Trying to converge to an equilibrium flow

November 20, 2023

Throughout, Vickrey queuing model is used.

Iterative Model

TBD

$$h_P^{(i+1)}(\theta) = \left(1 - \alpha(d_P^{(i)}(\theta))\right) \cdot h_P^i(\theta) + \frac{\mathbb{I}_{\{d_P^{(i)}<\varepsilon\}}(\theta)}{\sum_{P'} \mathbb{I}_{\{d_{P'}^{(i)}<\varepsilon\}}(\theta)} \cdot \left(\sum_{P'} \alpha(d_P^{(i)}(\theta)) \cdot h_P^i(\theta)\right).$$

Dynamic Replicator Model

Let $\mathcal{P}$ be a collection of s-t paths. At time θ, each path P receives a fraction $h_P(\theta)$ of the total inflow $u(\theta)$. We are considering the replication dynamic:

$$\dot{h}_P = R \cdot h_P \cdot a_P, \quad \forall P \in \mathcal{P},$$

where $R > 0$ is a constant, $a_P = \phi_P - \sum_{P' \in \mathcal{P}} h_{P'} \cdot \phi_{P'}$ is advantage function based on fitness ϕ.

Role of the fitness can be played by negative signed average experienced travel time of the particles:

$$\phi_P^{a.t.}(\theta, h) = -\frac{1}{F_P^+(\theta)} \left(\int_0^\theta F_P^+(\psi)d\psi - \int_0^\theta F_P^-(\psi)d\psi \right).$$

Above, $F_P^+(\cdot)$, $F_P^-(\cdot)$ denote functions of accumulated path in- and outflow.

$\tau = 1, \nu = 2$
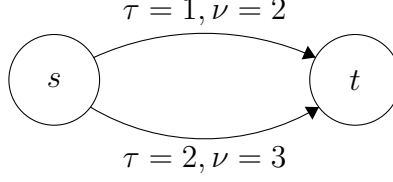
$\tau = 2, \nu = 3$

s

t

Figure 1: Simple network

Another option is negative signed last available travel time:

$$\phi_P^{l.t.}(\theta, h) = -1 \cdot \begin{cases} \theta, & 0 \le \theta < T_P(0) \\ \theta - T_P^{-1}(\theta), & \theta \ge T_P(0) \end{cases},$$

$T_P(\cdot)$ is path exit time function.

Finally, utilising constant predictors for queues, one can predict path exit times $\hat{T}_P$ and use negative predicted travel time:

$$\phi_P^{p.t.}(\theta, h) = -(\hat{T}_P(\theta) - \theta).$$

If path P has only one edge, then $\hat{T}_P(\cdot) \equiv T_P(\cdot)$.

Considered Instance

Consider simple network with two parallel edges on Figure 1 and inflow $u(\theta) = 5$. Let upper edge have index 0 and lower 1.

The equilibrium flow is achieved as follows: all the inflow is redirected to the shorter edge, until both edges achieve equal costs, then the distribution is proportional to the capacities.

$$h_0^*(\theta) = \begin{cases} 1, & 0 \le \theta < \frac{2}{3} \\ \frac{2}{5}, & \theta \ge \frac{2}{3} \end{cases}, \quad h_1^*(\theta) = 1 - h_0^*(\theta).$$

Numerical Approximation

Figures 5, 6 and 7 show numerically computed inflow shares and fitness of replicator flow with initial condition $h_0(0) = h_1(0) = \frac{1}{2}$. The following approximation was used:
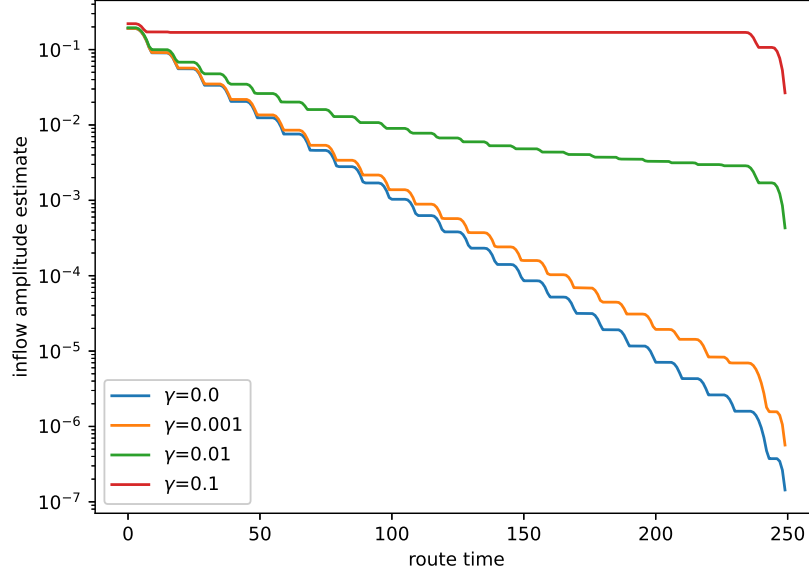
2

Figure 2: Predicted travel time with logit regularization $\lambda = 1$

$$h_P(\theta + \Delta\theta) \approx \frac{h_P(\theta) \cdot e^{R \cdot a_P \cdot \Delta\theta}}{\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta}}.$$

By construction of the initial dynamic $\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta} = 1 + o(\Delta\theta)$. Normalisation is required to remain on simplex.

Stabilizing Dynamics

Logit Regularization

One can use modify the fitness using logit regularizer:

$$\phi_P^{\text{reg}}(\theta, h) = \phi_P(\theta, h) - \lambda e^{-\gamma \cdot \theta} \ln h_P(\theta),$$

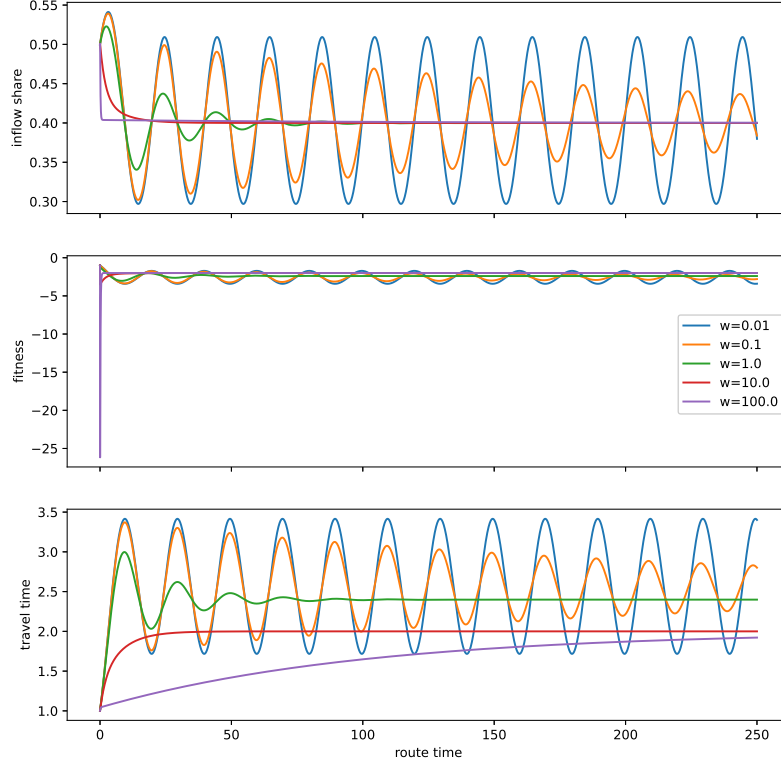where $\lambda > 0$ is a scaling constant, $\gamma > 0$ is regularization decay.

Figure 3: $\phi = -\hat{t}^{2\Delta\theta}(\theta + w), R = 0.1, \Delta\theta = 0.01$

Projected Travel Times

If we use use projected queue values $\hat{q}_e^{\delta}(\theta') = q_e(\theta) + \frac{q_e(\theta) - q_e(\theta - \delta)}{\delta} \cdot (\theta' - \theta)$ to estimate upcoming travel times, we can use fitness $\phi^{\text{proj t.t.}}(\theta, h) = -\hat{t}^{\delta}(\theta + w)$ for different values of projection window w. See Figure 2.
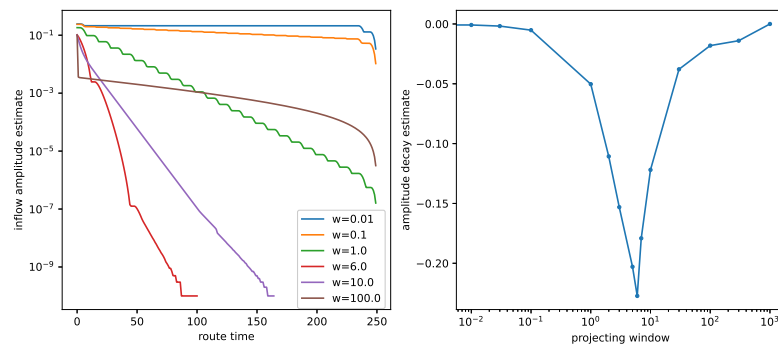
Unstable dynamics
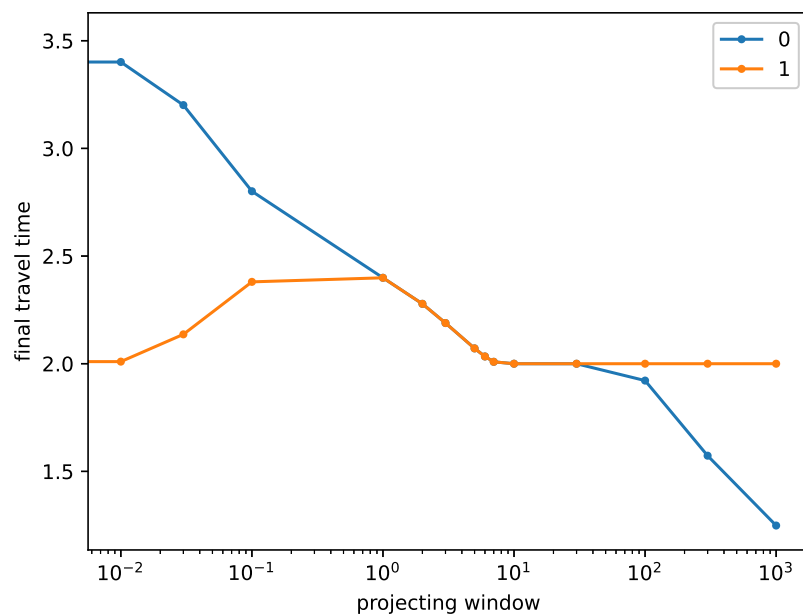
Figure 4: Inflow amplitudes for path 0
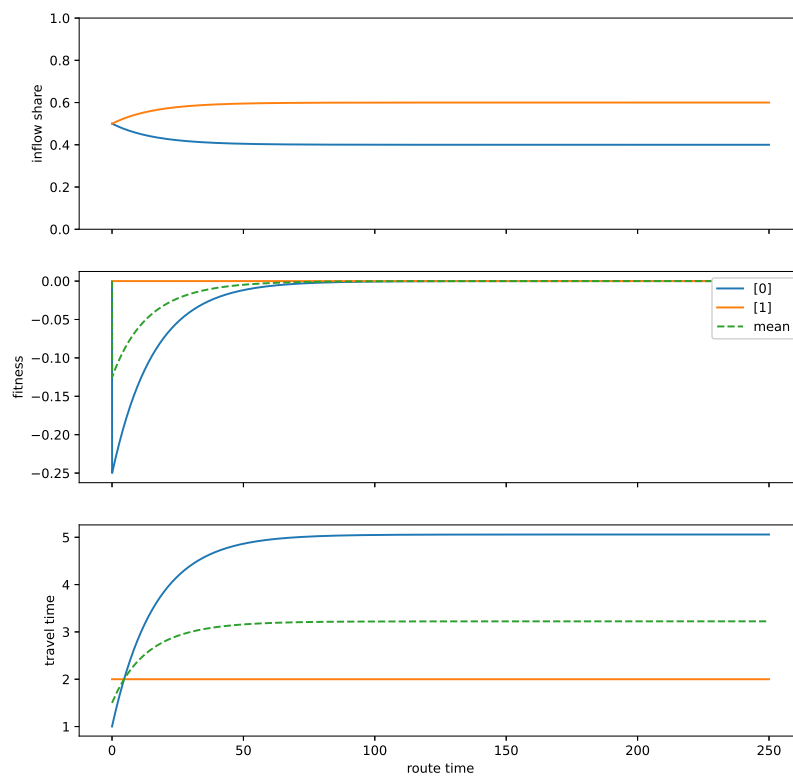

Figure 5: Final travel times

5

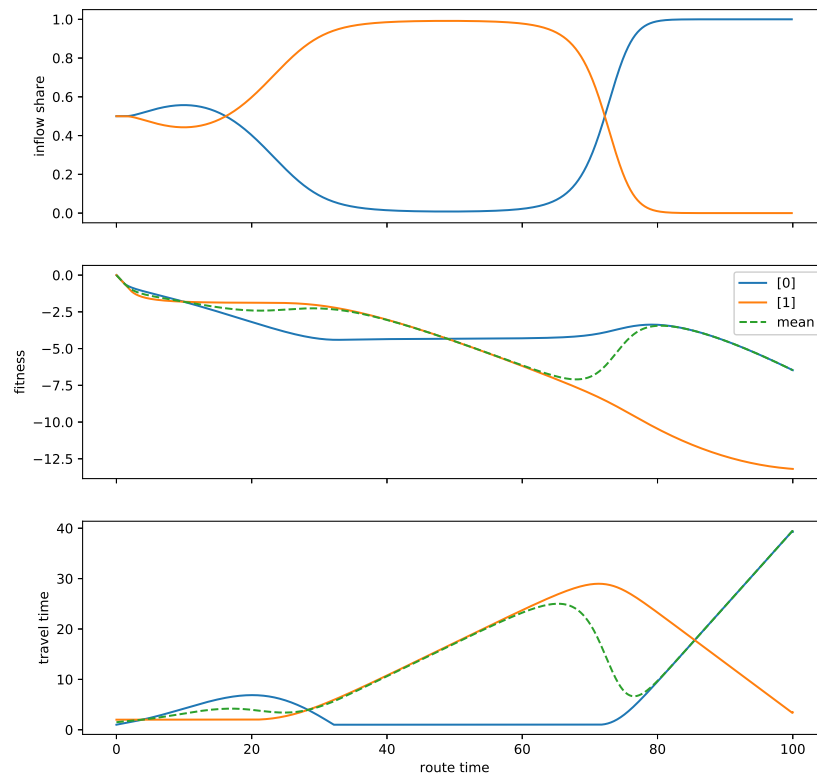Figure 6: Travel time gradient as fitness

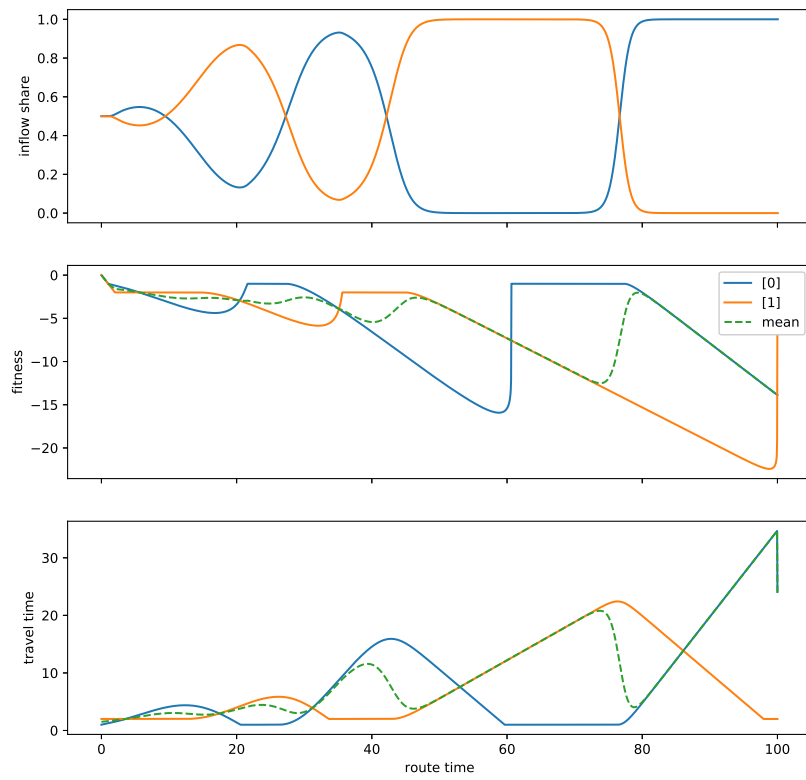Figure 7: Fluctuations, average travel time

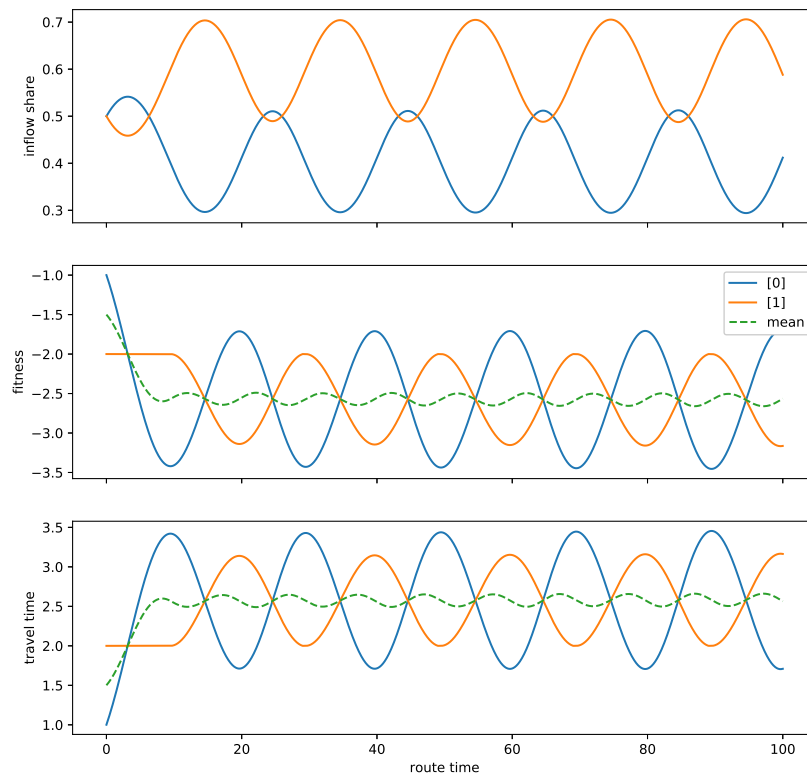Figure 8: Fluctuations, last travel time

Figure 9: Fluctuations, predicted travel time

Proof ?

Necessary condition for stability: $\forall P: \phi_P(\theta, h) = \sum_{P'} h_{P'}(\theta)\phi_{P'}(\theta, h)$.
In cases of average travel time and last travel, influence of inflows on fitnesses is delayed for at least τ_0.

Sufficient condition for stability for non-aggregated travel times: queues have to not change for the period of delay. This guarantees that fitnesses don't change once equality is reached.

Even when we use undelayed travel times, inflows tend to get in antiphase with fitnesses. In this example, we need inflows to equalize with edge capacities at the moment when both path have same costs.