Trying to converge to an equilibrium flow

November 2, 2023

Throughout, Vickrey queuing model is used.

Iterative Model

TBD

$$h_P^{(i+1)}(\theta) = \left(1 - \alpha(d_P^{(i)}(\theta))\right) \cdot h_P^i(\theta) + \frac{\mathbb{I}_{\{d_P^{(i)} < \varepsilon\}}(\theta)}{\sum_{P'} \mathbb{I}_{\{d_{P'}^{(i)} < \varepsilon\}}(\theta)} \cdot \left(\sum_{P'} \alpha(d_P^{(i)}(\theta)) \cdot h_P^i(\theta)\right).$$

Dynamic Replicator Model

Let $\mathcal{P}$ be a collection of s-t paths. At time θ, each path P receives a fraction $h_P(\theta)$ of the total inflow $u(\theta)$. We are considering the replication dynamic:

$$\dot{h}_P = R \cdot h_P[\phi_P - \sum_{P' \in \mathcal{P}} h_{P'} \phi_{P'}],\ \forall P \in \mathcal{P},$$

where R is a constant. Role of the fitness ϕ is played by average experienced travel time of the particles:

$$\phi_P^{a.t.}(\theta, h) = \frac{1}{F_P^+(\theta)} \left(\int_0^\theta F_P^+(\psi) d\psi - \int_0^\theta F_P^-(\psi) d\psi\right).$$

Above, $F_P^+(\cdot)$, $F_P^-(\cdot)$ denote functions of accumulated path in- and outflow.

Another option is to use last available travel time:

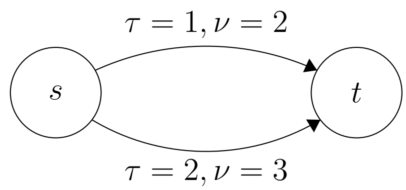

Figure 1: Simple network

$$\phi_P^{l.t.}(\theta, h) = \begin{cases} \theta, & 0 \le \theta < T_P(0) \\ \theta - T_P^{-1}(\theta), & \theta \ge T_P(0) \end{cases},$$

$T_P(\cdot)$ is path exit time function.

Unstable instance

Consider simple network with two parallel edges on Figure 1 and inflow $u(\theta) = 5$. Let upper edge have index 0 and lower 1.

The equilibrium flow is achieved as follows: all the inflow is redirected to the shorter edge, until both edges achieve equal costs, then the distribution is proportional to the capacities.

$$h_0^*(\theta) = \begin{cases} 1, & 0 \le \theta < \frac{2}{3} \\ \frac{2}{5}, & \theta \ge \frac{2}{3} \end{cases}, \quad h_1^*(\theta) = 1 - h_0^*(\theta).$$

Numerical approximation

Figures 2 and 3 show numerically computed inflow shares and fitness of replicator flow with initial condition $h_0(0) = h_1(0) = \frac{1}{2}$. The following approximation was used:

$$h_P(\theta + \Delta\theta) \approx h_P(\theta) \cdot e^{A[\phi_P - \sum_{P'} h_{P'} \phi_{P'}] \cdot \Delta\theta}$$

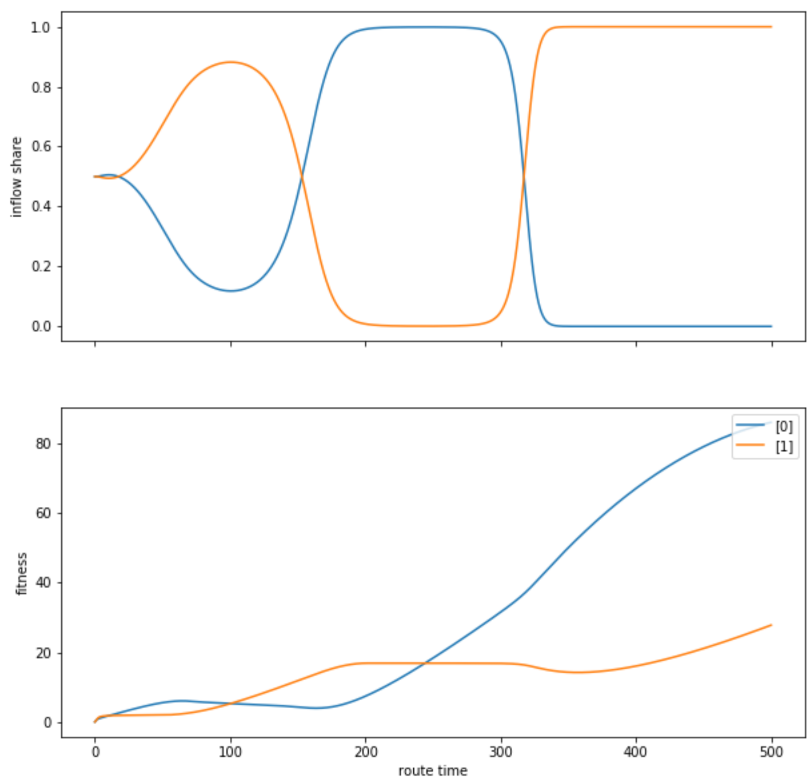

Figure 2: Fluctuations, average travel time

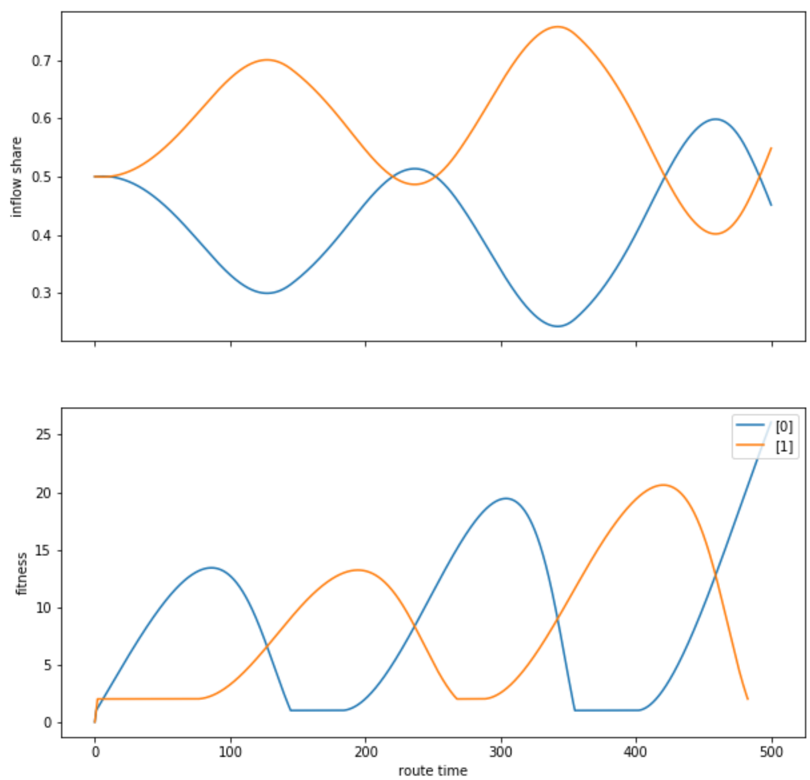

Figure 3: Fluctuations, last travel time

Proof ?

Necessary condition for stability: $\forall P: \ \phi_P(\theta, h) = \sum_{P'} h_{P'}(\theta)\phi_{P'}(\theta, h)$.
In both cases, influence of inflows on fitnesses is delayed for at least τ_0. So, stability condition can be maintained only if it was already present on the moment of last influence, and hence should have been fulfilled from the very start.