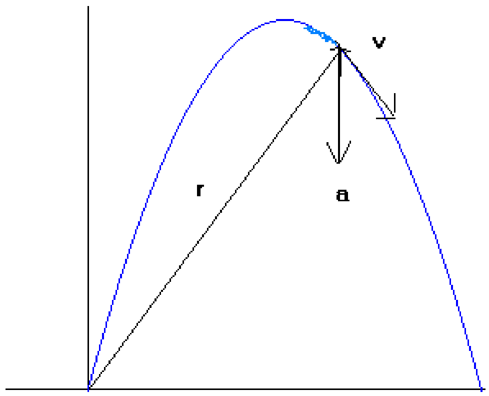
13.2 Modeling Projectile Motion

To derive equations for projectile motion, we assume that the projectile is moving along in a vertical plane and that the only force acting on the projectile is the constant force of gravity, which always points straight downward.



We assume that the projectile is launched from the origin at time t = 0 into the first quadrant with an initial velocity $\overrightarrow{v_0}$. If $\overrightarrow{v_0}$ makes an angle α with the horizontal and the initial speed of the projectile is $v_0 = |\overrightarrow{v_0}|$, then

$$\overrightarrow{v_0} = (v_0 cos\alpha)\vec{\imath} + (v_0 sin\alpha)\vec{\jmath} \text{ and } \overrightarrow{r_0} = \vec{0}$$

By Newton's Second Law of Motion $\sum F_i = m\vec{a}$, so

$$m\vec{a} = (-mg)\vec{\jmath}$$
$$\vec{a} = -g\vec{\jmath}$$
$$\frac{d^2\vec{r}}{dt^2} = -g\vec{\jmath}$$

Integrating twice and using the fact that $\vec{v}(0) = (v_0 cos\alpha)\vec{\imath} + (v_0 sin\alpha)\vec{\jmath}$ and $\vec{r}(0) = \vec{0}$, we get

$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{\jmath} + \overrightarrow{v_0}t + \overrightarrow{r_0}$$
$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{\jmath} + \big((v_0 cos\alpha)\vec{\imath} + (v_0 sin\alpha)\vec{\jmath}\big)t + \vec{0}$$
$$\vec{r}(t) = (v_0 cos\alpha)t\vec{\imath} + \left(-\frac{1}{2}gt^2 + (v_0 sin\alpha)t\right)\vec{\jmath}$$

Ideal Projectile Motion Equation

$$\vec{r}(t) = (v_0 cos\alpha)t\vec{\imath} + \left(-\frac{1}{2}gt^2 + (v_0 sin\alpha)t\right)\vec{\jmath}$$

The angle α is the projectile's launch angle. The horizontal and vertical component's of position give the parametric equations

$$x = (v_0 cos\alpha)t \text{ and } y = -\tfrac{1}{2}gt^2 + (v_0 sin\alpha)t$$

Where x is the distance downrange and y is altitude of the projectile at time t.

Height, Flight Time, and Range

The projectile reaches its highest point when its vertical velocity is zero, that is, when

$$\frac{dy}{dt} = v_0 \sin\alpha - gt = 0 \quad \text{or} \quad t = \frac{v_0 \sin\alpha}{g}$$

For this value of time, the altitude of the projectile is

$$y_{\max} = v_0 \sin\alpha\left(\frac{v_0 \sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2 = \frac{\left(v_0 \sin\alpha\right)^2}{2g}$$

To find when the projectile lands when fired over horizontal ground, we set the vertical component equal to zero and solve for t.

$$-\frac{1}{2}gt^2 + \left(v_0 \sin\alpha\right)t = 0$$

$$t\left(-\frac{1}{2}gt + \left(v_0 \sin\alpha\right)\right) = 0$$

$$t = 0 \quad \text{or} \quad -\frac{1}{2}gt + \left(v_0 \sin\alpha\right) = 0$$

$$t = 0 \quad \text{or} \quad -\frac{1}{2}gt = -v_0 \sin\alpha$$

$$t = 0 \quad \text{or} \; t = \frac{2v_0 \sin\alpha}{g}$$

To find the projectile's range, we find the value of the horizontal component when $t = \frac{2v_0 \sin\alpha}{g}$

$$x = \left(v_0 \cos\alpha\right)t = \left(v_0 \cos\alpha\right)\left(\frac{2v_0 \sin\alpha}{g}\right) = \frac{v_0^2 2\sin\alpha\cos\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}$$

The range is largest when $\sin 2\alpha = 1$ or when $2\alpha = 90^0, or\ \alpha = 45^0$

Height, Flight Time, and Range for Ideal Motion.

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

- Maximum Height $y_{\max} = \frac{\left(v_0 \sin\alpha\right)^2}{2g}$
- Flight Time $t = \frac{2v_0 \sin\alpha}{g}$
- Range $x = \frac{v_0^2 \sin 2\alpha}{g}$

Examples:

1. Find the muzzle speed of a gun whose maximum range is 24.5 km.

Solution: Since this the maximum range, we know the launch angle is $\alpha = 45^0$. We use the expression for range:

$$x = \frac{v_0^2 \sin 2\alpha}{g} \quad \text{where } g = 9.8 \, {}^{m}\!/_{\sec^2}$$

$$24500 = \frac{v_0^2 \sin\left(2\left(45^0\right)\right)}{9.8}$$

$$24500 = \frac{v_0^2 \sin\left(90^0\right)}{9.8}$$

$$24500 = \frac{v_0^2}{9.8}$$

$$v_0^2 = 240{,}100$$

$$v_0 = \sqrt{240{,}100} = 490 \, {}^{m}\!/_{s}$$

2. A baseball is thrown from the stands 32 ft above the field at an angle of 30^0 from the horizontal. When and how far away will the ball strike the ground if its initial speed is 32 ft/sec.

Solution:

$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + \overrightarrow{v_0}t + \overrightarrow{r_0}$$

$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + (v_0 cos\alpha\vec{i} + v_0 \sin\alpha\vec{j})t + \overrightarrow{r_0}$$

$$\vec{r}(t) = -\frac{1}{2}(32)t^2\vec{j} + (32cos30^0\vec{i} + 32\sin 30^0\vec{j})t + 32\vec{j}$$

$$\vec{r}(t) = -16t^2\vec{j} + \left(16\sqrt{3}\vec{i} + 16\vec{j}\right)t + 32\vec{j}$$

$$\vec{r}(t) = \left(16\sqrt{3}t\right)\vec{i} + (-16t^2 + 16t + 32)\vec{j}$$

The baseball will strike the ground when the vertical component of position is zero, that is, when

$$-16t^2 + 16t + 32 = 0$$

$$-16\left(t^2 - t - 2\right) = 0$$

$$-16(t-2)(t+1) = 0$$

$$\text{when } t = 2 \text{ seconds}$$

The range will be

$$x = 16\sqrt{3}t$$

$$= \left(16\sqrt{3}\right)2$$

$$= 55.4 \; ft$$

3. A spring gun at ground level fires a golf ball at an angle of 45^0. The ball lands 10 m away.

a) What was the ball's initial speed?

b) For the same initial speed, find the two firing angles that make the range 6 m.

Solution:

$$x=\frac{v_0^2\sin 2\alpha}{g}$$

$$\frac{v_0^2\sin\left(2\left(45^0\right)\right)}{9.8}=10$$

$$\frac{v_0^2}{9.8}=10$$

$$v_0^2=980$$

$$v_0=\sqrt{980}=31.3\,{}^{m}\!/_{s}$$

b)

$$x=\frac{v_0^2\sin 2\alpha}{g}$$

$$\frac{980\sin 2\alpha}{9.8}=6$$

$$10\sin 2\alpha=6$$

$$\sin 2\alpha=\frac{3}{5}$$

$$2\alpha=\sin^{-1}\left({}^{3}\!/_{5}\right)$$

$$2\alpha=36.87^0 \quad \text{and its supplement } 180^0-36.87^0=143.13^0$$

$$\text{so } \alpha=18.4^0 \ \ or \ \ \alpha=71.6^0$$

4. A golf ball is hit with an initial velocity of 116 ft/sec at an angle of elevation of 45^0 from the tee to a green that is elevated 45 ft above the tee. Assuming the pin, 369 ft downrange, does not interfere with the flight of the ball, where will the ball land in relation to the pin?

Solution: We need to determine when the elevation of the ball is 45 ft. When this happens, on the way down, the ball will be on the green.

$$y=-\frac{1}{2}gt^2+(v_0 sin\alpha)t$$

$$-\frac{1}{2}(32)t^2+\left(116\sin 45^0\right)t=45$$

$$-16t^2+82.02t-45=0$$

Using the quadratic formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-82.02 \pm \sqrt{82.02^2 - 4(-16)(-45)}}{2(-16)}$$

$t = 0.624 \text{ sec } or\ t = 4.5 \text{sec}$

The ball will be aloft for 4.5 seconds. The ball will have traveled

$x = (v_0 cos\alpha)t$

$x = (116 \cos 45^0)4.5$

$x = 369.11$ feet or 0.11 feet from the pin.

5. A baseball is hit when it is 2.5 ft above the ground. It leaves the bat with an initial velocity of 145 ft/sec at a launch angle of 23^0. At the instant the ball is hit, an instantaneous gust of wind blows against the baseball, adding a component of $-14\vec{\imath}\ (ft/sec)$ to the baseball's initial velocity. A 15-ft high fence lies 300 ft from home plate in the direction of the flight.

a) Find a vector equation for the path of the baseball.

b) How high does the baseball go, and when does it reach its maximum height?

c) Find the range and flight time of the baseball, assuming the ball is not caught.

d) Did the batter hit a home run?

Solution

$a)\quad \vec{r}(t) = -\frac{1}{2}gt^2\vec{\jmath} + \overrightarrow{v_0}t + \overrightarrow{r_0}$ where $\overrightarrow{v_0} = (-14 + 145\cos 23^0)\vec{\imath} + (145\sin 23^0)\vec{\jmath}\ and\ \overrightarrow{r_0} = 2.5\vec{\jmath}$

$\vec{r}(t) = -\frac{1}{2}(32)t^2\vec{\jmath} + \left((-14 + 145\cos 23^0)\vec{\imath} + (145\sin 23^0)\vec{\jmath}\right)t + 2.5\vec{\jmath}$

$\vec{r}(t) = -16t^2\vec{\jmath} + (119.47\vec{\imath} + 56.66\vec{\jmath})\,t + 2.5\vec{\jmath}$

$\vec{r}(t) = (119.47t)\vec{\imath} + (-16t^2 + 56.66t + 2.5)\vec{\jmath}$

b) The baseball will reach its maximum height when the vertical component of the velocity is zero.

$-32t + 56.66 = 0$

$$t = \frac{56.66}{32} = 1.771 \text{sec}$$

$$y_{\max} = -16(1.771)^2 + 56.66(1.771) + 2.5$$
$$= 52.66\ ft$$

c) The range of the baseball occurs when the vertical component of its position is zero.

$-16t^2 + 56.66t + 2.5 = 0$

Using the quadratic formula, we get t = 3.585 sec for the flight time and

$$x = 119.47t$$
$$= 119.47(3.585) = 428.3\ ft$$

For the range.

d) The batter would have hit a home run, provided the baseball clears the fence. To determine this, we find the time the baseball is in the air when the vertical component is equal to the height of the fence.

$-16t^2 + 56.66t + 2.5 = 15$

Using the quadratic formula, we get t = 3.3 sec. And at this time, the baseball's horizontal component of position is

$$\begin{aligned} x &= 119.47t \\ &= 119.47(3.3) = 394\ ft \end{aligned}$$

Yes, the batter hit a home run