Calculus 1

Olivér Kiss

Mathematics and Informatics Pre-session for Business Analytics

2018

Topics

- Sequences
- Limits
- Differentiation

Sequences

- A sequence is an enumerated collection of numbers
- The usual notation for the nth element of sequence a is a_n
- Example: The sequence of prime numbers.

$$a_1 = 2$$
$$a_2 = 3$$
$$a_3 = 5$$
$$\vdots$$

- We often define sequences by the rule a certain element is calculated. Example:

$$a_n = n/2$$

List the first 5 elements of this sequence!

Types of sequences

Finite and infinite

- A finite sequence has a finite number of elements
- An infinite sequence has infinitely many elements

Increasing or decreasing

- A sequence is monotonically increasing if $a_{n+1} \geq a_n \quad \forall n$
- A sequence is monotonically decreasing if $a_{n+1} \leq a_n \quad \forall n$

Boundedness

- If $\exists\ N$ such that $a_n < N \quad \forall n$ the sequence is bounded from above
- If $\exists\ M$ such that $a_n > M \quad \forall n$ the sequence is bounded from below

Give an example for each type!

Limit of a sequence

The limit of a sequence is a number that the terms of a sequence "tend to". The notation is

$$a_n \to A$$

or

$$\lim_{n \to \infty} a_n = A$$

Examples:

- $a_n = 5 \quad \Longrightarrow \quad a_n \to 5$
- $a_n = \frac{1}{n} \quad \Longrightarrow \quad \lim_{n \to \infty} a_n = 0$

Convergence

If a sequence has a limit, it is called convergent. If it does not, it is divergent.

Formal definition of convergence

A sequence a_n converges to A if $\forall \varepsilon > 0 \quad \exists N$ such that $\forall n > N$ it holds that $|a_n - A| < \varepsilon$.

Examples:

- $a_n = n$ is divergent
- $a_n = \frac{(-1)^n}{n}$ is convergent, $\lim\limits_{n \to \infty} a_n = 0$

Properties of limits

- $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n \quad \forall c$
- $\lim\limits_{n\to\infty}(a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, provided that $\lim_{n\to\infty} b_n \neq 0$
- $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p \quad \forall p > 0$

Let's look at some examples!

$$\lim_{n\to\infty}\frac{n^2-3}{n^3-2}=\lim_{n\to\infty}\frac{\frac{1}{n^3}(n^2-3)}{\frac{1}{n^3}(n^3-2)}$$

$$\lim_{n\to\infty}\frac{\frac{1}{n^3}(n^2-3)}{\frac{1}{n^3}(n^3-2)}=\lim_{n\to\infty}\frac{\left(\frac{1}{n}-\frac{3}{n^3}\right)}{\left(1-\frac{2}{n^3}\right)}$$

$$\lim_{n\to\infty}\frac{\left(\frac{1}{n}-\frac{3}{n^3}\right)}{\left(1-\frac{2}{n^3}\right)}=\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n}-\frac{3}{n^3}\right)}{\lim\limits_{n\to\infty}\left(1-\frac{2}{n^3}\right)}$$

$$\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n}-\frac{3}{n^3}\right)}{\lim\limits_{n\to\infty}\left(1-\frac{2}{n^3}\right)}=\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)-\lim\limits_{n\to\infty}\left(\frac{3}{n^3}\right)}{\lim\limits_{n\to\infty}(1)-\lim\limits_{n\to\infty}\left(\frac{2}{n^3}\right)}$$

$$\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)-\lim\limits_{n\to\infty}\left(\frac{3}{n^3}\right)}{\lim\limits_{n\to\infty}(1)-\lim\limits_{n\to\infty}\left(\frac{2}{n^3}\right)}=\frac{0-0}{1-0}=0$$

$$\lim_{n\to\infty} n^3 - 2 = \lim_{n\to\infty} \frac{\frac{1}{n^2}(n^3-2)}{\frac{1}{n^2}(n^2-3)}$$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}(n^3-2)}{\frac{1}{n^2}(n^2-3)} = \lim_{n\to\infty} \frac{\left(n-\frac{2}{n^2}\right)}{\left(1-\frac{3}{n^2}\right)}$$

$$\lim_{n\to\infty} \frac{\left(n-\frac{2}{n^2}\right)}{\left(1-\frac{3}{n^2}\right)} = \frac{\lim\limits_{n\to\infty}\left(n-\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty}\left(1-\frac{3}{n^2}\right)}$$

$$\frac{\lim\limits_{n\to\infty}\left(n-\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty}\left(1-\frac{3}{n^2}\right)} = \frac{\lim\limits_{n\to\infty}(n) - \lim\limits_{n\to\infty}\left(\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty}(1) - \lim\limits_{n\to\infty}\left(\frac{3}{n^2}\right)}$$

$$\frac{\lim\limits_{n\to\infty}(n) - \lim\limits_{n\to\infty}\left(\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty}(1) - \lim\limits_{n\to\infty}\left(\frac{3}{n^2}\right)} = \frac{\infty - 0}{1 - 0} = \infty$$

Thus this sequence is divergent.

$$\lim_{n\to\infty} \frac{1+(-1)^n}{2}$$

Notice that there are two alternating terms: 0 and 1. Thus this sequence doesn't have a limit.

Also good to know

It is not always obvious how to calculate the limit of a sequence. E.g:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

There are some more advanced ways to calculate limits that we don't cover, but they are also good to know:

- Stolz–Cesàro theorem
- L'Hôpital's rule

Solve the following problems

1. $\lim\limits_{n\to\infty} \frac{n^4+5n^3+3n^2-2}{3n^4-6}$
2. $\lim\limits_{n\to\infty} \frac{5}{n+1} + \frac{n}{n+1}$
3. $\lim\limits_{n\to\infty} b^n$ depending on the value of b.
4. $\lim\limits_{n\to\infty} \frac{1}{n(\sqrt{n^2-1}-n)}$
5. $\lim\limits_{n\to\infty} \sqrt[n]{5}$
6. $\lim\limits_{n\to\infty} \ln\left(\frac{1}{n}\right)$
7. $\lim\limits_{n\to\infty} e^{-n}$

Sidenote: Series

Roughly speaking a series is the sum of the elements of a sequence.

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \dots$$

There is one series that you should remember: the geometric series. The sum of a sequence defined by

$$a_n = a \cdot b^n$$

where $b < 1$ is given by

$$\sum_{i=1}^{\infty} a_i = \frac{ab}{1-b}$$

What is the sum of the following sequences?

- $a_n = \frac{3}{5^n}$
- $a_n = 0.5^n$

Limits of functions

Just like for sequences, we can define the limits for functions.

Definition

A function $f(x)$ has a limit L when x approaches to p IF for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x that satisfies $|x - p| < \delta$ it holds that $|f(x) - L| < \varepsilon$. The notation is

$$\lim_{x \to p} f(x) = L$$

Example: $f(x) = 3x$. Calculate $\lim_{x \to 3} f(x)$. Let's guess this limit first!

Limits of functions

Example: $f(x) = 3x$. Calculate $\lim_{x \to 3} f(x)$. Now let's understand the definition.

- We claim that $\lim_{x \to 3} f(x) = 9$
- Let's have any positive number ε
- There should exist a δ for any ε that if we are in the δ neighborhood of 3, the function value is always closer to 9 than ε
- We can compute this δ depending on ε.

$$|f(x) - 9| < \varepsilon \implies -\varepsilon < f(x) - 9 < \varepsilon \implies -\varepsilon + 9 < f(x) < \varepsilon + 9 \implies$$

$$-\varepsilon + 9 < 3x < \varepsilon + 9 \implies -\frac{\varepsilon}{3} + 3 < x < \frac{\varepsilon}{3} + 3 \implies |x - 3| < \frac{\varepsilon}{3} = \delta$$

- Let's say $\varepsilon = 6$. It implies that $\delta = \frac{6}{3} = 2$, that is, if we are in the $(3 - 2, 3 + 2)$ interval, the function value should always be closer to 9 than 6.

Limits of functions

- We don't really want to use the formal definition in most cases to find the limits.
- The graphical approach often helps.
- An important property: For continuous functions the limit is the same as the value of the function.
- We can also use the following properties:

$$\lim_{x\to p}(f(x)+g(x)) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$$

$$\lim_{x\to p}(f(x)-g(x)) = \lim_{x\to p} f(x) - \lim_{x\to p} g(x)$$

$$\lim_{x\to p}(f(x)\cdot g(x)) = \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x)$$

$$\lim_{x\to p}(f(x)/g(x)) = \lim_{x\to p} f(x) / \lim_{x\to p} g(x)$$

Examples

Find $\lim_{x\to 5} e^{x-3}$. Notice that this is a standard exponential function, which is continuous. Thus

$$\lim_{x\to 5} e^{x-3} = e^{5-3} = e^2$$

Find $\lim_{x\to 0} \ln(x)$. Now notice, that $ln(0)$ is not defined. However the $\ln(x)$ function is monotonically increasing, thus as we get closer and closer to zero, it's value gets closer and closer to minus infinity. Thus

$$\lim_{x\to 0} \ln(x) = -\infty$$

Examples

Find $\lim\limits_{x\to\infty} \frac{x^4-2x^3+x-3}{x^5-2x}$

$$\lim_{x\to\infty} \frac{x^4-2x^3+x-3}{x^5-2x} = \lim_{x\to\infty} \frac{\frac{1}{x^5}\left(x^4-2x^3+x-3\right)}{\frac{1}{x^5}\left(x^5-2x\right)}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^5}\left(x^4-2x^3+x-3\right)}{\frac{1}{x^5}\left(x^5-2x\right)} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^2}+\frac{1}{x^4}-\frac{3}{x^5}}{1-\frac{2}{x^4}}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^2}+\frac{1}{x^4}-\frac{3}{x^5}}{1-\frac{2}{x^4}} = \frac{\lim\limits_{x\to\infty}\frac{1}{x}-\lim\limits_{x\to\infty}\frac{2}{x^2}+\lim\limits_{x\to\infty}\frac{1}{x^4}-\lim\limits_{x\to\infty}\frac{3}{x^5}}{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}\frac{2}{x^4}}$$

$$\frac{\lim\limits_{x\to\infty}\frac{1}{x}-\lim\limits_{x\to\infty}\frac{2}{x^2}+\lim\limits_{x\to\infty}\frac{1}{x^4}-\lim\limits_{x\to\infty}\frac{3}{x^5}}{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}\frac{2}{x^4}} = \frac{0-0+0-0}{1-0} = 0$$

Examples

Find $\lim\limits_{x \to 2} \frac{3x^2+3x-18}{x-2}$

$$\lim_{x \to 2} \frac{3x^2+3x-18}{x-2} = \lim_{x \to 2} \frac{3(x^2+x-6)}{x-2}$$

$$\lim_{x \to 2} \frac{3(x^2+x-6)}{x-2} = \lim_{x \to 2} \frac{3(x+3)(x-2)}{x-2}$$

$$\lim_{x \to 2} \frac{3(x+3)(x-2)}{x-2} = \lim_{x \to 2} 3(x+3) = 3 \cdot 5 = 15$$

Solve the following problems

1. $\lim_{x \to 0} (3 + 2x^2)$
2. $\lim_{x \to -1} \frac{3+2x}{x-1}$
3. $\lim_{x \to 1} \frac{x^2+7x-8}{x-1}$
4. $\lim_{x \to \infty} \frac{x^3-3x^2+x-5}{3x^3+5x^2-2}$
5. $\lim_{x \to 1} \frac{x^2-1}{x-1}$
6. $\lim_{h \to 0} \frac{\sqrt{h+1}-1}{h}$
7. $\lim_{x \to 5} \frac{3x^2-9x-30}{x-5}$

Differentiation

- We are often interested in the slope of the tangent line of a curve at a given point.
- To get this, we use differentiation.
- It is especially useful in case of optimization problems.
- Why? Consider for example the case when you are looking for the maximum of $f(x) = 3 - x^2$.
- What is the slope of the tangent line at the maximum point?

Differentiation

- The first differential $f'(x_0)$ of a function $f(x)$ at a given point x_0 is given by the limit:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- Notice that $\frac{f(a)-f(b)}{a-b}$ is the slope of the section connecting the function at a and b.
- What we do here, is we get these two points closer and closer.
- Once they are infinitesimally close, it gives the slope of the tangent line.

Differentiation

Our workhorse function will be $f(x) = x^2$. Let's find $f'(1)$. By definition:

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1}$$

$$\lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} x + 1 = 2$$

Differentiation

Still working with $f(x) = x^2$, let's find $f'(2)$. By definition:

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2}$$

$$\lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2} = \lim_{x \to 2} x + 2 = 4$$

Solve the following problems

Still working with $f(x) = x^2$ find

1. $f'(5)$
2. For any general x_0 find $f'(x_0)$

A bit more difficult problem: Consider now $g(x) = x^3$.

1. First find $g'(2)$
2. Now find $g'(-2)$
3. For any general x_0 try to find $g'(x_0)$

The derivative function

- We have shown that $f'(x_0) = 2x_0$ if $f(x) = x^2$
- We have also shown that $g'(x_0) = 3x_0^2$ if $g(x) = x^3$
- These are the first derivative functions, that give the derivative of a function at any point.
- The usual notation is either $f'(x)$ or

$$\frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x}$$

- Let's find it for the general power function $f(x) = x^n$

Properties of derivative functions

- $(\alpha f + \beta g)' = \alpha f' + \beta g'$
- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Solve the following problems

Find the derivative of the following functions:

- $f(x) = x^3 + 2x^2 - x$
- $g(x) = (x^2 + 2)(x - 4)$
- $h(x) = \frac{x^{12} - 15x^2}{x - 5}$

Some additional useful derivatives

- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} a^x = a^x \ln(a)$
- $\frac{d}{dx} \ln(x) = \frac{1}{x}, \quad \forall\, x > 0$
- $\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$

Solve the following problems

Find the derivative of the following functions:

- $f(x) = \frac{x^2}{\ln x}$
- $g(x) = e^x(x^3 - x^2)$
- $h(x) = \frac{5^x}{x^2-2}$