Achieving Equillibria in Vickrey's model

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Problem Setting

- Directed graph $G = (V, E)$, source $s \in V$, destination $t \in V$
- Planning horizon $[0, \infty)$, time $\theta \geq 0$
- Strategy space $\Lambda^\theta(r) = \{h(\theta) \in \mathbb{R}^{\mathcal{P}} | \ \sum_{p' \in \mathcal{P}} h_{p'}(\theta) = r\}$
- Path-delay operator according to Vickrey's model:

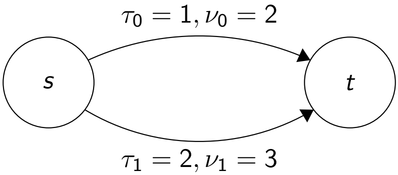
$$\begin{aligned} \Psi : (L^2_+[0,\theta])^{\mathcal{P}} &\to (L^2_+[0,\theta])^{\mathcal{P}} \\ h &\mapsto \Psi(\cdot, h) \end{aligned}$$

- Dynamic equilibrium $h^*(\theta) \in \Lambda^\theta(r)$:

$$\langle \Psi(\theta, h^*), h^* \rangle \leq \langle \Psi(\theta, h^*), h \rangle \ \forall h \in \Lambda^\theta(r)$$

- Goal: construct a dynamic for $h(\cdot)$, s.t.

$$\|\Psi(\theta, h) - \Psi(\theta, h^*)\|_2 \overset{\theta \to \infty}{\to} 0$$



$$\nu_0 < r < \nu_0 + \nu_1$$

$$h_0^*(\theta) = \begin{cases} r, & 0 \le \theta < \theta^s \\ \nu_0, & \theta \ge \theta^s \end{cases} \qquad h_1^*(\theta) = r - h_0^*(\theta)$$

$$\Psi_0(\theta, h^*) = \begin{cases} \tau_0 + \frac{r-\nu_0}{\nu_0}\theta, & 0 \le \theta < \theta^s \\ \tau_1, & \theta \ge \theta^s \end{cases} \qquad \Psi_1(\theta, h^*) \equiv \tau_1$$

$$\theta^s = (\tau_1 - \tau_0)\frac{\nu_0}{r - \nu_0}$$

Replicator dynamics

- Fitness $(\phi_p(\theta, h))_{p\in\mathcal{P}}$, s.t. higher fitness corresponds to "better" path
- Replicator equation for $p \in \mathcal{P}$:

$$\dot{h}_p = h_p \cdot K(\phi_p - \overline{\phi})$$

$K > 0$ is a scaling constant, $\overline{\phi}(\theta, h) := \langle \phi(\theta, h), \frac{1}{r} h(\theta) \rangle$ is average fitness

- By construction $\sum_{p\in\mathcal{P}} \dot{h}_p = 0 \quad \Rightarrow \quad h(\theta) \in \Lambda^\theta(r)$
- We hope to decrease the fitnesses gap by redirecting the inflow to "better" paths

$$\mathrm{Gap}_\phi(\theta, h) := \max_{h' \in \Lambda^\theta(r)} \langle \phi(\theta, h), h' - h(\theta) \rangle = r(\phi_{max}(\theta, h) - \overline{\phi}(\theta, h))$$

- Integrating the equations allows to write them as Dual Averaging Dynamic:

$$y(\theta) = \log(h(0)) + K \cdot \int_0^\theta \phi(\theta', h)d\theta'$$
$$h(\theta) = Q_\lambda(y(\theta))$$

- Q_λ is logit choice map on $\Lambda^\theta(r)$:

$$Q_\lambda(v) := \arg \max_{h' \in \Lambda^\theta(r)} \left[\langle v, h' \rangle - \sum_{p \in \mathcal{P}} h'_p \log h'_p \right] = r \cdot \mathrm{SoftMax}(v)$$

- Condition for correct stabilization: $\phi(\infty, h^*) = -\Psi(\infty, h^*)$
- Average delay, true delay, projected delay may be considered as fitnesses

$$\phi^{\text{avg}}(\theta, h) := -\frac{1}{w}\int_{\theta-w}^{\theta} \Psi(\theta', h)d\theta'$$
$$\phi^{\text{true}}(\theta, h) := -\Psi(\theta, h)$$
$$\phi^{\text{proj}}(\theta, h) := -\left(\Psi(\theta, h) + w\dot{\Psi}(\theta, h)\right)$$

- Numerical experiments show that only projected delays lead to convergence
- The Gap function is still not monothone

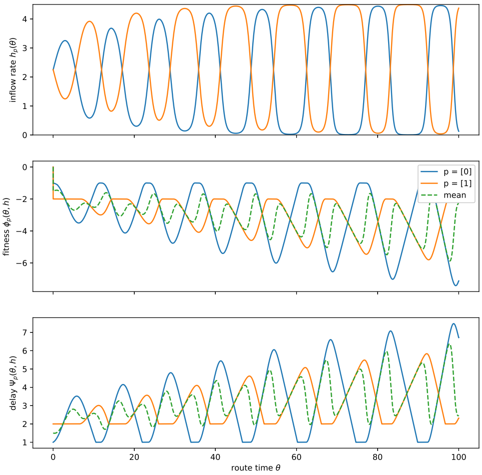


Figure: Average delays

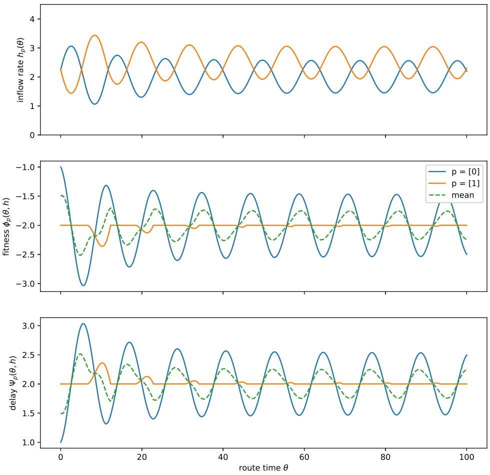


Figure: True delays

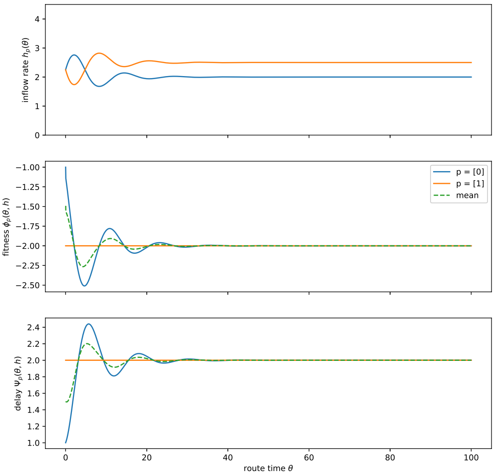


Figure: Projected delays

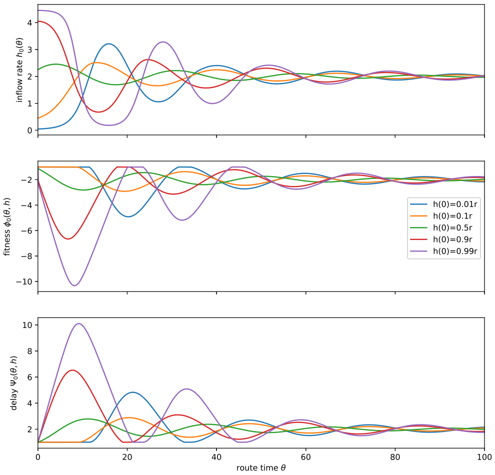


Figure: Global stability in experiments

Analysis

We have an autonomous system $\dot{x} = f(x), \quad f(0) = 0$ with Lipshitz RHS

Theorem (Global asymptotic stability)

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, s.t. :

$$V(0) = 0 \textit{ and } \forall x \neq 0 : \; V(x) > 0,$$

$$\|x\|_2 \to \infty \quad \Rightarrow \quad V(x) \to \infty,$$

$$\forall x \neq 0 : \; \dot{V} = \nabla V(x) \cdot f(x) < 0$$

Then $x = 0$ is globally asymptotically stable.

Theorem (Local asymptotic stability)

Let $A = \frac{\partial f}{\partial x}(0)$ be Jacobian matrix at $x = 0$. If we have $\mathrm{Re}(\lambda_i(A)) < 0$ for all eigenvalues, then $x = 0$ is locally asymptotically stable.

- Consider a network with n parallel links $\{(\tau_i, \nu_i)\}_{i=1}^{n}$
- Introduce queue activity indicator $\mathbb{I}_{q_i} := \begin{cases} 1, & q_i > 0 \text{ or } h_i \geq \nu_i \\ 0, & \text{otherwise} \end{cases}$
- Fitnesses based on projected delays:

$$\phi_i^{proj} = -(\Psi_i + w\dot{\Psi}_i) = -\left(\tau_i + \frac{q_i}{\nu_i} + w\frac{h_i - \nu_i}{\nu_i}\mathbb{I}_{q_i}\right)$$

- Assumptions:
  1. strict order $\tau_1 < \cdots < \tau_n$
  2. $\exists k^* \leq n: \; \sum_{i=1}^{k^*-1} \nu_i < r < \sum_{i=1}^{k^*} \nu_i$
- There is a unique equilibrium configuration:

$$\mathbb{I}_{q_i} = \begin{cases} 1, & i < k^* \\ 0, & i \geq k^* \end{cases} \quad \Psi_i^* = \begin{cases} \tau_{k^*}, & i \leq k^* \\ \tau_i, & i > k^* \end{cases} \quad h_i^* = \begin{cases} \nu_i, & i < k^* \\ r - \sum_{i<k^*} h_i^*, & i = k^* \\ 0, & i > k^* \end{cases}$$

- Any other "instantenious" equillibrium is unstable, i.e. some queues vanish

- Denote

$$\begin{aligned} \Delta q_i &:= \nu_i(\Psi_i - \Psi_i^*), && 1 \le i < k^* \\ \Delta h_i &:= h_i - h_i^*, && 1 \le i \le n \end{aligned}$$

- Assuming same active queues and utilizing $\sum_i \Delta h_i = 0$, we linearize the system near the equilibrium :

$$\frac{d}{d\theta}\begin{bmatrix} \Delta q \\ \Delta h_{<k^*} \\ \Delta h_{k*} \\ \Delta h_{>k^*} \end{bmatrix} \approx \boldsymbol{A} \begin{bmatrix} \Delta q \\ \Delta h_{<k^*} \\ \Delta h_{k*} \\ \Delta h_{>k^*} \end{bmatrix},$$

$$\boldsymbol{A} = -\frac{K}{r}\begin{bmatrix} 0 & -\frac{r}{K}\mathbf{I} & 0 & 0 \\ r\boldsymbol{I} - h_{<k^*} \cdot \mathbf{1}^T & w(r\boldsymbol{I} - h_{<k^*} \cdot \mathbf{1}^T) & 0 & -h_{<k^*} \cdot \tau_{>k^*}^T \\ -h_{k^*} \cdot \mathbf{1}^T & 0 & wh_{k^*} & -h_{k^*}(\tau_{>k^*} - w\mathbf{1})^T \\ 0 & 0 & 0 & r(\mathrm{diag}(\tau_{>k^*}) - \tau_{k^*}\boldsymbol{I})) \end{bmatrix}$$

- Finding eigenvalues by $\boldsymbol{A}v = \lambda v$ with $v = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$
  1. $c, d = 0$:
     $-\frac{K}{r}(r\boldsymbol{I} - h_{<k^*} \cdot \boldsymbol{1}^T)a = \mu a \Rightarrow \quad \mu = -K$ or $\mu = -K(1 - \frac{1}{r}\boldsymbol{1}^T h_{<k^*}) < 0$
     $\lambda^2 = \mu(1 + w\lambda) \Rightarrow Re(\lambda) < 0$
  2. $c \neq 0$: $h_{k^*} > 0 \Rightarrow \lambda < 0$
  3. $d \neq 0$: $\tau_j > \tau_{k^*}$ for $j > k^* \Rightarrow \lambda < 0$
- We have $\mathrm{Re}(\lambda(\boldsymbol{A})) < 0$, which guarantees local asymptotic stability

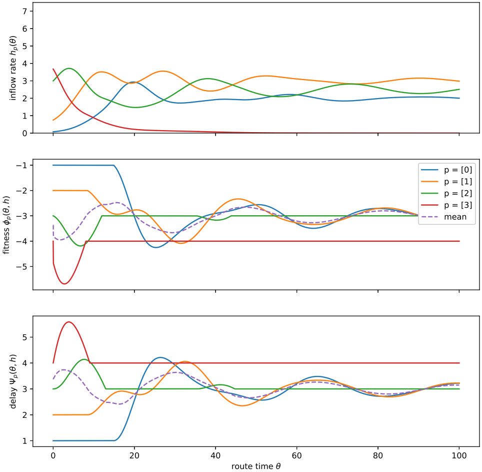


Figure: Multiple links

- Suppose we only have two links 0 and 1 with $\tau_0 < \tau_1$.
- Denoting $\psi(\theta, h) := \Psi_0(\theta, h) - \Psi_1(\theta, h)$ allows to simplify the equations:

$$\dot{h}_0 = -\frac{K}{r} h_0(r - h_0)(\psi + w\dot{\psi})$$

$$\dot{\psi} = \frac{h_0 - \nu_0}{\nu_0}\mathbb{I}_{q_0} + \frac{h_0 - (r - \nu_1)}{\nu_1}\mathbb{I}_{q_1}$$

- Equillibrium $\Leftrightarrow \psi, \dot{\psi} = 0$

- Introduce inflow potentials

$$\Pi_0(h_0) := \int_{\nu_0}^{h_0} \frac{r^{\frac{h-\nu_0}{\nu_0}}}{h(r-h)} dh, \quad \Pi_1(h_0) := \int_{r-\nu_1}^{h_0} \frac{r^{\frac{h-(r-\nu_1)}{\nu_1}}}{h(r-h)} dh$$

- Candidate for Lyapunov function

$$V(\psi, h_0) = \frac{1}{2}\psi^2 + \frac{1}{K}\left(\Pi_0 \mathbb{I}_{q_0} + \Pi_1 \mathbb{I}_{q_1}\right)$$

1. $V = 0$ only at the equilibrium, else $V > 0$;
2. $V \to \infty$ if $|\psi| \to \infty$ or $h_0 \to 0, 1$;
3. For each queue configuration we have

$$\dot{V} = \frac{\partial V}{\partial \psi}\dot{\psi} + \frac{\partial V}{\partial h_0}\dot{h}_0 = -w\dot{\psi}^2 \leq 0$$

- Problem: switching between cases

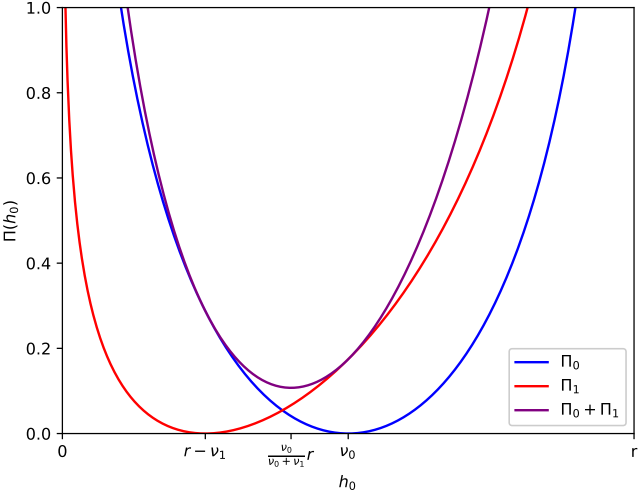
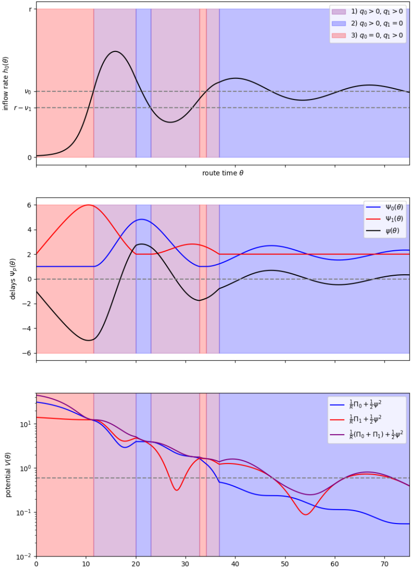


Figure: Inflow potentials


inflow rate $h_0(\theta)$
r
ν_0
$r-\nu_1$
0
route time θ
1) $q_0 > 0$, $q_1 > 0$
2) $q_0 > 0$, $q_1 = 0$
3) $q_0 = 0$, $q_1 > 0$
delays $\Psi_p(\theta)$
6
4
2
0
−2
−4
−6
$\Psi_0(\theta)$
$\Psi_1(\theta)$
$\psi(\theta)$
potential $V(\theta)$
10^1
10^0
10^{-1}
10^{-2}
0
10
20
30
40
50
60
70
$\frac{1}{R}\Pi_0 + \frac{1}{2}\psi^2$
$\frac{1}{R}\Pi_1 + \frac{1}{2}\psi^2$
$\frac{1}{R}(\Pi_0 + \Pi_1) + \frac{1}{2}\psi^2$

- Max demand: $r = \nu_0 + \nu_1$
- Three different "attractors" merge into one

$$r - \nu_1 = r\frac{\nu_0}{\nu_0 + \nu_1} = \nu_0$$

- Denote $Q := q_0 + q_1$ and observe that

$$\dot{Q} = \dot{q_0} + \dot{q_1} = (h_0 - \nu_0)\,\mathbb{I}_{q_0} + ((r - h_0) - \nu_1)\,\mathbb{I}_{q_1} \geq 0$$

- Queues grow and can't vanish

$$\Psi_e(\theta, h) - \Psi_e(\theta, h^*) \overset{\theta\to\infty}{\to} c > 0$$

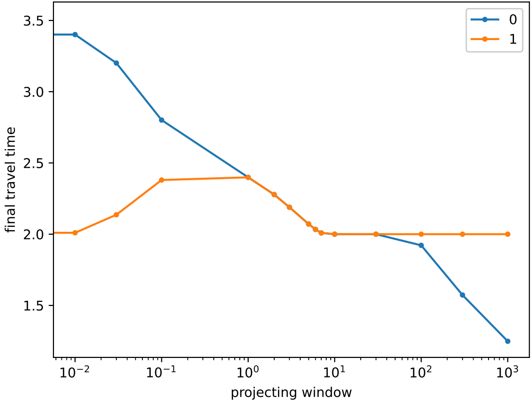


Figure: Max demand convergence

Further directions

- Showing global stability
  1. Patterns in case switching
  2. Max demand
- Considering more complex networks
- Other options for fitnesses