

# Trying to converge to an equilibrium flow

December 23, 2023

Throughout, Vickrey queuing model is used.

## Iterative Model

TBD

$$h_P^{(i+1)}(\theta) = \left(1 - \alpha(d_P^{(i)}(\theta))\right) \cdot h_P^i(\theta) + \frac{\mathbb{I}_{\{d_P^{(i)} < \varepsilon\}}(\theta)}{\sum_{P'} \mathbb{I}_{\{d_{P'}^{(i)} < \varepsilon\}}(\theta)} \cdot \left(\sum_{P'} \alpha(d_{P'}^{(i)}(\theta)) \cdot h_{P'}^i(\theta)\right).$$

## Dynamic Replicator Model

Let  $\mathcal{P}$  be a collection of  $s$ - $t$  paths. At time  $\theta$ , each path  $P$  receives a fraction  $h_P(\theta)$  of the total inflow  $u(\theta)$ . We are considering the replication dynamic:

$$\dot{h}_P = R \cdot h_P \cdot a_P, \quad \forall P \in \mathcal{P},$$

where  $R > 0$  is a constant,  $a_P = \phi_P - \sum_{P' \in \mathcal{P}} h_{P'} \cdot \phi_{P'}$  is advantage function based on fitness  $\phi$ .

Role of the fitness can be played by negative signed average experienced travel time of the particles:

$$\phi_P^{a.t.}(\theta, h) = -\frac{1}{F_P^+(\theta)} \left( \int_0^\theta F_P^+(\psi) d\psi - \int_0^\theta F_P^-(\psi) d\psi \right).$$

Above,  $F_P^+(\cdot)$ ,  $F_P^-(\cdot)$  denote functions of accumulated path in- and outflow.

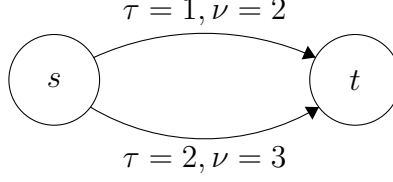


Figure 1: Simple network

Another option is negative signed last available travel time:

$$\phi_P^{l.t.}(\theta, h) = -1 \cdot \begin{cases} \theta, & 0 \leq \theta < T_P(0) \\ \theta - T_P^{-1}(\theta), & \theta \geq T_P(0) \end{cases},$$

$T_P(\cdot)$  is path exit time function.

Finally, utilising constant predictors for queues, one can predict path exit times  $\hat{T}_P$  and use negative predicted travel time:

$$\phi_P^{p.t.}(\theta, h) = -(\hat{T}_P(\theta) - \theta).$$

If path  $P$  has only one edge, then  $\hat{T}_P(\cdot) \equiv T_P(\cdot)$ .

## Considered Instance

Consider simple network with two parallel edges on Figure 1. Let upper edge have index 0 and lower 1. We consider inflow  $u(\theta) \equiv U$ .

The equilibrium flow is achieved as follows: all the inflow is redirected to the shorter edge, until both edges achieve equal costs, then the distribution is proportional to the capacities.

$$h_0^*(\theta) = \begin{cases} 1, & 0 \leq \theta < (\tau_1 - \tau_0) \frac{\nu_0}{U - \nu_0} \\ \frac{\nu_0}{U}, & \theta \geq (\tau_1 - \tau_0) \frac{\nu_0}{U - \nu_0} \end{cases}, \quad h_1^*(\theta) = 1 - h_0^*(\theta).$$

## Numerical Approximation

show numerically computed inflow shares and fitness of replicator flow with initial condition  $h_0(0) = h_1(0) = \frac{1}{2}$ . The following approximation was used:

$$h_P(\theta + \Delta\theta) \approx \frac{h_P(\theta) \cdot e^{R \cdot a_P \cdot \Delta\theta}}{\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta}}.$$

By construction of the initial dynamic  $\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta} = 1 + o(\Delta\theta)$ .  
 Normalisation is required to remain on simplex.

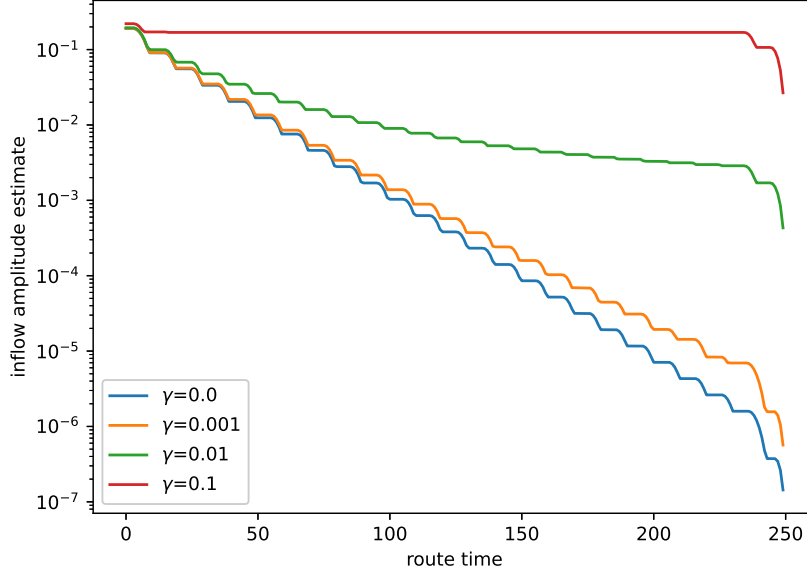


Figure 2: Predicted travel time with logit regularization  $\lambda = 1$

## Stabilizing Dynamics (max demand)

### Logit Regularization

One can use modify the fitness using logit regularizer:

$$\phi_P^{\text{reg}}(\theta, h) = \phi_P(\theta, h) - \lambda e^{-\gamma \cdot \theta} \ln h_P(\theta),$$

where  $\lambda > 0$  is a scaling constant,  $\gamma > 0$  is regularization decay. Figure 2 shows amplitude drops for different

### Projected Travel Times

If we use projected queue values  $\hat{q}_e^\delta(\theta') = q_e(\theta) + \frac{q_e(\theta) - q_e(\theta - \delta)}{\delta} \cdot (\theta' - \theta)$  to estimate upcoming travel times, we can use fitness

$$\phi^{\text{proj t.t.}}(\theta, h) = -(t_P(\theta) + w \cdot \frac{\partial t_P}{\partial \theta}(\theta)) \approx -(\tau_P + \frac{1}{\nu_P} \hat{q}_e^\delta(\theta + w)).$$

Projection window  $w$  is a paramater.

## Analysis

Denote  $\Phi(\theta) := t_0(\theta) - t_1(\theta)$ ,  $\mathbb{I}_{q_e}(\theta) := \begin{cases} 1, & q_e(\theta) > 0 \text{ or } f_e^+(\theta) \geq \nu_e \\ 0, & \text{otherwise} \end{cases}$ .

Queue  $q_e$  "activates" at the moment  $\theta_{q_e}$  s.t.  $f_e^+(\theta_{q_e}) = \nu_e$  and deactivates at the moment  $\theta'_{q_e}$  s.t.  $\int_{\theta_{q_e}}^{\theta'_{q_e}} (f_e^+(z) - \nu_e) dz = 0$ .

We can write the derivative of  $\Phi$  as

$$\dot{\Phi} = \left( \frac{Uh_0}{\nu_0} - 1 \right) \mathbb{I}_{q_0} - \left( \frac{U(1-h_0)}{\nu_1} - 1 \right) \mathbb{I}_{q_1}$$

and substitute in replication DE with projected travel times as fitnesses:

$$\dot{h}_0 = Rh(\phi_0 - h_0\phi_0 - (1-h_0)\phi_1) = Rh_0(1-h_0)(\phi_0 - \phi_1) = -Rh_0(1-h_0)(\Phi + w\dot{\Phi})$$

There are 4 different cases:

$$1. \mathbb{I}_{q_0} = 1, \mathbb{I}_{q_1} = 1$$

$$h_0 = \frac{\dot{\Phi} + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} \xrightarrow{\dot{\Phi} \rightarrow 0} \frac{\nu_0}{\nu_0 + \nu_1}, \quad 1 - h_0 = \frac{\frac{U}{\nu_0} - \dot{\Phi}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}}.$$

$$\frac{\ddot{\Phi}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = -R \frac{\dot{\Phi} + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} \cdot \frac{\frac{U}{\nu_0} - \dot{\Phi}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} \cdot (\Phi + w\dot{\Phi})$$

$$\ddot{\Phi} \approx -k(\Phi + w\dot{\Phi}), \quad k = R \frac{U}{\nu_0 + \nu_1}$$

$$2. \mathbb{I}_{q_0} = 1, \mathbb{I}_{q_1} = 0$$

$$h_0 = \frac{\dot{\Phi} + 1}{\frac{U}{\nu_0}} \xrightarrow{\dot{\Phi} \rightarrow 0} \frac{\nu_0}{U}, \quad 1 - h_0 = \frac{\frac{U}{\nu_0} - 1 - \dot{\Phi}}{\frac{U}{\nu_0}}.$$

$$\frac{\ddot{\Phi}}{\frac{U}{\nu_0}} = -R \frac{\dot{\Phi} + 1}{\frac{U}{\nu_0}} \cdot \frac{\frac{U}{\nu_0} - 1 - \dot{\Phi}}{\frac{U}{\nu_0}} \cdot (\Phi + w\dot{\Phi})$$

$$\ddot{\Phi} \approx -k(\Phi + w\dot{\Phi}), \quad k = R(1 - \frac{\nu_0}{U})$$

$$3. \mathbb{I}_{q_0} = 0, \mathbb{I}_{q_1} = 1$$

$$h_0 = \frac{\dot{\Phi} + \frac{U}{\nu_1} - 1}{\frac{U}{\nu_1}} \xrightarrow{\dot{\Phi} \rightarrow 0} \frac{U - \nu_1}{U}, \quad 1 - h_0 = \frac{1 - \dot{\Phi}}{\frac{U}{\nu_1}}.$$

$$\frac{\ddot{\Phi}}{\frac{U}{\nu_1}} = -R \frac{\dot{\Phi} + \frac{U}{\nu_1} - 1}{\frac{U}{\nu_1}} \cdot \frac{1 - \dot{\Phi}}{\frac{U}{\nu_1}} \cdot (\Phi + w\dot{\Phi})$$

$$\ddot{\Phi} \approx -k(\Phi + w\dot{\Phi}), \quad k = R(1 - \frac{\nu_1}{U})$$

$$4. \mathbb{I}_{q_0} = 0, \mathbb{I}_{q_1} = 0$$

$$\frac{d}{d\theta} \ln \frac{h_0}{1 - h_0} = \frac{\dot{h}_0}{h_0(1 - h_0)} = R(\tau_1 - \tau_0)$$

$$h_0(\theta) = \sigma \left( \ln \frac{h_0(\theta_0)}{1 - h_0(\theta_0)} + R(\tau_1 - \tau_0)(\theta - \theta_0) \right) \xrightarrow{\theta \rightarrow \infty} 1, \quad \sigma(x) = \frac{1}{1 + e^{-x}}$$

For equation  $\ddot{\Phi} + k(\Phi + w\dot{\Phi}) = 0$  we can estimate the solution decay rate:

$$w \leq \frac{2}{\sqrt{k}} : \quad t_0 - t_1, h_0 - h^* \propto e^{-\frac{k w}{2} \theta}$$

$$w > \frac{2}{\sqrt{k}} : \quad t_0 - t_1, h_0 - h^* \propto e^{-\frac{k w - \sqrt{(k w)^2 - 4k}}{2} \theta}$$

## Case switching

We have invariants  $\Phi$ , and  $h_0$ . Below we study the derivative switch  $\dot{\Phi} \rightarrow \dot{\Phi}'$ .

- 1  $\rightarrow$  2. Necessary condition:  $h_0 > 1 - \frac{\nu_1}{U}$ .

$$\frac{\dot{\Phi} + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = \frac{\dot{\Phi}' + 1}{\frac{U}{\nu_0}}$$

$$\dot{\Phi}' = \frac{\dot{\Phi} - \frac{\nu_0 + \nu_1 - U}{\nu_1}}{\frac{\nu_0 + \nu_1}{\nu_1}}$$

We are either close to equilibrium, or decrease  $|\dot{\Phi}|$ :

$$-\frac{\nu_0 + \nu_1 - U}{\nu_0} < \dot{\Phi} < \frac{\nu_0 + \nu_1 - U}{\nu_0 + 2\nu_1} \Rightarrow |\dot{\Phi}'| < \frac{\nu_0 + \nu_1 - U}{\nu_0}$$

$$\dot{\Phi} \geq \frac{\nu_0 + \nu_1 - U}{\nu_0 + 2\nu_1} \Rightarrow |\dot{\Phi}'| \leq |\dot{\Phi}|$$

- 1  $\rightarrow$  3. Necessary condition:  $h_0 < \frac{\nu_0}{U}$ .

$$\frac{\dot{\Phi} + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = \frac{\dot{\Phi}' + \frac{U}{\nu_1} - 1}{\frac{U}{\nu_1}}$$

$$\dot{\Phi}' = \frac{\dot{\Phi} + \frac{\nu_0 + \nu_1 - U}{\nu_0}}{\frac{\nu_0 + \nu_1}{\nu_0}}$$

$|\dot{\Phi}|$  decreases:

$$\dot{\Phi} > \frac{\nu_0 + \nu_1 - U}{\nu_1} \Rightarrow |\dot{\Phi}'| < |\dot{\Phi}|$$

- 2  $\rightarrow$  1. Necessary condition:  $h_0 = 1 - \frac{\nu_1}{U}$ .

$$\frac{\dot{\Phi} + 1}{\frac{U}{\nu_0}} = \frac{\dot{\Phi}' + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = 1 - \frac{\nu_1}{U}$$

$$\dot{\Phi} = \dot{\Phi}' = -\frac{\nu_0 + \nu_1 - U}{\nu_0}$$

- 3  $\rightarrow$  1. Necessary condition:  $h_0 = \frac{\nu_0}{U}$ .

$$\frac{\dot{\Phi} + \frac{U}{\nu_1} - 1}{\frac{U}{\nu_1}} = \frac{\dot{\Phi}' + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = \frac{\nu_0}{U}$$

$$\dot{\Phi} = \dot{\Phi}' = \frac{\nu_0 + \nu_1 - U}{\nu_1}$$

- 2/3  $\rightarrow$  4  $\rightarrow$  2. Necessary condition:  $h_0 = \frac{\nu_0}{U}$ .

Since there are no queues during case 4, we have  $\Phi = \Phi' = \tau_0 - \tau_1$ .

Also,  $\dot{\Phi}' = 0$ , so we come closer to the equilibrium

## Lyapunov Theory

Our differential equation has form  $\ddot{\Phi} = f(\Phi, \dot{\Phi})$ , where  $f$  is a polynomial. We view it as autonomous system with Lipshitz RHS

$$\frac{d}{d\theta} \begin{bmatrix} \Phi \\ \dot{\Phi} \end{bmatrix} = \begin{bmatrix} \dot{\Phi} \\ f(\Phi, \dot{\Phi}) \end{bmatrix}$$

In this case Lyapunov Theory is applicable

**Theorem 1** (*Global asymptotic stability*) Let  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuously differentiable function, s.t. :

$$V(0,0) = 0 \text{ and } \forall (\Phi, \dot{\Phi}) \neq (0,0) : V(\Phi, \dot{\Phi}) > 0,$$

$$\sqrt{\Phi^2 + \dot{\Phi}^2} \rightarrow \infty \Rightarrow V(\Phi, \dot{\Phi}) \rightarrow \infty,$$

$$\forall (\Phi, \dot{\Phi}) \neq (0,0) : \dot{V}(\Phi, \dot{\Phi}) = \frac{\partial V}{\partial \Phi} \dot{\Phi} + \frac{\partial V}{\partial \dot{\Phi}} f(\Phi, \dot{\Phi}) < 0$$

Then  $(\Phi, \dot{\Phi}) = (0,0)$  is globally asymptotically stable.

**Theorem 2** (*Local asymptotic stability*) Let  $A = \begin{bmatrix} 0 & 1 \\ \frac{\partial f}{\partial \Phi}(0,0) & \frac{\partial f}{\partial \dot{\Phi}}(0,0) \end{bmatrix}$ . If we have  $\text{Re}(\lambda_i(A)) < 0$  for both eigenvalues, then  $(\Phi, \dot{\Phi}) = (0,0)$  is locally asymptotically stable.

## Local stability

Linearized equation  $\ddot{\Phi} = \frac{\partial f}{\partial \Phi}(0,0) \cdot \Phi + \frac{\partial f}{\partial \dot{\Phi}}(0,0) \cdot \dot{\Phi} = -k\Phi - kw\dot{\Phi}$ .

$$\det \left( \begin{bmatrix} -\lambda & 1 \\ -k & -kw - \lambda \end{bmatrix} \right) = \lambda^2 + kw\lambda + k; \quad D = k^2w^2 - 4k.$$

Since  $k, w > 0$ , both eigenvalues have negative real parts for any  $D$ . Hence, we always have local asymptotic stability.



### Sufficient conditions for global stability

In case  $i \in \{1, 2, 3\}$  we can write the equation as

$$\ddot{\Phi} = -C_i h_0 (1 - h_0) (\Phi + w \dot{\Phi}).$$

Denote

$$C := \min_{i \in \{1, 2, 3\}} C_i = \min \left\{ R \left( \frac{U}{\nu_0} + \frac{U}{\nu_1} \right), R \frac{U}{\nu_0}, R \frac{U}{\nu_1} \right\} = \frac{RU}{\max\{\nu_0, \nu_1\}}.$$

Further, assume that

$$\forall \theta \geq 0 : \quad h_0(\theta) \in [a, b] \text{ with } a > 0, b < 1$$

and denote  $m := \min_{x \in [a, b]} x(1 - x) > 0$ .

Consider  $V(\Phi, \dot{\Phi}) = \Phi^2 + (\Phi + w \dot{\Phi})^2 > 0$ .

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial \Phi} \dot{\Phi} + \frac{\partial V}{\partial \dot{\Phi}} \cdot \left( -C_i h_0 (1 - h_0) (\Phi + w \dot{\Phi}) \right) = \\ &= \left( 4\Phi \dot{\Phi} + 2w \dot{\Phi}^2 \right) - 2w C_i h_0 (1 - h_0) (\Phi + w \dot{\Phi})^2 \leq \left( 4\Phi \dot{\Phi} + 2w \dot{\Phi}^2 \right) - 2w C m (\Phi + w \dot{\Phi})^2 = \\ &= 2w \left( \frac{1}{w^2} - C m \right) (\Phi + w \dot{\Phi})^2 - \frac{2}{w} \Phi^2. \end{aligned}$$

We can guarantee  $\dot{V} < 0$  if  $w \geq \frac{1}{\sqrt{Cm}}$ .

For  $R = 0.1, U = 4.5, [a, b] = [0.01, 0.99]$  we need  $w \geq \frac{1}{\sqrt{\frac{0.1 \cdot 4.5}{3} \cdot 0.01 \cdot 99}} \approx 25.95$ .

Suppose we have  $V = \Phi^2 + (\Phi + w \dot{\Phi})^2 \leq B^2$ . Then it follows that  $|\Phi| \leq B$ ,  $|\dot{\Phi}| \leq \frac{\sqrt{2B}}{w}$ . I

If we can reach  $V = B^2$  s.t.  $\frac{\sqrt{2B}}{w} = \frac{\nu_0 + \nu_1 - U}{\nu_0}$  with case switching phase, it guarantees that we stop case switching and converge to zero.

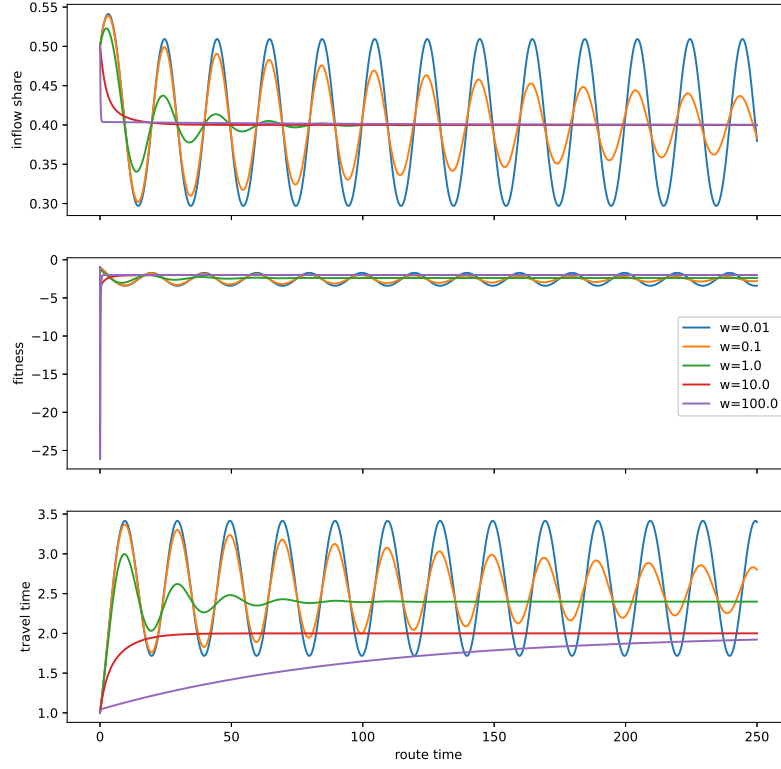


Figure 3:  $U = 5, R = 0.1, \Delta\theta = 0.01$

### Max demand ( $U = \nu_0 + \nu_1$ )

Case 4 not possible; Case 1-3 all converge to  $h^* := \frac{\nu_0}{U}$ .

During the dynamic, queues may form

See Figure 3. Figures 4 and 5 show inflow amplitude decays and final travel times for different projection windows.

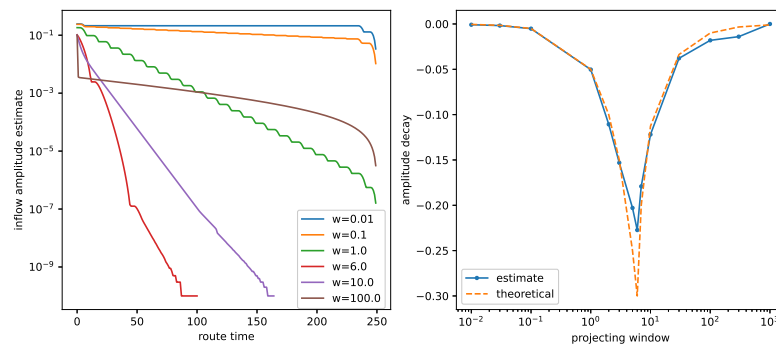


Figure 4: Inflow amplitudes for path 0

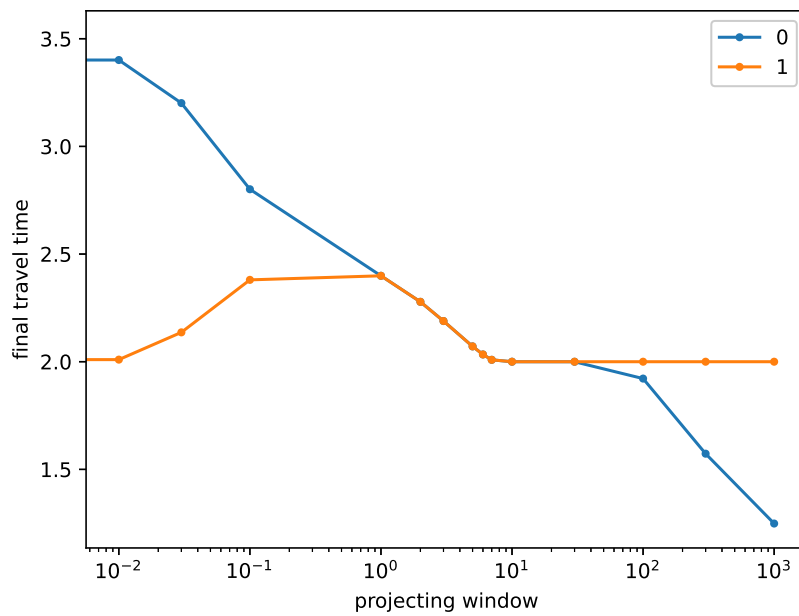


Figure 5: Final travel times

### Medium Demand ( $\nu_0 < U < \nu_0 + \nu_1$ )

Different cases have different equilibria:

$$\frac{\nu_0}{U} > \frac{\nu_0}{\nu_0 + \nu_1} > \frac{U - \nu_1}{U}$$

Figure 6 shows how cases switch until amplitude drops enough and we end up in case  $q_0 > 0, q_1 = 0$ .

Figure 7 shows dynamics for the faster edge 0 with different projection windows. Both fitnesses eventually stabilize to values  $\phi = -\tau_1$  (negative cost of slower edge).

Now assume that edge 0 operates on full capacity and edge 1 always operates below capacity, i.e.

$$\dot{t}_0(\theta) = \frac{U \cdot h_0(\theta) - \nu_0}{\nu_0}, \quad \dot{t}_1(\theta) \equiv 0.$$

Sufficient conditions of the solution to fulfill the assumptions:

$$\Phi(\theta) - \Phi(0) \geq 0; \quad \dot{\Phi}(\theta) \geq -\frac{\nu_0 + \nu_1 - U}{\nu_0}.$$

Figure 8 shows that that theoretical decay estimate is good.

Figure 9 shows the difference between theoretical and observed dynamics.  
Used parameters:  $U = 4.5; \Delta\theta = 0.01; h_0(0) = 0.5; R = 0.1; w = 1.0$

Figures 10 and 11 show dynamics for edge 0 with parameters  $U = 4.5; \Delta\theta = 0.01; R = 0.1$  for different initial share with  $w = 1.0$  and  $w = \frac{2}{\sqrt{k}} = 8.49$

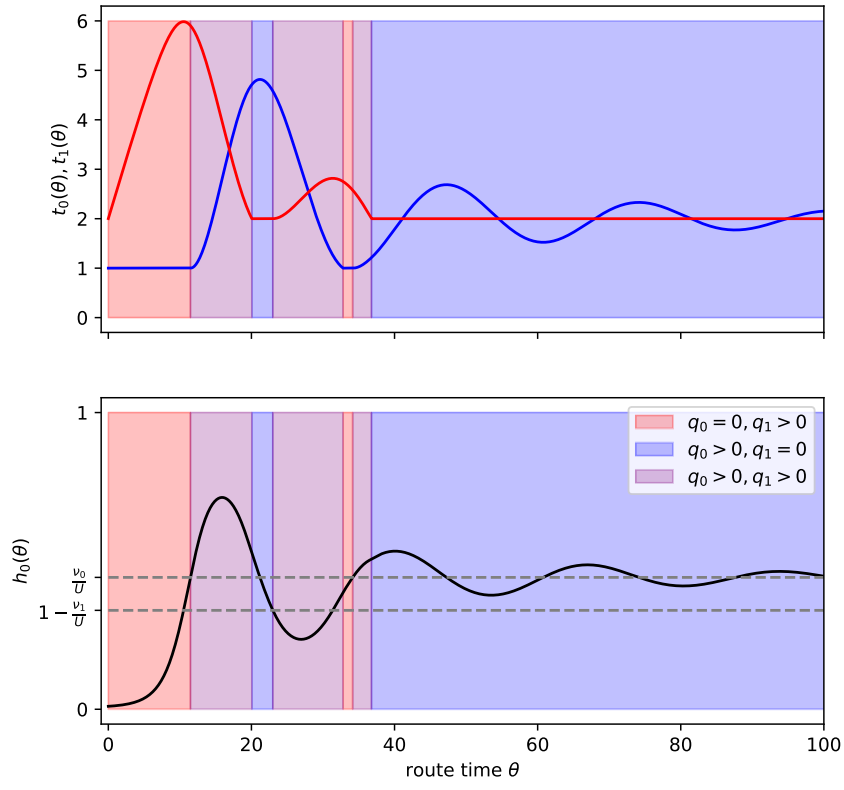


Figure 6: Case switching

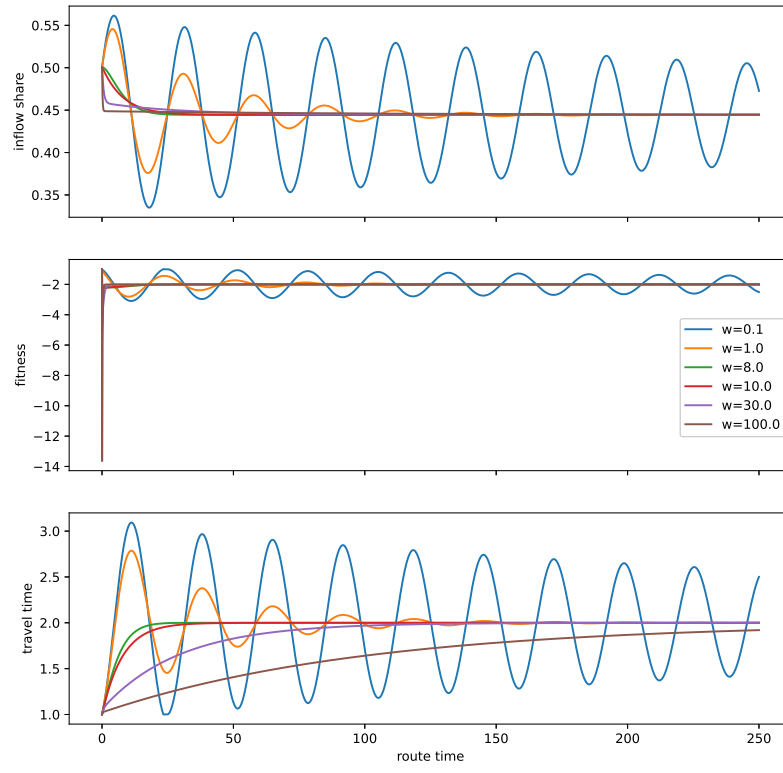


Figure 7: Medium demand.  $U = 4.5$ ;  $\Delta\theta = 0.01$ ;  $R = 0.1$

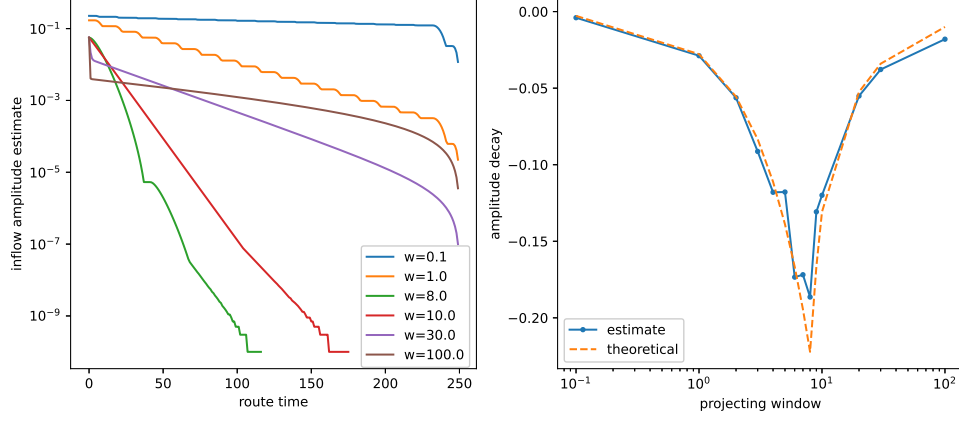


Figure 8: Amplitudes decay

## Unstable dynamics

Figures 12, 13 and 14 show how using bad fitness leads to divergence.

### Proof ?

Necessary condition for stability:  $\forall P : \phi_P(\theta, h) = \sum_{P'} h_{P'}(\theta) \phi_{P'}(\theta, h)$ .

In cases of average travel time and last travel, influence of inflows on fitnesses is delayed for at least  $\tau_0$ .

Sufficient condition for stability for non-aggregated travel times: queues have to not change for the period of delay. This guarantees that fitnesses don't change once equality is reached.

Even when we use undelayed travel times, inflows tend to get in antiphase with fitnesses. In this example, we need inflows to equalize with edge capacities at the moment when both path have same costs.

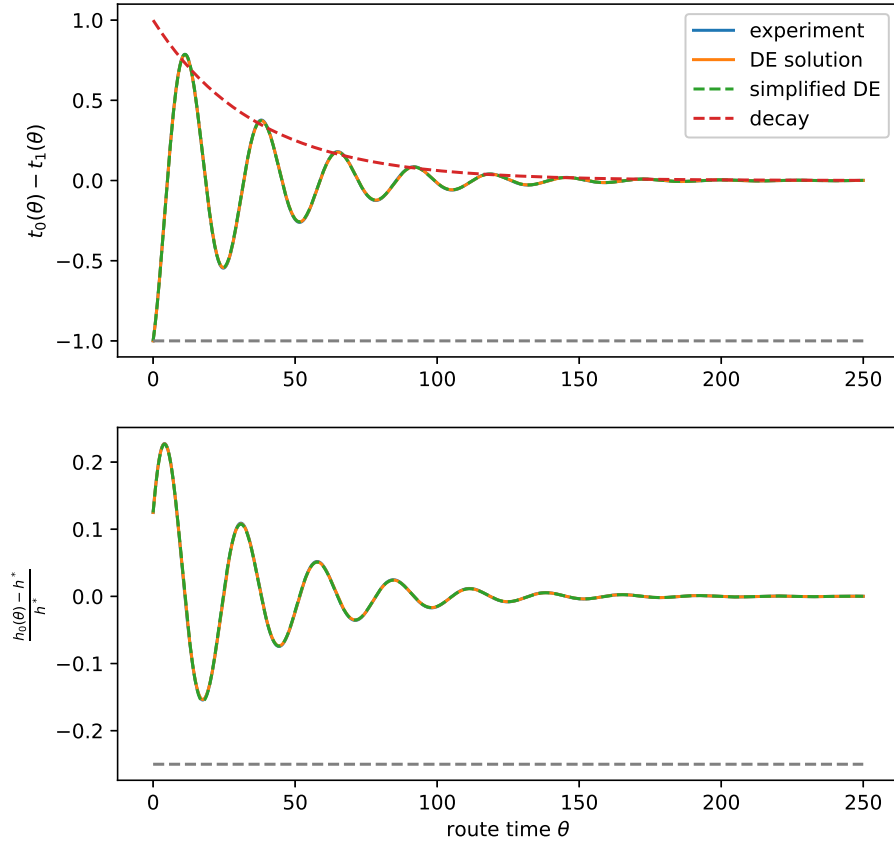


Figure 9: Solution of DEs compared to the experiment; gray dashed lines show illustrate fulfillment of assumptions



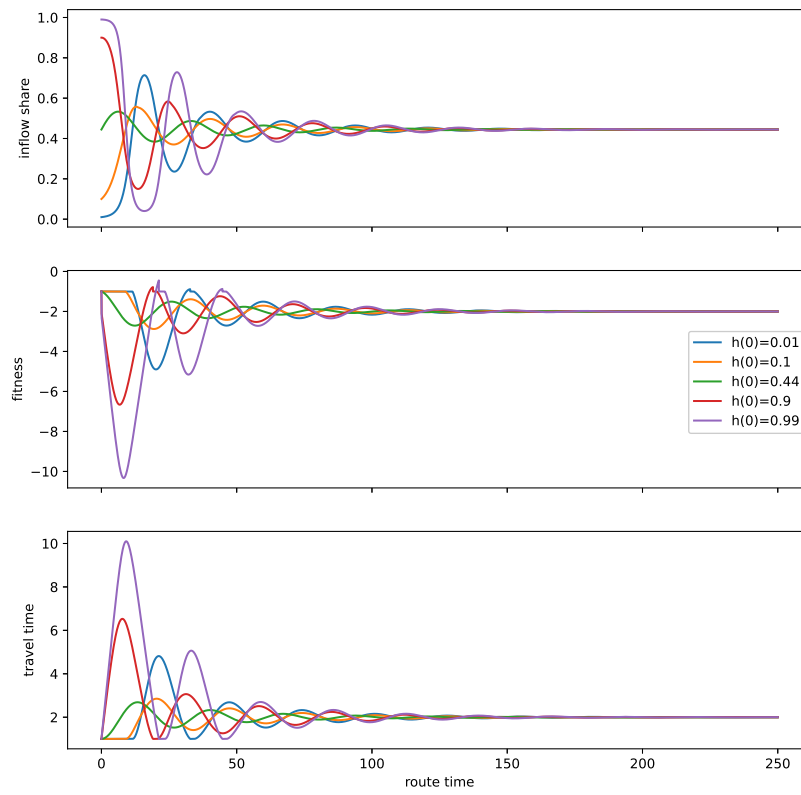


Figure 10:  $w = 1.0$

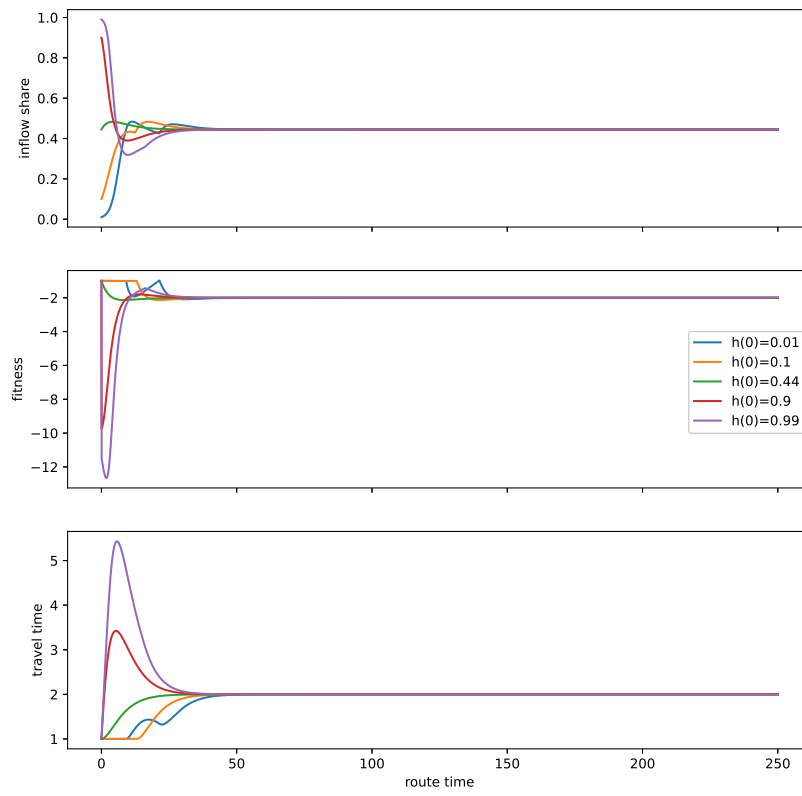


Figure 11:  $w = \frac{2}{\sqrt{k}} = 8.49$

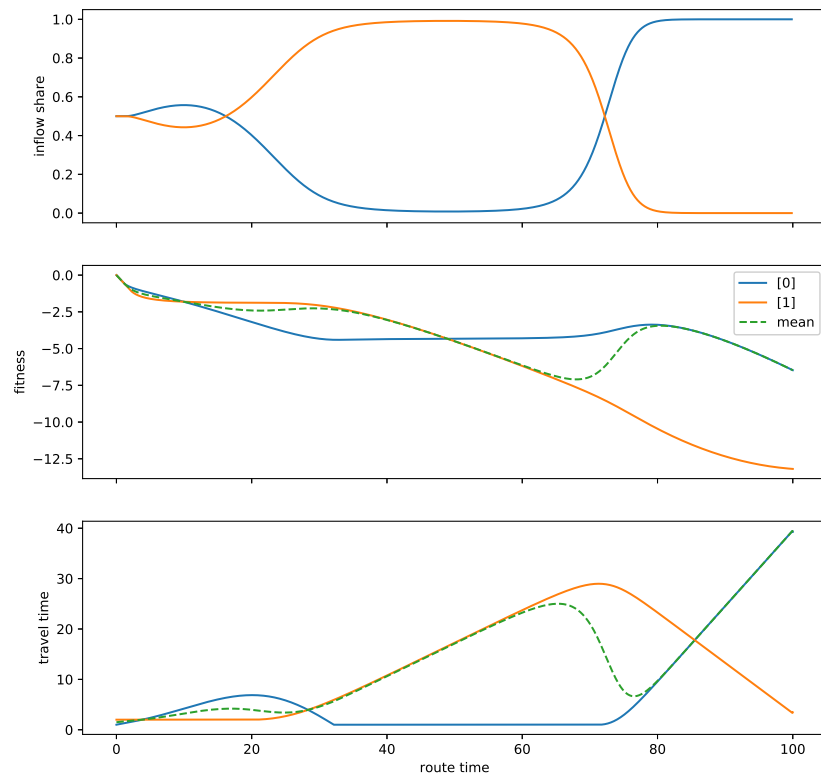


Figure 12: Fluctuations, average travel time

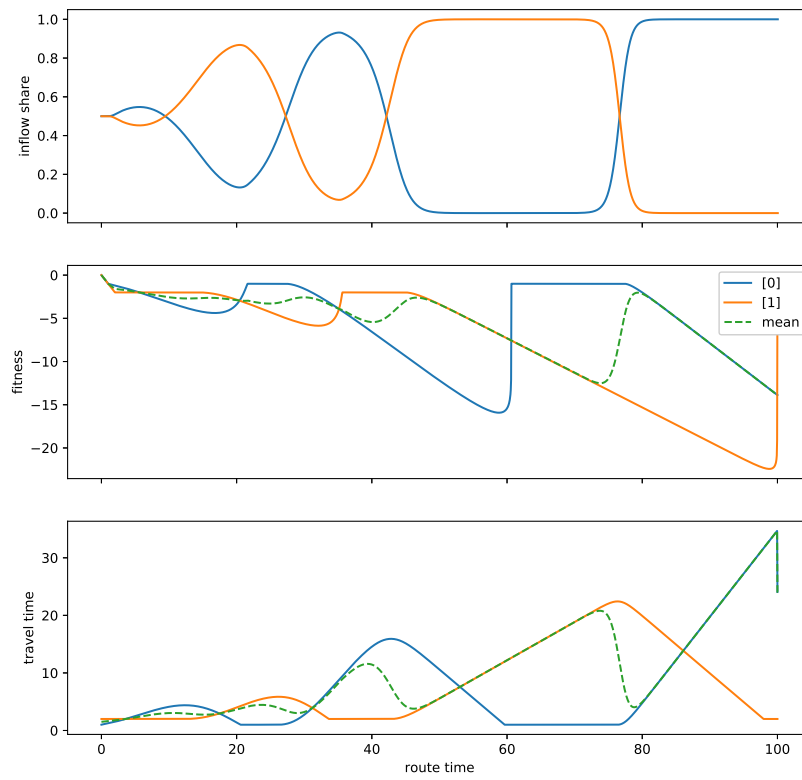


Figure 13: Fluctuations, last travel time

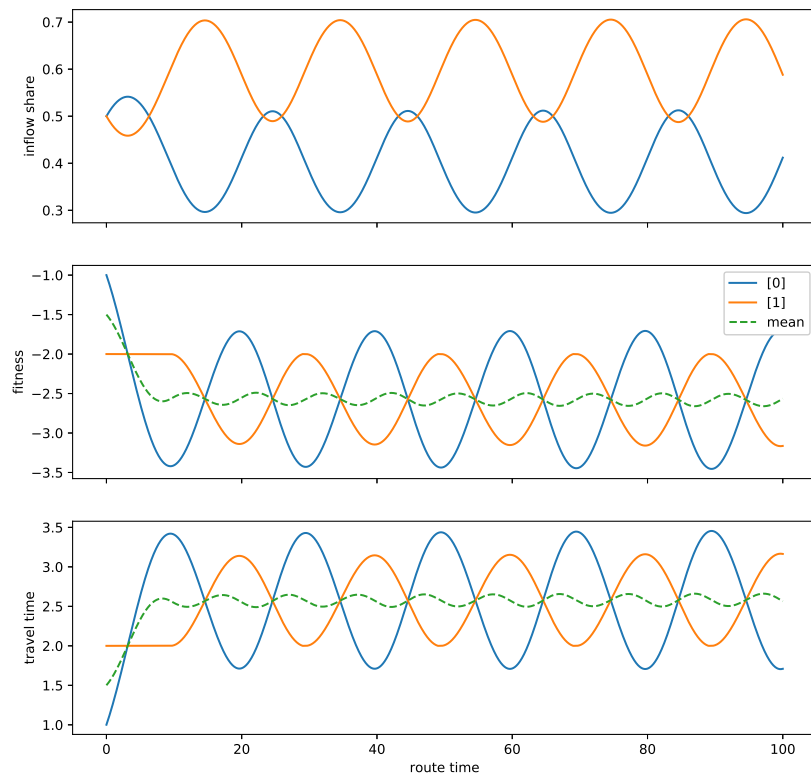


Figure 14: Fluctuations, predicted travel time