Trying to converge to an equilibrium flow

December 1, 2023

Throughout, Vickrey queuing model is used.

Iterative Model

TBD

$$h_P^{(i+1)}(\theta) = \left(1 - \alpha(d_P^{(i)}(\theta))\right) \cdot h_P^i(\theta) + \frac{\mathbb{I}_{\{d_P^{(i)} < \varepsilon\}}(\theta)}{\sum_{P'} \mathbb{I}_{\{d_{P'}^{(i)} < \varepsilon\}}(\theta)} \cdot \left(\sum_{P'} \alpha(d_P^{(i)}(\theta)) \cdot h_P^i(\theta)\right).$$

Dynamic Replicator Model

Let $\mathcal{P}$ be a collection of s-t paths. At time θ, each path P receives a fraction $h_P(\theta)$ of the total inflow $u(\theta)$. We are considering the replication dynamic:

$$\dot{h}_P = R \cdot h_P \cdot a_P, \quad \forall P \in \mathcal{P},$$

where $R > 0$ is a constant, $a_P = \phi_P - \sum_{P' \in \mathcal{P}} h_{P'} \cdot \phi_{P'}$ is advantage function based on fitness ϕ.

Role of the fitness can be played by negative signed average experienced travel time of the particles:

$$\phi_P^{a.t.}(\theta, h) = -\frac{1}{F_P^+(\theta)} \left(\int_0^\theta F_P^+(\psi) d\psi - \int_0^\theta F_P^-(\psi) d\psi\right).$$

Above, $F_P^+(\cdot)$, $F_P^-(\cdot)$ denote functions of accumulated path in- and outflow.

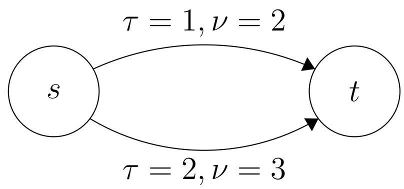

Figure 1: Simple network

Another option is negative signed last available travel time:

$$\phi_P^{l.t.}(\theta, h) = -1 \cdot \begin{cases} \theta, & 0 \le \theta < T_P(0) \\ \theta - T_P^{-1}(\theta), & \theta \ge T_P(0) \end{cases},$$

$T_P(\cdot)$ is path exit time function.

Finally, utilising constant predictors for queues, one can predict path exit times $\hat{T}_P$ and use negative predicted travel time:

$$\phi_P^{p.t.}(\theta, h) = -(\hat{T}_P(\theta) - \theta).$$

If path P has only one edge, then $\hat{T}_P(\cdot) \equiv T_P(\cdot)$.

Considered Instance

Consider simple network with two parallel edges on Figure 1. Let upper edge have index 0 and lower 1. We consider inflow $u(\theta) \le 5 = \nu_0 + \nu_1$ with two options: $u(\theta) = 5$(max demand) and $u(\theta) = 4.5 > \nu_0$ (medium demand).

The equilibrium flow is achieved as follows: all the inflow is redirected to the shorter edge, until both edges achieve equal costs, then the distribution is proportional to the capacities.

$$h_0^*(\theta) = \begin{cases} 1, & 0 \le \theta < \frac{2}{3} \\ \frac{2}{5}, & \theta \ge \frac{2}{3} \end{cases}, \quad h_1^*(\theta) = 1 - h_0^*(\theta).$$

Numerical Approximation

show numerically computed inflow shares and fitness of replicator flow with initial condition $h_0(0) = h_1(0) = \frac{1}{2}$. The following approximation was used:

$$h_P(\theta + \Delta\theta) \approx \frac{h_P(\theta) \cdot e^{R \cdot a_P \cdot \Delta\theta}}{\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta}}.$$

By construction of the initial dynamic $\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta} = 1 + o(\Delta\theta)$. Normalisation is required to remain on simplex.

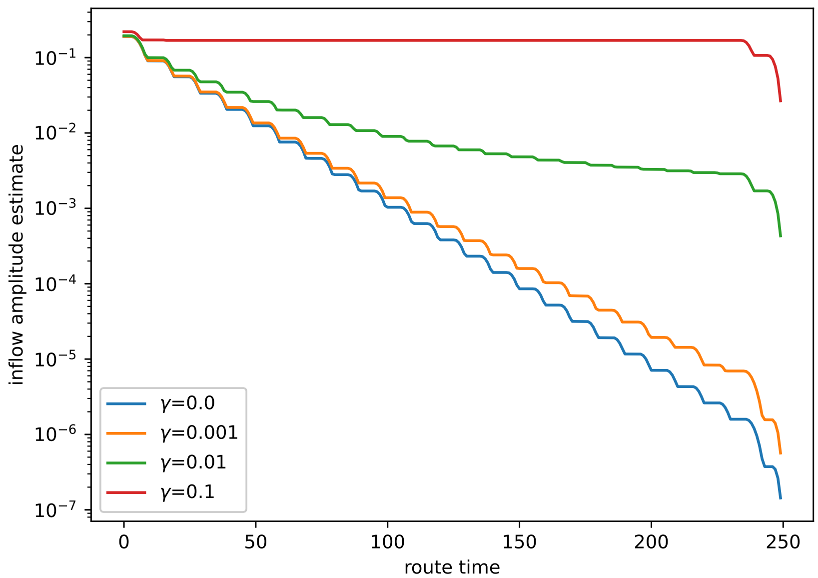

Figure 2: Predicted travel time with logit regularization $\lambda = 1$

Stabilizing Dynamics (max demand)

Logit Regularization

One can use modify the fitness using logit regularizer:

$$\phi_P^{\text{reg}}(\theta, h) = \phi_P(\theta, h) - \lambda e^{-\gamma \cdot \theta} \ln h_P(\theta),$$

where $\lambda > 0$ is a scaling constant, $\gamma > 0$ is regularization decay. Figure 2 shows

Projected Travel Times

If we use projected queue values $\hat{q}_e^{\delta}(\theta') = q_e(\theta) + \frac{q_e(\theta) - q_e(\theta - \delta)}{\delta} \cdot (\theta' - \theta)$ to estimate upcoming travel times, we can use fitness

$$\phi^{\text{proj t.t.}}(\theta, h) = -\hat{t}^{\delta}(\theta + w) \approx -(t_P(\theta) + \frac{\partial t_P}{\partial \theta}(\theta) \cdot w).$$

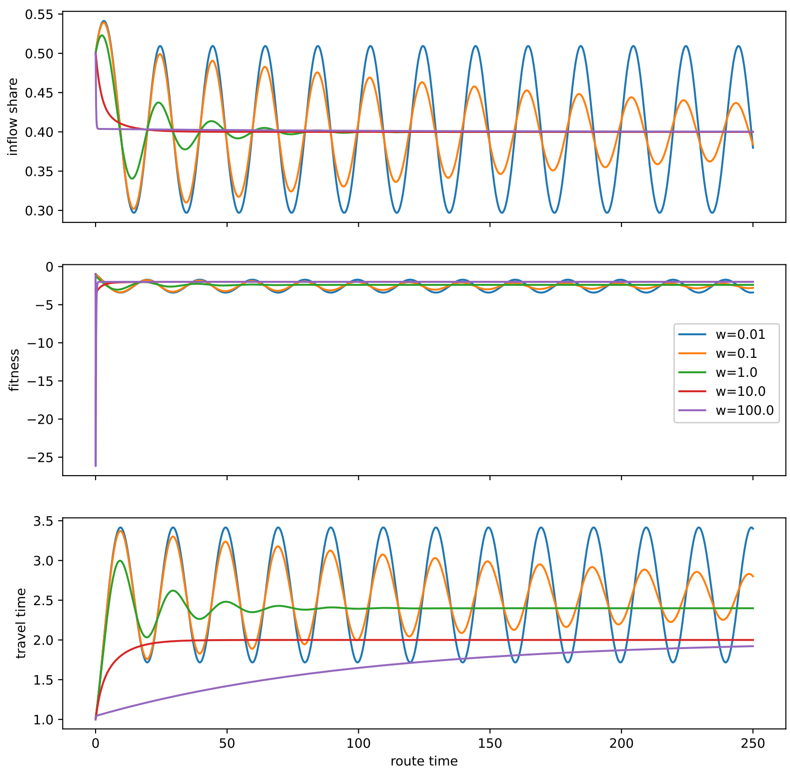

Figure 3: $\phi = -\hat{t}^{2\Delta\theta}(\theta + w), R = 0.1, \Delta\theta = 0.01$

Projection window w is a paramater. See Figure 3. Figures 4 and 5 show inflow amplitude decays and final travel times for different projection windows.

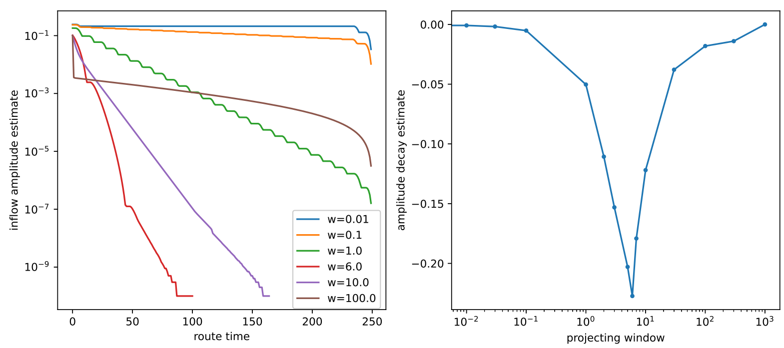

Figure 4: Inflow amplitudes for path 0

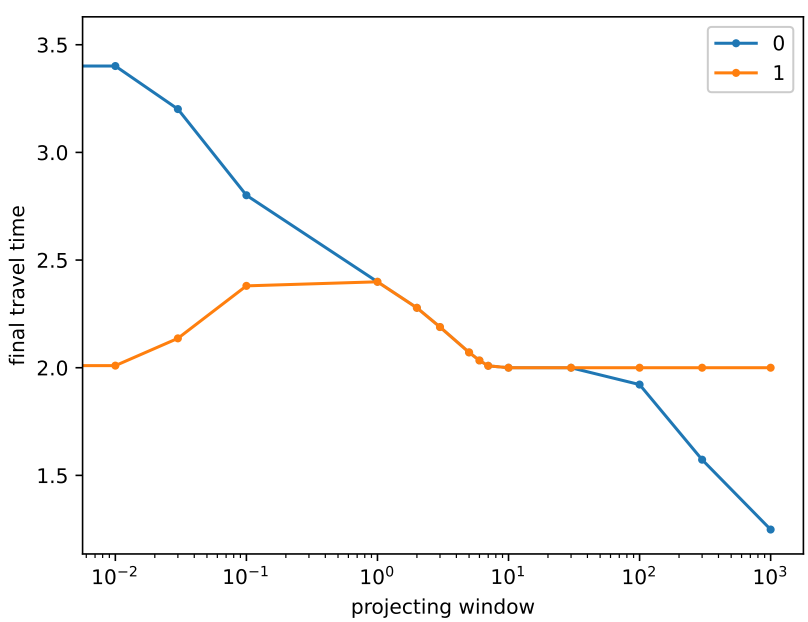

Figure 5: Final travel times

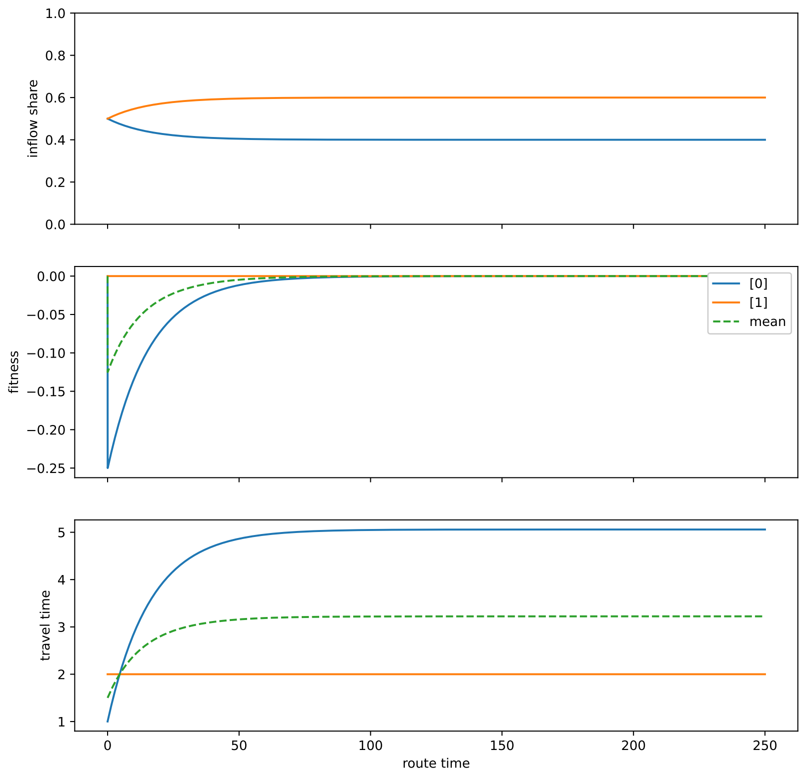

Figure 6: Travel time gradient as fitness

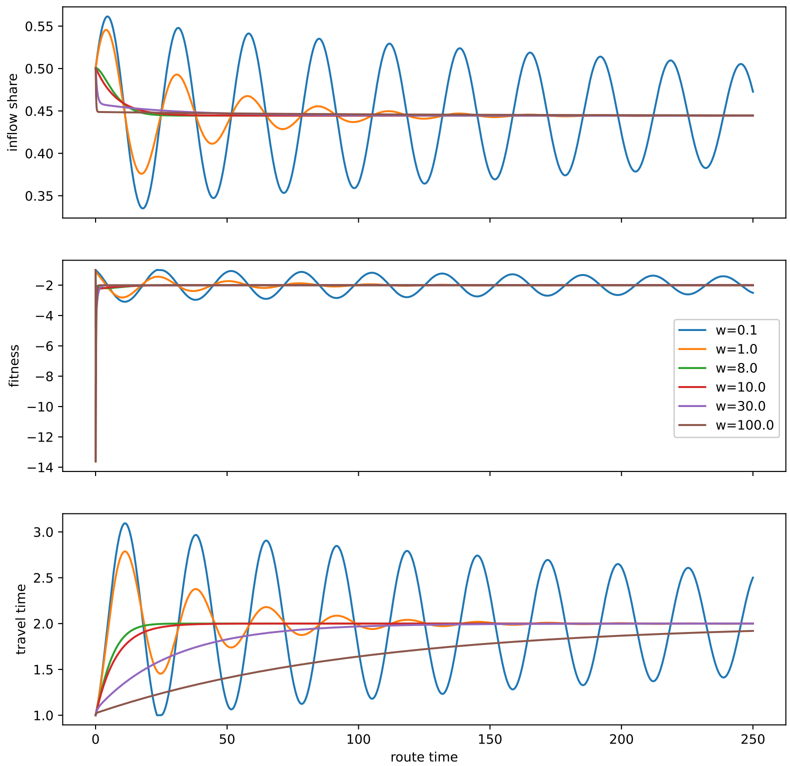

Figure 7: Medium demand

Stabilizing Dynamics (medium demand)

Projected Travel Times

Figure 7 shows dynamics for the faster edge 0 with different projection windows. Both fitnesses eventually stabilize to values $\phi = -\tau_1$ (negative cost of slower edge). It also appears that the queue doesn't form on edge 1, so it has constant fitness $\phi_1(\theta) \equiv -\tau_1$.

Now assume that edge 0 operates on full capacity and edge 1 always operates below capacity, i.e.

$$q_0(\theta) = \int_0^\theta (U \cdot h_0(z) - \nu_0) dz, \quad q_1(\theta) \equiv 0.$$

This allows to write the replication equation

$$\dot{h}_0 = R \cdot h_0 \cdot (\phi_0 - h_0 \phi_0 - (1 - h_0)\phi_1) = -R \cdot h_0 \cdot (1 - h_0) \cdot \left(\tau_0 + \frac{1}{\nu_0}(q_0 + w \cdot \frac{\partial q_0}{\partial \theta}) - \tau_1 \right)$$

in the form

$$\ddot{\Phi} = -R \cdot (\Phi + w\dot{\Phi}) \cdot h^*(1 + \dot{\Phi}) \cdot (1 - h^*(1 + \dot{\Phi})),$$

where the following notation is used:

$$h^* := \frac{\nu_0}{U}, \quad \Phi(\theta) := t_0(\theta) - t_1(\theta) = \int_0^\theta \frac{h_0(z) - h^*}{h^*} dz + \tau_0 - \tau_1.$$

We have $h_0(\theta) = h^*(1 + \dot{\Phi}(\theta))$ and the assumptions can be written as

$$\Phi(\theta) - \Phi(0) \geq 0; \quad \dot{\Phi}(\theta) \geq \frac{U - \nu_1 - \nu_0}{\nu_0}.$$

If we further assume that $\dot{\Phi}$ is small (i.e. h_0 is close to h^*), we can make a simplification and analyze the linear equation

$$\ddot{F} = -R \cdot (F + w\dot{F}) \cdot h^*(1 - h^*) \quad \Leftrightarrow \quad \ddot{F} + kw\dot{F} + kF = 0, \quad k = Rh^*(1 - h^*).$$

This yields a theoretical estimate of decay rate:

$$w \leq \frac{2}{\sqrt{k}}: \quad h_0 - h^* \propto e^{-\frac{kw}{2}\theta}$$

$$w > \frac{2}{\sqrt{k}}: \quad h_0 - h^* \propto e^{-\frac{kw - \sqrt{(kw)^2 - 4k}}{2}\theta}$$

Figure 8 shows that the estimate is good in case $w > \frac{2}{\sqrt{k}}$.

Figure 9 shows the difference between theoretical and observed dynamics. Used parameters: $U = 4.5; \Delta\theta = 0.01; h_0(0) = 0.5; R = 0.1; w = 1.0$

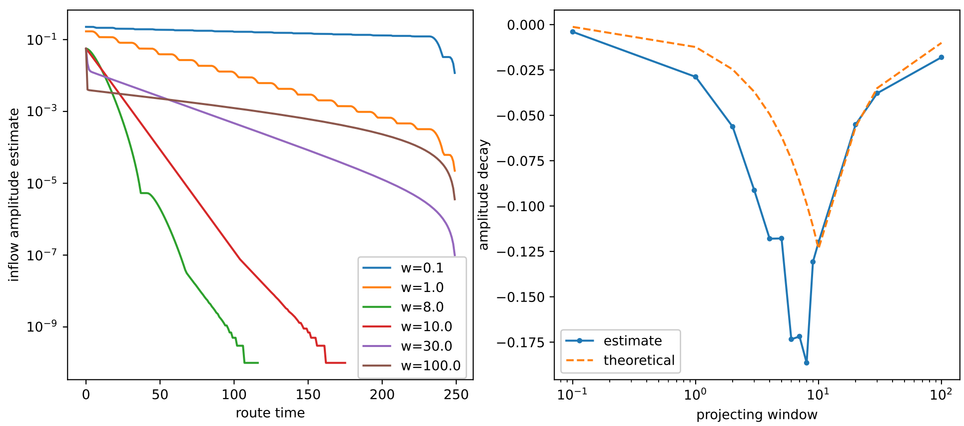

Figure 8: Amplitudes decay

Unstable dynamics

Figures 10, 11 and 12 show how using bad fitness leads to divergence.

Proof ?

Necessary condition for stability: $\forall P: \ \phi_P(\theta, h) = \sum_{P'} h_{P'}(\theta)\phi_{P'}(\theta, h)$.
In cases of average travel time and last travel, influence of inflows on fitnesses is delayed for at least τ_0.

Sufficient condition for stability for non-aggregated travel times: queues have to not change for the period of delay. This guarantees that fitnesses don't change once equality is reached.

Even when we use undelayed travel times, inflows tend to get in antiphase with fitnesses. In this example, we need inflows to equalize with edge capacities at the moment when both path have same costs.

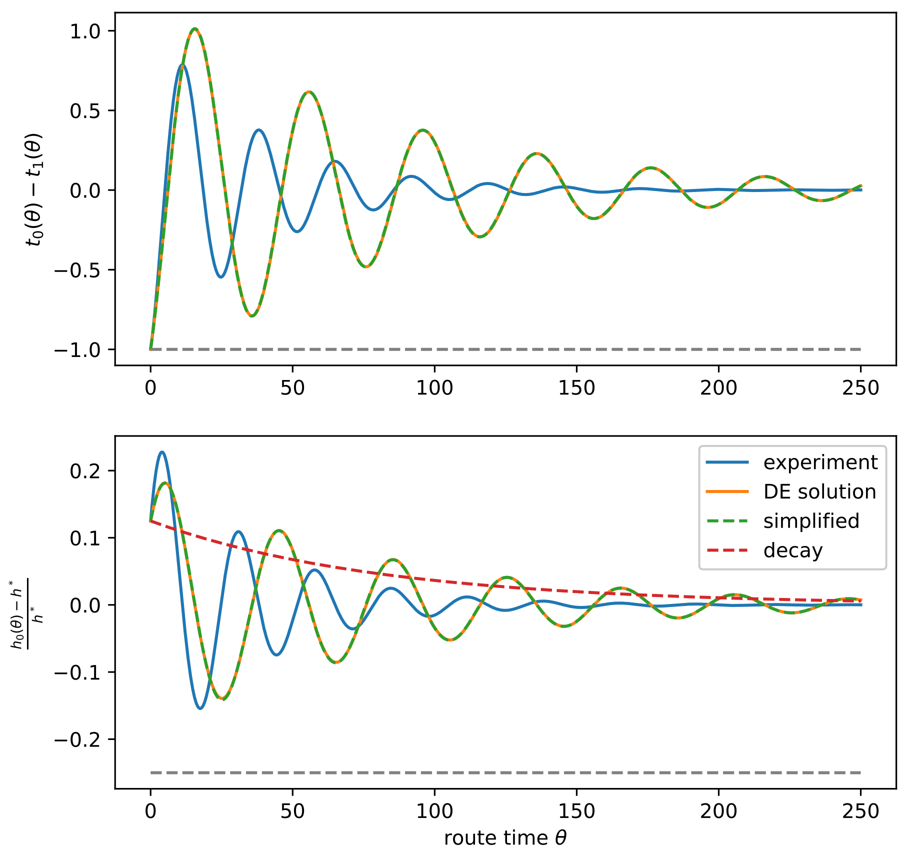

Figure 9: Solution of DEs compared to the experiment; gray dashed lines show illustrate fulfillment of assumptions

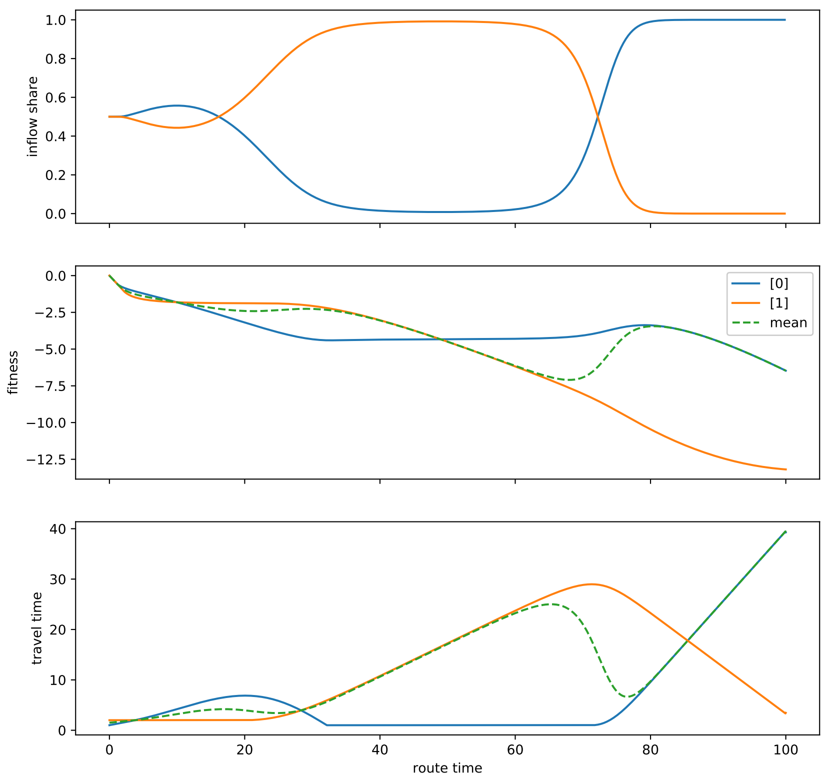

Figure 10: Fluctuations, average travel time

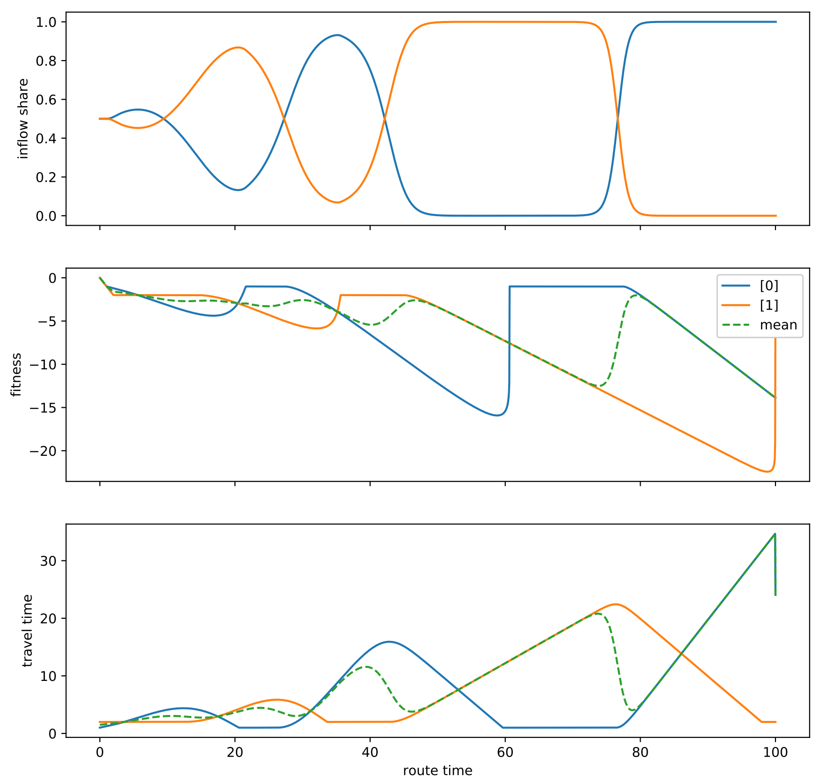

Figure 11: Fluctuations, last travel time

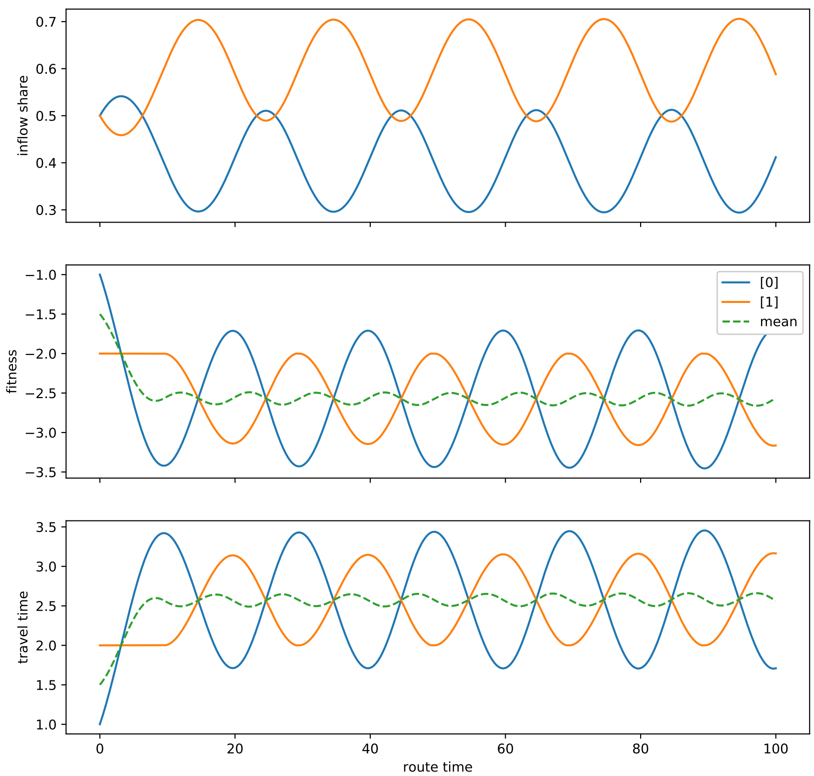

Figure 12: Fluctuations, predicted travel time