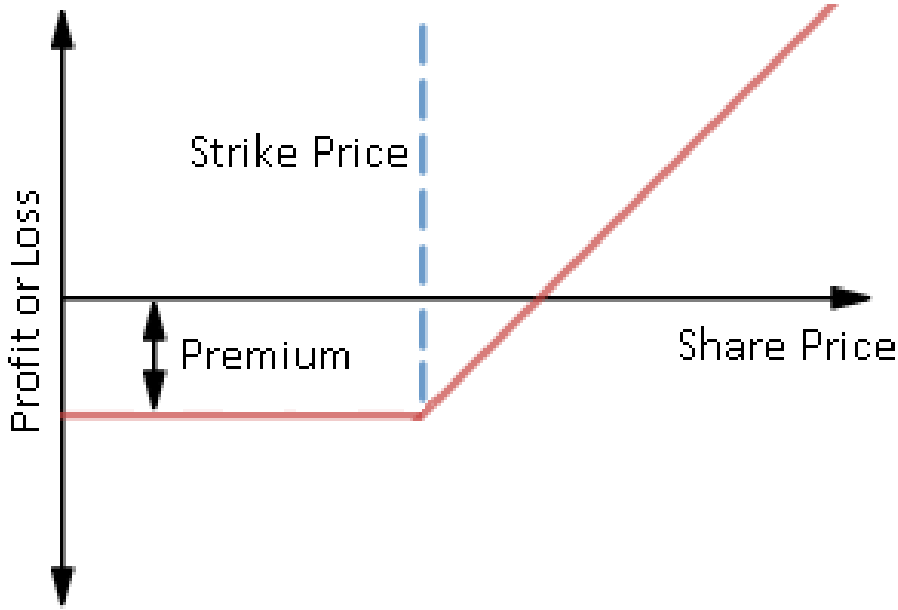

Figure 2.2: An illustration of how the strike price and premium affects the profit or loss of a call option.

The pay-off of a call option can be expressed algebraically as follows:

$$C(S,t) = max(S - K, 0)$$

The profit is equal to the pay-off minus the premium as follows:

$$Profit = C(S,t) - Premium$$

Option Sensitivities - The Greek Letters

The option sensitives, commonly referred to as the greeks or the greek letters, are different risk measurements for options. Each risk measurement is the derivative of the option value with respect to an underlying paramter [15]. The following are three common sensitives used in finance:

- The delta, $\Delta = \frac{\partial V}{\partial S}$. The delta measures the rate of change of the option value with respect to the change in the underlying asset [17]. Changes in the option value will induce a change proportional to the delta in the underlying asset.

  The delta value is important when hedging a portfolio because it tells the hedger how sensitive the position is to fluctuations in the underlying asset. If the delta is equal to zero, then changes in the underlying asset will have no effect on the option value. This is referred to as delta-neutrality. [17]. Delta-neutrality will be illustrated with the following example, where it will be assumed that a stock is currently trading to AUD 10 and an option is currently valued AUD 1. An investor writes call options on 200 shares and purchases 100 stock shares. If the stock price increases by AUD 1, a profit equal to $AUD1 \cdot 100 = AUD$ 100 would be made from the stocks. At the same time, the increase of the stock price would increase the option value by $\Delta \cdot AUD1 = AUD0.5$. Consequentely, a total loss of $0.5 \cdot 200 = AUD100$ would have been suffered from the options. Since there is a gain and a loss of $AUD100$, the total change of the position's value is equal to zero. This is a delta netural position since the changes in the underlying asset does not affect the total value of the position. However, in practice, one cannot hedge-and-forget a position. Static hedging does not work because of changes in the underlying asset price over time. Dynamic hedging is preferable in order to successfully hedge a position over a longer period of time. The hedge must be rebalanced periodically in order to sucessfuly hedge a position. [17]. Figure 2.3 illustrates how the delta value variates for different underlying asset prices.

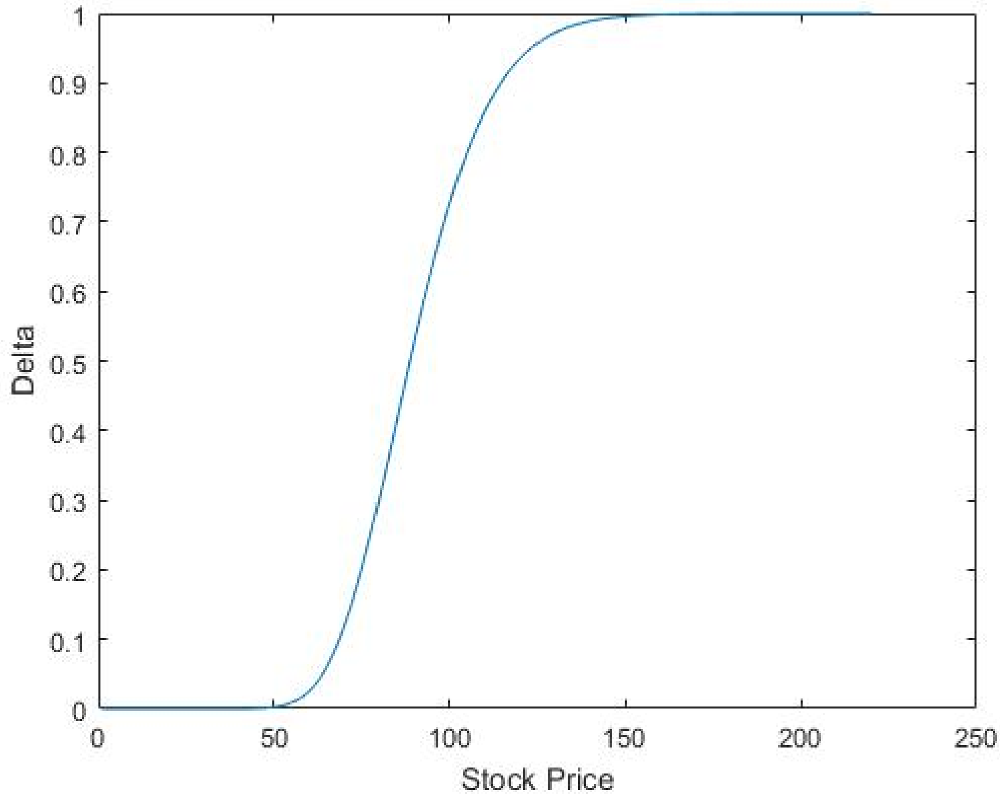

Figure 2.3: An illustration of the variations in delta for different stock prices. The parameters for the option with the stock as an underlying asset are the following: K=100, T=1, $\sigma = 0.2$ and r=0.1

As illustrated in figure 2.3, the delta is affected by changes in the underlying asset. These changes require a periodical hedge rebalancing. For European options, the delta value can be calculated with the following equation [15]:

$$Call\ option : e^{-qT}\phi(d_1)$$

$$Put\ option : e^{-qT}[1 - \phi(d_1)]$$

$$where\ d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2}(T - t)}{\sigma\sqrt{T - t}}$$

- The gamma, $\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}$. The gamma measures the changes of the Δ with respect to the underlying asset [17]. This measurement is of great importance when delta-hedging a position because the gamma can be used to protect the delta-hedge against fluctuations in the underlying asset price. Successfully protecting a hedge against fluctuations in the underlying asset will result in a better hedged delta between each periodical rebalancing. A portfolio that is perfectly protected against variations in the underlying asset price can be referred to as gamma-neutral.

A gamma and delta hedging approach assumes constant volatility, which is not the case in practice. The volatility in the underlying asset changes over time and a gamma and delta hedging approach alone would not perfectly hedge a portfolio. In figure 2.4, the effects of different volatilities on the delta are illustrated.

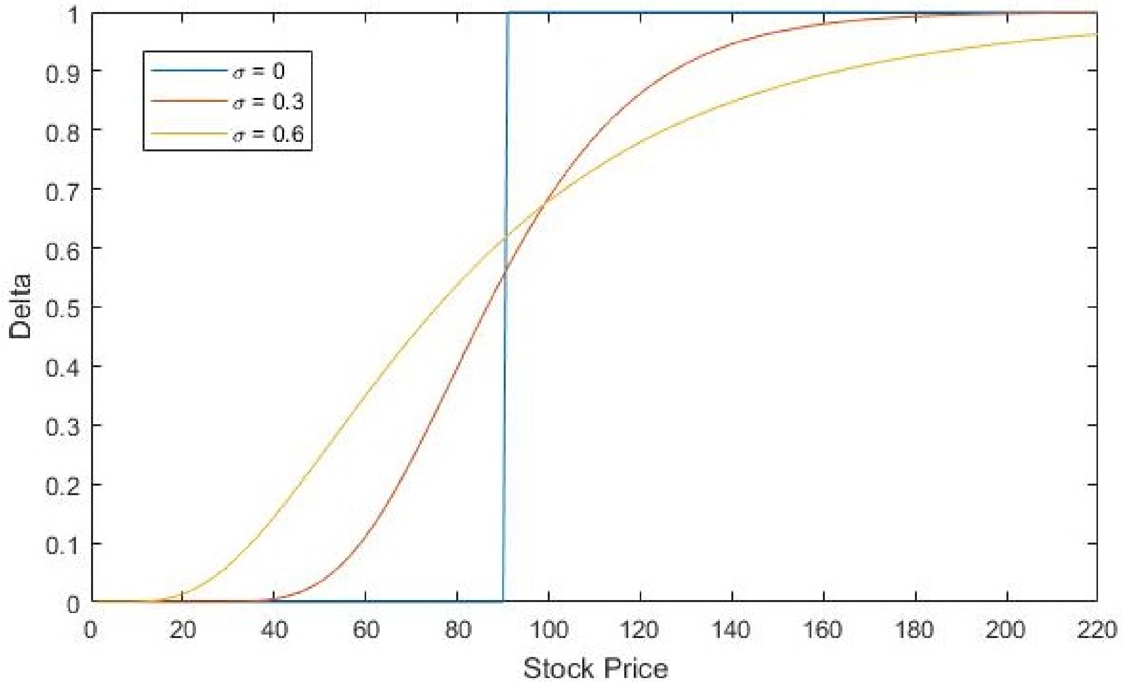

Figure 2.4: An illustration of how a variation in the volatility affects the delta. The constant parameters for the option are the following: K=100, T=1 and r=0.1. The volatility is $\sigma = 0$ for the first option, $\sigma = 0.3$ for the second option and $\sigma = 0.6$ for the third option.

As illustrated in figure 2.4, only one rebalancing needs to be done when the volatility is equal to zero. For the volatility equal to 0.3, rebalancing is needed for stock prices varying between 40 and 200. When the volatility is equal to 0.6, rebalancing is needed for a broader range of underlying asset prices. In addition, a hedge with a lower volatility will be better hedged between each periodical rebalancing compared to a hedge with a higher volatility. The strategy to hedge against the volatility is called vega hedging.

- The vega, $\nu = \frac{\partial V}{\partial \sigma}$. The vega is used to hedge against fluctuations in the volatility [17]. If the vega has a highly positive value or a highly negative value, then the position is very sensitive to fluctuations in the volatility. A vega close to zero indicates that the portfolio is insensitive to variations in the volatility.

The ideal hedging would be if all the option sensitivities were hedged neutral. However, this is normally not possible. A gamma-neutral portfolio will generally not be vega-neutral, for instance. In addition to this, there is a trade-off between transaction costs and how often one choses to rebalance a hedging position, which limits the hedger.

American Options

American options are options with an additional right for the holder of the contract. The option can be exercised at any time prior to or on the day of expiration. An American option can therefore be worth more than a European option because of this additional right. The American option can never be worth less than a European option since the American option will have the same pay-off as a European option if it is not exercised prior to the time of expiration, but the additional right of being able to exercise it early will make it possible to obtain a better pay-off in some cases [32]. Assuming that two different portfolios hold one call option each with a stock as the underlying asset. The strike price is AUD 30 and the current stock price is AUD 30. The time of expiration is in two months and a discrete dividend will be distributed after one month. Binomial trees will be used to illustrate how an American option can have a higher value than a corresponding European option. Portfolio A holds a European call option and portfolio B holds an American call option. The considered option has the following parameters: $\sigma = 0.2$, $r = 0.05$ and $q = AUD\ 1$. The following values are obtained for the up and down movements:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{\sqrt{\frac{1}{12}}}$$
$$d = \frac{1}{u} = e^{-\sqrt{\frac{1}{12}}}$$
$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{-0.05\cdot\frac{1}{12}} - e^{-\sqrt{\frac{1}{12}}}}{e^{\sqrt{\frac{1}{12}}} - e^{-\sqrt{\frac{1}{12}}}} = 0.44...$$

The binomial tree obtained for portfolio A can be seen in figure 2.5.

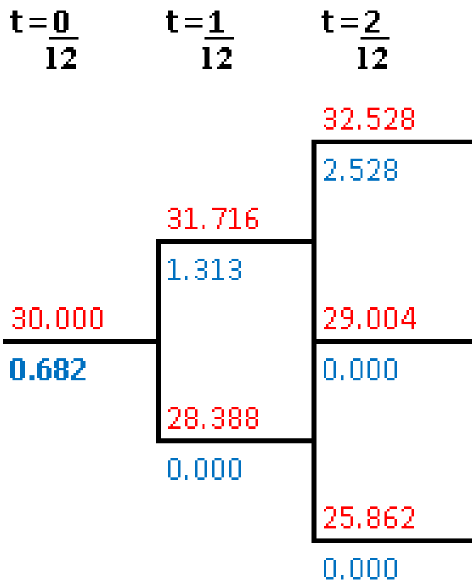

Figure 2.5: A two-step binomial tree for a European option with the parameters: $\sigma = 0.2$, $r = 0.05$, $S = 30$, $K = 30$, $T = \frac{2}{12}$ and $q = AUD\ 1$. The dividend is discrete and is distributed at $t = \frac{1}{12}$. The red number indicate the stock price movement and the blue prices are the option value for that stock price at that given time.

The binomial tree obtained for portfolio B can be seen in figure 2.6.

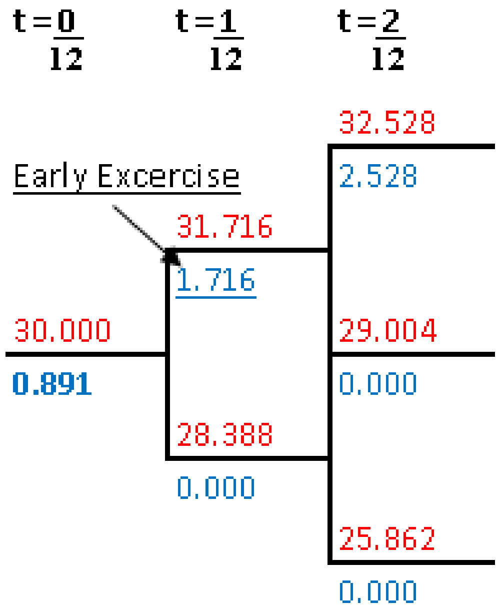

Figure 2.6: A two-step binomial tree for an American option with the parameters: $\sigma = 0.2$, $r = 0.05$, $S = 30$, $K = 30$, $T = \frac{2}{12}$ and $q = AUD\ 1$. The dividend is discrete and is distributed at $t = \frac{1}{12}$. The red number indicate the stock price movement and the blue prices are the option value for that stock price at that given time.

As illustrated, the value of the different options differ despite the fact that they both hold two options with the same settings, i.e. strike price, underlying asset and time of expiration. Portfolio B, which holds an American option, is worth 31% more than portfolio A. The early exercise right before the dividend yield is the reason why the American option has a higher value and this is an important relationship between American call options and European call options, since they will always have the same value if the underlying asset does not pay any dividends.

Marketable collateral

Marketable collateral refers to the use of financial assets as collateral for a loan. For an asset to be considered marketable, it must be sold in regulated markets at a fair market value [8]. To fulfill these requirements, the asset must have a high liquidity and the spread between buyers and sellers must be small. A loan with a financial asset as collateral is a good alternative to increase the liquidity without selling parts of one's portfolio. Henceforth, stocks will be considered as the asset used as collateral. The way stock loans work is that the lender, often a bank or a private firm, offers a loan in exchange for having custody of the collateral stocks. In addition, the loan can also include agreements on lending limits and loan-to-value ratios [22]. The purpose of this is to manage the risk for the lender. In the loan agreement, the lender may have the right to sell the stocks if changes in the price of the collateral affects the limits in the agreement.

Furthermore, there exists two types of loans which will be explained more in depth in the next section. The two types of loans are recourse loans and non-recourse loans.

Different Types of Stock Loans

Recourse Loan

This type of loan is common for home loans in Europe [23], but is also starting to appear as an alternative for loans with stocks as collateral. Recourse loans give the lender the right to collect the debt from all the borrower's assets if the collateral does not cover the loaned amount plus accumulated interest [30]. This means that the loss is not limited for the borrower. In the financial industry, this type of loan exists as a stock loan option for private investors. Avanza and Nordned are examples of brokerage firms which offer this type of loans [2] [3]. The stock loan offered by Avanza will be used to illustrate how recourse stock loans can work in practice.

To be able to use the service, one must apply for credit. Once a credit has been approved, one may use the credit for stock loans and no interest will be charged as long as the credit is not used. Interest will start to accumulate when the investor acquires stocks on credit. The stocks purchased on credit will then be considered as the collateral. The interest rate on the loan will differ depending on the risk associated with the collateral stock. Some selected stocks are eligible for lower interest rates. Furthermore, one must meet the diversification requirements and also not exceed the loan-to-portfolio ratio (LTP):

$$LTP = \frac{Borrowed\ Amount + Accumulated\ Interest}{Current\ Portfolio\ Value} \tag{2.1}$$

If the LTP limit is exceeded, a warning will be sent to the borrower who will be given a reasonable amount of time to either deposit cash or sell some assets. If the borrower fails to meet the requirements, then the brokerage firm will sell the collateral assets and make sure the loan agreement is fulfilled.

Recourse loans will not be the focus in this report and will therefore not be valued. The type of loans that will be valued and elaborated upon in this thesis are non-recourse stock loans.

Non-Recourse Loan

The main difference between non-recourse loans and recourse loans is the degree of power the lender has to collect the debt. For non-recourse loans, the lender can only seize the collateral if the borrower does not pay back the borrowed amount. The lender cannot go after any other assets to seek compensation [30], which is the case with recourse loans. Because of this, the loss is limited for the borrower for non-recourse loans.

This type of loan is riskier for the lender which causes the lender to execute more thorough assessments and have stricter requirements on the borrower. One measurement used to manage risk for this type of loan is the loan-to-value ratio (LTV) [22]:

$$LTV = \frac{Borrowed\ Amount + Accumulated\ Interest}{Value\ of\ Collateral} \tag{2.2}$$

A high LTV ratio is associated with a higher risk for the lender. For stock loans, the LTV increases if the collateral stock decreases in price. If the LTV is larger than 1, it means that the value of the collateral is smaller than the borrowed amount. In this case, the borrower might choose to surrender the stocks and cannot be held liable for returning the borrowed amount that is not covered by the collateral. In the case where the stocks increase in value, the borrower can sell the stock, repay the loan and gain a profit equal to the difference between the stock price and the principal plus the accumulated interest [20]. For loans where stocks are used as collateral, the borrower's profit can be expressed as follows:

$$S - (Qe^{\gamma t})$$

The lender will make a profit from the accumulated interest and fees.

Arbitrage

Arbitrage is a term defining the use of imbalances between different markets and a profit completely free from risk can be made [31]. There are arbitrage opportunities when an asset does not have the same price on all markets where it is traded. There is also an arbitrage opportunity if an asset with a known future price is not traded at the discounted price to the risk-free interest rate. Arbitrage can mathematically be defined as that the $(n+1)$ dimensional portfolio $\theta(t) = \theta_0(t), ..., \theta_n(T)$ and must satisfy the following conditions for all expiration times $T > 0$ [17]

$$\begin{cases} V^\theta(0) = 0 \\ V^\theta(T)) \geq 0 \\ P(V^\theta(T) > 0) > 0 \end{cases} , where\, V^\theta\, is\, the\, value\, of\, the\, portfolio\, and\, P\, denotes\, the\, probability.$$

An example of how an arbitrage opportunity can be exploited in an exchange market will now be illustrated with an example where all transaction costs and spreads between buyers and sellers are ignored:
Assuming that the currencies Australian Dollar and Swedish Kronor are currently trading at AUD 6 / SEK 1 in the currency exchange market and that a particular stock is traded to 10 AUD per stock in the Australian market and to 90 SEK per stock in the Swedish market. An arbitrage opportunity exists since an arbitrageur can borrow 10 AUD, buy one stock and sell it in the Swedish market for 90 SEK. After converting the profit back to AUD and repaying the loan, the arbitrageur will have a risk-free profit of 5 AUD.

When arbitrage is possible, arbitrageurs will always try to fully exploit the opportunity. In the example above, arbitrageurs would not limit their profit to one share. Instead the would buy as many stocks as possible to maximize their risk-free profit. The consequences that would follow from this would be that the demand for borrowing AUD and the demand on the shares in the Australian market would increase. Simultaneously, the supply of the stock in the Swedish market would increase. This would eliminate the arbitrage opportunity since the increased demand will increase the stock price in the Australian market and the increased demand on borrowing AUD would increase the price of the currency. At the same time, the increased supply would decrease the stock price in the Swedish market. The arbitrage opportunity would consequently be completely eliminated.

Since arbitrage opportunities instantly get eliminated, it is reasonable to assume that no arbitrage opportunities exist when calculating option- and stock loan values. Consequently, all calculations in this report are written under the assumption that arbitrage does not exist.

Efficient-Market Hypothesis

The efficient-market hypothesis (EMH) states that there are certain conditions which must be satisfied in order for a market to be efficient. Eugene Famas introduced the idea that stocks trade at their true value at any given time in an efficient market, which implies that it is not possible to buy disvalued assets with whom an investor can obtain excess return [14]. The following conditions must be satisfied in order for a market to be efficient according to Fama [13]:

- Arbitrage does not exist.

- All information is freely available. The market price factor in all available information. Existing information cannot be used to create excess return.
- The market value assets rationally, even though single individuals might act irrationally. The idea is that the net effect of all investors will result in a rational market. If an irrational investor creates arbitrage opportunities, then rational investors will exploit the opportunity.

Different forms of efficient markets are identified by the EMH. Markets are mainly divided into three different forms: the weak efficient market, the semi-strong efficient market and the strong efficient market [14].

The weak market is characterized by a lack of predictability. Prices only reflect past information and future prices are assumed to follow random walks with no predictable patterns. Therefore, one can assume that excess return cannot be earned using strategies based on historical data. A technical analysis is consequently useless, while a fundamental analysis can be used successfully to find disvalued stocks since all information is not effectively factored in the price.

The semi-strong efficient market factor in all publicly available information in the stock prices. Firm-specific information, as well as macro-economic information are factored in. Prices reflect historical data as well as future data. Examples of future information are annual reports, analyst reports and prognoses. New information is reflected immediately. For semi-strong efficient markets, one can assume that neither technical analysis or fundamental analysis can be utilized to obtain excess returns. Only non-available information can be used to obtain excess return.

The strong efficient market is a market where all information is available, including information considered as insider information. Due to the nature of this efficiency-form, one cannot utilize strategies to beat the market, since all stocks are traded at their true value.

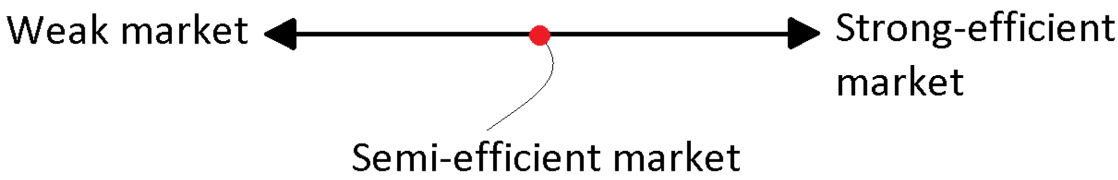

Figure 2.7: An illustration of how the market-efficiency varies from weak to strong. The level of efficiency is a scale rather than being seither weak, semi-strong or strong. The red dot shows an semi-efficient market, which in most cases is the highest level of efficiency a market will have in practice.

Different markets vary in efficiency, but one can assume that a market can be semi-strong at best [5], e.g. due to legislation around insider information. Additionally, the variation of efficiency can be seen as a scale rather than as being either strong, semi-strong or weak. Large cap markets tend to be closer to semi-strong in figure 2.7 while Small Cap markets tend to be weaker. Therefore, in real life, it is possible to obtain excess return. Weaker markets offer more opportunities to find disvalued stocks. In this thesis, all calculations will be carried on under the assumptions of markets being strongly-efficient.

Random Walk Hypothesis

The Random Walk Hypothesis (RWH) is a theory that suggests that stocks take random path. According to the RWH, prices of assets will change randomly as new information arrives randomly [7] [26]. This is illustrated in figure 2.8.

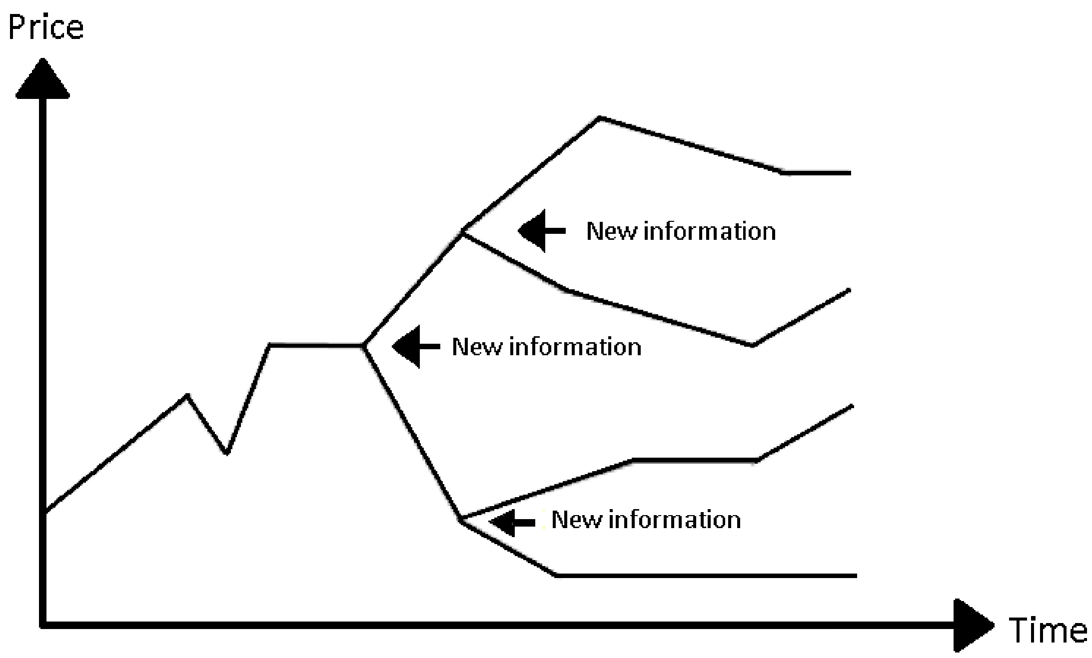

Figure 2.8: An illustration of the Random Walk Hypothesis. The random arrival ofnew information will induce a random movement in the price of the asset.

The RWH is connected to the efficient-market hypothesis [29], since it assumes that assets are correctly priced and only the random arrival of information will affect the price. A random walk is a discrete-time process but have been approached mathematically in continuous time with Brownian motions. Brownian motions are commonly referred to as continuous-time random walks and will be further explained in the next chapter.

Chapter 3

Mathematical Background

Brownian Motion

A Brownian motion, or a continuous-time random walk, is a stochastic process that originally was used to explain the random walk of particles in fluids. The idea of Brownian motions was first introduced by Robert Brown in 1827 [27], but was not completely explained until 1905 by Albert Einstein [1]. It was not until 1965 that the first serious attempts to apply Brownian motions to the field of financial mathematics were made [25]. These attempts were made by Paul Samuelson with his studies of Geometric Brownian Motions that described log-normally distributed returns from assets. This is also called a wiener process, which is a special case of Markov stochastic processes. The first property that must be fulfilled in order for a variable z to follow a wiener process is the following [17]:

$$\Delta z = \phi\sqrt{\Delta t}, \ \textit{where } \phi \textit{ is a standard normal distribution.} \tag{3.1}$$

Furthermore, the following property must be fulfilled [17]:

$$\Delta z \ at \ different \ time \ steps \ \Delta t \ are \ independent.$$

The first property indicates that Δz is normally distributed with $E[\Delta z] = 0$ and $Var[\Delta z] = (\sqrt{\Delta z})^2 = \Delta z$. The second property indicates that z has Markovian properties.

The change of the mean for each time step is called the drift rate and the variance is called the variance rate [17]. In this case, the drift rate is equal to zero and the variance rate is equal to one. When the drift rate is equal to zero, means that any expected future value of z is equal to the current value. When the variance rate is equal to 1, it means that for a time interval T, the change in z will be equal to T. From this, a generalized Wiener process can be expressed for x in terms of dz:

$$dx = \alpha dt + \beta dz \tag{3.2}$$

The variables α and β are constants. The first term indicates the expected drift rate for each time step and the second term adds variability to the path of the process. Considering a small time step Δt and combining equation 3.2 with equation 3.1 will give the following equation:

$$\Delta x = \alpha \Delta t + \beta \phi \sqrt{\Delta t} \tag{3.3}$$

For this equation, the normal distribution ϕ has the mean $E[\Delta x] = a\Delta t$ and the variance $Var[\Delta x] = b^2 \Delta t$. From this, it follows that the expected drift rate is equal to α for each time step and the added variability to the path is equal to β^2 for each time step. To apply this on stock prices, one cannot assume that the drift rate and the variance rate are constants.

A formula for the stock price movement with the assumption of no noise can be obtained by considering equation 3.3 with the noise removed, i.e. set $dz = 0$. Considering a short interval of time Δt under the assumption that the expected change in the stock price ΔS is equal to $\mu S \Delta t$, one obtains the following equation:

$$\begin{aligned} &\Delta S = \mu S \Delta t \\ &As\ \Delta t \to 0,\ dS = \mu S dt \Leftrightarrow \frac{dS}{S} = \mu dt \end{aligned} \tag{3.4}$$

Equation 3.4 will be integrated to obtain a formula for the stock price under the assumption that variability or noise is non-existent:

$$\begin{aligned} &\int_{S_0}^{S_T} \frac{dS}{S} = \int_0^T \mu dt \\ &\Rightarrow [\ln S]_{S_0}^{S_T} = [\mu t]_0^T \\ &\Rightarrow \ln S_T - \ln S_0 = \mu T \\ &\Rightarrow \ln \frac{S_T}{S_0} = \mu T \\ &\Rightarrow \frac{S_T}{S_0} = e^{\mu T} \\ &\Rightarrow S_T = S_0 e^{\mu T},\ where\ S_T \geq 0\ and\ S_0 \geq 0 \end{aligned} \tag{3.5}$$

In other words, if there exists no uncertainties, the stock at time T would grow with the factor $e^{\mu T}$ from the current price S_0. This formula is valid for risk-free assets, but iis not valid for other assets as noise or variability must be included. The variable σ which denotes the standard deviation will be introduced under the assumption that the variability is the same regardless of the stock price. From this it follows that the standard deviation σ should be proportional to the stock price. The following equation is obtained when combining this with equation 3.3 and equation 3.4:

$$dS = \mu S dt + \sigma S dz \tag{3.6}$$

Equation 3.6 is known as the geometric Brownian motion and is the most common model for stock price behaviour. The standard deviation σ is commonly referred to as the volatility and μ as the expected rate of return for the stock. From a risk-neutral approach, μ is equal to the risk-free interest rate r. [17]

Heat Equation

The heat equation models the diffusion of heat in a continuous medium [18]. This model is one of the most successfully implemented models in applied mathematics. The heat equation have the following important features [32]:

- The heat equation is a linear equation.
- The heat equation is a second order partial differential equation.
- The heat equation is a parabolic equation and changes made at a particular point will therefore have an instantaneous effect on everywhere else in the system.

The homogeneous heat equation is defined as follows [32]:

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} \tag{3.7}$$

The heat equation is in forward time and in this thesis, it will be used to solve the Black-Scholes partial differential equation. The Black-Scholes partial differential equation will be reduced into the heat equation and then solved using its fundamental solution, which will transform the problem into an integral.

The Heat Kernel

The heat kernel is a fundamental solution for a homogeneous heat equation with a given initial data in some point. In this thesis, the heat kernel will be used to solve the heat equation in order to obtain an analytical solution of the Black-Scholes partial differential equation. The fundamental solution is defined as follows [11]:

$$\begin{gathered} u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)dy \\ \begin{cases} \frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} \\ u(x,0) = g(x) \end{cases} \end{gathered} \tag{3.8}$$

Normal Distribution

The normal distribution function is an important distribution in probability theory and is commonly used for modelling asset returns. The normal distribution is used in the Black-Scholes model to value European options. The normal distribution depends on two parameters [28]:

- The mean, $\mu \in \boldsymbol{R}$, which is the expectation of the normal distribution.
- The variance, $\sigma^2 > 0$, which measures the magnitude of the spread from the mean.

In the Black-Scholes formula, it is the normal cumulative distribution that is used. The cumulative distribution, usually denoted as $\phi(X)$, is the probability that X will be equal to or less than x, expressed as $F_X(x) = P(X \leq x)$. The normal cumulative distribution function is defined with the following equation [28]

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \tag{3.9}$$

The normal distribution is a symmetric distribution, which means that it is mirrored around a vertical line of symmetry. This is clearer if one look at the illustration of a normal distribution in figure 3.1.

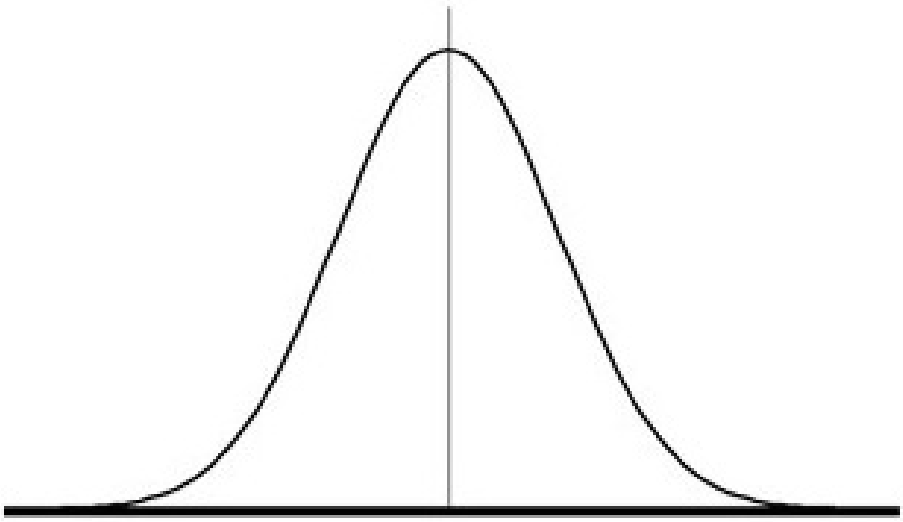

Figure 3.1: An illustration of a normal distribution.

From this it follows that there is a relationship between any given points with the same distance to the vertical line. This relationship is defined with the following equation [28]:

$$\phi(x) = 1 - \phi(-x) \tag{3.10}$$

Black-Scholes Model Derivation

The Black-Scholes partial differential equation will be used to derive an analytical formula for European call, and put options. The following assumptions are made when deriving the Black-Scholes equation:

- The asset price follows a geometric random walk.
- The risk free interest rate and the volatility are known during the lifetime of the option.
- There are no transaction costs.
- There are no arbitrage.

From the first assumption, it follows that the asset price follows the model from equation 3.6:

$$dS = \sigma S dz + \mu S dt$$

Since it is known that dS is a number, then it follows that:

$$dS^2 = (\sigma S dz + \mu S dt)^2 = \sigma^2 S^2 dz^2 + 2\sigma\mu S^2 dt dz + \mu^2 S^2 dt^2 \tag{3.11}$$

The following convergence is true [32]:

$$dz^2 \to dt \; as \; dt \to 0 \tag{3.12}$$

From equation 3.12, it follows that:

$$dS^2 = \sigma^2 S^2 dt \ when \ dt \rightarrow 0 \tag{3.13}$$

Assuming that V(S,t) is a function of the asset price and the time. If the asset price S is varied by a small change dS, then the function V will also change by a small amount. Using Taylor series expansions, this can be expressed as:

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2} \cdot \frac{\partial^2 V}{\partial S^2} \delta S^2 + ..., \tag{3.14}$$

Combining equation 3.14 with equation 3.6 and equation 3.11, one obtains the following equation:

$$dV = \sigma S \frac{\partial V}{\partial S} dz + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) \tag{3.15}$$

This is Ito's lemma as a small change in the random variable relates to the small change in the function [32].

The value of one portfolio consisting of one stock can be expressed with the function $V(S, t)$ [17]:

$$\Pi = V - \Delta S \tag{3.16}$$

For each time step, the change in the value of the portfolio can be expressed with the following equation:

$$d\Pi = dV - \Delta dS \tag{3.17}$$

Inserting the value for dS from equation 3.6 and the value dV from equation 3.15 into equation 3.17 gives the following equation:

$$d\Pi = \sigma S(\frac{\partial V}{\partial S} - \Delta)dz + (\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S)dt \tag{3.18}$$

To eliminate eliminate the random component $\sigma S(\frac{\partial V}{\partial S} - \Delta)$, the delta will be set to:

$$\Delta = \frac{\partial V}{\partial S} \tag{3.19}$$

This results in the following equation:

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt \tag{3.20}$$

The return on the invested amount Π must grow to the risk free interest rate in order to eliminate arbitrage. Consequently, the following equality must be true $d\Pi = r\Pi dt$ for each time step. From this, combined with equation 3.20, equation 3.19 and equation 3.16, the following equation is obtained:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3.21}$$

This partial differential equation is known as the Black-Scholes equation [17].

European Call Option

The Black-Scholes partial differential equation for a European call option with Value C(S,t) is defined with the following equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{3.22}$$

The Black-Scholes partial differential equation [equation 3.22] has the following boundary and final conditions for a call option:

$$\begin{aligned} &C(0,t) = 0 \\ &C(S,t) = S \ when \ S \to \infty \\ &C(S,T) = max(S-K,0) \end{aligned} \tag{3.23}$$

Equation 3.22 and 3.23 will be combined to obtain an analytical solution for valuing European call options. The first step to solving the Black-Scholes partial differential equation is to classify the differential equation in order to be able to choose an efficient solution strategy. The following general formula for the classification of second order partial differential equations will be used for this purpose [28]:

$$\begin{aligned} &au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0 \\ &b^2 - 4ac < 0 \to elliptic \\ &b^2 - 4ac = 0 \to parabolic \\ &b^2 - 4ac > 0 \to hyperbolic \end{aligned} \tag{3.24}$$

The following values from the Black-Scholes partial differential equation can now be identified for the constants a, b and c in Equation 3.24:

$$a = \frac{1}{2}\sigma^2 S^2$$

$$b = 0$$

$$c = 0$$

This results in $b^2 - 4ac = 0$, which implies that the partial differential equation can be classified as parabolic. An efficient strategy for solving parabolic partial differential equations is to reduce the differential equation into the heat equation. The first step of this reduction is to make a substitution of variables with the purpose of reducing the Black-Scholes partial differential equation into the heat equation. The Black-Scholes differential equation is in backward time with the initial data given at t=T. This must be reversed. Furthermore, the S^2 term and the S term must be eliminated. To achieve that, a function whose first derivative contains $\frac{1}{S}$ and whose second derivative contains $\frac{1}{S^2}$ is needed. The following new variables are introduced:

$$\begin{cases} t = T - \frac{\tau}{\frac{1}{2}\sigma^2} \\ S = Ke^x \\ C(S,t) = Kv(x,\tau) \end{cases} \rightarrow \begin{cases} \tau = \frac{\sigma^2}{2}(T-t) \\ x = \ln\frac{S}{K} \\ v(x,\tau) = \frac{1}{K}C(S,t) \end{cases} \tag{3.25}$$

The following derivatives can now be calculated from the change of variables from equation 3.25:

$$\frac{\partial C}{\partial t} = K\frac{\partial v}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = -K\frac{\partial v}{\partial \tau} \cdot \frac{\sigma^2}{2}$$

$$\frac{\partial C}{\partial S} = K\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x} \cdot \frac{1}{S} \tag{3.26}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}(\frac{\partial C}{\partial S}) = \frac{\partial}{\partial S}(K\frac{\partial v}{\partial x}\frac{1}{S}) = -K\frac{\partial v}{x} \cdot \frac{1}{S^2} + K\frac{\partial}{\partial S}(\frac{\partial v}{\partial x})\frac{1}{S} =$$

$$= -K\frac{\partial v}{x} \cdot \frac{1}{S^2} + K\frac{\partial}{\partial x}(\frac{\partial v}{\partial x})\frac{\partial x}{\partial v}\frac{1}{S} = -K\frac{\partial v}{x} \cdot \frac{1}{S^2} + K\frac{\partial^2 v}{\partial x^2} \cdot \frac{1}{S^2}$$

The terminal condition can now be calculated for the function $v(x,\tau)$ from equation 3.25:

$$v(x,0) = \frac{1}{K}C(S,T) = \frac{1}{K}max(Ke^x - K, 0) = max(e^x - 1, 0) \tag{3.27}$$

The derivatives from equation 3.26 are then substituted into the Black-Scholes partial differential equation [equation 3.22]:

$$-K\frac{\partial v}{\partial \tau} \cdot \frac{\sigma^2}{2} + \frac{\sigma^2}{2}S^2(-K\frac{\partial v}{\partial x} \cdot \frac{1}{S^2} + K\frac{\partial^2}{\partial x^2} \cdot \frac{1}{S}) + rS(K\frac{\partial v}{x} \cdot \frac{1}{S}) - rKv = 0 \tag{3.28}$$

Terms are cancelled in equation 3.28 and the common factor K is factorized using the distributive law backwards. This results in the following:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv,\ where\ k = \frac{r}{\frac{\sigma^2}{2}} \tag{3.29}$$

The first order derivative and the zero-order derivative must be eliminated in equation 3.29 to fully reduce the Black-Scholes partial differential equation into the heat equation. This will be done using an ansatz for the solution to the function v:

$$\begin{aligned}
&Ansatz: v = e^{\alpha x+\beta\tau}u(x,\tau) \\
&v_\tau = \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}u_\tau \\
&v_x = \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}u_x \\
&v_{xx} = \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}u_x + e^{\alpha x+\beta\tau}u_{xx}
\end{aligned} \tag{3.30}$$

The ansatz and its derivatives from equation 3.30 are substituted into equation 3.29. The following is then obtained:

$$\begin{aligned}
&\beta u + u_\tau = \alpha^2 u + 2\alpha u_x + u_{xx} + (k-1)(\alpha u + u_x) - kv \\
&\rightarrow u_\tau = u_{xx} + u_x(2\alpha + (k-1)) + u(\alpha + (k-1)\alpha - k - \beta)
\end{aligned} \tag{3.31}$$

The constants α and β are set to $\alpha = -\frac{k-1}{2}$ and $\beta = -\frac{(k+1)^2}{4}$ in order to eliminate the unwanted terms. The full reduction of the Black-Scholes partial differential equation into the heat equation is obtained when inserting the constants into equation 3.31 and the terminal condition is obtained from the ansatz in equation 3.30:

$$\begin{aligned}
&u_t = u_{xx} \\
&u(x,0) = e^{-\alpha x}v = e^{x\frac{k-1}{2}}max(e^x - 1, 0)
\end{aligned} \tag{3.32}$$

At this point, the Black-Scholes partial differential equation is fully reduced into the heat equation, and the heat kernel fundamental solution can be used to solve the heat equation. When combining equation 3.32 with the fundamental solution [equation

3.8], the following integral is obtained:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds$$

The following substitution of variable is made to solve the integral:

$$z = \frac{(s-x)}{\sqrt{2\tau}} \rightarrow dz = -\frac{1}{\sqrt{w\tau}} dx$$

The following is obtained from the change of variables:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(z\sqrt{2\tau}+x) e^{-\frac{z^2}{2}} dz$$

The following is obtained from equation 3.32:

$$z > -\frac{x}{\sqrt{2\tau}} \; for \; u_0 > 0$$
$$\rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz = I_1 - I_2$$

The integrals I_1 and I_2 will be solved separately.

The terms containing z will be separated from the terms which do not contain z in the exponents:

$$I_1 : \frac{k+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2} = -\frac{1}{2}(z^2 - \sqrt{2\tau}(k+1)z) + \frac{k+1}{2}x =$$
$$= -\frac{1}{2}(z^2 - \sqrt{2\tau}(k+1)z + \tau\frac{(k+1)^2}{2}) + \frac{k+1}{2}x + \tau\frac{(k+1)^2}{4} =$$
$$= -\frac{1}{2}(z - \sqrt{\frac{\tau}{2}}(k+1))^2 + \frac{k+1}{2}x + \frac{(k+1)^2}{4}\tau$$

$$\rightarrow I_1 = \frac{e^{\frac{(k+1)x}{2}+\frac{(k+1)^2\tau}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(z-\sqrt{\frac{\tau}{2}}(k+1)^2)} dz$$

$$\begin{aligned} I_2 : \frac{k-1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2} &= -\frac{1}{2}(z^2-\sqrt{2\tau}(k-1)z) + \frac{k-1}{2}x = \\ = -\frac{1}{2}(z^2-\sqrt{2\tau}(k-1)z+\tau\frac{(k-1)^2}{2}) &+ \frac{k-1}{2}x + \tau\frac{(k-1)^2}{4} = \\ = -\frac{1}{2}(z-\sqrt{\frac{\tau}{2}}(k-1))^2 + \frac{k-1}{2}x &+ \frac{(k-1)^2}{4}\tau \end{aligned}$$

$$\rightarrow I_2 = \frac{e^{\frac{(k-1)x}{2}+\frac{(k-1)^2\tau}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(z-\sqrt{\frac{\tau}{2}}(k-1)^2)} dz$$

Another change of variable is made to simplify the equations:

$$I_1 : y = z - \sqrt{\frac{\tau}{2}}(k+1) \qquad \partial y = \partial z$$

$$\rightarrow I_1 = \frac{e^{\frac{(k+1)x}{2}+\frac{(k+1)^2\tau}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}/k+1)}^{\infty} e^{-\frac{y^2}{2}} dz$$

$$I_2 : y = z - \sqrt{\frac{\tau}{2}}(k-1) \qquad \partial y = \partial z$$

$$\rightarrow I_2 = \frac{e^{\frac{(k-1)x}{2}+\frac{(k-1)^2\tau}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}-\sqrt{\frac{\tau}{2}}/k-1)}^{\infty} e^{-\frac{y^2}{2}} dz$$

The equations can now be rewritten using the cumulative distribution function of a normal distribution [equation 3.9]. When doing so, the following solutions for I_1 and

I_2 are obtained:

$$I_1 = e^{\frac{(k+1)x}{2}+\frac{(k+1)^2\tau}{4}}\phi(d_1),\ where\ d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$$
$$I_2 = e^{\frac{(k-1)x}{2}+\frac{(k-1)^2\tau}{4}}\phi(d_2),\ where\ d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k-1)$$

The variables will now be changed back to their original variables:

$$\tau = \frac{\sigma^2}{2}(T-t)$$
$$x = ln\frac{S}{K}$$
$$v(x,\tau) = \frac{1}{K}C(S,t)$$
$$C(S,t) = Kv(x,\tau)$$

The following analytical solution is obtained for the value of a call option:

$$C(S,t) = S\phi(d_1) - Ke^{-r(T-t)}\phi(d_2)$$
$$where,$$
$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} = d_1 - S\sqrt{T-t} \tag{3.33}$$

European Put Option

The Black-Scholes partial differential equation for a European put option with value P(S,t) is defined with the following equation:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \tag{3.34}$$

The Black-Scholes partial differential equation [equation 3.34] has the following boundary and final conditions for a put option:

$$\begin{aligned}
&P(0,t) = Ke^{-r(T-t)} \\
&P(S,t) = 0 \ when \ S \to \infty \\
&P(S,T) = max(K-S,0)
\end{aligned} \tag{3.35}$$

The strategy for solving this partial differential equation is similar to the solution strategy of the call option since the only difference is the boundary conditions. Therefore, a reduction of the equation into the heat equation will be an efficient strategy. The same changes of variables as for the solution for the call option are made and the same differential equations are obtained:

$$\begin{cases} t = T - \frac{\tau}{\frac{1}{2}\sigma^2} \\ S = Ke^x \\ P(S,t) = Kv(x,\tau) \end{cases} \to \begin{cases} \tau = \frac{\sigma^2}{2}(T-t) \\ x = ln\frac{S}{K} \\ v(x,\tau) = \frac{1}{K}P(S,t) \end{cases}$$

$$\frac{\partial P}{\partial t} = K\frac{\partial v}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = -K\frac{\partial v}{\partial \tau} \cdot \frac{\sigma^2}{2}$$

$$\frac{\partial P}{\partial S} = K\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial S} = K\frac{\partial v}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 P}{\partial S^2} = \frac{\partial}{\partial S}(\frac{\partial P}{\partial S}) = \frac{\partial}{\partial S}(K\frac{\partial v}{\partial x} \cdot \frac{1}{S}) = -K\frac{\partial v}{x} \cdot \frac{1}{S^2} + K\frac{\partial}{\partial S}(\frac{\partial v}{\partial x})\frac{1}{S} =$$

$$= -K\frac{\partial v}{x} \cdot \frac{1}{S^2} + K\frac{\partial}{\partial x}(\frac{\partial v}{\partial x})\frac{\partial x}{\partial v}\frac{1}{S} = -K\frac{\partial v}{x} \cdot \frac{1}{S^2} + K\frac{\partial^2 v}{\partial x^2} \cdot \frac{1}{S^2}$$

The terminal condition for a put option is different from the terminal condition for a call option because of the difference in the boundary and final conditions [equation 3.35]. From the boundary and final conditions for a put option one obtains the following terminal condition:

$$v(x,0) = \frac{1}{K}P(S,T) = \frac{1}{K}max(K - Ke^x, 0) = max(1 - e^x, 0) \tag{3.36}$$

The following is obtained when substituting the derivatives into the Black-Scholes partial differential equation [equation 3.34]:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \ where\ k = \frac{r}{\frac{\sigma^2}{2}} \tag{3.37}$$

To fully reduce the Black-Scholes partial differential equation into the heat equation, the same ansatz as before is used. This results in the following equations:

$$\begin{aligned} &Ansatz: v = e^{\alpha x + \beta \tau}u(x,\tau) \\ &v_\tau = \beta e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}u_\tau \\ &v_x = \alpha e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}u_x \\ &v_{xx} = \alpha^2 e^{\alpha x + \beta \tau}u + 2\alpha e^{\alpha x + \beta \tau}u_x + e^{\alpha x + \beta \tau}u_{xx} \end{aligned} \tag{3.38}$$

The ansatz and its derivatives from equation 3.38 are substituted into equation 3.37. The following is then obtained:

$$\begin{aligned} &\beta u + u_\tau = \alpha^2 u + 2\alpha u_x + u_{xx} + (k-1)(\alpha u + u_x) - kv \\ &\rightarrow u_\tau = u_{xx} + u_x(2\alpha + (k-1)) + u(\alpha + (k-1)\alpha - k - \beta) \end{aligned} \tag{3.39}$$

Choosing α and β in the same manner as before reduces the Black-Scholes partial differential equation for put options into the heat equation:

$$\begin{aligned} &u_t = u_{xx} \\ &u(x,0) = e^{\alpha x} v = e^{x\frac{k-1}{2}} max(1-e^x, 0) \end{aligned} \tag{3.40}$$

The following is obtained when applying the fundamental solution [equation 3.8]:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz = I_1 - I_2$$

$$\textit{The integrals } I_1 \textit{ and } I_2 \textit{ will be solved separately.}$$

Separating the terms containing z from the terms which do not contain z gives the following integrals:

$$I_1 = \frac{e^{\frac{(k-1)x}{2} + \frac{(k-1)^2\tau}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x}{\sqrt{2\tau}}} e^{-\frac{1}{2}(z-\sqrt{\frac{\tau}{2}}(k-1)^2)} dz$$

$$I_2 = \frac{e^{\frac{(k+1)x}{2} + \frac{(k+1)^2\tau}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x}{\sqrt{2\tau}}} e^{-\frac{1}{2}(z-\sqrt{\frac{\tau}{2}}(k+1)^2)} dz$$

Another change of variables is made:

$$\begin{aligned} &I_1 : y = z - \sqrt{\frac{\tau}{2}}(k-1) \qquad \partial y = \partial z \\ &\to I_1 = \frac{e^{\frac{(k-1)x}{2} + \frac{(k-1)^2\tau}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x}{\sqrt{2\tau}} - \sqrt{\frac{\tau}{2}}/k-1)} e^{-\frac{y^2}{2}} dz \end{aligned}$$

$$\begin{aligned} &I_2 : y = z - \sqrt{\frac{\tau}{2}}(k+1) \qquad \partial y = \partial z \\ &\to I_2 = \frac{e^{\frac{(k+1)x}{2} + \frac{(k+1)^2\tau}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x}{\sqrt{2\tau}} - \sqrt{\frac{\tau}{2}}/k+1)} e^{-\frac{y^2}{2}} dz \end{aligned}$$

The integrals can now be rewritten by using the cumulative distribution function of a normal distribution [equation 3.9]. The following solutions are obatined for I_1 and I_2:

$$I_1 = e^{\frac{(k-1)x}{2}+\frac{(k-1)^2\tau}{4}}\phi(-d_2),\ where\ d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k-1)$$
$$I_2 = e^{\frac{(k+1)x}{2}+\frac{(k+1)^2\tau}{4}}\phi(-d_1),\ where\ d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$$

The variables will now be changed back into their original variables:

$$\tau = \frac{\sigma^2}{2}(T-t)$$
$$x = ln\frac{S}{K}$$
$$v(x,\tau) = \frac{1}{K}C(S,t)$$
$$P(S,t) = Kv(x,\tau)$$

The following analytical solution is obtained for the value of a put option:

$$\begin{aligned} &P(S,t) = Ke^{-r(T-t)}\phi(-d_2) - S\phi(-d_1)\\ &where,\\ &d_1 = \frac{ln(\frac{S}{K}) + (r+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\\ &d_2 = \frac{ln(\frac{S}{K}) + (r-\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} = d_1 - S\sqrt{T-t} \end{aligned} \tag{3.41}$$

Put-Call Parity

The put-call parity is an important relationship between the prices of European put options and European call options, if they have the same strike price, the same underlying asset and the same time to maturity. This relationship must always be valid, since arbitrage will exist if it is not. The put-call parity is defined by the following mathematical equation [17]:

$$C(t) - P(t) = S - Ke^{-r(T-t)} \tag{3.42}$$

Expressed in words, the difference between the price of a call option and a put option is equal to the stock price minus the discounted strike price. The put-call parity formula will now be confirmed using the analytical solutions obtained in equation 3.33 and equation 3.41. According to equation 3.42, the following equation can be used to calculate the put option value:

$$P(t) = C(t) + Ke^{-r(T-t)} - S$$

Equation 3.33 will now be inserted instead of $C(t)$. The following is obtained:

$$P(t) = S\phi(d_1) - Ke^{-r(T-t)}\phi(d_2) + Ke^{-r(T-t)} - S$$

Common factors are then factorized:

$$\rightarrow P(t) = Ke^{-r(T-t)}(1 - \phi[d_2]) - S(1 - \phi[d_1])$$

The terms $1 - \phi[d_2]$ and $1 - \phi[d_1]$ can from equation 3.10 be rewritten to $\phi[-d_2]$ and $\phi[-d_1]$. The following is obtained:

$$\rightarrow P(t) = Ke^{-r(T-t)}\phi[-d_2] - S\phi[-d_1]$$

The equation above is exactly the analytical solution for put options [equation 3.41].

Hence the put-call parity is confirmed.

The put-call parity is only valid for European options. However, it is possible to get some results for American Options. The following is valid for American options [17]

$$S_0 - K \leq C(t) - P(t) \leq S_0 - Ke^{-r(T-t)} \tag{3.43}$$

Including a Continuous Dividend Yield

To extend the equations and formulas to include a continuous dividend yield, one must look at how the stock prices were modelled with the geometric Brownian motion. Merton suggested that if dividends are expressed continuously, then one can discount the stock price at the dividend yield rate and insert this into the Black-Scholes formula [21]. Considering that there must not exist arbitrage, it is reasonable to assume that dividends are proportional to the stock price because the asset price must decrease by the same amount of the dividend payment. This will only have an affect on the drift ratio. A dividend paying stock will therefore pay $qSdt$ for each time step dt instead of Sdt. This change will result in the following random walk for the asset price:

$$dS = \mu qSdt + \sigma Sdz \tag{3.44}$$

Solving this using the same framework as for a non-dividend paying asset will lead to the following partial differential equation which includes a continuous dividend yield:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0 \tag{3.45}$$

The following analytical solution will be obtained for a call option:

$$
\begin{aligned}
&C(S,t) = Se^{-qT}\phi(d_1) - Ke^{-r(T-t)}\phi(d_2)\\
&where,\\
&d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\\
&d_2 = \frac{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} = d_1 - S\sqrt{T-t}
\end{aligned}
\tag{3.46}
$$

The following analytical solution will be obtained for a put option:

$$
\begin{aligned}
&P(S,t) = Ke^{-r(T-t)}\phi(-d_2) - Se^{-qT}\phi(-d_1)\\
&where,\\
&d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\\
&d_2 = \frac{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} = d_1 - S\sqrt{T-t}
\end{aligned}
\tag{3.47}
$$

Chapter 4

Finite Difference Method

The finite difference methods are important methods used for calculating partial differential equations. The advantage of these methods is that they are easy to implement. In financial mathematics, the explicit method were first introduced to value derivatives by Brennan and Schwartz [10]. Hull and White continued by explaining the similarities between the explicit method and the tree pricing models, e.g. binomial trees [16]. The disadvantage of the explicit method was its instability. A stable finite difference method was later introduced by Courtadon with the Crank-Nicholson method. The Crank-Nicholson method is not only unconditionally stable, but it also has a more accurate representation of the behaviours of a stock price for larger steps [12].

Implementing the Finite Difference Method

Finite difference approximations are about replacing derivatives with approximations in order to rewrite a differential equation as an algebraic equation. The derivatives are approximated with a Taylor polynomial which is defined with the following equation

[28]:

$$f(x_0 + h) = f(x_0) + f^{'}(x_0)h + \frac{f^{''}(x_0)}{2!}h^2 + ... + \frac{f^n(x_0)}{n!}h^n + R_n(x)$$

When approximating the first derivative one obtains the following equation from the Taylor polynomial:

$$f(x_0 + h) = f(x_0) + f^{'}(x_0)h + R_1(x)$$

By rearranging the equation, the following equation is obtained when approximating the first derivative:

$$f^{'}(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{R_1(x)}{h}$$

R_1 is the difference between the approximation and the actual value. This value will approach zero as h approaches zero. The following approximation is obtained for the first derivative as h approaches zero:

$$f^{'}(x) = \frac{f(x + h) - f(x)}{h} + O(h) \tag{4.1}$$

Three different approximations are common:

- Forward approximation
- Backward approximation
- Central approximation

The difference between the different approximations will be explained using figure

4.1.

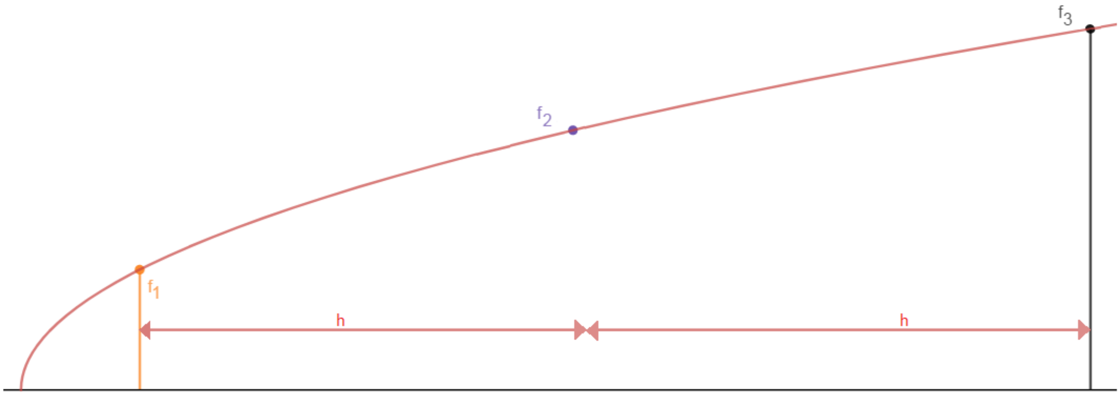

Figure 4.1: An illustration of a graph with three different points. The distance between each point is equal to h.

How to implement the different approximations to approximate the points in figure 4.1 is illustrated with the equations below:

- Forward approximation of f_1': $f(x+h) = \frac{f(x+h)-f(x)}{h} \rightarrow f_1' = \frac{f_2-f_1}{h}$
- Backward approximation of f_3': $f(x-h) = \frac{f(x)-f(x-h)}{h} \rightarrow f_3' = \frac{f_3-f_2}{h}$
- Central approximation of f_2': $f(x) = \frac{f(x+h)-f(x-h)}{2h} \rightarrow f_2' = \frac{f_3-f_1}{2h}$

In figure 4.2, a graphical interpretation of the approximations above is provided.

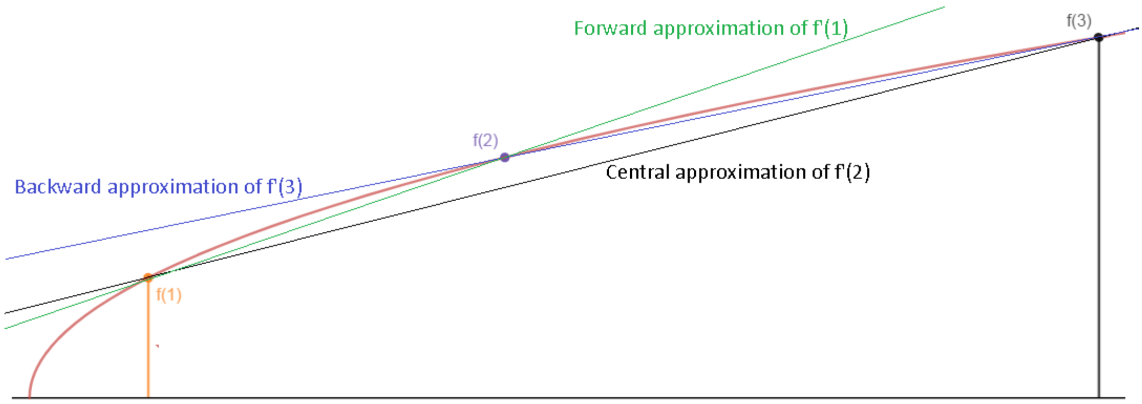

Figure 4.2: An illustration of how the different types of finite difference methods approximate a given point. A forward approximation is approximating f_1' with a secant line through f_1 and f_2. A backward approximation is approximating f_3' with a secant line through f_2 and f_3. A central approximation is approximating f_2' with a secant line through f_1 and f_3.

The Black-Scholes partial differential equation also includes one second derivative and needs to be approximated with the taylor polynomial. This derivative will be approximated from the sum of the forward approximation and the backward approximation. From the Taylor polynomial, the following is obtained for a forward approximation:

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + ... \tag{4.2}$$

The following is obtained for a backward approximation:

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - ... \tag{4.3}$$

The following equation is obtained for the the second derivative when combining equation 4.2 with equation 4.3:

$$f(x+h)+f(x-h)=[f(x)+hf^{'}(x)+\frac{1}{2}h^2f^{''}(x)+...]+[f(x)-hf^{'}(x)+\frac{1}{2}h^2f^{''}(x)-...]=$$
$$=f(x)+f(x)+hf^{'}(x)-hf^{'}(x)+\frac{1}{2}h^2f^{''}(x)+\frac{1}{2}h^2f^{''}(x)+...$$
$$\to f(x+h)+f(x-h)=2f(x)+h^2f^{''}(x)+R_2(x)\to$$
$$\to f^{''}(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}-\frac{R_2(x)}{h^2}$$

After rearranging the equation, one obtains the following approximating of the second order derivative:

$$f^{''}(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}-\frac{R_2(x)}{h^2}$$

R_2 is the difference between the approximation and the actual value. This value will approach zero as h approaches zero. The following approximation is obtained for the second order derivative as h approaches zero:

$$f^{''}(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}+O(h^2) \tag{4.4}$$

The approximation of the second derivative will have a negative impact on the convergence of the numerical algorithms because it has larger truncation error compared to the approximations of the first order derivatives.

Applying the Finite Difference Method on Option Pricing

To implement the finite difference method on the Black-Scholes partial differential equation, a price-time mesh will be introduced, as shown in figure 4.3 [32]. The vertical axis in the mesh will resemble the stock prices, while the horizontal axis will resemble the time. Each point in the mesh will have a horizontal index j and a vertical index i. Each point in the mesh is the option price for a certain time and a certain stock price. At any given point, $j\delta S$ is equal to the stock price and $i\delta t$ is equal to the time. There will also exist three boundary conditions for the mesh, and it is with these that all the calculations can be made. A put option will be used to illustrate how the boundary conditions will be calculated. First of all, there exists an upper boundary where the option values for maximal stock value in the mesh S_max will be calculated. S_max will be chosen large enough so that the put option is equal to zero according to the pay-off function $max(K-S,0)$. For this boundary condition, the option value will therefore be equal to zero for all times in the mesh. For the lower boundary conditions, the stock price will be equal to zero ($S=0$). Consequently, the option price will be equal to K for this boundary condition according to the pay-off function. Therefore, for this boundary condition, the option value will be equal to the discounted value of the strike price for all times in the mesh. The righter boundary condition, which is the option price at expiration date, will be calculated using the pay-off function. The time is known at expiration, $t=T$, and the option can be calculated for all the different stock prices for this boundary condition. To obtain the prices at $t=0$, the option prices will be calculated for each time step backwards from $t=T$.

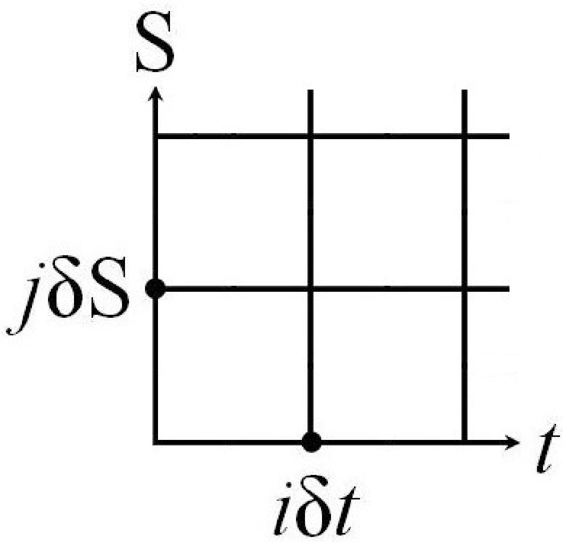

Figure 4.3: An illustration of the price-time mesh used for implementing the finite difference method. The x-axis is divided into i number of steps, with the distance δt between each step. The y-axis is divided into j number of steps, with the distance δS between each step.

Applying the finite difference approximations on the derivatives in the Black-Scholes partial differential equation on the price-time mesh gives the following approximations of the derivatives:

Forward approximation:

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\delta t} \tag{4.5}$$

Backward approximation:

$$\frac{\partial f}{\partial t} = \frac{f_{i,j} - f_{i-1,j}}{\delta t} \tag{4.6}$$

Central approximations:

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\delta S} \tag{4.7}$$

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i-1,j}}{2\delta t} \tag{4.8}$$

Second Derivative (central approximation):

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\delta S)^2} \tag{4.9}$$

These approximations of the derivatives will be used to rewrite the Black-Scholes partial differential equation. The Black-Scholes equation will for each of the different finite difference methods be approximated by a different algebraic equation. The methods that will be used are:

- The explicit method
- The implicit method
- The Crank-Nicholson method

Explicit Method

The explicit method is the easiest finite difference method to implement and it has the fastest alghoritm, but it is also the most instable method [24]. The method calculates the option prices for each time step using known quantities from the previous time step. The calculations made with the explicit method only uses already known quantities and will therefore only solve linear equations for each time step.

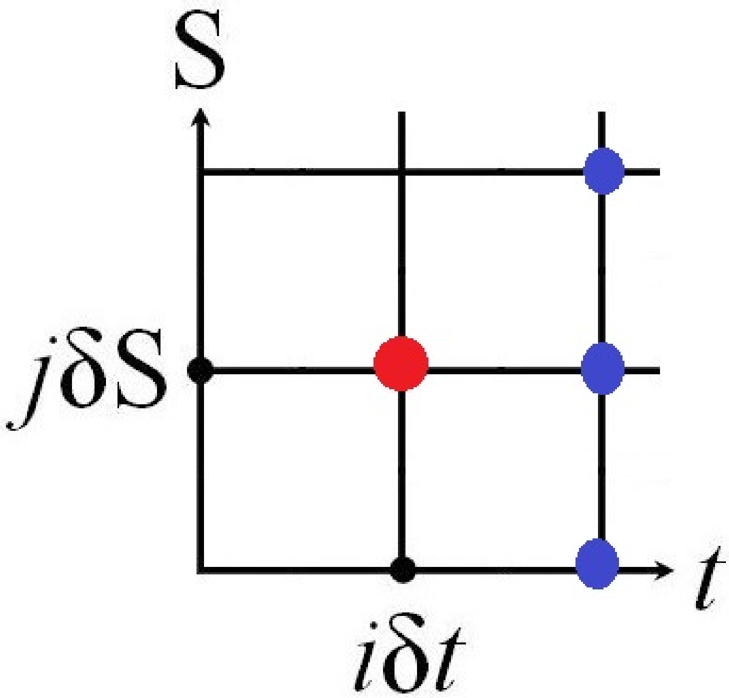

Figure 4.4: Explicit method: An illustration of how the blue, known values, are used to calculate the red, unknown option value, backward in time.

In order for the explicit method to converge, the option value must not increase for each iteration backwards in time.This can be expressed as:

$$f_{i,j} \leq f_{i+1,j} \Leftrightarrow \frac{f_{i,j}}{f_{i+1,j}} \leq 1$$

The Black-Scholes partial differential equation will now be rewritten using the explicit method. A backward approximation will be used for approximating $\frac{\partial f}{\partial t}$ and a central approximation will be used for approximating $\frac{\partial f}{\partial S}$. The approximation from equation 4.9 will be used for the second derivative. Additionally, S will be replaced by $j\delta S$. When inserting the approximations from equation 4.6, 4.7 and 4.9 together with $j\delta S$ into the Black-Scholes partial differential equation, the following equation is obtained:

$$\frac{f_{i,j} - f_{i-1,j}}{\delta t} + rj\delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\delta S} + \frac{1}{2}\sigma^2 j^2 (\delta S)^2 \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\delta S)^2} - rf_{i,j} = 0$$

This can be simplified to:

$$f_{i-1,j} = [\frac{1}{2}\delta t(\sigma^2 j^2 - rj)]f_{i,j-1} + [1 - \delta t(\sigma^2 j^2 + r)]f_{i,j} + [\frac{1}{2}\delta t(\sigma^2 j^2 + rj)]f_{i,j+1}$$

This indicates that calculating the option price at $T - \delta t$ is explicitly dependent on already known information. The coefficients in front of $f_{i,j-1}$, $f_{i,j}$ and $f_{i,j+1}$ will be called a_j, b_j and c_j respectively. This equation does at this point only depend on the indexes in the price-time mesh. The following equation for calculating each point in the price-time mesh is obtained for the explicit method:

$$f_{i-1,j} = a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} \tag{4.10}$$

Implicit Method

The implicit method is more stable than the explicit method. However, it requires larger computations [24]. This method does not solely depend on quantities from the previous state, instead it combines both the current and last state in an equation system.

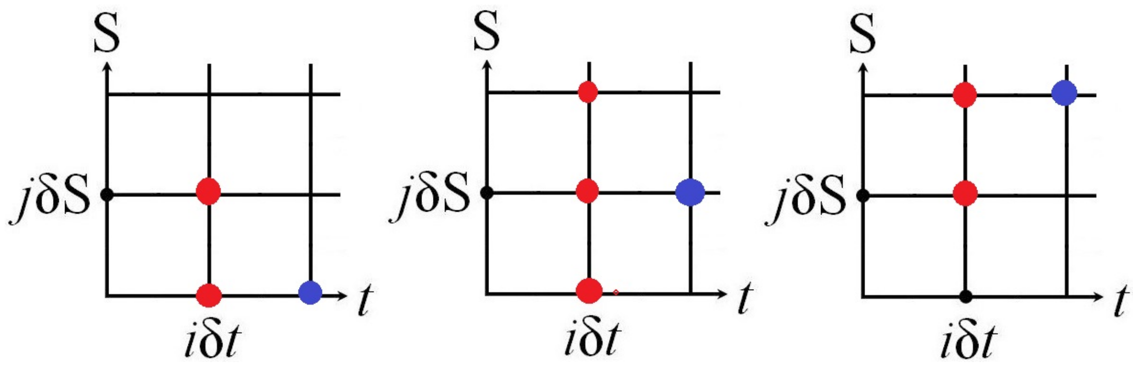

Figure 4.5: Implicit method: An illustration of which points in the mesh are used for each iteration in the implicit method. Known values are indicated by a blue colour and unknown values are indicated by a red colour.

For the implicit method, a forward approximation will be used for approximating $\frac{\partial f}{\partial t}$. The other derivatives will be approximated in the same way as with the explicit method. When inserting the approximations from equation 4.5, 4.7 and 4.9 together with $j\delta S$ into the Black-Scholes partial differential equation, the following equation is obtained:

$$\frac{f_{i+1,j} - f_{i,j}}{\delta t} + rj\delta S\frac{f_{i,j+1} - f_{i,j-1}}{2\delta S} + \frac{1}{2}\sigma^2 j^2(\delta S)^2\frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\delta S)^2} - rf_{i,j} = 0$$

This can be simplified to:

$$f_{i+1,j} = [\frac{1}{2}\delta t(rj - \sigma^2 j^2)]f_{i,j-1} + [1 + \delta t(\sigma^2 j^2 + r)]f_{i,j} + [-\frac{1}{2}\delta t(rj + \sigma^2 j^2)]f_{i,j+1}$$

Changing the index from $f_{i+1,j}$ to $f_{i,j}$ gives the following equation:

$$f_{i,j} = [\frac{1}{2}\delta t(rj - \sigma^2 j^2)]f_{i-1,j-1} + [1 + \delta t(\sigma^2 j^2 + r)]f_{i-1,j} + [-\frac{1}{2}\delta t(rj + \sigma^2 j^2)]f_{i-1,j+1}$$

The coefficients are then replaced with a_j b_j and c_j. As a result, the following equation is obtained for the implicit method:

$$f_{i,j} = a_j f_{i-1,j-1} + b_j f_{i-1,j} + c_j f_{i-1,j+1} \tag{4.11}$$

This can be formulated in a matrix form, which will be the form used for the numerical calculations:

$$\begin{aligned} &F_i = BF_{i-1} \\ &where \\ &F_i = \begin{bmatrix} f_{i,1} \\ f_{i,2} \\ \vdots \\ \vdots \\ f_{i,M-1} \end{bmatrix} \\ &B = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 \\ 0 & a_3 & b_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & a_{M-1} & b_{M-1} \end{bmatrix} \end{aligned} \tag{4.12}$$

Crank-Nicholson Method

The Crank-Nicholson method is an implicit method that is weighted between the explicit method and the implicit method [24].

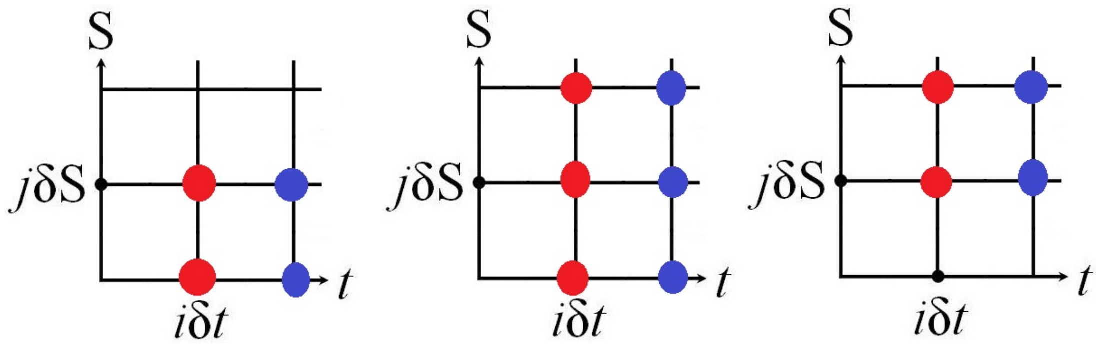

Figure 4.6: Crank-Nicholson method: An illustration of which points in the mesh are used for each iteration in the Crank-Nicholson method. Known values are indicated by a blue colour and unknown values are indicated by a red colour.

The advantages of this method is that it is stable and that the convergence of the accuracy is faster. For a Crank-Nicholson approximation, a central approximation will be used for approximating $\frac{\partial f}{\partial t}$. The other derivatives will be approximated in the same way as before. When inserting the approximations from equation 4.7, 4.8 and 4.9 together with $j\delta S$ into the Black-Scholes partial differential equation, the following equation is obtained:

$$\begin{aligned}
&[-\frac{\delta t}{4}(\sigma^2 j^2 - rj)]f_{i-1,j-1} + (1 - [-\frac{\delta t}{2}(\sigma^2 j^2 + r)])f_{i-1,j} - [\frac{\delta t}{4}(\sigma^2 j^2 + rj)]f_{i-1,j+1} = \\
&= [\frac{\delta t}{4}(\sigma^2 j^2 - rj)]f_{i,j-1} + (1 + [-\frac{\delta t}{2}(\sigma^2 j^2 + r)])f_{i,j} + [\frac{\delta t}{4}(\sigma^2 j^2 + rj)]f_{i,j+1}
\end{aligned}$$

The expressions inside the square brackets will be replaced with the coefficients a_j b_j and c_j. The following equation is obtained:

$$-a_j f_{i-1,j-1} + (1 - b_j)f_{i-1,j} - c_j f_{i-1,j+1} = a_j f_{i,j-1} + (1 + b_j)f_{i,j} + c_j f_{i,j+1} \quad (4.13)$$

This can be formulated in a Matrix form, which will be the form used for the numerical calculations:

$$BF_{i-1} = CF_i$$

where

$$B = \begin{bmatrix} 1-b_1 & -c_1 & 0 & \cdots & 0 \\ -a_2 & 1-b_2 & -c_2 & \cdots & 0 \\ 0 & -a_3 & 1-b_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -c_{M-2} \\ 0 & 0 & \cdots & -a_{M-1} & 1-b_{M-1} \end{bmatrix}$$

$$F_i = \begin{bmatrix} f_{i,1} \\ f_{i,2} \\ \vdots \\ \vdots \\ f_{i,M-1} \end{bmatrix} \tag{4.14}$$

$$C = \begin{bmatrix} 1+b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & 1+b_2 & c_2 & \cdots & 0 \\ 0 & a_3 & 1+b_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & a_{M-1} & 1+b_{M-1} \end{bmatrix}$$

Choice of Method

The performance of the three different finite difference methods will be compared and the best one will be used for valuing American options and stock loans. According to theory, the following should be true about each method [17] [24] [32]:

- The Explicit method is a fast method due to its nature of calculating new values solely from previously known values. The disadvantage with this method is that it is unstable. The explicit method is stiff for many problems.

- The Implicit method is a stable method due to its nature of finding new values by solving equations involving both the current state at time t and the next state at time $t + \Delta t$. The disadvantage with this method is that it requires many computations.
- The Crank-Nicholson method is a combination of the explicit method and the implicit method. The advantages of this method is its stability and its fast convergence of accuracy.

According to theory, the Crank-Nicholson method should perform better compared to the explicit method and the implicit method. To confirm this, the three different methods will be compared when valuing a European option. Table 4.1 shows the accuracy and computational time for each method.

		Explicit Method			Implicit Method			Crank-Nicholson		
S	**True Value**	Value	Accuracy	Time	Value	Accuracy	Time	Value	Accuracy	Time
80	1.8594	1.8591	99.99%	0.0241	1.8599	99.97%	0.2063	1.8595	100.00%	0.0302
85	3.2136	3.2128	99.98%	0.0242	3.2135	100.00%	0.2093	3.2136	100.00%	0.0296
90	5.0912	5.0900	99.98%	0.0243	5.0906	99.99%	0.2108	5.0913	100.00%	0.0287
95	7.5109	7.5094	99.98%	0.0241	7.5098	99.98%	0.2091	7.5112	100.00%	0.0266
100	10.4506	10.4488	99.98%	0.0241	10.4491	99.99%	0.1950	10.4515	99.99%	0.0247
105	13.8579	13.8562	99.99%	0.0256	13.8564	99.99%	0.2123	13.8590	99.99%	0.0299
110	17.6630	17.6616	99.99%	0.0265	17.6616	99.99%	0.2295	17.6642	99.99%	0.0341
115	21.7905	21.7892	99.99%	0.0277	21.7894	100.00%	0.2418	21.7919	99.99%	0.0355
120	26.1690	26.1681	100.00%	0.0296	26.1682	100.00%	0.2816	26.1705	99.99%	0.0402

Table 4.1: A comparison between the performance of the explicit method, implicit method and the Crank-Nicholson method for a European option with $K = 100$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. The computational time is the average computational time for 100 trials. Explicit: dt = 0.0001 and ds = 1. Implicit: dt = 0.001 and ds = 0.5. Crank-Nicholson: dt = 0.01 and ds = 0.5.

In table 4.1, the three different finite difference methods are compared. The values used for δt and δS are the values that gives a good trade-off between computational time and accuracy. From table 4.1, the following conclusions can be drawn:

- The explicit method is the fastest method, despite the fact that it uses a larger price-time mesh.
- A long computational time was required for obtaining a good accuracy when using the implicit method.
- The Crank-Nicholson method shows a good performance for a relatively small price-time mesh.
- The accuracy was best for the Crank-Nicholson method, despite the fact that it

uses a smaller price-time mesh.

In order to evaluate the methods more accurately, the performance of each method has been measured. The accuracy has been measured for different time steps when the stock price step is constant and set to 1.

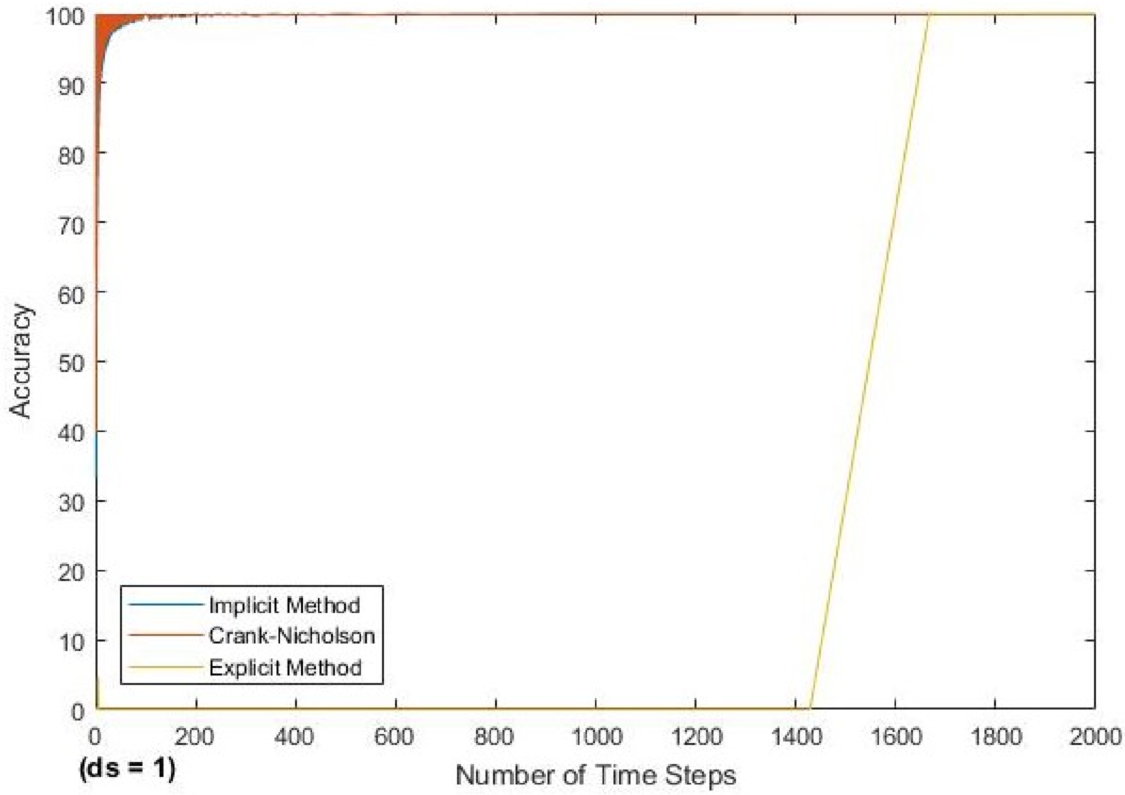

Figure 4.7: A comparison between the explicit method, the implicit method and the Crank-Nicholson method. The plot illustrates the accuracy for a constant $\delta S = 1$ for an increasing number of time steps. The performance is measured on a European option with $K = 100$, $S = 80$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$.

From figure 4.7, one can conclude that the explicit method should be excluded because of its instability. Furthermore, it appears that the Crank-Nicholson method shows a slightly better performance than the implicit method. Although, further measurements are necessary to draw any certain conclusions. Further measurements have therefore been made in order to determine whether the Crank-Nicholson method or the implicit method is more accurate. Figure 4.8 illustrates the accuracy performance between the Crank-Nicholson method and the Implicit Method.

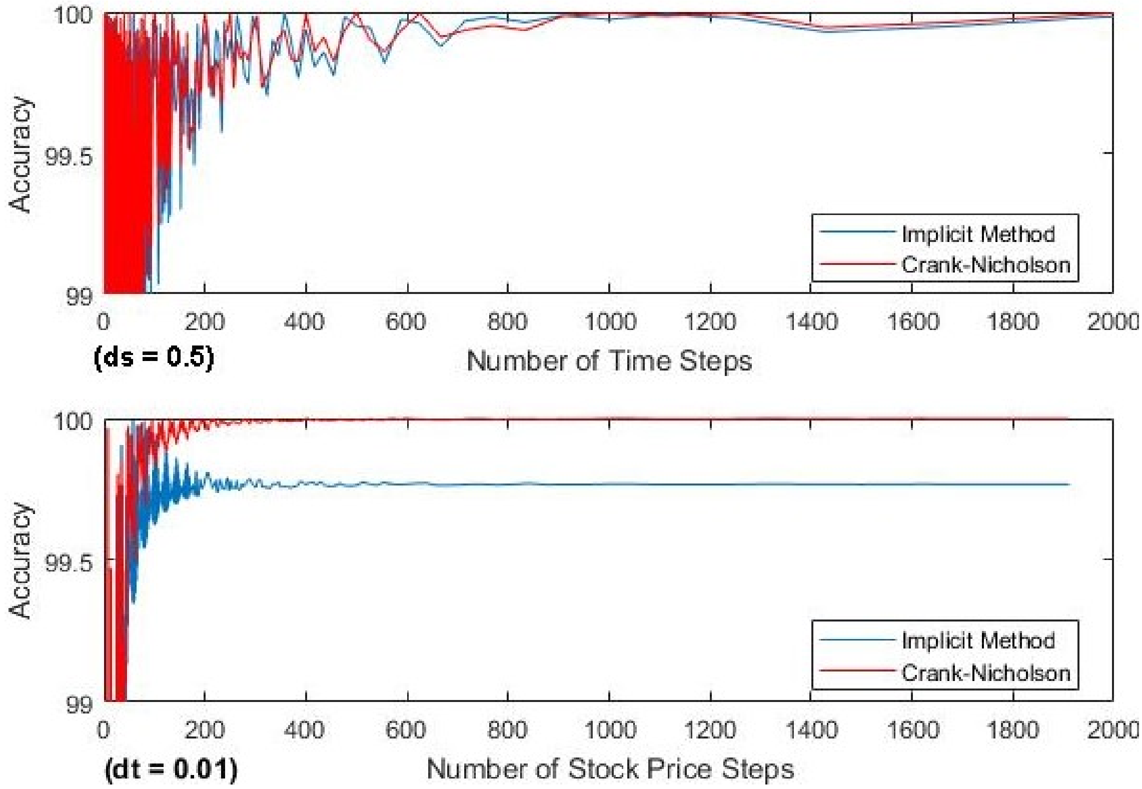

Figure 4.8: A comparison on the accuracy between the Crank-Nicholson method and the Implicit method. The upper plot illustrated the accuracy for a constant $\delta S = 0.5$ with an increasing number of time steps. The lower plot illustrates the accuracy for a constant $\delta t = 0.01$ with an increasing number of Stock Price steps. The performance is measured on a European option with $K = 100$, $S = 80$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$.

From figure 4.8, one can conclude that the Crank-Nicholson method is more accurate and converges faster. From the upper plot, the stock price step δS is constant and set to $\delta S = 0.5$. The performance is then measured for different time steps. The accuracy performance is only slightly better for the Crank-Nicholson Method. In the lower plot, the time step δt is constant and set to $\delta t = 0.01$. The performance is then measured for different stock price steps. In the lower plot, there is a clear difference in performance for the different methods. The Crank-Nicholson converges faster than the implicit method and also has a better accuracy when converging.

From this, one can conclude that the Crank-Nicholson method outperformed the implicit method and it will therefore be used for valuing American option and stock loans.

Chapter 5

American Options

An American option is a financial derivative that is similar to the European option. The main difference is that American options include an additional right to exercise the option at any given time during the option's lifetime [17]. This difference creates a lot of difficulties when pricing American options. As shown earlier, the European style option can be calculated analytically. This is not possible for the American option because of its non-linear features. This problem with non-linearity comes from the fact that the optimal exercise boundary must be computed to effectively price the option. Ahn and Wilmott describes it as a price-maximization strategy [4]. There are two main strategies to handle this problem. One is to price the option from the risk-neutral valuation at discrete time steps, while the other is to use discrete time and discrete asset prices under the Black-Scholes equation. The latter strategy will be used in this report in order to find the value of an American call option.

To be able to value American call options under the Black-Scholes partial differential equation, the following boundary conditions must be added to the conditions for a European call option [32]:

$$
\begin{aligned}
&C(S_f(t), t) = max(S_f(t) - K, 0) \\
&\frac{\partial C}{\partial S}(S_f(t), t)) = 1
\end{aligned}
\tag{5.1}
$$

The first condition in equation 5.1 is added because of the additional right to exercise the American option at any given time during the lifetime of the option. This condition will take the intrinsic value of the option into account. The second condition is introduced because a delta value close to the system is necessary.

In addition to that, the condition that $C(S, t) = max(S - K, 0)$ for a European call option will be replaced by the following inequality:

$$
C(S, t) \geq max(S - K, 0) \tag{5.2}
$$

This is necessary because the American call option can be larger than the value of a European option because of the possibility of an early exercise.

Pricing American Call Options

When implementing the Crank-Nicholson method on a American call option, one has to take the possibility of an early exercise into account when computing the values in the price-time mesh. For the European call option case, the option value at each point in the mesh was calculated by the equation $max(S - K, 0)$. The difference now is that there is another possibility, which is the possibility of an early exercise. To include this, the optimal exercise boundary must be taken into account by considering the time value in the case of an early exercise in the price-time mesh.

Optimal Exercise Boundary

The optimal exercise boundary is the boundary that includes all the optimal exercise prices at different times during an option's lifetime. The equation that will be used to obtain the optimal exercise boundary in this report is the following [31]:

$$C(S_f(t), t) = S_f(t) - K \tag{5.3}$$

This equation is derived from the fact that the American call option will reach its optimal exercise price at the first contact with the intrinsic value. This optimal exercise price changes with time. For the finite difference method, the optimal exercise boundary is found by setting the time to a constant value and calculating all the different stock prices in the price-time mesh. The stock price that is equal to the option value for that constant time is the optimal exercise point for that specific time. This is done for very small time steps until the time of expiration. When making these calculations, the option value will not be exactly equal to the intrinsic value. Therefore, a tolerance will be allowed and the following equation will be used to find the optimal exercise boundary:

$$C(S_f(t), t) = S_f(t) - K \pm tolerance \tag{5.4}$$

Chapter 6

Stock Loan

A non-recourse loan is a contract between two parties, the borrower and the lender. The borrower, who owns a share of a stock, obtains a loan from the lender with the share as collateral. The borrower can regain control of the stock at any time by repaying the principal plus the accumulated interest [30]. Alternatively, the borrower can surrender the stock. The borrower can then no longer be held liable for repaying the loan [20]. A stock loan is a good alternative to increase the liquidity without selling the stocks if there were to be an expected growth in the stock prices. Stock loans can also be used to hedge the market because if [20]:

- The stock decreases in price, the investor can walk away instead of repaying the loan.
- The stock increases in price, the investor can regain the stock by repaying the loan.

The lender will make a profit from fees and accumulated interest. Stock loans can also be used to leverage the returns, i.e. increase the volatility of a portfolio.

The Connection between American Call Options and Stock Loans

Stock loans behave much like American call options. The borrower can be seen as the holder of a call option and the lender can be seen as the writer of the same option. The link between American options and stock loans can be explained with the following connections between their pay-offs:

If the stock price increases [20]

American Option If the underlying stock decreases in price and the American option expires exercised, then the loss is limited to the premium.

Stock Loan If the collateral stock decreases in price and the borrower surrender the collateral, then the loss is limited to the value of the collateral. This is because the borrower can no longer be held liable for repaying the loan.

If the stock price decreases [20]:

American Option If the underlying stock increases in price and the option is exercised early, then the pay-off will be equal to the difference between the value of the stock and the strike price.

Stock Loan If the collateral stock increases in price, the borrower can sell the collateral stock, repay the loan and keep the profit. The pay-off for a stock loan will be equal to the difference between the collateral value and the principal plus accumulated interest.

This link between an American option and a stock loan makes it possible to use the Black-Scholes partial differential equation, with some modification, to value stock loans. The most significant difference between an American option and a stock loan is that there exists no strike price for stock loans, instead the principal plus the accumulated interest will determine the pay-off.

Deriving the Stock Loan Value from American Call Options

Assuming that a borrower lends the amount Q at the interest rate γ with one share valued S as collateral. The borrower will at any given time have to repay the loan plus the accumulated interest, which can be expressed mathematically as $Qe^{\gamma t}$ [20]. This value will determine the profit, similarly to how the strike price determines the profit for an American call option. The difference between valuing stock loans compared to American options is that the the value of the principal plus accumulated interest is time dependant while the strike price is not. If this is taken into consideration, the following formulation of the stock loan problem is obtained under the Black-Scholes equation [20]:

$$\frac{\partial L}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 L}{\partial S^2} + (r-q)S\frac{\partial L}{\partial S} - rL = 0 \tag{6.1}$$

Solving equation 6.1 with the following boundary and final conditions will solve the stock loan problem [20]:

$$\begin{cases} L(0,t) = 0 \\ L(S,T) = max(S - Qe^{\gamma T}, 0) \\ L(S_f(t),t) = S_{f_1}(t) - Qe^{\gamma t} \\ \frac{\partial L}{\partial S}(S_{f_1}(t),t) = 1 \end{cases} \tag{6.2}$$

The two last lines in equation 6.2 are defining the optimal exit boundary which is the equivalence to the optimal exercise boundary for American call options.

Loan Value using American Call Options

Because of the similarities between American options and stock loans, the algorithm for calculating American call options can, after some modifications, be adapted to an algorithm that values stock loans. The difference between an American option and a stock loan is that the strike price will be replaced by the principal plus accumulated interest. For an American option, it is the strike price that decide the pay-off of a call option, while the principal plus accumulated interest will decides the pay-off for a stock loan. This will be illustrated with an example:

1. Assuming that an investor holds an American call option on a stock with the current stock price AUD 25 and the strike price AUD 25. At the time of expiration, six months later, the stock price has increased to 30 AUD. The value of the option is equal to $30 - 25 = AUD$ 5.

2. Assuming that an investor loans AUD 25 and buys a stock with the current stock price AUD 25 and the strike price AUD 25. The loan interest rate is 10 percent. At the time of expiration of the loan, six months later, the stock has increased to AUD 30. The investor sells the stock in order to repay the loan. The pay-off in this case is equal to $30 - 25e^{0.1 \cdot \frac{1}{2}} \approx AUD3.72$.

As illustrated, the equivalence to the strike price for stock loans is growing with time, which will affect the pay-off. When implementing this with the finite difference method, this is taken into account in the optimal exit boundary. For each small time step in the price-time mesh, accumulated interest will increase. Implementing this to the Crank-Nicholson algorithm will successfully calculate the stock loan value.

Optimal Exit Price

The optimal exit price for a stock loan is similar to the optimal exercise boundary for an American call option. The optimal exit price is the prices at which is optimal to sell the collateral stock and repay the loan. The difference between the optimal exit price and the optimal exercise boundary is that the optimal exercise boundary depends on the constant strike price while the optimal exit price depends on the principal plus accumulated interest $Qe^{\gamma t}$. The optimal exit price will be calculated with the following equation:

$$L(L_f(t), t) = S_f(t) - Qe^{\gamma t} \pm tolerance \tag{6.3}$$

Chapter 7

Results and Discussion

The obtained results has been computed using the codes in the appendix.

All results were calculated on a computer with an *Intel Core i7 CPU, 3.40 GHz* with *8GB* of *RAM*.

American Options

The first results are for an American Call option. The following parameters were used:

- Exercise Price: $K = 100$
- Time to Maturity: $T = 1$
- Volatility of the underlying asset: $\sigma = 0.2$
- Risk free interest rate: $r = 0.05$
- Continuous dividend yield: $q = 0.1$

- $\delta t = 0.0001$

- $\delta S = 0.5$

For the optimal exercise boundary, the following parameters are used for the δt and the δS instead:

- $\delta t = 0.001$

- $\delta S = 0.1$

- $Tolerance = 10^{-200}$

Option Value

The American call option value have been calculated for stock prices ranging from 80 to 120. The computational time is the average of 100 computations. The following results were obtained:

S	Option Value	Average Computational Time
80	0.6956	0.2134 seconds
85	1.3534	0.2143 seconds
90	2.3890	0.2165 seconds
95	3.8909	0.2180 seconds
100	5.9283	0.2210 seconds
105	8.5468	0.2374 seconds
110	11.7702	0.2646 seconds
115	15.6058	0.2871 seconds
120	20.0519	0.3079 seconds

Table 7.1: Results for American call value using the Crank-Nicholson method. The average computational times is for 100 calculations. Parameters used: $K = 100$, $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $T = 1$, $\delta t = 0.0001$, $\delta S = 0.5$

As it can be found, the computational time increases when the stock prices increase. This is because the price-time mesh gets larger when the stock price increases. Furthermore, it can be found that the option price converges with the pay-off function. For the case when the stock price is equal to $S = 120$, the option value is almost equal to the pay-off function $max(S - K, 0) = 20$. The convergence of the American call option with the pay-off function is illustrated in figure 7.1.

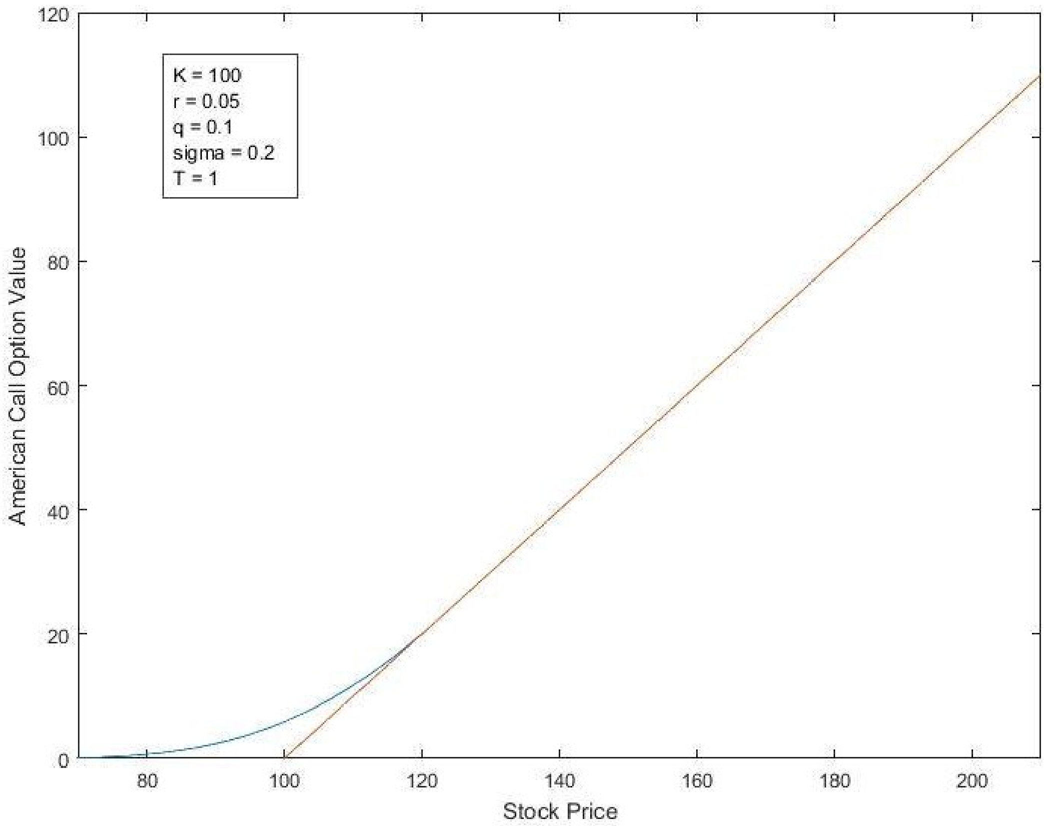

Figure 7.1: An illustration of the value of an American call option for varying stock prices at $t = 0$. The blue line is the value of the American option and the red line is the pay-off function. The parameters of the option are the following: $K = 100$, $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $T = 1$, $\delta t = 0.0001$, $\delta S = 0.5$.

The blue line in figure 7.1 resembles the value of an American call option and the red line is the pay-off function $max(S - K, 0)$. In the figure, it can found that the value of an American call option converges with the pay-off function for increasing stock prices. It can also be found that the American call option is always worth more or equal to the pay-off function. This can be explained from the boundary condition for an American option:

$$C(S,t) = S \ when \ S \to \infty$$

As the stock prices increase and become very large, the call value is equal to the S and hence the option value will be equal to the pay-off function.

Optimal Exercise Boundary

The gain from an early exercise differs over time. The optimal prices for an early exercise are listed in table 7.2 for different times:

Optimal Exercise Prices	
t	Optimal Exercise Price
0	121.70
0.25	120.10
0.50	117.90
0.75	114.35
1	100.00

Table 7.2: Optimal exercise prices for an American call option with the parameters: $K = 100$, $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$.

As it can be found in table 7.2, the optimal exercise price converges with the strike price. This can be explained by the loss of the option's time value as the maturity is approached. At expiry, the time value will be equal to zero, and hence the intrinsic value will determine the value of the option. Consequently, at expiration date, the optimal exercise price is equal to the strike price. The loss of time value as the option approaches the expiration date is illustrated in the plot of the optimal exercise boundary in figure 7.2.

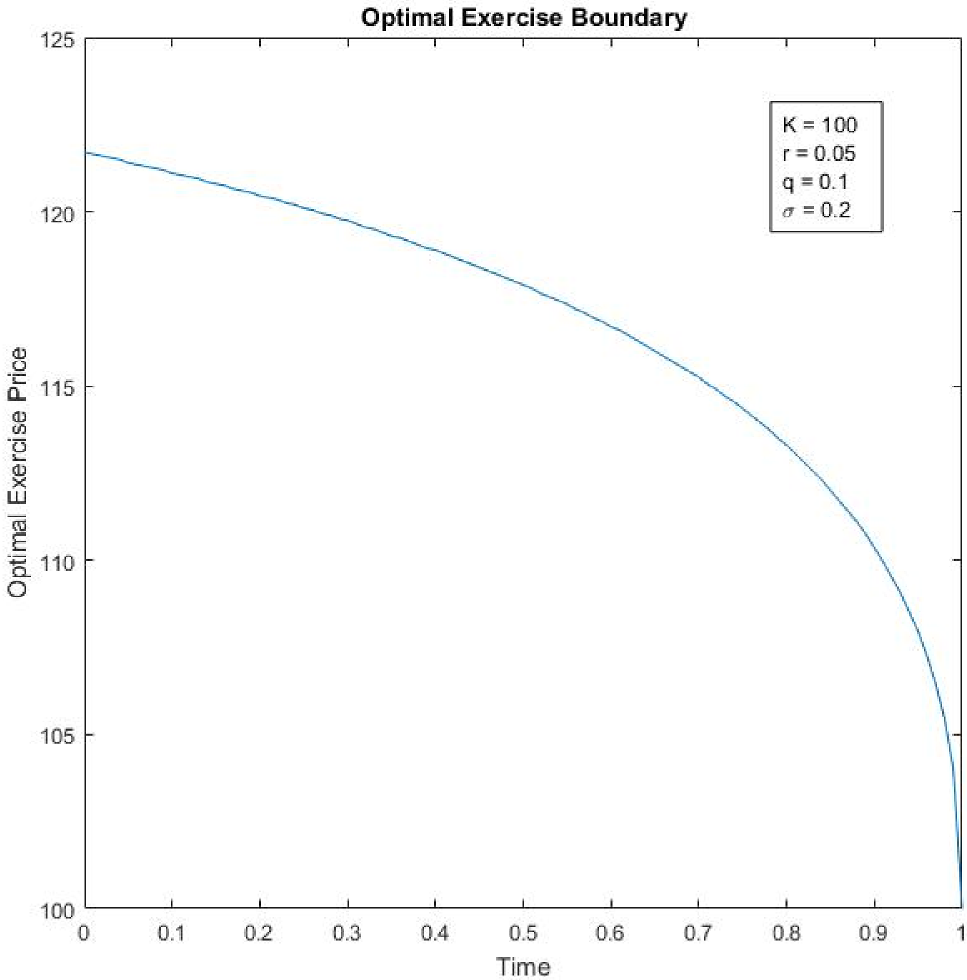

Figure 7.2: An illustration of the optimal exercise boundary of an American call option with the following parameters: $K = 100$, $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$.

In figure 7.2 it can be found the optimal exercise price decreases over time. This is explained, as mentioned, by the fact that the time value decrease as the option approaches the expiration date. Furthermore, the optimal exercise price is equal to the strike price at maturity, since the time value will be equal to zero at $t = T$.

Optimal Exercise Boundary - Varying Volatility Ceteris Paribus

From figure 7.3, it can be found that the optimal exercise price increases with an increased volatility. The exercise price is equal to the strike price at maturity.

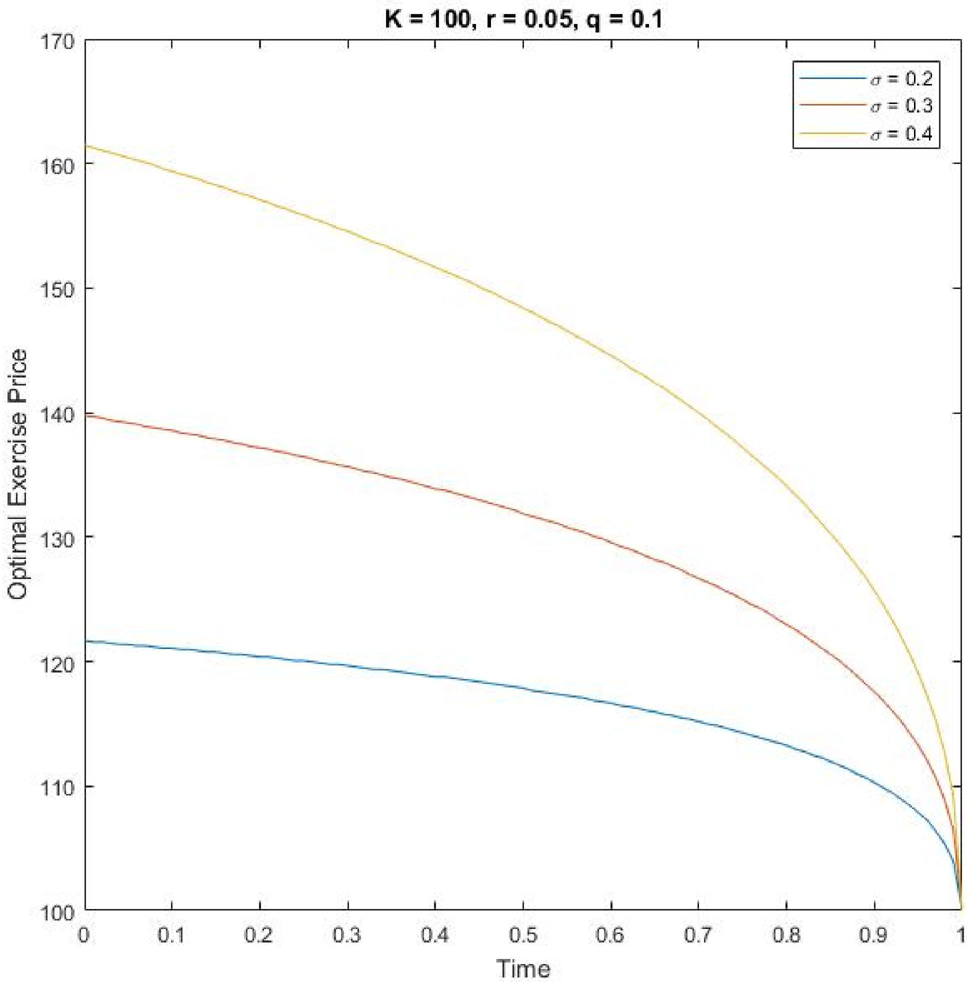

Figure 7.3: An illustration of the optimal exercise boundary of an American call option for varying volatilities. The following parameters are used: $K = 100$, $r = 0.05$, $q = 0.1$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$. The blue line have the volatility $\sigma = 0.2$, the red line $\sigma = 0.3$ and the yellow line $\sigma = 0.4$.

The explanation to why the optimal exercise price increase when the volatility increases is because the time value of the option will grow because of an increased probability of a profitable stock movements. This is true since the the loss is limited to the premium. As the option approaches maturity, the time value will approach zero and hence the optimal exercise price at expiration date will be equal to the strike price.

Optimal Exercise Boundary - Varying Risk Free Interest Rate Ceteris Paribus

From figure 7.4, it can be found that the optimal exercise price increases with an increased risk free interest rate. The exercise price is equal to the strike price at maturity.

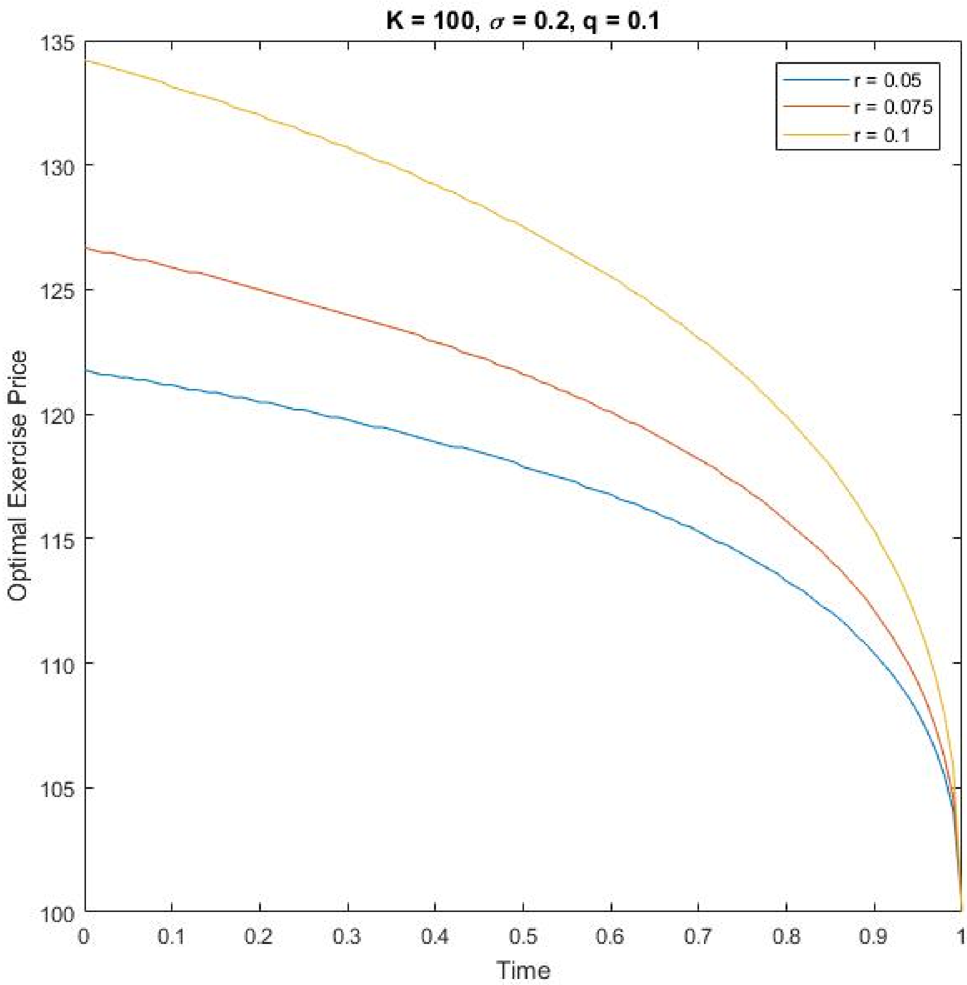

Figure 7.4: An illustration of the optimal exercise boundary of an American call option for varying risk free interest rates. The following parameters are used: $K = 100$, $\sigma = 0.2$, $q = 0.1$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$. The blue line have the risk free interest rate $r = 0.05$, the red line $r = 0.075$ and the yellow line $r = 0.1$.

The explanation to why the optimal exercise prices increase when the risk free interest rate increase is because the value of today's money increase. This means that the time value have increased. The logic behind this is that for a higher risk-free interest rate, if the option is exercised early, the cash from selling the option's underlying asset after an exercise can be invested in risk-free assets with a return equal to the risk-free interest rate. A higher risk-free interest rate will result in an increased return. As the option approaches maturity, the time value will approach zero and hence the optimal exercise price at expiration date will be equal to the strike price.

Optimal Exercise Boundary - Varying Dividend Rate Ceteris Paribus

From figure 7.5, it can be found that the optimal exercise price decreases with an increased dividend yield rate. The exercise price is equal to the strike price at maturity.

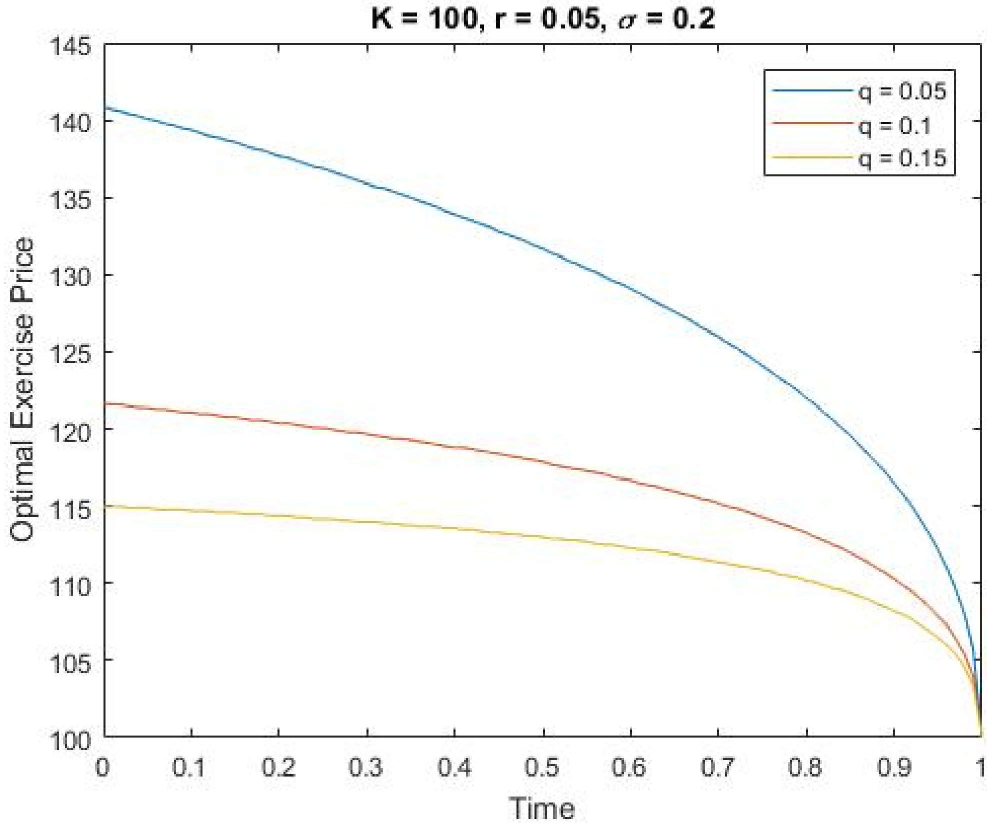

Figure 7.5: An illustration of the optimal exercise boundary of an American call option for varying dividend yield rates. The following parameters are used: $K = 100$, $\sigma = 0.2$, $r = 0.05$, $T = 1$, $\delta t = 0.001$, $\delta S = 0.1$. The blue line have the dividend yield rate $q = 0.05$, the red line $q = 0.1$ and the yellow line $q = 0.15$.

The explanation to why the optimal exercise prices decrease when the dividend yield rate increases is because the stock price will decrease. A decreasing stock price will result in a decreased call option value because of the pay-off function for a call option. As the option approaches maturity, the time value will approach zero and hence the optimal exercise price at expiration date will be equal to the strike price.

Stock Loan

The results in this section are for the valuation of a stock loan with the following parameters:

- Borrowed Amount: $Q = 0.4$
- Loan interest rate: $\gamma = 0.1$
- Risk free interest rate: $r = 0.06$
- Continuous Dividend Yield: $q = 0.1$
- Volatility of the underlying asset: $\sigma = 0.4$
- $\delta t = 0.01$
- $\delta S = 0.001$
- $Tolerance = 10^{-3}$

Loan Values

The loan value for different stock prices have been computed and is presented are table 7.3.

Loan Value for Different Principals t = 0 **Q (Principal) = 0.2**		
S	Loan Value	Growth of the Loan Value
0.4	0.2007	49.3%
0.5	0.2998	
Q (Principal) = 0.3		
S	Loan Value	Growth of the Loan Value
0.4	0.1343	57.6%
0.5	0.2117	
Q (Principal) = 0.4		
S	Loan Value	Growth of the Loan Value
0.4	0.0949	64.5%
0.5	0.1561	

Table 7.3: Stock loan values for different principals. The following parameters were used: $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $T = 5$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$

In table 7.3, it can be found that the level of leverage affects the loan value. For the same growth in the stock price (25%), the increase of the loan value will be different depending on the level of leverage. The loan value growth for different principals that can be found in table 7.3 are the following:

- **Q = 0.2**: Loan value growth = 49.3%
- **Q = 0.3**: Loan value growth = 57.6%
- **Q = 0.4**: Loan value growth = 64.5%

The growth of the loan value increases when the ratio between the stock value and the loan value increases. This is explained by the leverage's effect on returns because the borrowed amount will increase in value as well, and after repaying the loan, the investor can keep the additional profit from the loan. Assuming that an investment grows from 100 AUD to 120 AUD. If there is no leverage, the return in percentage will be equal to $\frac{120-100}{100} = 20\%$. If instead assuming that the initial investment consisted of a debt equal to AUD 50, then the return in percentage would be equal to $\frac{120-100}{50} = 40\%$. As illustrated with the example, an increased leverage results in an increased growth of the return.

Optimal Exit Price (T=20 years)

The optimal exit price for the stock loan with an maturity of 5 years can be found in table 7.4.

Optimal Exit Prices for T = 20	
t	Optimal Exit Price
0	0.94
5	1.54
10	2.45
15	3.68
20	2.96

Table 7.4: Optimal exit prices for a stock loan with the parameters: $T = 20$, $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$

From table 7.4, it can be found that the optimal exit price increase at $t = 0$, reach a maximum, and then decrease until $t = T$. The optimal exit price at maturity is equal to the principal plus the accumulated interest $Qe^{\gamma t} = 0.4e^{0.1 \cdot 20} \approx 2.96$. The optimal exit prices corresponding to table 7.4 are shown in figure 7.6.

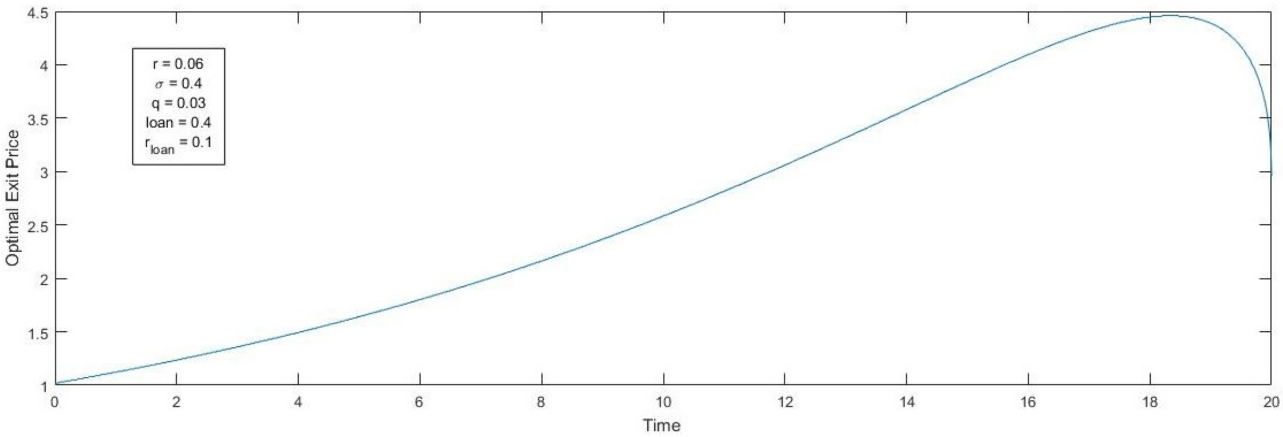

Figure 7.6: An illustration of the optimal exit boundary of a stock loan with the following parameters: $T = 20$, $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$

In figure 7.6, it can be found that the optimal exit prices initially increase with time, reach a maximum price and then decrease. The shape of the optimal exit price graph can be explained by the fact that there are two time-dependent factors for a stock loan, compared to an call option where there only exist one time-dependent factor. For a stock loan, the time value will be determined by two different factors:

- The risk-free interest rate. An increased risk-free interest rate results in an increased time value. Therefore, as the maturity is approached, there will be a loss in the increase of the time value. Therefore, the risk-free interest rate will decrease the time value over time.

- The loan interest rate. An increased loan interest rate results in a decreased time value. Therefore, as the maturity is approached, there will be a loss of in decrease of the time value. Therefore, the loan interest rate will increase the time value over time.

It is because of this that the time value increase with time in figure 7.6. The optimal exit price will reach a maximum and from there decrease because the total significance of the time value will decrease with time. Instead, there will be an increase in the significance of the intrinsic value as the expiration date is approached. The optimal exit price will therefore from the maximum point approach the principal plus the accumulated interest.

Optimal Exit Price (T=5 years)

The optimal exit price for the value of a stock loan with a maturity of 5 years can be found in table 7.5.

Optimal Exercise Prices	
t	Optimal Exercise Price
0	0.84
1	0.89
2	0.94
3	0.97
4	0.96
5	0.66

Table 7.5: Optimal exit prices for a stock loan with the parameters: $T = 5$, $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$

The optimal exit prices corresponding to table 7.5 are shown in figure 7.7.

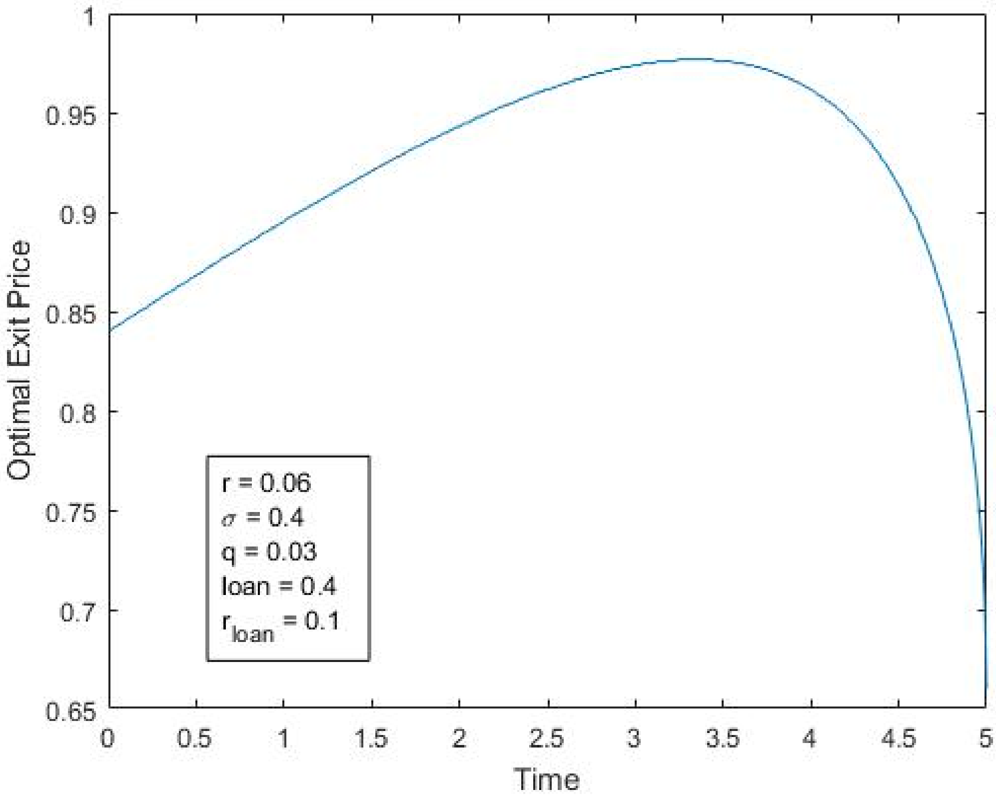

Figure 7.7: An illustration of the optimal exit boundary of a stock loan with the following parameters: $T = 5$, $r = 0.06$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$

The exit price grows initially and then reaches a maximum, from where it will approach the principal plus the accumulated interest.

Optimal Exit Boundary - Varying Volatility Ceteris Paribus

From figure 7.8, it can be found that the optimal exit price increases with an increased volatility. The exit prices for the different volatilities are equal to the principal plus accumulated interest at maturity.

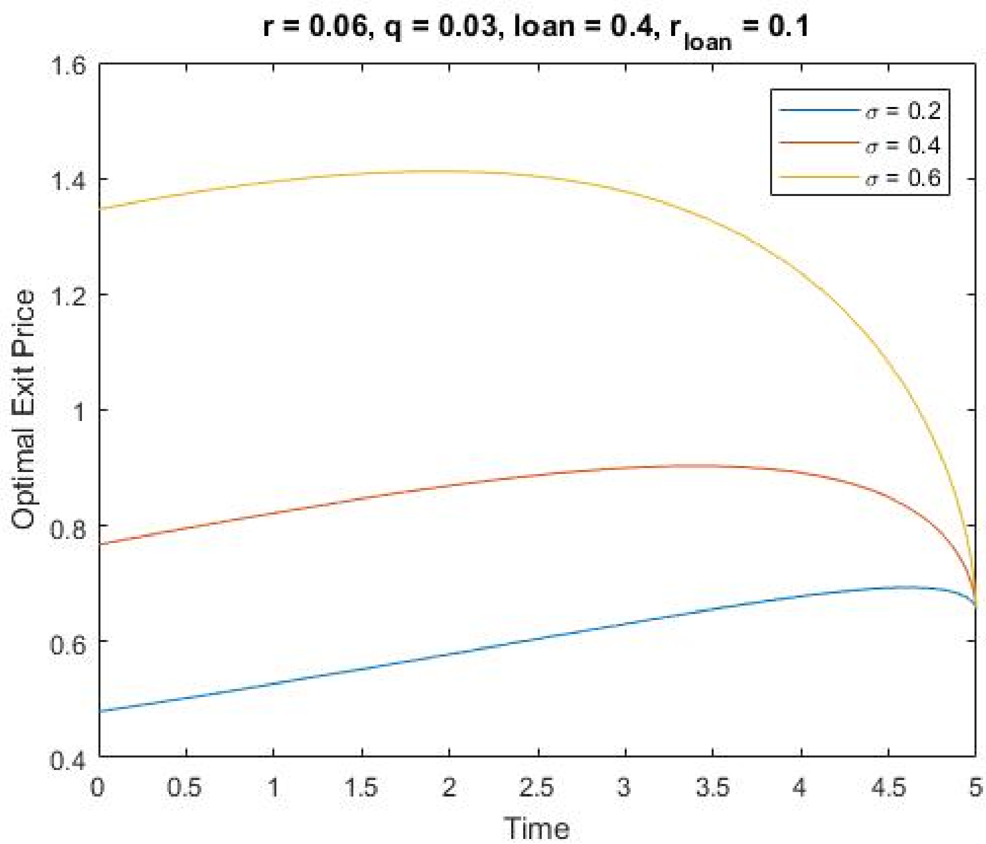

Figure 7.8: An illustration of the optimal exit boundary of a stock loan for varying volatilities. The following parameters are used: $T = 5$, $r = 0.06$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$. The blue line have the volatility $\sigma = 0.2$, the red line $\sigma = 0.4$ and the yellow line $\sigma = 0.6$.

The explanation to why the optimal exit prices increase when the volatility increases is because the time value of the option will grow because of an increase in the probability of a profitable stock movement. This is true since the loss is limited to the stock value at $t = 0$. As the option approaches maturity, the time value converges to zero, and hence the optimal exit price at expiration date will be equal to the principal plus accumulated interest.

Optimal Exit Boundary - Varying Risk Free Interest Rate Ceteris Paribus

From figure 7.9, it can be found that the optimal exit price increases with an increased risk free interest rate. The exit prices for the different risk free interest rates are equal to the principal plus accumulated interest at maturity.

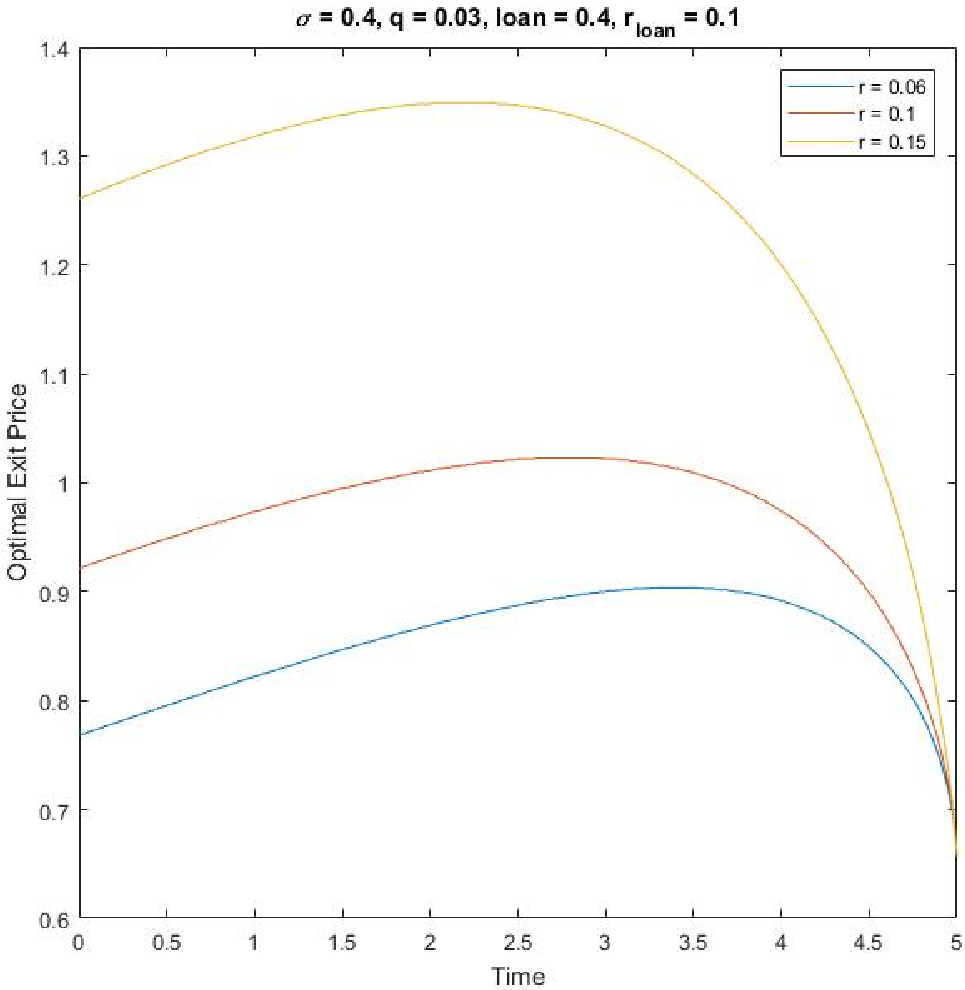

Figure 7.9: An illustration of the optimal exit boundary of a stock loan for varying risk free interest rates. The following parameters are used: $T = 5$, $\sigma = 0.4$, $q = 0.03$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$. The blue line have the risk fre interest rate $r = 0.06$, the red line $r = 0.1$ and the yellow line $r = 0.15$.

The explanation to why the optimal exit prices increases when the risk free interest rate increases is because the value of today's money increase. This means that the time value have increased. The logic behind this is that for a higher risk-free interest rate, the profit when selling the stock and repaying the loan can can be invested in a risk-free asset with a return equal to the risk-free interest rate. A higher risk-free interest rate will result in an increased risk-free return. As the stock loan approaches maturity, the time value will approach zero and hence the optimal exit price at expiration date will be equal to the principal plus the accumulated interest.

Optimal Exit Boundary - Varying Dividend Rate Ceteris Paribus

From figure 7.10, one can see that the optimal exit price decreases with an increased dividend yield. The exit prices for the different dividend yields are equal to the principal plus accumulated interest at maturity.

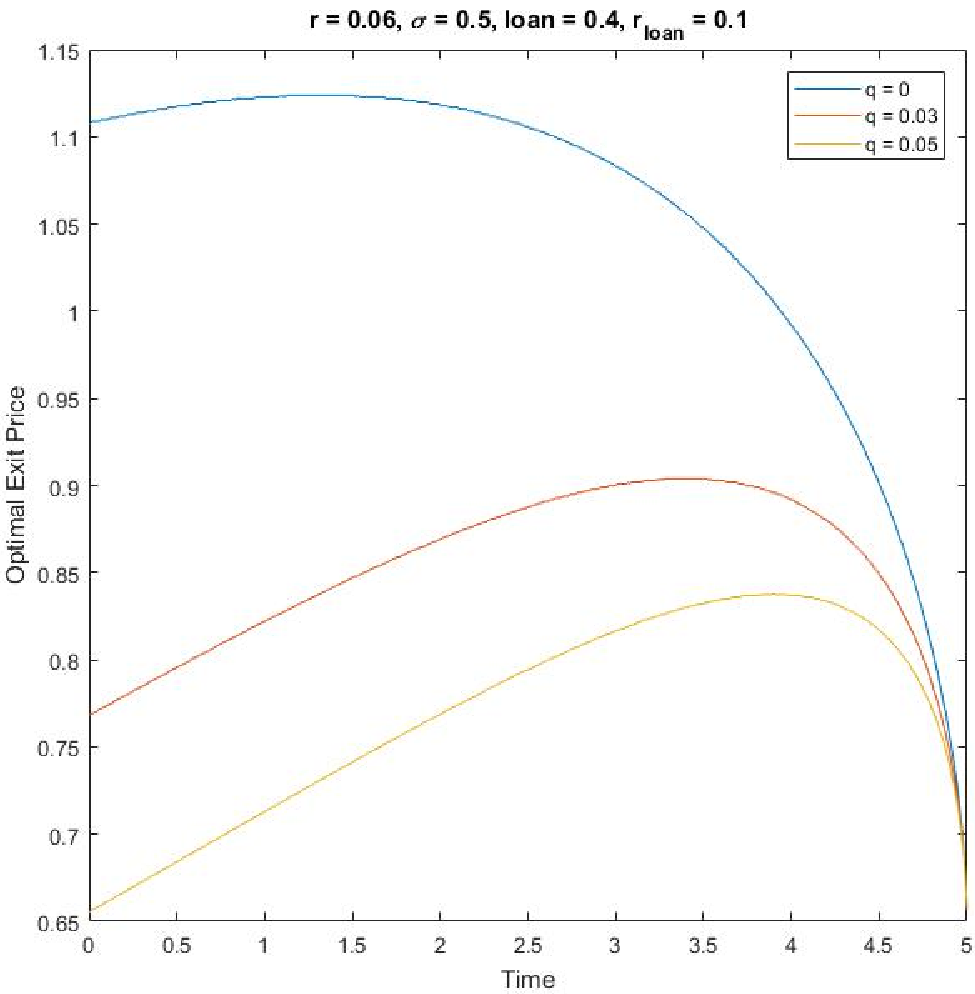

Figure 7.10: An illustration of the optimal exit boundary of a stock loan for varying dividend yield rates. The following parameters are used: $T = 5$, $\sigma = 0.4$, $r = 0.06$, $Q = 0.4$, $\gamma = 0.1$, $\delta S = 0.001$, $\delta t = 0.01$. The blue line have the dividend yield rate $q = 0$, the red line $q = 0.03$ and the yellow line $q = 0.05$.

The explanation to why the optimal exit prices decrease when the dividend yield rate increases is because the stock price will decrease. A decreasing stock price will result in a decreased stock value because of the pay-off function for a stock loan. As the stock loan approaches maturity, the time value will approach zero and hence the optimal exit price at expiration date will be equal to the principal plus accumulated interest.

For a non-dividend paying underlying asset, there exists no optimal exercise price for an American option but as it can be found in figure 7.10, there exists an optimal exit boundary for a stock loan with a non-dividend paying asset as collateral. This is because the stock loan value depends on the time-dependent principal plus accumulated interest instead of the constant strike price.

Optimal Exit Boundary - Varying Loan Interest Rate Ceteris Paribus

From figure 7.11, it can be found that the optimal exit price decreases for an increased loan interest rate at the time $t = 0$. The growth of the time value will increase with a higher loan interest rate. Therefore, at the time of expiration $t = T$, the exit prices will increase for an increased loan interest rate since they will approach the principal plus the accumulated interest rate.

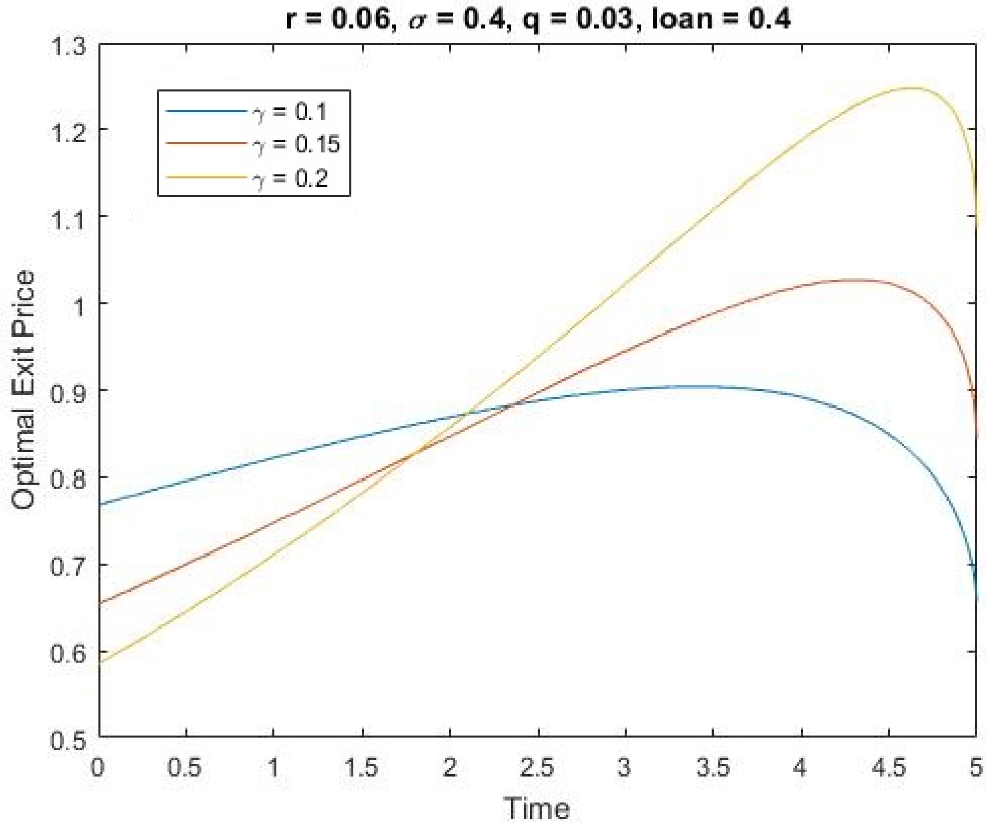

Figure 7.11: An illustration of the optimal exit boundary of a stock loan for varying loan interest rates. The following parameters are used: $T = 5$, $\sigma = 0.4$, $r = 0.06$, $Q = 0.4$, $q = 0.03$, $\delta S = 0.001$, $\delta t = 0.01$. The blue line have the loan interest rate $\gamma = 0.1$, the red line $\gamma = 0.15$ and the yellow line $\gamma = 0.2$.

This can be explained by the fact that the loan interest rate have an oppositional effect of the risk-free interest rate on the time value. At time $t = 0$, there is 5 year left until the time of expiration, which means that interest will be accumulated during five more years, which is an expense. Because of this, a higher loan interest rate will result in a lower optimal exit price. With time, the time value will increase from the loan interest rate because the time left , on which interest will be accumulated, have decreased. Because of this, a higher loan interest rate will result in a larger growth of time value as the time of expiration is approached. Consequently, the optimal exit price $t = T$ will be greater for a larger loan interest rate. The optimal exit prices from figure 7.11 at the time of expiration can be explained with the following calculations:

- Yellow line ($\gamma = 0.2$): Optimal exit price at expiration date $= 0.4e^{0.2 \cdot 5} \approx 1.09$
- Red line ($\gamma = 0.15$): Optimal exit price at expiration date $= 0.4e^{0.15 \cdot 5} \approx 0.85$
- Blue line ($\gamma = 0.1$): Optimal exit price at expiration date $= 0.4e^{0.1 \cdot 5} \approx 0.66$

Chapter 8

Conclusion and Future Work

In this thesis, American options have been valued with the Crank-Nicholson method under the Black-Scholes equation. Furthermore, the connection between stock loans and American options were used to value non-recourse loans with stocks as collateral.

The conclusions that can be drawn from the analyses in this thesis are that it is never optimal to exercise an American call option if the underlying asset does not pay dividends. For American options paying dividends, it is always optimal to exercise an American call option at the first contact between the option value and its intrinsic value, i.e. $C(S,t) = S_f - K$, where S_f is the optimal exercise price. The optimal exercise price will change over time, since the time value will decrease as the time of expiration is approached. The optimal exercise boundary shows the different optimal exercise prices during a specific options' lifetime. Furthermore, it was found that the optimal exercise price increases when the following parameters increases:

- Volatility
- Risk-free interest rate

An increasing dividend yield rate did on the other hand decrease the optimal exercise price. A higher risk-free interest rate will decrease time value faster over time.

The conclusions that can be drawn from the analyses of the stock loans is that an increased leverage increase the growth of the stock value. In other words, having a highly leveraged portfolio will result in a increased growth in value. The leverage will increase the overall volatility of the portfolio. Furthermore, it is always optimal to exercise a stock loan at the first contact between the stock loan value and its intrinsic loan value, i.e. $L(S,t) = S_f - Qe^{\gamma t}$, where S_f is the optimal exit price. Important to note for stock loans is that the loan interest rate γ will affect the time value and on the contrary to an American option, the time value can increase with time for stock loans. The effect of the loan interest rate is the opposite of the effect of the risk-free interest rate.

The different optimal prices for a stock loan can be shown in a plot of the optimal exit boundary. The optimal exit boundary show the different optimal exit prices during the lifetime of the loan for a specific portfolio of stocks used as collateral. Furthermore, the optimal exit price will increase when the volatility increases and it will decrease when the dividend yield rate increases. When time $t = 0$, i.e. today, the optimal exit price will increase when the risk-free interest rate is increased and it will decrease when the stock loan value is increased. With time, the risk-free interest rate will decrease the time value and the loan interest rate will increase the time value. Consequently, at time $t = T$, i.e. at expiration date, the optimal exit price will decrease when the risk-free interest rate is increased and it will increase when the loan interest rate is increased.

Future research opportunities exist for formulating a stock loan problem for recourse loans. The loss is not limited for recourse loans and valuing this type of loan might provide results of interest. Furthermore, there exists future research opportunities for valuing stock lending for short selling. Similar to recourse loans, the loss is not limited for this type of lending and results from short selling might be of interest as short selling is common amongst hedge funds.

Appendix A

European Options

```
% Input: S0 = current stock price, K = strike price,
% r = risk-free interest rate, T = time to expiration,
% sigma = volatility, ds = stock price step
% dt = time step
% fdmmethod = 'CRANK' or 'IMPLICIT' or 'EXPLICIT',
% optiontype = 'CALL' or 'PUT'

%Output: optionvalue = The option value

function optionvalue = VanillaEuropeanOption(S0,K,r,T,sigma,ds,dt,...
    fdmmethod,optiontype)

s1 = {'CRANK','IMPLICIT','EXPLICIT'};
s2 = fdmmethod;
fdmmethod_L = strcmp(s1,s2);
z1 = {'CALL','PUT'};
```

```
z2 = optiontype;
optiontype_L = strcmp(z1,z2);

if S0 < 0 || K < 0 || T <= 0 || r < 0 || sigma < 0 || ds < 0 ...
        || dt < 0 || sum(fdmmethod_L) ~= 1 || sum(optiontype_L) ~= 1
    fprintf('Invalid input parameters.');
    optionvalue = NaN;
    return;
end

smax = 2*max(S0,K)*exp(r*T);
M = round(smax/ds);
N = round(T/dt);

mesh = zeros(M+1,N+1);
S = linspace(0,smax,M+1);
j = 0:M;
i = 0:N;

switch optiontype
    case 'CALL'
        mesh(:,N+1) = max(S-K,0);
        mesh(1,:) = 0;
        mesh(M+1,:) = (smax-K) * exp(-r * dt * (N-i));

    case 'PUT'
```

```
        mesh(:,N+1) = max(K-S,0);
        mesh(1,:) = K * exp(-r * dt * (N-i));
        mesh(M+1,:) = 0;
end

switch fdmmethod
    case 'CRANK'
        a = (dt/4) *(sigma^2 * (j.^2) - r * j);
        b = -(dt/2) *(sigma^2 * (j.^2) + r);
        c = (dt/4) *(sigma^2 * (j.^2) + r * j);

        C = -diag(a(3:M),-1) + diag(1-b(2:M)) - diag(c(2:M-1),1);
        [L,U] = lu(C);
        D = diag(a(3:M),-1) + diag(1+b(2:M)) + diag(c(2:M-1),1);

        lostval = zeros(size(D,2),1);
        for i = N:-1:1
            if length(lostval)>1
                lostval(1) = a(2) * (mesh(1,i)+mesh(1,i+1));
                lostval(end) = c(end) * (mesh(end,i)+mesh(end,i+1));
            else
                lostval = lostval(1)+lostval(end);
            end

            mesh(2:M,i) = U \ (L \ (D * (mesh(2:M,i+1)+lostval)));
        end
```

```
case 'IMPLICIT'
    a = 0.5 * (r * dt * j - sigma^2 * dt * (j.^2));
    b = 1 + sigma^2 * dt * (j.^2) + r * dt;
    c = -0.5 * (r * dt * j + sigma^2 * dt * (j.^2));

    B = diag(a(3:M),-1) + diag(b(2:M)) + diag(c(2:M-1),1);
    [L,U] = lu(B);

    lostval = zeros(size(B,2),1);
    for i = N:-1:1
        lostval(1) = -a(2)*mesh(1,i);
        lostval(end) = -c(end)*mesh(end,i);
        if length(lostval) == 1
            lostval = -a(2)*mesh(1,i)-c(end)*mesh(end,i);
        end

        mesh(2:M,i) = U \ (L \ (mesh(2:M,i+1) + lostval));
    end

case 'EXPLICIT'
    a = 0.5 * dt * (sigma^2 * j - r) .* j;
    b = 1 - dt * (sigma^2 * j.^2 + r);
    c = 0.5 * dt * (sigma^2 * j + r) .* j;

    for i = N:-1:1
```

```
            for j = 2:M
                mesh(j,i) = a(j) * mesh(j-1,i+1) ...
                    + b(j) * mesh(j,i+1) + c(j) * mesh(j+1,i+1);
            end
        end
end

optionvalue = max(interp1(S,mesh(:,1),S0,'spline'),0);
```

Appendix B

Performance Measurement for varying δS

```
% Input: S = current stock price, K = strike price,
% r = risk-free interest rate, T = time to expiration,
% sigma = volatility, dt = time step
% start = intial stock price step,
% final = final stock price step,
% step = decrease in ds for each calculation,
% truevalue = true analytical european option value,
% fdmmethod = 'CRANK' or 'IMPLICIT' or 'EXPLICIT',
% optiontype = 'CALL' or 'PUT'

%Output: ds = number of stock price steps,
%Accuracy = The corresponding accuracy

function [ds, Accuracy] = accuracy_varying_ds(S,K,r,T,sigma,dt,...
```

```
    start ,final ,step ,truevalue ,fdmmethod,optiontype)

if start<0 || step < 0 || final < 0 || start < final
    fprintf('Invalid input parameters.');
    price = NaN;
    return;
end

Accuracy = [];
ds = [];
DS = start;
c = 1;
while DS >= final
    smax = 2*max(S,K)*exp(r*T);
    ds(1,c) = smax/DS;
    CallImp = VanillaEuropeanOption(S,K,r,T,sigma,DS,dt,...
        fdmmethod,optiontype);
    Accuracy(1,c) = CallImp;
    DS = DS - step;
    c = c + 1;
end
Accuracy = 100*max(1-abs(Accuracy-truevalue)/truevalue,0);
plot(ds, Accuracy)
end
```

Appendix C

Performance Measurement for varying δt

```
% Input: S = current stock price, K = strike price,
% r = risk-free interest rate, T = time to expiration,
% sigma = volatility, ds = stock price step
% start = intial stock price step,
% final = final stock price step,
% step = decrease in ds for each calculation,
% truevalue = true analytical european option value,
% fdmmethod = 'CRANK' or 'IMPLICIT' or 'EXPLICIT',
% optiontype = 'CALL' or 'PUT'

%Output: dt = number of time steps,
%Accuracy = The corresponding accuracy

function [dt, Accuracy] = accuracy(S,K,r,T,sigma,ds,start,final,...
```

```
    step,truevalue,fdmmethod,optiontype)

if start<0 || step < 0 || final < 0 || start < final
    fprintf('Invalid input parameters.');
    price = NaN;
    return;
end

Accuracy = [];
dt = [];
DT = start;
c = 1;
while DT >= final
    dt(1,c) = DT;
    CallImp = VanillaEuropeanOption(S,K,r,T,sigma,ds,DT,...
        fdmmethod,optiontype);
    Accuracy(1,c) = CallImp;
    DT = DT - step;
    c = c + 1;
end
dt = T./dt;
Accuracy = 100*max(1-abs(Accuracy-truevalue)/truevalue,0);
plot(dt, Accuracy)
end
```

Appendix D

American Options

```
% Input: S0 = current stock price, K = strike price,
% r = risk-free interest rate, T = time to expiration,
% sigma = volatility, ds = stock price step
% q = the dividend yield rate, dt = time step
% plotvalue = true for plot, false for not plot
% plotoptimal = true for plot, false for not plot,
% plotperpetual = true for plot, false for not plot.
% optimaltolerance = tolerance for optimal exercise boundary

%Output: optionvalue = The option value

function price = CrankNicolson(S0,K,r,T,sigma,ds,dt,q,...
    plotvalue,plotoptimal,plotperpetual,optimaltolerance)

if S0 < 0 || K < 0 || T <= 0 || r < 0 || sigma < 0....
    || ds < 0 || dt < 0 || q < 0
```

```
    fprintf('Invalid input parameters.');
    optionvalue = NaN;
    return;
end

smax = 2*max(S0,K)*exp(r*T);
M = round(smax/ds);
N = round(T/dt);

mesh = zeros(M+1,N+1);
S = linspace(0,smax,M+1);
t = linspace(0,T,N+1);
j = 0:M; %ds*j = s
i = 0:N; %dt*i = t

mesh(:,end) = max(S-K,0);
mesh(1,:) = 0;
mesh(end,:) = max(smax-K,0)*exp(-r*t);

a = (dt/4) *(sigma^2 * (j.^2) - (r-q) * j);
b = -(dt/2) *(sigma^2 * (j.^2) + r);
c = (dt/4) *(sigma^2 * (j.^2) + (r-q) * j);

C = -diag(a(3:M),-1) + diag(1-b(2:M)) - diag(c(2:M-1),1);
D = diag(a(3:M),-1) + diag(1+b(2:M)) + diag(c(2:M-1),1);
[L,U] = lu(C);
```

```
for i = N:-1:1
    optprice = U \ (L \ (D * mesh(2:M,i+1)));
    mesh(2:M,i) = max(optprice, S(2:end-1)'-K);
end
price = max(interp1(S,mesh(:,1),S0),0);

if plotoptimal == true
    beta = (-(r-q-0.5*sigma^2)+...
        sqrt((r-q-0.5*sigma^2)^2+2*sigma^2 *r))/sigma^2;
    OptimalExercise(mesh,t,K,S,T,optimaltolerance,beta,plotperpetual)
end

hold on

if plotvalue == true
    plot(S(1:end),mesh(1:end,1))
    hold on
    plot(S,max(S-K,-5))
end
end
```

Appendix E

Optimal Exercise Boundary

```
% Input: mesh = price-time mesh, t = times from mesh,
% K = strike price, S = stock prices from mesh,
% T = time to expiration,
% optimaltolerance = tolerance for optimal exercise boundary
% beta = constant value for perpetual
% plotperpetual = true for plot and false for not plot

%Output: Sf = optimal exercise prices

function Sf = OptimalExercise(mesh,t,K,S,T,optimaltolerance,...
    beta,plotperpetual)
Sf = zeros(1,length(t));
tolerance = optimaltolerance;
for i = 1:length(t)
    Sf(i) = S(find(abs(mesh(:,i)+K-S')< tolerance, 1, 'first'));
end
```

```
x=0:0.01:T;
SF = interp1(t,Sf,x);
plot(x,SF)
hold on
if plotperpetual == true
    perpetual = (beta* K )/(beta-1);
    perpetual = perpetual * ones(1, length(t));
    plot(t,perpetual)
end
end
```

Appendix F

Stock Loan

```
% Input: S0 = current stock price, r = risk-free interest rate,
% T = time to expiration, sigma = volatility,
% ds = stock price step, dt = time step
% q = the dividend yield rate, Q = principal
% loan_r = loan interest rate
% optimaltolerance = tolerance for optimal exercise boundary,
% plotexit = true for plot, false for not plot

%Output: loanvalue = The loan value

function loanvalue = CrankNicolson(S0,r,T,sigma,ds,dt,q,Q,loan_r...
    ,optimaltolerance,plotexit)

if S0 < 0 || T <= 0 || r < 0 || sigma < 0 || loan_r < 0....
    || ds < 0 || dt < 0 || q < 0 || Q < 0 || optimaltolerance < 0
    fprintf('Invalid input parameters.');
```

```
    optionvalue = NaN;
    return;
end

smax = 2*max(S0,Q)*exp(r*T);
M = round(smax/ds);
N = round(T/dt);

mesh = zeros(M+1,N+1);
S = linspace(0,smax,M+1);
t = linspace(0,T,N+1);
j = 0:M;
i = 0:N;

mesh(:,end) = max(S-(Q * exp(loan_r * T)),0);
mesh(1,:) = 0;
mesh(end,:) = (smax - (Q * exp(loan_r * dt * (N-i)))) .*...
    exp(-r * dt * (N-i));

a = (dt/4) *(sigma^2 * (j.^2) - (r-q) * j);
b = -(dt/2) *(sigma^2 * (j.^2) + r);
c = (dt/4) *(sigma^2 * (j.^2) + (r-q) * j);

C = -diag(a(3:M),-1) + diag(1-b(2:M)) - diag(c(2:M-1),1);
D = diag(a(3:M),-1) + diag(1+b(2:M)) + diag(c(2:M-1),1);
[L,U] = lu(C);
```

```
for i = N:-1:1
    mesh(2:M,i) = max(U \ (L \ (D * (mesh(2:M,i+1)))),S(2:end-1)'...
        -(Q * exp(loan_r * i * dt)));
end
loanvalue = max(interp1(S,mesh(:,1),S0),0);

if plotexit == true
OptimalExit(mesh,t,Q,loan_r,S,T,optimaltolerance,dt)
end
end
```

Appendix G

Optimal Exit Price

```
% Input: mesh = price−time mesh, t = times from mesh
% Q = principal, loan_r = loan interest rate
% S = stock prices from mesh, T = time to expiration
% optimaltolerance = tolerance for optimal exit boundary
% dt = delta time constant in mesh

%Output: Sf = optimal exit prices

function Sf = OptimalExit(mesh,t,Q,loan_r,S,T,optimaltolerance,dt)
    Sf = zeros(1,length(t));
    eps_star = optimaltolerance;
    for i = 1:length(t)
        Sf(i) = S(find(abs(mesh(:,i)+ Q * exp(loan_r * i*dt)−S')< ...
            eps_star, 1, 'first'));
    end
    x=0:0.01:T;
```

```
    SF = interp1(t,Sf,x,'spline');
    plot(x,SF)
end
```

References

[1] Einstein A. Uber die von der molekularkinetischen theorie der warme geforderte bewegung von in ruhenden flussigkeiten suspendierten teilchen. *Annalen der Physik*, 1905.

[2] Avanza Bank AB. Ansokan om kredit.

[3] Nordnet Bank AB. Villkor for ranterabatt.

[4] Fink J. Ahn M. and Fink K.E. On exercising american options: The risk of making more money than you expected. *Wilmott Magazine*, 2003.

[5] Khan A.M. Conformity with large speculators: A test of efficiency in the grain futures market. *Atlantic Economic Journal*, 14(3), 1986.

[6] Aristotle. Politics. Book II.

[7] L. Bahelier. Theorie de la speculation. *Annales Scientifiques de lEcole Normale Superieure*, pages 21–86, 1900.

[8] European Central Bank. Marketable assets.

[9] Fischer Black and Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, pages 637–654, 1973.

[10] M. J. Brennan and E. S. Schwartz. Finite difference methods and jump processes arising in the pricing of contingenet claims: A synthesis. *Journal of Financial and Quantitative Analysis*, 1978.

[11] K. Bryan. Greens functions and the heat equation.

[12] G. Curtardon. A more accurate finite difference approximation for the valuation of options. *Journal of Financial and Quantitative Analysis*, 1982.

[13] Fama F.E. The behavior of stock market prices. *Journal of Business*, 38(1), 1965.

[14] Fama F.E. Efficient capital markets: A review of theory and empirical work. *The Journal of Finance*, 25(2), 1970.

[15] E. Haug. *Complete Guide to Option Pricing Formulas*. McGraw Hill, 2007.

[16] J. Hull and A. White. Valuing derivative securities using the explicit finite difference method. *Journal of Financial and Quantative Analysis*, 1990.

[17] John C. Hull. *Options,Futures and Other Derivatives*. Pearson Education Inc, 2012.

[18] Cannon J.R. *The OneDimensional Heat Equation*. Cambridge University Press, 1984.

[19] Kwok Y. K. *Mathematical Models of Financial Derivatives*. Springer, 2008.

[20] Putri E.R.M. Lu X. Semi-analytic valuation of stock loans with finite maturity. *Elsevier B.V.*, 2015.

[21] R. C. Merton. Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 1973.

[22] Reserve Bank of New Zealand. Loan-to-value ratio restrictions.

[23] The Congress of the United States. Fannie mae, freddie mac, and the federal role in the secondary mortgage market, 2010.

[24] Brandimarte P. *Numerical Methods in Finance and Economics*. Wiley, 2006.

[25] Samuelson P. Rational theory of warrant pricing. *Industrial Management Review*, 6(2), 1965.

[26] Samuelson P. Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, 6(2), 1972.

[27] Brown R. A brief account of microscopical observations made in the months of june, july and august, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *Philosophical Magazine*, 1, 1827.

[28] Westergren B. Rade L. *Mathematics Handbook - for Science and Engineering*. Studentlitteratur AB, 2003.

[29] J. Regnault. *Calcul des Chances et Philosophie de la Bourse*. Mallet-Bachelier and Castel., 1863.

[30] Kitchen D. White B. Recourse vs. nonrecourse: Commercial real estate financing which one is right for you?

[31] DeWynne J. Willmott P. and Howison S. Option pricing: Mathematical models and computation. Oxford Financial Press, 1993.

[32] DeWynne J. Willmott P. and Howison S. *The Mathematics of Financial Derivatives*. Cambridge University Press, 2008.

TRITA TRITA-MAT-K 2017:21
ISRN -KTH/MAT/K--17/21--SE