Trying to converge to an equilibrium flow

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Throughout, Vickrey queuing model is used.

Iterative Model

TBD

$$h_P^{(i+1)}(\theta) = \left(1 - \alpha(d_P^{(i)}(\theta))\right) \cdot h_P^i(\theta) + \frac{\mathbb{I}_{\{d_P^{(i)} < \varepsilon\}}(\theta)}{\sum_{P'} \mathbb{I}_{\{d_{P'}^{(i)} < \varepsilon\}}(\theta)} \cdot \left(\sum_{P'} \alpha(d_P^{(i)}(\theta)) \cdot h_P^i(\theta)\right).$$

Dynamic Replicator Model

Let $\mathcal{P}$ be a collection of s-t paths. At time θ, each path P receives a fraction $h_P(\theta)$ of the total inflow $u(\theta)$. We are considering the replication dynamic:

$$\dot{h}_P = R \cdot h_P \cdot a_P, \qquad \forall P \in \mathcal{P},$$

where $R > 0$ is a constant, $a_P = \phi_P - \sum_{P' \in \mathcal{P}} h_{P'} \cdot \phi_{P'}$ is advantage function based on fitness ϕ.

Role of the fitness can be played by negative signed average experienced travel time of the particles:

$$\phi_P^{a.t.}(\theta, h) = -\frac{1}{F_P^+(\theta)} \left(\int_0^\theta F_P^+(\psi) d\psi - \int_0^\theta F_P^-(\psi) d\psi\right).$$

Above, $F_P^+(\cdot)$, $F_P^-(\cdot)$ denote functions of accumulated path in- and outflow.

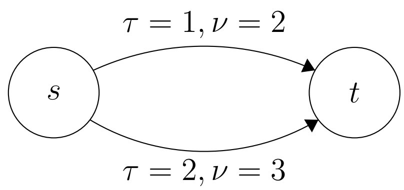

Figure 1: Simple network

Another option is negative signed last available travel time:

$$\phi_P^{l.t.}(\theta, h) = -1 \cdot \begin{cases} \theta, & 0 \le \theta < T_P(0) \\ \theta - T_P^{-1}(\theta), & \theta \ge T_P(0) \end{cases},$$

$T_P(\cdot)$ is path exit time function.

Finally, utilising constant predictors for queues, one can predict path exit times $\hat{T}_P$ and use negative predicted travel time:

$$\phi_P^{p.t.}(\theta, h) = -(\hat{T}_P(\theta) - \theta).$$

If path P has only one edge, then $\hat{T}_P(\cdot) \equiv T_P(\cdot)$.

Unstable instance

Consider simple network with two parallel edges on Figure 1 and inflow $u(\theta) = 5$. Let upper edge have index 0 and lower 1.

The equilibrium flow is achieved as follows: all the inflow is redirected to the shorter edge, until both edges achieve equal costs, then the distribution is proportional to the capacities.

$$h_0^*(\theta) = \begin{cases} 1, & 0 \le \theta < \frac{2}{3} \\ \frac{2}{5}, & \theta \ge \frac{2}{3} \end{cases}, \quad h_1^*(\theta) = 1 - h_0^*(\theta).$$

Numerical approximation

Figures 2, 3 and 4 show numerically computed inflow shares and fitness of replicator flow with initial condition $h_0(0) = h_1(0) = \frac{1}{2}$. The following approximation was used:

$$h_P(\theta + \Delta\theta) \approx \frac{h_P(\theta) \cdot e^{R \cdot a_P \cdot \Delta\theta}}{\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta}}.$$

By construction of the initial dynamic $\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta} = 1 + o(\Delta\theta)$. Normalisation is required to remain on simplex.

Stabilizing dynamic

If we use try to estimate upcoming travel times using projected queue values $\hat{q}_e = q_e(\theta) + \frac{q_e(\theta) - q_e(\theta - \delta)}{\delta} \cdot w$, fluctuation decays over time, see Figure 5. Decay seems to be present only if $w > \Delta\theta$.

Proof ?

Necessary condition for stability: $\forall P: \ \phi_P(\theta, h) = \sum_{P'} h_{P'}(\theta) \phi_{P'}(\theta, h)$.
In cases of average travel time and last travel, influence of inflows on fitnesses is delayed for at least τ_0.

Sufficient condition for stability for non-aggregated travel times: queues have to not change for the period of delay. This guarantees that fitnesses don't change once equality is reached.

Even when we use undelayed travel times, inflows tend to get in antiphase with fitnesses. In this example, we need inflows to equalize with edge capacities at the moment when both path have same costs.

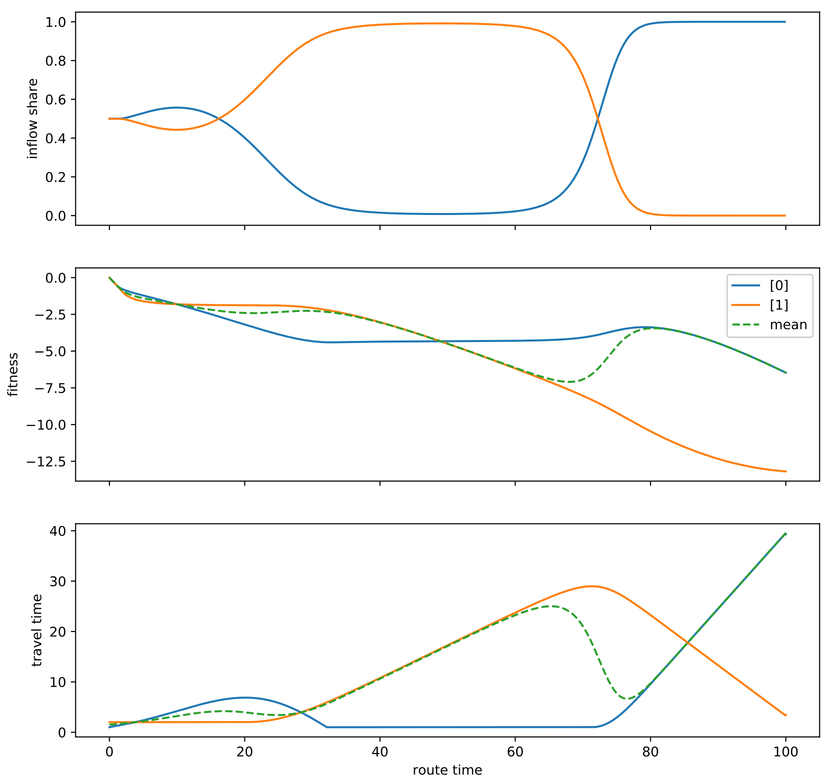

Figure 2: Fluctuations, average travel time

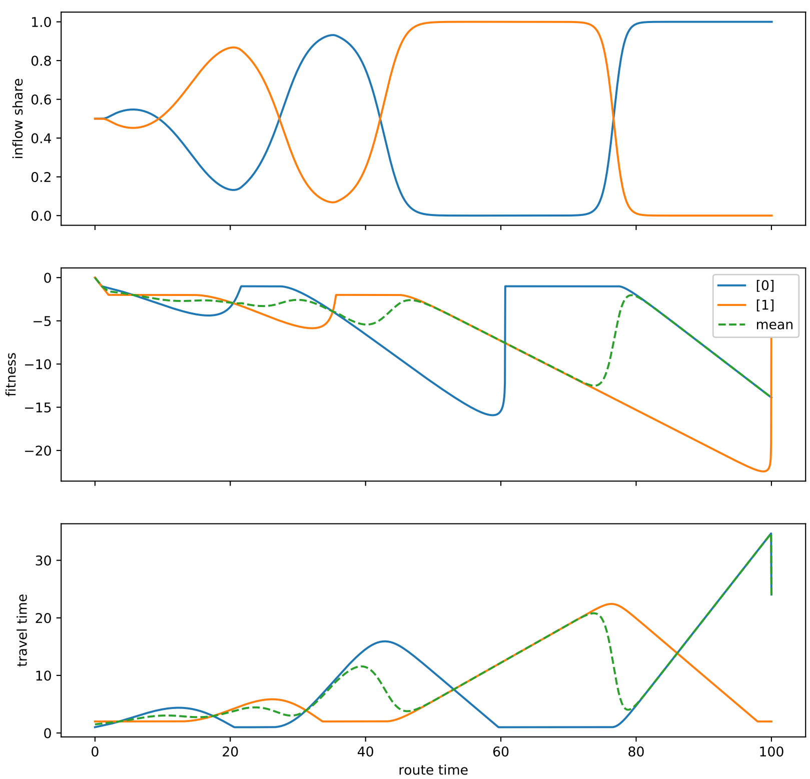

Figure 3: Fluctuations, last travel time

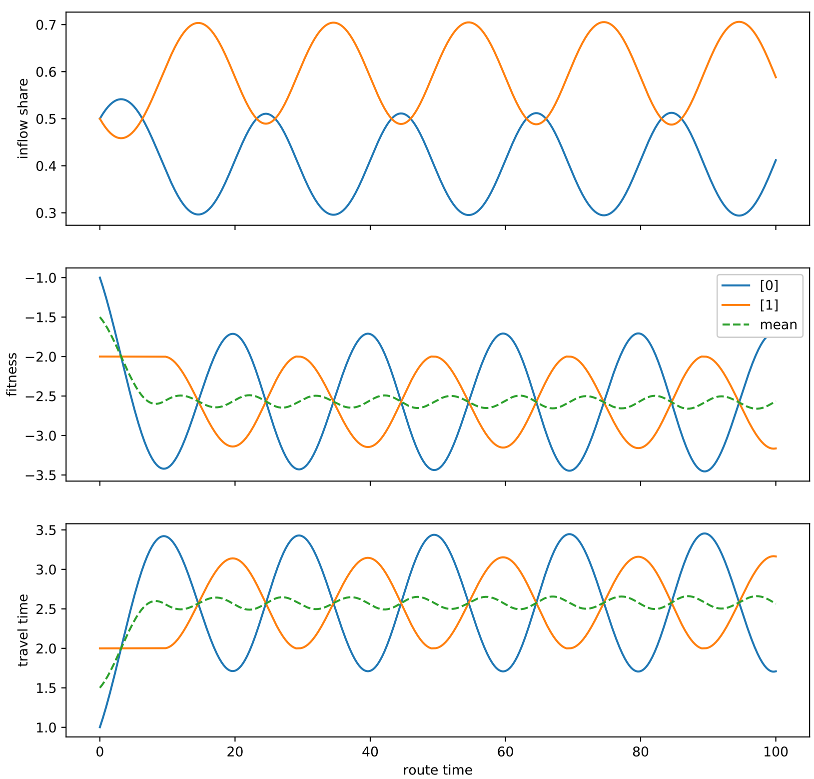

Figure 4: Fluctuations, predicted travel time

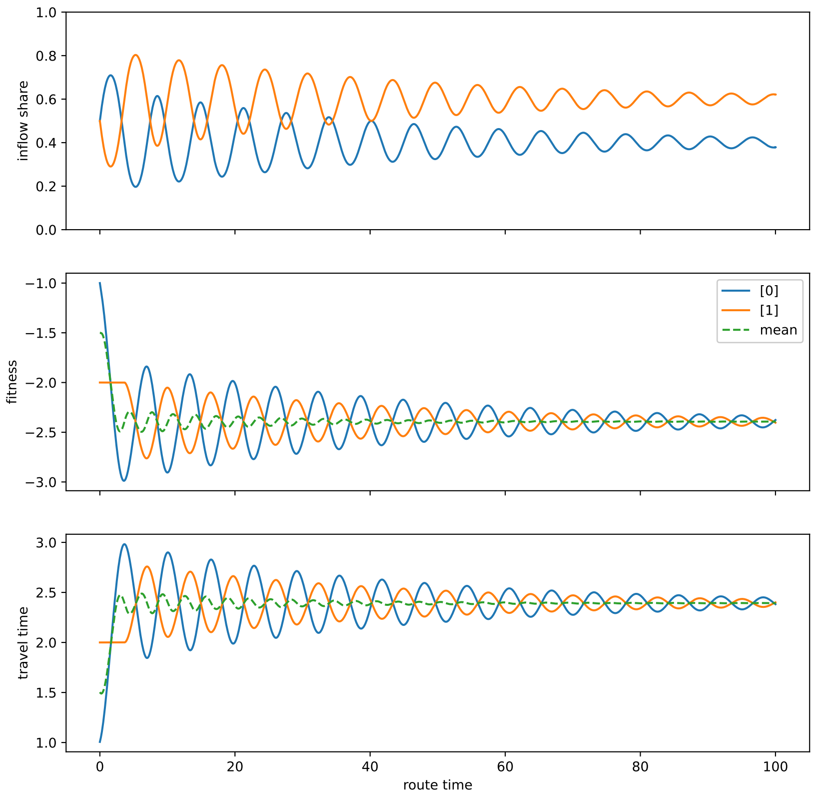

Figure 5: Stabilization, projected travel time