ME106 Project12 Report

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Contents

1 Problem Restatement

This problem is an example where Navier-Stokes equation is used. An oscillating plate is moving in the pattern $U = cos(500t)$, where U denotes its velocity. The plate is horizontal and infinitely long, where the domain of fluid is semi-infinite. With additional information, we can solve for the fluid field which varies with time.

Several assumptions have to be made before we can proceed to solve the problem:

- The fluid is incompressible
- The flow is always laminar
- The flow satisfies no-slip boundary conditions
- The fluid is Newtonian
- Gravity is not taken into consideration

There are some other assumptions closely related to the equations themselves. For the sake of convenience and perspicuity, they will be illustrated later in this report.

2 Part (a). Solve For the Velocity Field

First we write the Navier-Stokes Equation for this flow field. Then, with proper assumptions, the equation can be solved analytically. Last, we plot the velocity field.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y}) = -\frac{\partial P}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{2}$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}) = -\frac{\partial P}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{3}$$

And boundary conditions: no-slip boundary condition

$$u|_{y=0} = U \tag{4}$$

where $U = Re[e^{i500t}]$. Also, the condition that fluid cannot penetrate the wall

$$v|_{y=0} = 0 \tag{5}$$

Next we try to simplify these equations by vanishing all zero terms. Since the plate is indefinite, any value cannot vary along the x direction because of symmetry. Therefore, all terms with x will be eliminated. For equation 1, we have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0 \tag{6}$$

For equation 2 and 3, also divide every term by ρ thus μ turns into ν. Therefore we have

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} \tag{7}$$

$$\nu \frac{\partial^2 v}{\partial y^2} = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} \tag{8}$$

Note that with boundary condition $v|_{y=0} = 0$, and equation 6, we can integrate and conclude that $v = 0$ throughout the space. Hence the equations can be further simplified into

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{9}$$

$$v = 0 \tag{10}$$

These are the equations that we need to solve, and they happen to have a close-form solution. When t is very large and the system has entered a stable(but not steady) state, all fluid particles move in an oscillating manner and the angular frequency is $\omega = 500$. Therefore, we can suppose that the velocity u is of the form

$$u = Re[f(y)e^{i500t}] \tag{11}$$

where $f(y)$ is some function of y. Substitute this expression into equation 9, and we obtain

$$\frac{\partial^2 f}{\partial y^2} = \frac{500i}{\nu} f \tag{12}$$

The general solution for this 2nd order homogeneous differential equation is

$$f = C_1 e^{\sqrt{500i/\nu}y} + C_2 e^{-\sqrt{500i/\nu}y} \tag{13}$$

where C_1 and C_2 are two arbitrary constants. Next, we obtain C_1 and C_2 by applying our boundary condition in 4. Plugging in $y = 0$, we have

$$C_1 + C_2 = 1 \tag{14}$$

However, by inspection, if $C_1 \neq 0$, then the amplitude of the velocity will go to infinity as y goes to infinity, which cannot be the case for our viscous flow. This is because the viscosity damps the velocity, meaning the momentum cannot diffuse to that far without being damped greatly. Therefore, we get

$$C_1 = 0, C_2 = 1 \tag{15}$$

This yields

$$f(y) = e^{-\sqrt{500i/\nu}y} \tag{16}$$

Thus we have

$$u = Re[e^{-\sqrt{500i/\nu}y+i500t}], v = 0 \tag{17}$$

Plugging in all values and simplify everything in 17 and we obtain

$$u = e^{-3.94y} cos(500t - 3.94y), v = 0 \tag{18}$$

3 Part (b). Non-dimensionalize the Result

In this section we want to find two values, x_0 and v_0 that we can use to non-dimensionalize the final answer. These values should be characteristic velocity and length for this problem.

For velocity, the normalization term is chosen as the maximum velocity of the plate.

Note that the form of our solution 18 looks like a wave that is propagating with time, except that the amplitude of the wave is shrinking with time. The general form of a harmonic wave is

$$y = Acos(\omega t - \frac{2\pi}{\lambda}x) \tag{19}$$

Comparing this with 18, the wave length is determined to be 1.6 m. Therefore, we can choose this wave length as our normalization factor for height, namely 1.6 m.

The scaled velocity shows how fast the fluid is moving compared to the plate. The height shows the distance compared to the wave length of the wave that is propagating from the plate to the farther.

As such we have $v_0 = 1$ m/s and $x_0 = 1.6$m. Thus the scaled y-value turns into the multiple of the amplitude and the scaled x-value turns into the multiple of the plate's maximum velocity.

Non-dimensionalize our original equation with the two scales, namely $\hat{u} = \frac{u}{v_0}$ and $\hat{y} = \frac{y}{x_0}$, and we obtain

$$\hat{u} = e^{-2\pi\hat{y}}cos(500t - 2\pi\hat{y}) \tag{20}$$

where $\hat{u}$ stands for the non-dimensionalized velocity and $\hat{y}$ stands for the non-dimensionalized height.

4 Part (c). Plot Velocity versus Height

In this section, we plot the non-dimensional velocity versus the height in the fluid. The y-axis is the non-dimensional height and the x-axis is the non-dimensional velocity.

Note that because the time period for the wave is 0.0126 s, although we are asked to examine the fluid motion within 4 s, we may as well just examine everything within the first 0.0126 s.

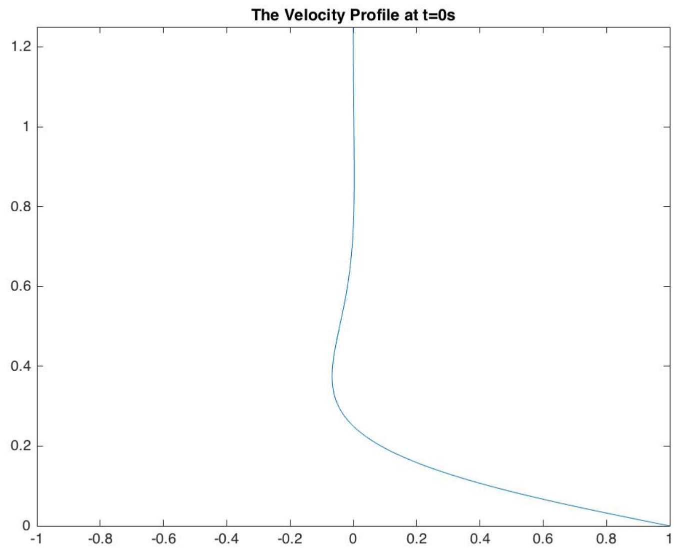

Figure 1: The Scaled Velocity Profile at t=0

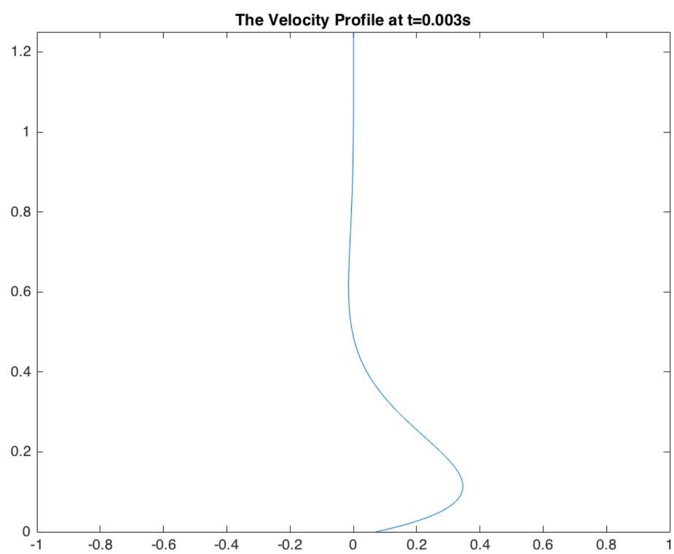

Figure 2: The Scaled Velocity Profile at t=0.003.

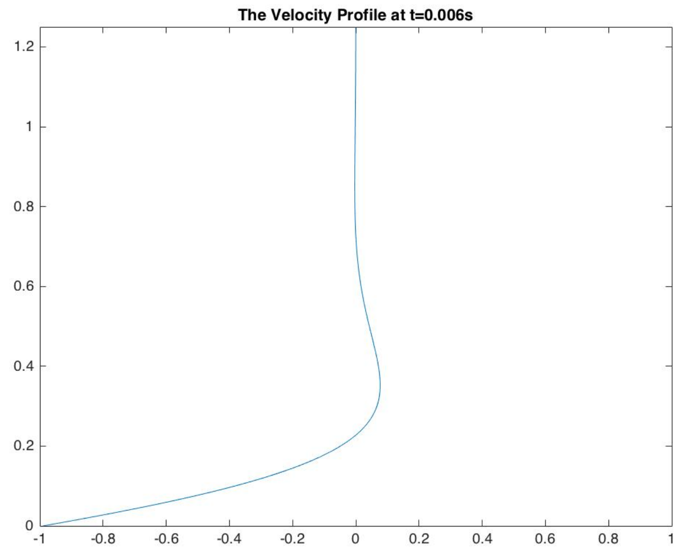

Figure 3: The Scaled Velocity Profile at t=0.006.

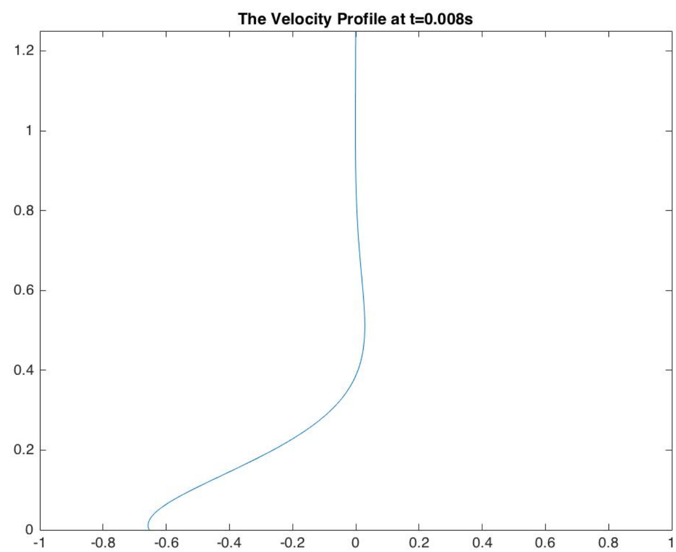

Figure 4: The Scaled Velocity Profile at t=0.008.

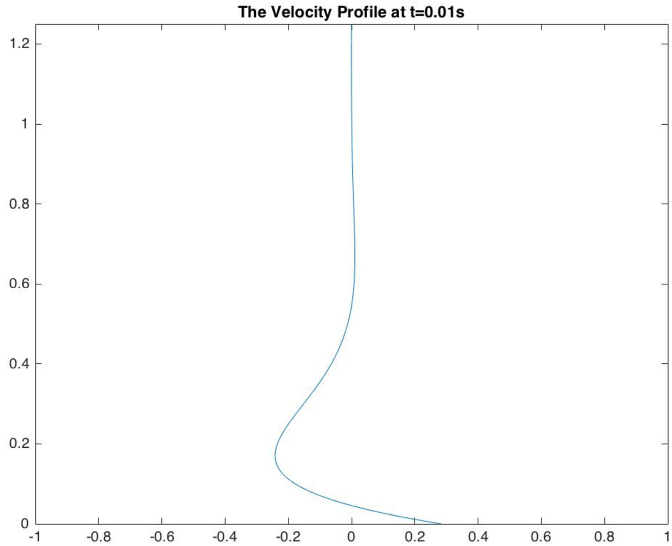

Figure 5: The Scaled Velocity Profile at t=0.01.

Combining them together into one graph, we can get an idea of how the velocity evolves with time.

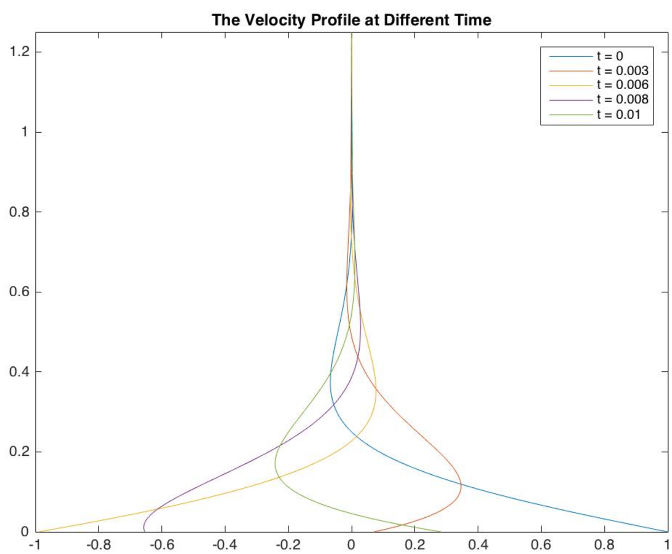

Figure 6: All plots combined

From these plots we can see some features that reflect the physical significances of it. The magnitude of the velocity is the same as the velocity of plate due to no-slip boundary condition(so the maximum nondimensional velocity is 1), and is often relatively high when y is **relatively** small. y takes the scale of 10^0 because the damping effect of the fluid is significant and the wave form propagating from the plate goes to zero very quickly, before one wave length. Fluid very far away will not be affected by the oscillation of the plate and will remain still. Also, the velocity profile evolves in a periodic pattern, which is obvious from our equation. This is also due to the fact that the plate is moving periodically, and that inertia exerts smaller influence on the motion of fluid particles than viscosity, so that the motion of the plate plays a key role in moving fluid particles along.

5 Explanation for the Code

The code for plotting is attached in appendix. The basic idea is to generate a function handle that calculates the velocity at a given time and height. Then we pass in the time, and the program generates a series of height from 0 to 1.25(undimensional height, equivalent to 2 m). Then the code will plot all values for this series of height at this time, and draw a graph accordingly.

Appendix: The code for plotting is as follows. Below is the code to plot everything at different time on the same graph.

```
%uy = @(y, t) ( real(exp( -sqrt(1i/500/16.1).*y + 500i*t )) ); % the velocity function u(y, t)
uy = @(y, t)(exp( -2*pi.*y).*cos(500*t - 2*pi.*y)); % the velocity function u(y, t)
y = linspace(0, 1.25, 1000); % create a series of height
ks = [0, 0.003, 0.006, 0.008, 0.01];
for k = ks
    u = uy(y, k); % calculate an array of velocity
    plot(u, y); % plot the velocity with respect to height
    hold on
end
title('The Velocity Profile at Different Time')
legend('t = 0','t = 0.003','t = 0.006','t = 0.008','t = 0.01')
xlim([-1 1]) % set x axis scale
ylim([0 1.25])
```