Functions of One Variable

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Topics

- What is a function?
- Linear functions
- Quadratic functions
- Polynomials
- Power functions
- Exponential functions
- Logarithmic functions

Functions

One variable is a function of another if the first depends upon the second. Examples:

- The circumference of the circle is 2π times its radius
- The distance is the elapsed time times the speed
- The total price is the sumproduct of quantities and unit prices

There are a few functions that we use regularly. We will cover these one by one today.

A bit more formally

- A function is a relation associating each element of a set with an element of another set
- The set of arguments (inputs) of a function is called the **domain**
- The set of values that a function can take is called the **image**
- The domain is often denoted by X and the image by Y
- A function can be defined by a set of ordered pairs (x, y) such that $x \in X$ and $y \in Y$
- Notation is either $f : X \to Y$ or $y = f(x)$

Examples

A function $f(fruit) = color$ that associates each fruit to its color:

- X is the set of fruits, Y is the set of colors
- $f(apple) = green$
- $f(lemon) = yellow$

A function $f : firm \to CEO$ that associates each firm with its CEO:

- X is the set of firms, Y is the set of people
- $f(apple) = Tim\ Cook$
- $f(facebook) = Mark\ Zuckerberg$

Numeric functions

- In most cases functions associate numbers with other numbers
- Example: square function, absolute value, logarithm of a number, etc.
- In these cases the domain and the image consists of certain types of numbers:
  - Natural numbers ($\mathbb{N}$): 1, 2, 3 ...
  - Integer numbers ($\mathbb{Z}$): ..., -3, -2, -1, 0, 1, 2, 3, ...
  - Real numbers ($\mathbb{R}$): Numbers with a decimal representation. E.g.: 0, 2.4, -54.23 etc.
- For example the square function $f(x) = x^2$ associates all real numbers with a non-negative real number. $f : \mathbb{R} \to \mathbb{R}_0^+$

Solve the following problems [SYD 2.2]

1. The total cost of producing x goods is given by $f(x) = 100\sqrt{x} + 500$. What is the cost of producing 16 goods? What is the cost of 100 goods?
2. Find the domain of $f(x) = \frac{1}{x+3}$
3. Find the domain of $\xi(x) = \sqrt{2x+4}$
4. Show that 5 is in the image of $f(x) = \frac{3x+6}{x-2}$. Also show that 3 is not in the image.

Linear functions

A linear relationship between y and y takes the form: $y = a + bx$ where a and b are constants. The graph of a linear function is called a line.

- a is the **intercept**
- b is the **slope**

Solve the following problem: The relationship between temperatures measured in Celsius and Fahrenheit is linear. 0°C is equivalent to 32°F and 100°C is the same as 212°F. Give the temperature in Fahrenheit as a function of the temperature in Celsius. Give the temperature in Celsius as a function of the temperature in Fahrenheit. Which temperature is measured by the same number on both scales?

Quadratic functions

A quadratic relationship between y and y takes the form: $y = ax^2 + bx + c$ where a, b and c are constants. The graph of a quadratic function is called a parabola.

- If $a > 0$ the parabola will be convex
- If $a < 0$ the parabola will be concave
- The graph will cross the x-axis at $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The minimum/maximum point of the graph is at $\frac{-b}{2a}$.
- The function has a minimum if it is convex
- It has a maximum if it is concave

Sidenote: Convexity

A function is convex if $\lambda f(a) + (1-\lambda)f(b) \geq f(\lambda a + (1-\lambda)b) \quad \forall a, b, \lambda \in (0,1)$.

- Graphically: connect any two points of the graph. The section you get should be above the graph.
- Mathematically: Choose any two elements of the domain. Any convex combination of the function evaluated at these two elements should be at least as much as the function evaluated at the same convex combination of the same two elements.

A function is concave if $\lambda f(a) + (1-\lambda)f(b) \geq f(\lambda a + (1-\lambda)b) \quad \forall a, b, \lambda \in (0,1)$.

- Graphically: connect any two points of the graph. The section you get should be below the graph.
- Mathematically: Choose any two elements of the domain. Any convex combination of the function evaluated at these two elements should be at most as much as the function evaluated at the same convex combination of the same two elements.

Solve the following problems

Plot the following functions:

1. $f(x) = x^2 - 3x + 2$
2. $f(x) = -5x^2 + 10x - 3.2$

Decide whether the following functions are convex:

1. $f(x) = 2^x$
2. $f(x) = x^2$

Polynomials

A logical step forward is the cubic function: $y = ax^3 + bx^2 + c^x + d$. This can look considerably more complicated. E.g.: $f(x) = -x^3 + 4x^2 - x - 6$.
We can generalize such functions that sum up the multiples of powers of a variable. These are called polynomials. A polynomial of degree n is defined by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where each a_i is a constant and $a_n \not= 0$.

- An important property: Each polynomial of degree n has n roots.
- Although there may be multiplicities and some might be complex.

Solve the following problems

We will use WolframAlpha to check some of the properties of polynomials. Let's take a look at $f(x) = -x^3 + 4x^2 - x - 6$, $g(x) = x^2 + 1$ and $h(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$

1. Check the roots of each polynomial.
2. How many inflection points do they have?

Power functions

A power function is defined as:

$$f(x) = ax^b$$

where a and b are constants. Some examples where it might come handy:

- The flow of blood through the heart (l/s) is proportional to $x^{0.7}$ where x is the weight.
- The migration speed of animals is also well captured by a power function of their weight.
- $S = 4.84V^{2/3}$ gives an approximation for the surface of a ball with volume V.

Solve the following problems

- Check the convexity of x^4, x^3, x^2, $x^1.5$, $x^0.5$, $x^{0.25}$ if $x > 0$. What can you infer?
- How does a power function with a negative exponent look like?

Exponential functions

An exponential function is defined as:

$$f(x) = ba^x$$

where a and b are constants. It is one of the most important functions in economics and business. Some examples:

- If you invest \$100 with an interest of 5%, the value of your investment in t years is given by $V(t) = 100 \cdot 1.05^t$
- If the half-life of a certain radioactive isotope is x, then after time t and initial units I_0 there will be $I(t) = I_0 \cdot (0.5)^{t/x}$ units remaining.

The natural exponential function

- The most important exponential function has an irrational base.
- This is the so-called Euler's number, or e.
- It is roughly $e = 2.71828$
- The exponential function with base e is called the natural exponential function.

$$f(x) = e^x$$

- We will understand it's importance later in calculus.
- Spoiler: It's most important property is that it's derivative is itself.

Euler's number

An account starts with \$1 and pays 100 percent interest per year. If the interest is credited once, at the end of the year, the value of the account at year-end will be \$2. What happens if the interest is computed and credited more frequently during the year?

- If it pays twice, we end up with $1 \cdot (1 + 0.5)^2 = \$2.25$
- If it pays four times, we end up with $1 \cdot (1 + 1/4)^4 = \$2.44$
- If it pays n times, we end up with $\$(1 + 1/n)^n$.
- What if it pays continuously? That is, if $n = \infty$.
- As it turns out, we will end up with exactly $\$e$.

Solve the following problems [SYD 5.3]

- Assume that you have \$100 savings. You invest it for 20 years. How much difference do you end up with depending on whether get a 2% or a 3% compound interest?
- How much would an investment of \$100 worth in a year if the interest payment was continuous and the interest was 100%?

Logarithmic functions

We often face equations of the form $b^x = a$ where a and b are constants. Let's start with cases where $b = e$. If $e^x = a$, we call x the natural logarithm of a and denote it by

$$\ln a$$

Thus $\ln a$ is the power of e we need to get a.

$$e^{\ln a} = a$$

Based on this definition find:

- $\ln 1$
- $\ln e$
- $\ln e^5$
- $\ln 1/e$
- $\ln -6$

Logarithmic functions

Some useful properties:

- $\ln(xy) = \ln x + \ln y$
- $\ln \frac{x}{y} = \ln x - \ln y$
- $\ln x^y = y \ln x$

Show that these formulas are correct!

Logarithmic functions

$$e^{\ln(xy)} = xy = e^{\ln x} e^{\ln y} = e^{\ln x + \ln y} \implies \ln(xy) = \ln x + \ln y$$

$$e^{\ln \frac{x}{y}} = \frac{x}{y} = \frac{e^{\ln x}}{e^{\ln y}} = e^{\ln x - \ln y} \implies \ln \frac{x}{y} = \ln x - \ln y$$

$$e^{\ln x^y} = x^y = \left(e^{\ln x}\right)^y = e^{y \ln x} \implies \ln x^y = y \ln x$$

Solve the following problems [SYD 8.2]

Solve for x:

- $5e^{-3x} = 16$
- $(1.08)^x = 10$
- $3^x 4^{x+2} = 8$

Logarithmic functions

We can define logarithms with other bases. $\log_a b$ defines the number satisfying

$$a^{\log_a b} = b$$

We call $\log_a b$ the logarithm of b with base a. The same properties hold (prove it!):

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^y = y \log_a x$

Also

- $\log_a b = \frac{\log_c b}{\log_c a}$

If you see $\log b$ without a base indicated it usually means a 10 base logarithm.

Solve the following problems [Beginning and Intermediate Algebra - Tyler Wallace]

Solve for x:

- $\log_5 125 = x$
- $\log_{125} 5 = x$
- $\log_{11}(x+5) = -1$
- $\log_7(-8r) = 1$

Rewrite to natural logarithm:

- $\log_2 133$
- $\log_{115} 56$