Trying to converge to an equilibrium flow

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Throughout, Vickrey queuing model is used.

Iterative Model

TBD

$$h_P^{(i+1)}(\theta) = \left(1 - \alpha(d_P^{(i)}(\theta))\right) \cdot h_P^i(\theta) + \frac{\mathbb{I}_{\{d_P^{(i)} < \varepsilon\}}(\theta)}{\sum_{P'} \mathbb{I}_{\{d_{P'}^{(i)} < \varepsilon\}}(\theta)} \cdot \left(\sum_{P'} \alpha(d_P^{(i)}(\theta)) \cdot h_P^i(\theta)\right).$$

Dynamic Replicator Model

Let $\mathcal{P}$ be a collection of s-t paths. At time θ, each path P receives a fraction $h_P(\theta)$ of the total inflow $u(\theta)$. We are considering the replication dynamic:

$$\dot{h}_P = R \cdot h_P \cdot a_P, \qquad \forall P \in \mathcal{P},$$

where $R > 0$ is a constant, $a_P = \phi_P - \sum_{P' \in \mathcal{P}} h_{P'} \cdot \phi_{P'}$ is advantage function based on fitness ϕ.

Role of the fitness can be played by negative signed average experienced travel time of the particles:

$$\phi_P^{a.t.}(\theta, h) = -\frac{1}{F_P^+(\theta)} \left(\int_0^\theta F_P^+(\psi) d\psi - \int_0^\theta F_P^-(\psi) d\psi\right).$$

Above, $F_P^+(\cdot)$, $F_P^-(\cdot)$ denote functions of accumulated path in- and outflow.

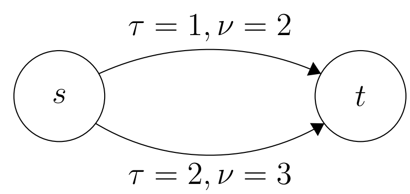

Figure 1: Simple network

Another option is negative signed last available travel time:

$$\phi_P^{l.t.}(\theta, h) = -1 \cdot \begin{cases} \theta, & 0 \leq \theta < T_P(0) \\ \theta - T_P^{-1}(\theta), & \theta \geq T_P(0) \end{cases},$$

$T_P(\cdot)$ is path exit time function.

Finally, utilising constant predictors for queues, one can predict path exit times $\hat{T}_P$ and use negative predicted travel time:

$$\phi_P^{p.t.}(\theta, h) = -(\hat{T}_P(\theta) - \theta).$$

If path P has only one edge, then $\hat{T}_P(\cdot) \equiv T_P(\cdot)$.

Considered Instance

Consider simple network with two parallel edges on Figure 1. Let upper edge have index 0 and lower 1. We consider inflow $u(\theta) \equiv U$.

The equilibrium flow is achieved as follows: all the inflow is redirected to the shorter edge, until both edges achieve equal costs, then the distribution is proportional to the capacities.

$$h_0^*(\theta) = \begin{cases} 1, & 0 \leq \theta < (\tau_1 - \tau_0)\frac{\nu_0}{U - \nu_0} \\ \frac{\nu_0}{U}, & \theta \geq (\tau_1 - \tau_0)\frac{\nu_0}{U - \nu_0} \end{cases}, \quad h_1^*(\theta) = 1 - h_0^*(\theta).$$

Numerical Approximation

show numerically computed inflow shares and fitness of replicator flow with initial condition $h_0(0) = h_1(0) = \frac{1}{2}$. The following approximation was used:

$$h_P(\theta + \Delta\theta) \approx \frac{h_P(\theta) \cdot e^{R \cdot a_P \cdot \Delta\theta}}{\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta}}.$$

By construction of the initial dynamic $\sum_{P'} h_{P'}(\theta) \cdot e^{R \cdot a_{P'} \cdot \Delta\theta} = 1 + o(\Delta\theta)$. Normalisation is required to remain on simplex.

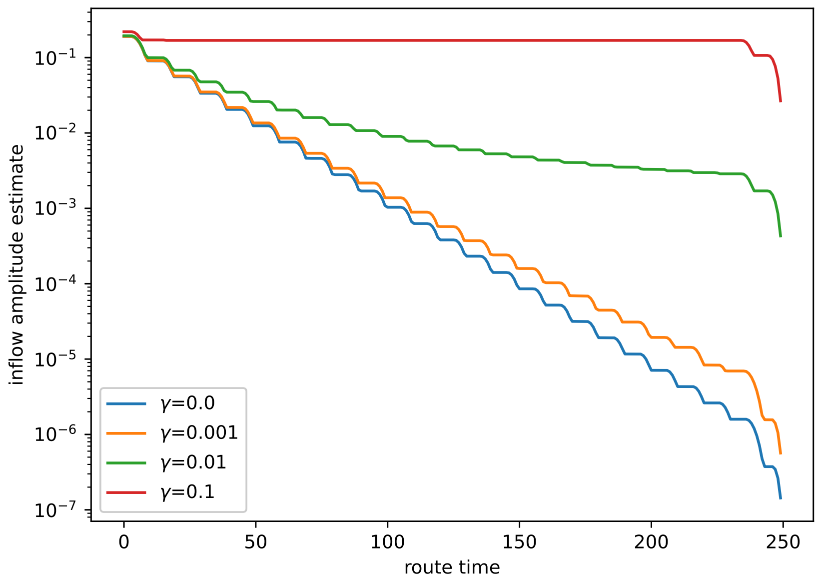

Figure 2: Predicted travel time with logit regularization $\lambda = 1$

Stabilizing Dynamics (max demand)

Logit Regularization

One can use modify the fitness using logit regularizer:

$$\phi_P^{\text{reg}}(\theta, h) = \phi_P(\theta, h) - \lambda e^{-\gamma\cdot\theta} \ln h_P(\theta),$$

where $\lambda > 0$ is a scaling constant, $\gamma > 0$ is regularization decay. Figure 2 shows amplitude drops for different

Projected Travel Times

If we use projected queue values $\hat{q}_e^\delta(\theta') = q_e(\theta) + \frac{q_e(\theta)-q_e(\theta-\delta)}{\delta} \cdot (\theta' - \theta)$ to estimate upcoming travel times, we can use fitness

$$\phi^{\text{proj t.t.}}(\theta, h) = -(t_P(\theta) + w \cdot \frac{\partial t_P}{\partial \theta}(\theta)) \approx -(\tau_P + \frac{1}{\nu_P}\hat{q}_e^\delta(\theta + w)).$$

Projection window w is a paramater.

Analysis

Denote $\Phi(\theta) := t_0(\theta) - t_1(\theta)$, $\mathbb{I}_{q_e}(\theta) := \begin{cases} 1, q_e(\theta) > 0 \text{ or } f_e^+(\theta) \geq \nu_e \\ 0, \text{ otherwise} \end{cases}$.

Queue q_e "activates" at the moment θ_{q_e} s.t. $f_e^+(\theta_{q_e}) = \nu_e$ and deactivates at first the moment θ'_{q_e} s.t. $\int_{\theta_{q_e}}^{\theta'_{q_e}} (f_e^+(z) - \nu_e)dz = 0$.

We can write the derivative of Φ as

$$\dot{\Phi} = \left(\frac{Uh_0}{\nu_0} - 1\right)\mathbb{I}_{q_0} - \left(\frac{U(1-h_0)}{\nu_1} - 1\right)\mathbb{I}_{q_1}$$

and substitute in replication DE with projected travel times as fitnesses:

$$\dot{h_0} = Rh(\phi_0 - h_0\phi_0 - (1-h_0)\phi_1) = Rh_0(1-h_0)(\phi_0 - \phi_1) = -Rh_0(1-h_0)(\Phi + w\dot{\Phi})$$

There are 4 different cases:

1. $\mathbb{I}_{q_0} = 1, \mathbb{I}_{q_1} = 1$

$$h_0 = \frac{\dot{\Phi} + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} \overset{\dot{\Phi}\to 0}{\to} \frac{\nu_0}{\nu_0 + \nu_1} =: h^{(1)}.$$

$$\frac{\ddot{\Phi}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = -R\frac{\dot{\Phi} + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} \cdot \frac{\frac{U}{\nu_0} - \dot{\Phi}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} \cdot (\Phi + w\dot{\Phi})$$

$$\ddot{\Phi} \approx -k_1(\Phi + w\dot{\Phi}), \quad k_1 := R\frac{U}{\nu_0 + \nu_1}$$

2. $\mathbb{I}_{q_0} = 1, \mathbb{I}_{q_1} = 0$

$$h_0 = \frac{\dot{\Phi} + 1}{\frac{U}{\nu_0}} \overset{\dot{\Phi}\to 0}{\to} \frac{\nu_0}{U} =: h^{(2)}.$$

$$\frac{\ddot{\Phi}}{\frac{U}{\nu_0}} = -R\frac{\dot{\Phi} + 1}{\frac{U}{\nu_0}} \cdot \frac{\frac{U}{\nu_0} - 1 - \dot{\Phi}}{\frac{U}{\nu_0}} \cdot (\Phi + w\dot{\Phi})$$

$$\ddot{\Phi} \approx -k_2(\Phi + w\dot{\Phi}), \quad k_2 := R(1 - \frac{\nu_0}{U})$$

3. $\mathbb{I}_{q_0} = 0, \mathbb{I}_{q_1} = 1$

$$h_0 = \frac{\dot{\Phi} + \frac{U}{\nu_1} - 1}{\frac{U}{\nu_1}} \overset{\dot{\Phi} \to 0}{\to} \frac{U - \nu_1}{U} =: h^{(3)}.$$

$$\frac{\ddot{\Phi}}{\frac{U}{\nu_1}} = -R\frac{\dot{\Phi} + \frac{U}{\nu_1} - 1}{\frac{U}{\nu_1}} \cdot \frac{1 - \dot{\Phi}}{\frac{U}{\nu_1}} \cdot (\Phi + w\dot{\Phi})$$

$$\ddot{\Phi} \approx -k_3(\Phi + w\dot{\Phi}), \quad k_3 := R(1 - \frac{\nu_1}{U})$$

4. $\mathbb{I}_{q_0} = 0, \mathbb{I}_{q_1} = 0$

$$\frac{d}{d\theta} \ln \frac{h_0}{1 - h_0} = \frac{\dot{h_0}}{h_0(1 - h_0)} = R(\tau_1 - \tau_0)$$

$$h_0(\theta) = \sigma\left(\ln \frac{h_0(\theta_0)}{1 - h_0(\theta_0)} + R(\tau_1 - \tau_0)(\theta - \theta_0)\right) \overset{\theta \to \infty}{\to} 1, \quad \sigma(x) = \frac{1}{1 + e^{-x}}$$

For equation $\ddot{\Phi} + k(\Phi + w\dot{\Phi}) = 0$ we can estimate the solution decay rate:

$$w \le \frac{2}{\sqrt{k}} : \quad t_0 - t_1, h_0 - h^* \propto e^{-\frac{kw}{2}\theta}$$

$$w > \frac{2}{\sqrt{k}} : \quad t_0 - t_1, h_0 - h^* \propto e^{-\frac{kw - \sqrt{(kw)^2 - 4k}}{2}\theta}$$

Case switching

We have invariants Φ, and h_0. Below we study the derivative switch $\dot{\Phi} \to \dot{\Phi}'$.

- $1 \to 2$. Necessary conditions: $h_0 > 1 - \frac{\nu_1}{U}$, $\Phi = \Phi' \ge \tau_0 - \tau_1$.

$$\frac{\dot{\Phi} + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = \frac{\dot{\Phi}' + 1}{\frac{U}{\nu_0}}$$

$$\dot{\Phi}' = \frac{\dot{\Phi} - \frac{\nu_0 + \nu_1 - U}{\nu_1}}{\frac{\nu_0 + \nu_1}{\nu_1}}$$

We are either close to equilibrium, or decrease $|\dot{\Phi}|$:

$$|\dot{\Phi}| < \frac{\nu_0 + \nu_1 - U}{\nu_0} \quad \Rightarrow \quad |\dot{\Phi}'| < \frac{\nu_0 + \nu_1 - U}{\nu_0}$$

$$\dot{\Phi} \geq \frac{\nu_0 + \nu_1 - U}{\nu_0} \quad \Rightarrow \quad |\dot{\Phi}'| \leq |\dot{\Phi}|$$

- $1 \to 3$. Necessary conditions: $h_0 < \frac{\nu_0}{U}$, $\Phi = \Phi' \leq \tau_0 - \tau_1$.

$$\frac{\dot{\Phi} + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = \frac{\dot{\Phi}' + \frac{U}{\nu_1} - 1}{\frac{U}{\nu_1}}$$

$$\dot{\Phi}' = \frac{\dot{\Phi} + \frac{\nu_0 + \nu_1 - U}{\nu_0}}{\frac{\nu_0 + \nu_1}{\nu_0}}$$

We are either close to equilibrium, or decrease $|\dot{\Phi}|$:

$$|\dot{\Phi}| < \frac{\nu_0 + \nu_1 - U}{\nu_1} \quad \Rightarrow \quad |\dot{\Phi}'| < \frac{\nu_0 + \nu_1 - U}{\nu_1}$$

$$\dot{\Phi} < -\frac{\nu_0 + \nu_1 - U}{\nu_1} \quad \Rightarrow \quad |\dot{\Phi}'| < |\dot{\Phi}|$$

- $2 \to 1$. Necessary condition: $h_0 = 1 - \frac{\nu_1}{U}$.

$$\frac{\dot{\Phi} + 1}{\frac{U}{\nu_0}} = \frac{\dot{\Phi}' + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = 1 - \frac{\nu_1}{U}$$

$$\dot{\Phi} = \dot{\Phi}' = -\frac{\nu_0 + \nu_1 - U}{\nu_0}$$

- $3 \to 1$. Necessary condition: $h_0 = \frac{\nu_0}{U}$.

$$\frac{\dot{\Phi} + \frac{U}{\nu_1} - 1}{\frac{U}{\nu_1}} = \frac{\dot{\Phi}' + \frac{U}{\nu_1}}{\frac{U}{\nu_0} + \frac{U}{\nu_1}} = \frac{\nu_0}{U}$$

$$\dot{\Phi} = \dot{\Phi}' = \frac{\nu_0 + \nu_1 - U}{\nu_1}$$

- $2 \to 4 \to 2$. Necessary conditions: $h_0 < \frac{\nu_0}{U}$, $\Phi = \Phi' = \tau_0 - \tau_1 < 0$.

$$\dot{\Phi} < 0,\ \dot{\Phi}' = 0 \quad \Rightarrow \quad |\Phi' + w\dot{\Phi}'| < |\Phi + w\dot{\Phi}|.$$

- $3 \to 4 \to 2$. Necessary conditions: $1 - \frac{\nu_1}{U} < h_0 < \frac{\nu_0}{U}$, $\Phi = \Phi' = \tau_0 - \tau_1 < 0$.

$$0 < \dot{\Phi} < \frac{\nu_0 + \nu_1 - U}{\nu_1},\ \dot{\Phi}' = 0.$$

Mupltiple links

Consider n parallel links $\{(\tau_i, \nu_i)\}_{i=1}^n$ with strict ordering of travel times $\tau_1 < \cdots < \tau_n$ and demand $\nu_1 < U < \sum_i \nu_i$.

$$\dot{h}_i = Rh_i(\phi_i - \sum_{j=1}^{n} h_j\phi_j) = Rh_i(1-h_i)\left(\phi_i - \overline{\phi_{-i}}\right), \quad i = 1 \ldots n$$

$$\overline{\phi_{-i}} := \sum_{j\neq i} \frac{h_j}{\sum_{j'\neq i} h_{j'}} \phi_j$$

Where we use negative projected travel times as fitnesses:

$$\phi_i = -(t_i + w\dot{t}_i) = -\left(\tau_i + \frac{q_i}{\nu_i} + w \cdot \left(\frac{Uh_i}{\nu_i} - 1\right)\mathbb{I}_{q_i}\right).$$

Suppose there exists $k^* \leq n$, s.t.

$$\sum_{i=1}^{k^*-1} \nu_i < U < \sum_{i=1}^{k^*} \nu_i.$$

Then one can show the following state is the equilibrium:

$$\mathbb{I}_{q_i} = \mathbb{I}_{[i<k^*]}; \quad t_i^* := \begin{cases} \tau_{k^*},\ i \leq k^* \\ \tau_i,\ i > k^* \end{cases} \quad ; \quad h_i^* = \begin{cases} \frac{\nu_i}{U}, i < k^* \\ 1 - \sum_{i<k^*} h_i^*,\ i = k^* \\ 0,\ i > k^* \end{cases} .$$

First show for $k^* = n$, then try to generalize

$$h_{>k^*} := \sum_{j>k^*} h_j; \quad \phi_{>k^*} := \sum_{j>k^*} \frac{h_j}{h_{>k^*}} \phi_j$$

Assuming same queue configuration, we can linearize the system near the equillibrium. Below, we denote $\Delta h := h - h^*$, $\Delta t := t - t^*$.

$$\dot{\Delta h}_i \approx -Rh_i^* \left(\left(\Delta t_i + \frac{w}{h_i^*}\Delta h_i\right) - \sum_{j<k^*} h_j^* \left(\Delta t_j + \frac{w}{h_j^*}\Delta h_j\right) - \sum_{j>k^*} \Delta h_j \tau_j\right),\ i < k^*$$

$$\dot{\Delta h}_{k^*} \approx -Rh^*_{k^*}\left(0 - \sum_{j<k^*} h^*_j \left(\Delta t_j + \frac{w}{h^*_j}\Delta h_j\right) - \sum_{j>k^*} \Delta h_j \tau_j\right)$$

$$\dot{\Delta h}_i \approx -R\Delta h_i(\tau_i - \tau_{k^*}),\ i > k^*$$

$$\dot{\Delta t}_i = \frac{1}{h^*_i}\Delta h_i,\ i < k^*; \quad \dot{\Delta t}_i = 0,\ i \geq k^*$$

$$\frac{d}{d\theta}\begin{bmatrix}\Delta h_{<k^*}\\ \Delta t_{<k^*}\\ \Delta h_{k*}\\ \Delta t_{k^*}\\ \Delta h_{>k^*}\\ \Delta t_{>k^*}\end{bmatrix} \approx -RA\begin{bmatrix}\Delta h_{<k^*}\\ \Delta t_{<k^*}\\ \Delta h_{k*}\\ \Delta t_{k^*}\\ \Delta h_{>k^*}\\ \Delta t_{>k^*}\end{bmatrix},$$

$$A = \begin{bmatrix} w(I - h^*_{<k^*}\mathbf{1}^T) & h^*_{<k^*}(I - (h^*_{<k^*})^T) & 0 & -h^*_{<k^*}h^*_{k^*} & -h^*_{<k^*}\tau^T_{>k^*} & 0\\ \frac{1}{h^*_{<k^*}} & 0 & 0 & 0 & 0 & 0\\ -h^*_{k^*}wh^*_{<k^*} & 0 & h^*_{k^*}\tau & b^T & h^*k^*(1-h^*k^*) & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \frac{w}{h^*}I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & (\tau - \tau_{k^*})I & 0 \end{bmatrix}$$

Lyapunov Theory

Our differential equation has form $\ddot{\Phi} = g(\Phi, \dot{\Phi})$, where g is a polynomial. We view it as autnomous system with Lipshitz RHS

$$\frac{d}{d\theta}\begin{bmatrix}\Phi\\ \dot{\Phi}\end{bmatrix} = \begin{bmatrix}\dot{\Phi}\\ g(\Phi, \dot{\Phi})\end{bmatrix}$$

In this case Lyapunov Theory is applicable.
In general, we consider autonomous system

$$\dot{x} = f(x), \quad f(0) = 0.$$

Theorem 1 *(Global asymptotic stability) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, s.t. :*

$$V(0) = 0 \textit{ and } \forall x \neq 0 : \; V(x) > 0,$$

$$\|x\|_2 \to \infty \quad \Rightarrow \quad V(x) \to \infty,$$
$$\forall x \neq 0 : \; \dot{V} = \nabla V(x) \cdot f(x) < 0$$

Then $x = 0$ is globally asymptotically stable.

Theorem 2 *(Local asymptotic stability) Let $A = \frac{\partial f}{\partial x}(0)$ be Jacobian matrix at $x = 0$. If we have $\mathrm{Re}(\lambda_i(A)) < 0$ for all eigenvalues, then $x = 0$ is locally asymptotically stable.*

Local stability

We consider the linearized equation

$$\ddot{\Phi} = \frac{\partial f}{\partial \Phi}(0,0) \cdot \Phi + \frac{\partial f}{\partial \dot{\Phi}}(0,0) \cdot \dot{\Phi} = -k\Phi - kw\dot{\Phi}.$$

Searching for eigenvalues yields

$$\det\left(\begin{bmatrix}-\lambda & 1\\ -k & -kw - \lambda\end{bmatrix}\right) = \lambda^2 + kw\lambda + k; \quad D = k^2w^2 - 4k.$$

$D \geq 0 : \quad \mathrm{Re}(\lambda_{1,2}) = -\frac{kw}{2} \pm \sqrt{\left(\frac{kw}{2}\right)^2 - k} < 0,$
$D < 0 : \quad \mathrm{Re}(\lambda_{1,2}) = -\frac{kw}{2} < 0.$
Hence, we always have local asymptotic stability.

Sufficient conditions for global stability (Quadratic potential)

In case $i \in \{1,2,3\}$ we can write the equation as

$$\ddot{\Phi} = -RC_i h_0(1-h_0)(\Phi + w\dot{\Phi}).$$

Denote

$$C := \min_{i\in\{1,2,3\}} C_i = \min\left\{\frac{U}{\nu_0} + \frac{U}{\nu_1}, \frac{U}{\nu_0}, \frac{U}{\nu_1}\right\} = \frac{U}{\max\{\nu_0,\nu_1\}}.$$

Further, assume that

$$\forall \theta \geq 0: \quad h_0(\theta) \in [a,b] \text{ with } a > 0, b < 1$$

and denote $m := \min_{x\in[a,b]} x(1-x) > 0$.
Consider Lyapunov function $V(\Phi,\dot{\Phi}) = \Phi^2 + (\Phi + w\dot{\Phi})^2 > 0$.

$$\dot{V} = \frac{\partial V}{\partial \Phi}\dot{\Phi} + \frac{\partial V}{\partial \dot{\Phi}} \cdot \left(-RC_i h_0(1-h_0)(\Phi + w\dot{\Phi})\right) =$$

$$= \left(4\Phi + 2w\dot{\Phi}\right)\dot{\Phi} - 2wRC_i h_0(1-h_0)(\Phi + w\dot{\Phi})^2 \leq$$

$$\leq 4\Phi\dot{\Phi} + 2w\dot{\Phi}^2 - 2wRCm(\Phi + w\dot{\Phi})^2 =$$

$$= 2w\left(\frac{1}{w^2} - RCm\right)(\Phi + w\dot{\Phi})^2 - \frac{2}{w}\Phi^2.$$

We can guarantee $\dot{V} < 0$ if $w \geq \frac{1}{\sqrt{RCm}}$.
For $R = 0.1, U = 4.5, [a,b] = [0.01, 0.99]$ we need $w \geq \frac{1}{\sqrt{0.1\cdot\frac{4.5}{3}\cdot 0.01\cdot .99}} \approx 25.95$.
If we have $V = \Phi^2 + (\Phi + w\dot{\Phi})^2 \leq B^2$, then it follows that $|\Phi| \leq B$, $|\dot{\Phi}| \leq \frac{\sqrt{2}B}{w}$.
Suppose we can reach $V = B^2$ s.t. $\frac{\sqrt{2}B}{w} = \frac{\nu_0+\nu_1-U}{\nu_0}$ Then we stop case switching?

Inflow potentials

Having the functional relation

$$\dot{\Phi}(h_0) = \left(\frac{Uh_0}{\nu_0} - 1\right)\mathbb{I}_{q_0} - \left(\frac{U(1-h_0)}{\nu_1} - 1\right)\mathbb{I}_{q_1} = C_i(h_0 - h^{(i)}),$$

we can construct the Lyapunov function

$$V(\Phi, \dot{\Phi}) = \frac{1}{2}\Phi^2 + \frac{1}{R}\int_{\dot{\Phi}(h)=0}^{h_0} \frac{\dot{\Phi}(h)dh}{h(1-h)} = \frac{1}{2}\Phi^2 + \frac{1}{R}\int_{h^{(i)}}^{h_0} \frac{C_i(h-h^{(i)})}{h(1-h)}dh > 0.$$

Then, for each case $i = 1, 2, 3$ we have

$$\dot{V} = \Phi \cdot \dot{\Phi} - \frac{\dot{\Phi}}{RC_i h_0(1-h_0)} \cdot RC_i h_0(1-h_0)(\Phi + w\dot{\Phi}) = -w\dot{\Phi}^2 < 0$$

The integral can be computed:

$$\int \frac{C_i(h_0 - h^{(i)})dh_0}{h_0(1-h_0)} = C_i\left(\ln\frac{1}{1-h_0} - h^{(i)}\ln\frac{h_0}{1-h_0}\right) + \text{const}$$

We introduce inflow potentials for the cases 2 and 3

$$\Pi_0 := \int_{\frac{\nu_0}{U}}^{h_0} \frac{\frac{U}{\nu_0}(h - \frac{\nu_0}{U})}{h(1-h)}dh, \quad \Pi_1 := \int_{\frac{U-\nu_1}{U}}^{h_0} \frac{\frac{U}{\nu_1}(h - \frac{U-\nu_1}{U})}{h(1-h)}dh.$$

One can observe that their sum plays a role of potential in the case 1

$$\Pi_0 + \Pi_1 = \int_{\frac{\nu_0}{\nu_0+\nu_1}}^{h_0} \frac{(\frac{U}{\nu_0} + \frac{U}{\nu_1})(h - \frac{\nu_0}{\nu_0+\nu_1})}{h(1-h)}dh + \Delta\Pi,$$

where we have a constant

$$\Delta\Pi := \int_{\frac{\nu_0}{\nu_0+\nu_1}}^{\frac{\nu_0}{U}} \frac{\frac{U}{\nu_0}(\frac{\nu_0}{U} - h)}{h(1-h)}dh + \int_{\frac{U-\nu_1}{U}}^{\frac{\nu_0}{\nu_0+\nu_1}} \frac{\frac{U}{\nu_1}(h - \frac{U-\nu_1}{U})}{h(1-h)}dh \geq 0,$$

with property $\Delta\Pi = 0 \Leftrightarrow U = \nu_0 + \nu_1$.
Hence, function $V = \frac{1}{2}\Phi^2 + \Pi_0\mathbb{I}_{q_0} + \Pi_1\mathbb{I}_{q_1}$ has following propreties:

- $V = 0$ only at the equillibrium, else $V > 0$;
- $V \to \infty$ if $|\Phi| \to \infty$ or $h_0 \to 0, 1$;
- $\dot{V} < 0$ during the cases $1, 2, 3$.

There still remain a problem of case switching. Namely, we need to show that particular case switches (e.g. $2 \to 1 \to 2$, $2 \to 1 \to 3 \to 1 \to 2$, and others) can't lead to an increase in V.

Max demand ($U = \nu_0 + \nu_1$)

Denote $Q := q_0 + q_1$ and observe that

$$\dot{Q} = \dot{q_0} + \dot{q_1} = (Uh_0 - \nu_0)\, \mathbb{I}_{q_0} + (U(1-h_0) - \nu_1)\, \mathbb{I}_{q_1}$$

We have $\dot{Q} = 0$ in case 1 and $\dot{Q} > 0$ in cases 2,3
Suppose we start with $h_0 < h^*$ in case 3.

$$\theta = s_0 = 0 : \quad h_0 - h^* < 0, \quad \Phi = \tau_0 - \tau_1 < 0 \quad \Rightarrow \quad \dot{h_0} > 0$$

We increase h_0 until value h^* is reached and switch to case 1.

$$q_0(s_1) = 0, \quad q_1(s_1) = \int_{s_0}^{s_1} U(h^* - h_0) d\theta$$

$$\theta = s_1 : \quad h_0 - h^* = 0, \quad \Phi = \tau_0 - (\tau_1 + \frac{q_1}{\nu_1}) < 0 \quad \Rightarrow \quad \dot{h_0} > 0$$

$$\frac{1}{2}\Phi^2(s_1) < \frac{1}{2}\Phi^2(s_0) + \frac{1}{R}\Pi_1(s_0)$$

There are two options: either we'll have $\Phi + w\dot{\Phi} = 0$

$$q_0(s_2) = \int_{s_1}^{s_2} U(h_0 - h^*) d\theta, \quad q_1(s_2) = q_1(s_1) - \int_{s_1}^{s_2} U(h_0 - h^*) d\theta$$

$$\theta = s_2 : \quad h_0 - h^* > 0, \quad \Phi = \Phi(s_1) + q_0(s_2)\left(\frac{1}{\nu_0} + \frac{1}{\nu_1}\right) = -wU\left(\frac{1}{\nu_0} + \frac{1}{\nu_1}\right)(h_0 - h^*)$$

$$\frac{1}{2}\Phi^2(s_2) + \frac{1}{R}(\Pi_0(s_2) + \Pi_1(s_2)) < \frac{1}{2}\Phi^2(s_1)$$

If q_1 vanishes, we would have

$$q_0(s_2) = \int_{s_1}^{s_2} U(h_0 - h^*) d\theta = q_1(s_1), \quad q_1(s_2) = q_1(s_1) - \int_{s_1}^{s_2} U(h_0 - h^*) d\theta = 0$$

See Figure 3. Figures 4 and 5 show inflow amplitude decays and final travel times for different projection windows.

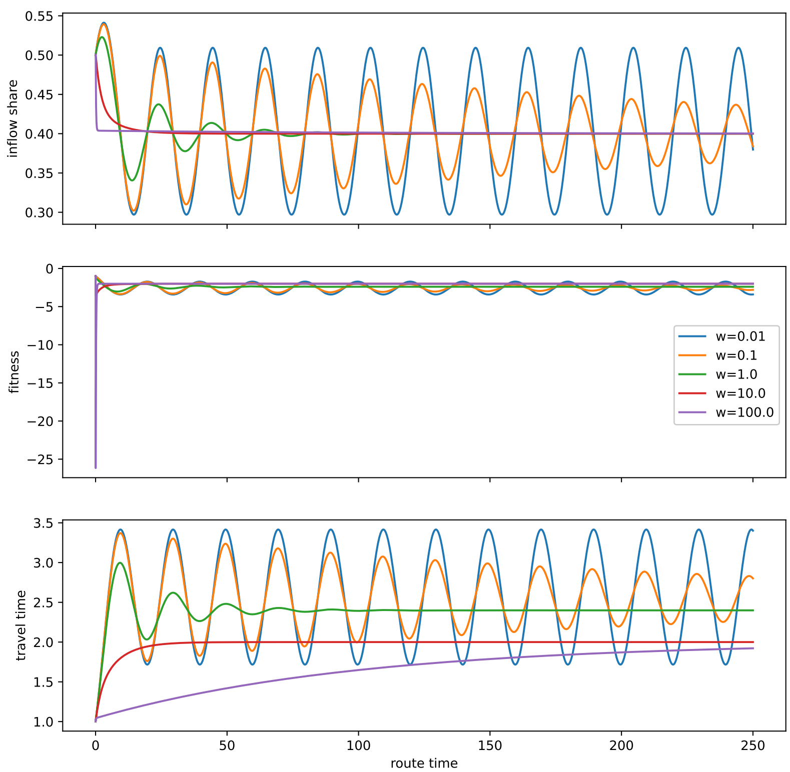

Figure 3: $U = 5, R = 0.1, \Delta\theta = 0.01$

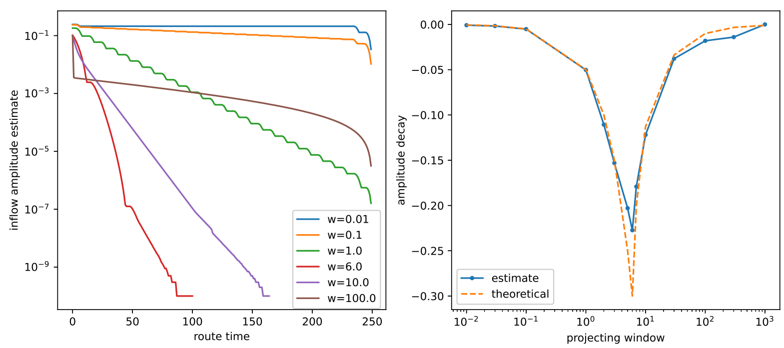

Figure 4: Inflow amplitudes for path 0

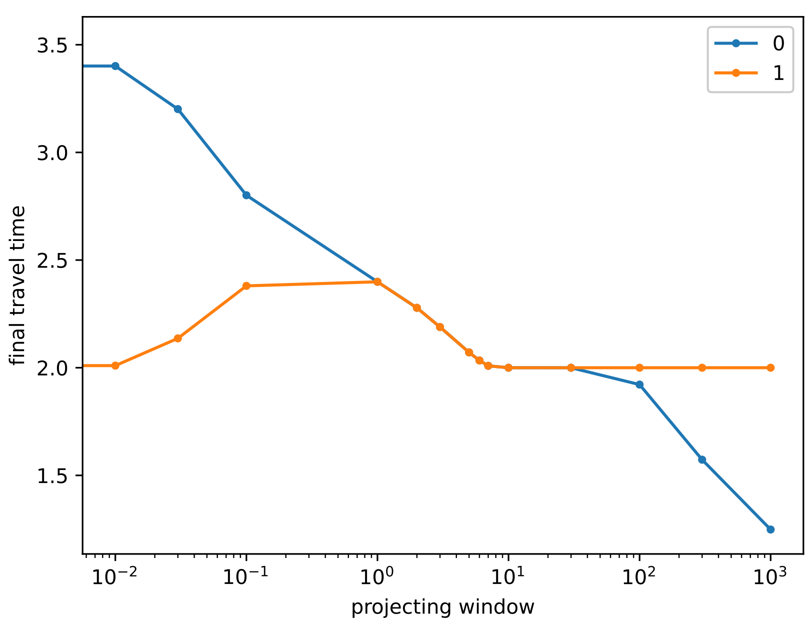

Figure 5: Final travel times

Medium Demand ($\nu_0 < U < \nu_0 + \nu_1$)

Different cases have different equilibria:

$$\frac{\nu_0}{U} > \frac{\nu_0}{\nu_0 + \nu_1} > \frac{U - \nu_1}{U}$$

Figure 6 shows how cases switch until amplitude drops enough and we end up in case $q_0 > 0, q_1 = 0$.

Figure 7 shows dynamics for the faster edge 0 with different projection windows. Both fitnesses eventually stabilize to values $\phi = -\tau_1$ (negative cost of slower edge).

Now assume that edge 0 operates on full capacity and edge 1 always operates below capacity, i.e.

$$\dot{t_0}(\theta) = \frac{U \cdot h_0(\theta) - \nu_0}{\nu_0}, \quad \dot{t_1}(\theta) \equiv 0.$$

Sufficient conditions of the solution to fulfill the assumptions:

$$\Phi(\theta) - \Phi(0) \geq 0; \quad \dot{\Phi}(\theta) \geq -\frac{\nu_0 + \nu_1 - U}{\nu_0}.$$

Figure 8 shows that that theoretical decay estimate is good.

Figure 9 shows the difference between theoretical and observed dynamics. Used parameters: $U = 4.5; \Delta\theta = 0.01; h_0(0) = 0.5; R = 0.1; w = 1.0$

Figures 10 and 11 show dynamics for edge 0 with parameters $U = 4.5; \Delta\theta = 0.01; R = 0.1$ for different initial share with $w = 1.0$ and $w = \frac{2}{\sqrt{k}} = 8.49$

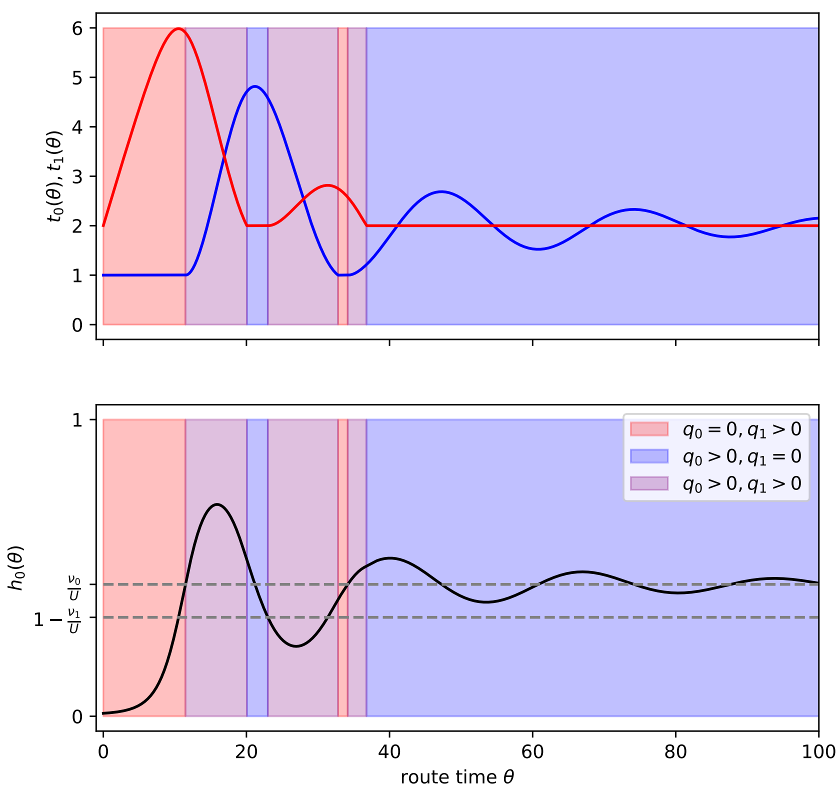

Figure 6: Case switching

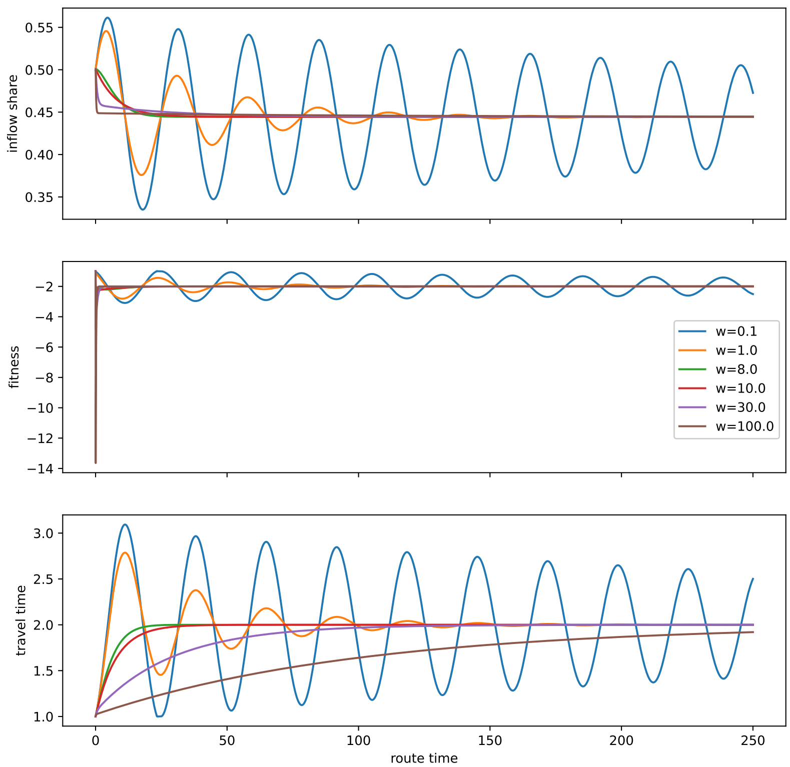

Figure 7: Medium demand. $U = 4.5; \Delta\theta = 0.01; R = 0.1$

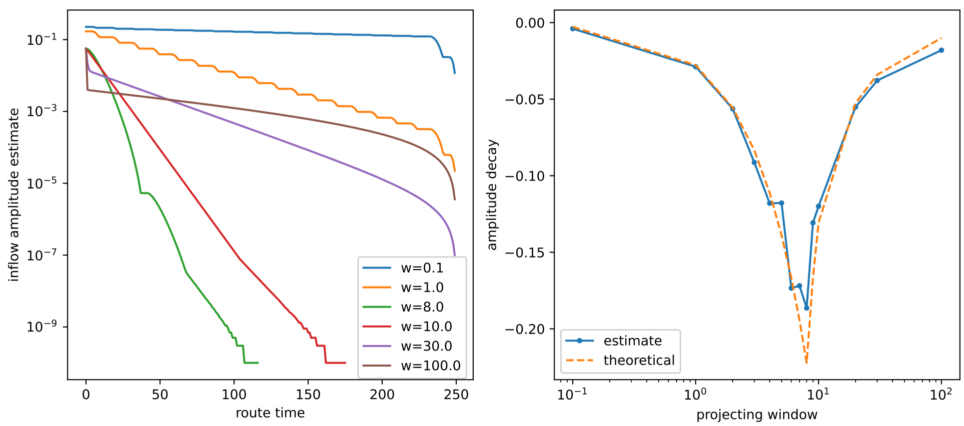

Figure 8: Amplitudes decay

Unstable dynamics

Figures 12, 13 and 14 show how using bad fitness leads to divergence.

Proof ?

Necessary condition for stability: $\forall P : \ \phi_P(\theta, h) = \sum_{P'} h_{P'}(\theta)\phi_{P'}(\theta, h)$.
In cases of average travel time and last travel, influence of inflows on fitnesses is delayed for at least τ_0.

Sufficient condition for stability for non-aggregated travel times: queues have to not change for the period of delay. This guarantees that fitnesses don't change once equality is reached.

Even when we use undelayed travel times, inflows tend to get in antiphase with fitnesses. In this example, we need inflows to equalize with edge capacities at the moment when both path have same costs.

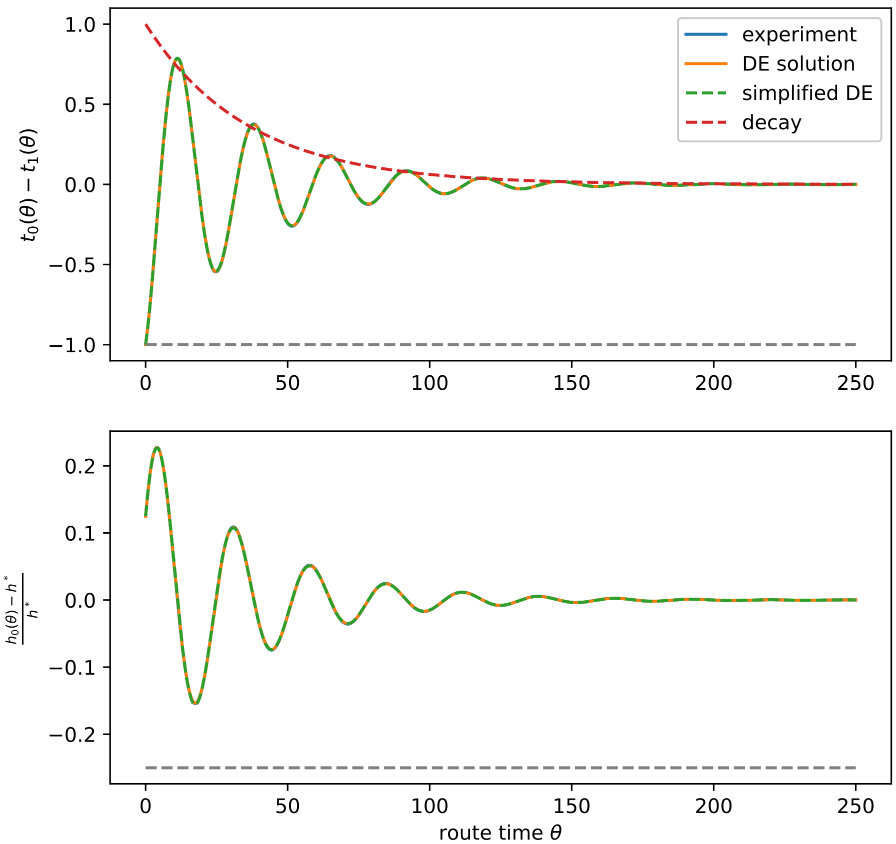

Figure 9: Solution of DEs compared to the experiment; gray dashed lines show illustrate fulfillment of assumptions

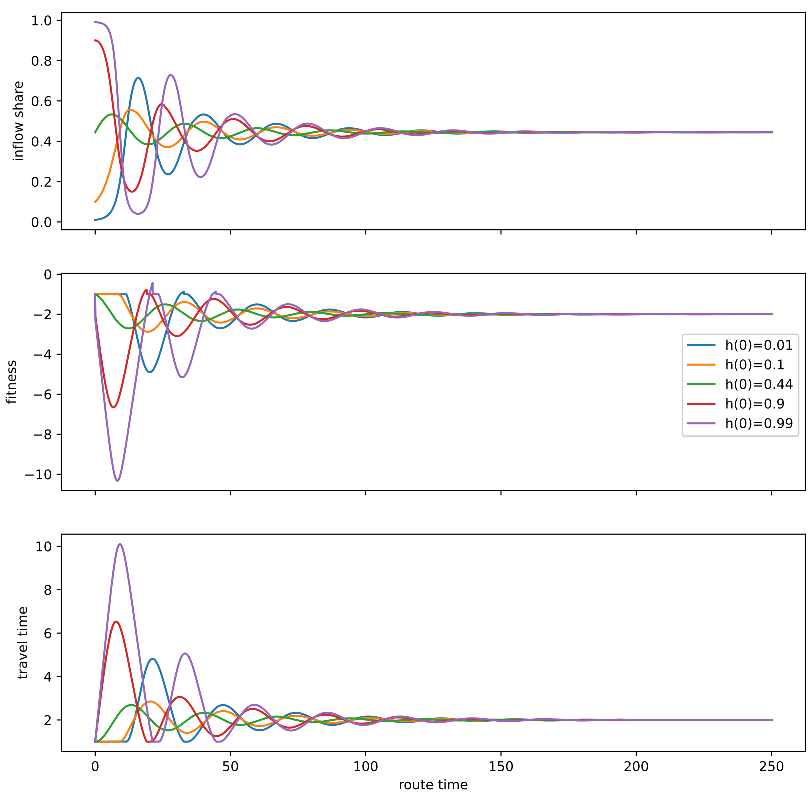

Figure 10: $w = 1.0$

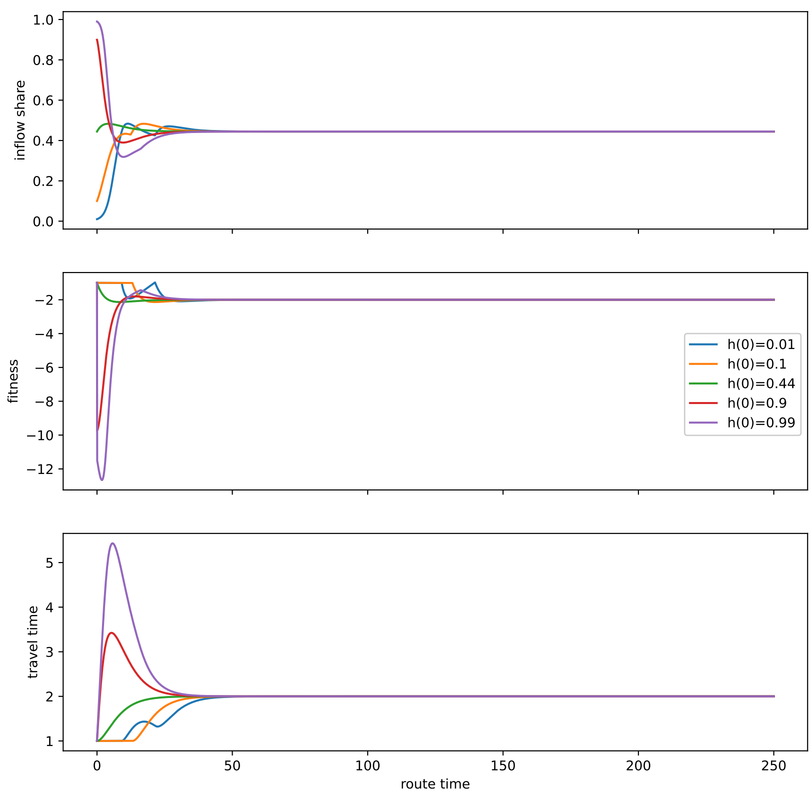

Figure 11: $w = \frac{2}{\sqrt{k}} = 8.49$

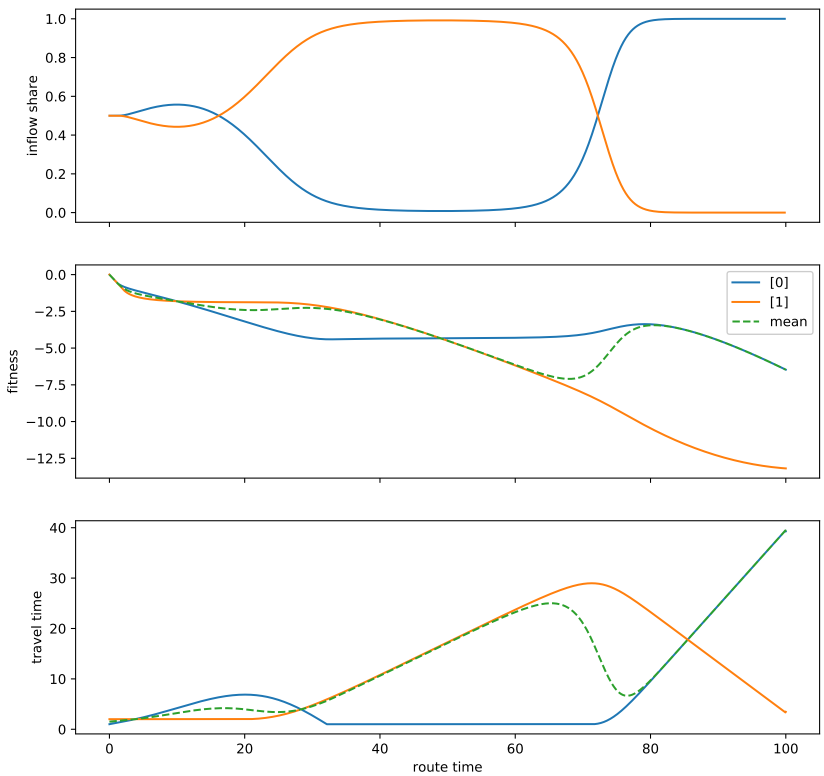

Figure 12: Fluctuations, average travel time

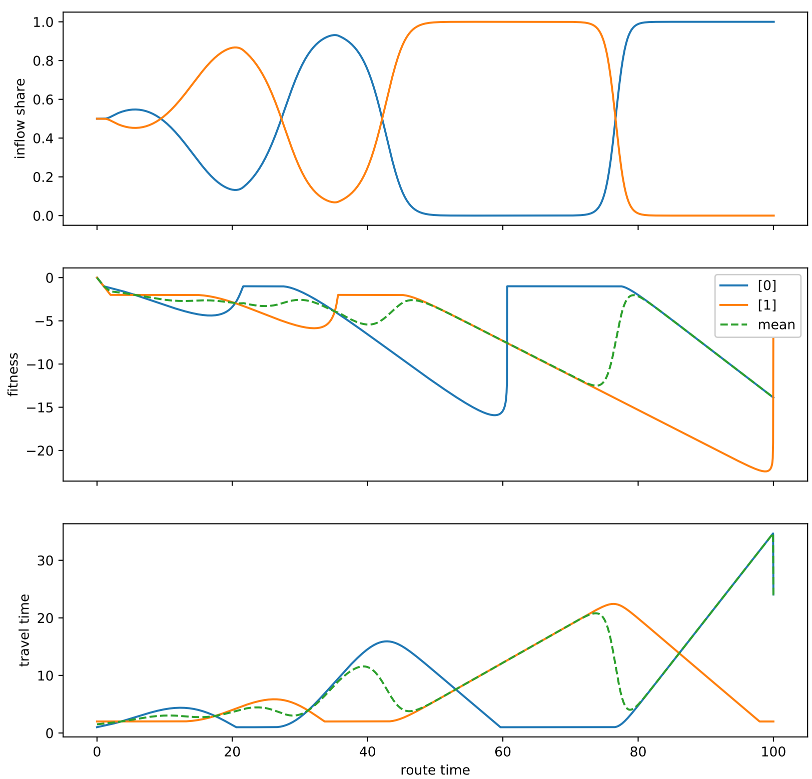

Figure 13: Fluctuations, last travel time

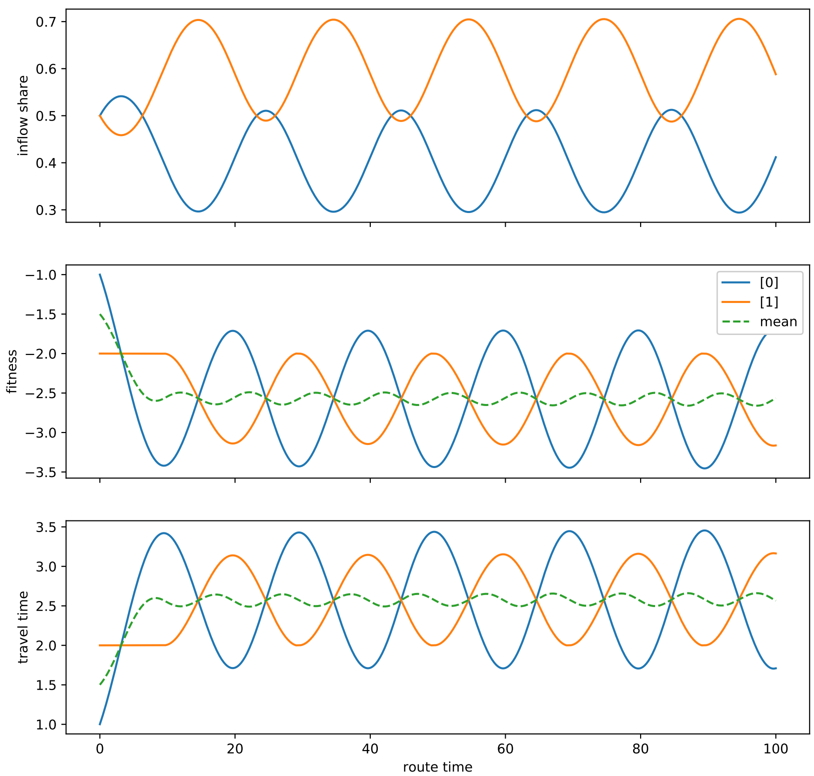

Figure 14: Fluctuations, predicted travel time