Experimental Comparison of (2,1)-Distance Oracles

(Eksperymentalne porównanie
(2,1)-wyroczni odległości)

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Abstract

Let $G = (V, E)$ be an undirected unweighted graph with $|V| = n$ vertices and $|E| = m$ edges. Let $\delta(u, v)$ denote the distance between vertices $u, v \in V$. An $(\alpha, \beta) - approximate$ distance oracle for G is a data structure that supports the following queries between pairs of vertices in G: given two vertices $u, v \in V$ it can compute a distance estimate $\hat{\delta}(u, v)$ that satisfies

$$\delta(u, v) \leq \hat{\delta}(u, v) \leq \alpha\delta(u, v) + \beta$$

What is remarkable that there exist $(\alpha, \beta) - approximate$ distance oracle that can take significantly less than quadratic space, but can answer distance queries in constant time, despite quadratic space structure required to retrieve exact distances. The $(3, 0) - approximate$ distance oracle by Thorup and Zwick is the most precise oracle that have $\beta = 0$ and takes subquadratic memory $- O(n^{3/2})$. By using the additive error $- \beta > 0$ we are able to decrease α to 2, using slightly more memory, but still subquadratic memory $- (2, 1)$ distance oracle by Baswana, Goyal and Sen takes only $O(n^{5/3})$ memory. In this thesis we will compare $(2, 1) - approximate$ distance oracles on different graph sets, looking at their memory use, query time and average error. We will also compare the preprocessing time.

Niech $G = (V, E)$ będzie nieskierowanym, nieważonym grafem, gdzie $n = |V|$ jest liczbą wierzchołków, a $m = |E|$ liczbą krawędzi. Oznaczmy jako $\delta(u, v)$ odległość pomiędzy wierzchołkami $u, v \in V$. $(\alpha, \beta) - przyblion$ wyrocznią odległości grafu G, gdzie $\alpha \geq 1$, $\beta \geq 0$ nazwiemy strukturę danych, która umożliwia następujące zapytania: Mając wierzchołki $u, v \in V$ możemy policzyć przybliżoną odległość $\hat{\delta}(u, v)$, która spełnia

$$\delta(u, v) \leq \hat{\delta}(u, v) \leq \alpha\delta(u, v) + \beta$$

Istnieją $(\alpha, \beta) - przyblione$ wyrocznie odległości, które zajmują mniej niż kwadratowo wiele pamięci, ale odpowiedzi na zapytania zwraca w stałym czasie, pomimo, że zwrócenie dokładnych odległości w stałym czasie wymaga struktury używającej kwadratowo wiele pamięci. $(3, 0) - przybliona$ wyrocznia odległości Thorupa i Zwicka jest najdokładniejszą wyrocznią, która ma $\beta = 0$ i dodatkowo zajmuje mniej niż kwadratowo wiele pamięci $- O(n^{3/2})$. Jednak używając błędu addytywnego $- \beta > 0$ jesteśmy w stanie zmniejszyć α do 2, zajmując trochę więcej pamięci, ale wciąż mniej niż kwadratowo wiele $- (2, 1) - przybliona$ wyrocznia odległości Baswana, Goyala i Sena zajmuje jedynie $O(n^{5/3})$ pamięci. W poniższej pracy porównamy kilka $(2, 1) - przyblionych$ wyroczni odległości na różnych zestawach grafów, porównamy ich zajmowaną pamięć, szybkość działania i średni błąd. Porównamy również czas, jaki zajmuje stworzenie wyroczni.

Contents

Chapter 1

Introduction

Given the (undirected) graph or a network, such as a road network or the internet, computing the distances is one of the most basic problems. However calculating the distances online is often too slow for practical use, so we want to preprocess the graph in a way that the distance queries could be answered quickly. If we want to answer the distance queries in constant time then we need $\Omega(n^{2-\epsilon})$ memory as shown in Chapter 2, which is often more than the memory needed to store the graph itself. Fortunately knowing the exact distances is not always necessary, as the data packets could be routed through the short enough paths, and for drivers knowing that the gas station or hotel is nearby is often enough. By dropping the requirement that the distances returned are exact, we are able to decrease both the memory needed and preprocessing time, while retaining the constant query time.

1.1 Summary of Related Work on Oracles

The first $(\alpha, \beta)-approximate$ distance oracle, the $(2k-1, 0)-approximate$ distance oracle was created in 2005 by Thorup and Zwick [4]. Their oracle works on weighted graphs, uses $O(kn^{1+1/k})$ memory and is constructed in $O(kmn^{1/k})$ expected time. They also provided a routing scheme, distance labels and spanner construction based on their oracle, improving the previous results. The oracle construction could be derandomized resulting in $\tilde{O}(knm)$ time and increased memory usage by factor of $\log n$. The derandomization was improved shortly after together with Roditty resulting in $\tilde{O}(kmn^{1/k})$ time and no extra memory [5]. For unweighted graphs, the expected preprocessing time was improved in 2006 by Baswana and Sen, who achieved $O(n^2)$ expected time [6].

In 2008 Baswana, Goyal and Sen have constructed $(2, 1)-approximate$ distance oracle for unweighted graphs [7], which by choosing different constants and using a hash table results in structure using $O(n^{5/3} \log n)$ memory and constructed in $O(mn^{2/3} \log n)$ expected time. They have also proposed using spanners in order to

reduce the preprocessing time to expected $\tilde{O}(n^2)$, but increasing β in the process.

The first $(2,1)-$*approximate* distance oracle for unweighted graphs with explicit subquadratic space was constructed in 2010 by Patrascu and Roditty [8]. Their oracle uses $O(n^{5/3})$ memory and is constructed in $O(mn^{2/3}+n^{7/3})$ expected time [1]. They also provide a modification for their oracle resulting in $(2,0)-$*approximate* distance oracle for weighted graphs, that uses $O(m^{1/3}n^{4/3})$ memory.

In 2011 $(2,1)-$*approximate* distance oracle by Baswana and Sen was generalized by Abraham and Gavoille, yielding $(2k-2,1)$ distance oracle [9]. This oracle uses $\tilde{O}(n^{1+2/(2k-1)})$ memory and is constructed in expected polynomial time. They also provided a routing scheme, distance labels based on their oracle, improving the previous results.

With clever use of spanners, the $(2,1)-$*approximate* distance oracle expected construction time was reduced to $\tilde{O}(n^2)$, using $O(n^{5/3}\log^2 n)$ memory by Sommer in 2016 [10]. Subsequently the expected construction time was reduced to $O(n^2)$, using $O(n^{5/3})$ memory by Knudsen [11].

1.2 Overview of the Thesis

The thesis is organized as follows:

In Chapter 2 we will discuss why $(3,0)-$*approximate* and $(2,1)-$*approximate* distance oracles have minimum possible error allowing for subquadratic space structure.

Next, in Chapter 3 we will present the basic $(3,0)-$*approximate* distance oracle by Thorup and Zwick [4].

Chapter 4 will be dedicated to how the $(3,0)$ multiplicative error barrier was breached by adding the additive error $-$ $(2,1)-$*approximate* distance oracle, at the expense of considering only unweighted graphs and increasing memory [7].

The next two Chapters will be dedicated to the different variations of the $(2,1)-$*approximate* distance oracle, in Chapter 5 we will discuss the theoretical differences, while in the Chapter 6 we will show the results of the tests on different graph sets.

The main contribution of this thesis is describing the oracles [4, 7, 8] in a way that they are all more similar to each other, expanding proofs, verifying claims and improving the bounds of a few steps basing on claims from papers. We also show how to adapt $(2,1)-$*approximate* distance oracle to weighted graphs $(2,h)-$*approximate* distance oracle, where h is the weight of the heaviest edge, as claimed possible by [10]. Finally, the $(\alpha,\beta)-$*approximate* distance oracles were implemented for unweighted

[1] In 2017, Knudsen claims that the expected construction time is actually $O(mn^{2/3})$ based on personal communication with Roditty [11, Chapter 4, Lemma 10].

graphs and tested on random graphs with a particular density. We also constructed unweighted graphs for which expected running time of $(\alpha, \beta)-$*approximate* distance oracle from Chapter 4 realizes the upper bound. This shows that both the memory bound and expected preprocessing time bound are tight.

1.3 Preliminaries

We consider only undirected graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$. Some of the ideas require the graph to be unweighted, particularly constant surplus oracles in Chapters 4, 5, 6.

The $\tilde{O}$, $\tilde{\Omega}$, $\tilde{\Theta}$ are the O, Ω, Θ counterparts, that hide log factors.

Definition 1.1. Surplus$-t$ estimate, for $t > 0$, is any estimate $\hat{\psi}$ of a value ψ that satisfies $\psi \leq \hat{\psi} \leq \psi + t$. It is also called the additive error estimate.

Definition 1.2. Stretch$-t$ estimate, for $t > 1$, is any estimate $\hat{\psi}$ of a value ψ that satisfies $\psi \leq \hat{\psi} \leq t\psi$. It is also called the multiplicative error estimate.

Definition 1.3. $(\alpha, \beta)-$approximation, for $\alpha \geq 1, \beta \geq 0$, is any estimate $\hat{\psi}$ of a value ψ that satisfies $\psi \leq \hat{\psi} \leq \alpha\psi + \beta$. We say that $(\alpha, \beta)-$approximation is both additive and multiplicative error estimate, when $\alpha > 1, \beta > 0$.

Usually, when $\beta = 0$ the graph is assumed to be weighted, and when $\beta > 0$ the graph is assumed to be weighted or β depends on the weight of the heaviest edge.

Definition 1.4. An $(\alpha, \beta)-$*approximate* distance oracle or in short $(\alpha, \beta)-$distance oracle is a data structure that for a given graph $G = (V, E)$ is able to return an $(\alpha, \beta)-$approximation of distance between any pair of vertices from G in constant time.

Definition 1.5. Single-Source Shortest Path or SSSP Problem consists of finding the distances between a given vertex v and all other vertices in the graph.

Definition 1.6. All-Pairs Shortest Path or APSP Problem consists of finding the distances between all pairs of vertices in the graph.

Definition 1.7. All-Pairs *Approximate* Shortest Path or APASP Problem consists of finding approximate distances between all pairs of vertices in the graph.

Definition 1.8. Vertex u has degree $\deg(u) = a$ when number of edges incident to u plus number of edges connecting u with itself (loops) amounts to a.[2]

[2]Every loop is also incident to its vertex.

Chapter 2

Theoretical Limitations

2.1 Space Lower Bound

Definition 2.1. A simple cycle is a path that has distinct edges and only the first and the last of its vertices are equal.

Definition 2.2. The girth of the graph is the length of its shortest simple cycle.

Theorem 2.3. *A graph with at least* $\frac{1}{2}n^{1+1/k}$ *edges has girth not greater than* $2k$.

Proof. We will prove the lemma for $k = 2$. The full proof can be found in Alon et al. [1].
Suppose that there exist graph G with $\frac{1}{2}n^{1+1/k}$ or more edges, that have girth at least $2k + 1$. In other words G does not have a simple cycle of length $2k$ or less.

Let $\mathrm{N}_a(v) = \{u \in V \mid \delta(v, u) = a\}$. For $2 \leq i \leq k$ we show that

$$|\mathrm{N}_i(v)| = \sum_{u \in N_{i-1}(v)} (\deg(u) - 1)$$

as

1. Every vertex u from set $\mathrm{N}_{i-1}(v)$ have exactly one edge (u, l) such that $\delta(v, l) \leq i - 1$. and
2. There are no two distinct edges $(u, l),\ (w, l)$ satisfying $u, w \in \mathrm{N}_{i-1}(v)$, $l \in V$.

We note that $u \in \mathrm{N}_i(v) \iff v \in \mathrm{N}_i(u)$, so for $2 \leq i \leq k$ we have:

$$\begin{aligned}\sum_{v \in V} |\mathrm{N}_i(v)| &= \sum_{v \in V} \sum_{u \in \mathrm{N}_{i-1}(v)} (\deg(u) - 1) = \\ &= \sum_{u \in V} \sum_{v \in \mathrm{N}_{i-1}(u)} (\deg(u) - 1) = \\ &= \sum_{u \in V} ((|\mathrm{N}_{i-1}(u)|)(\deg(u) - 1))\end{aligned}$$

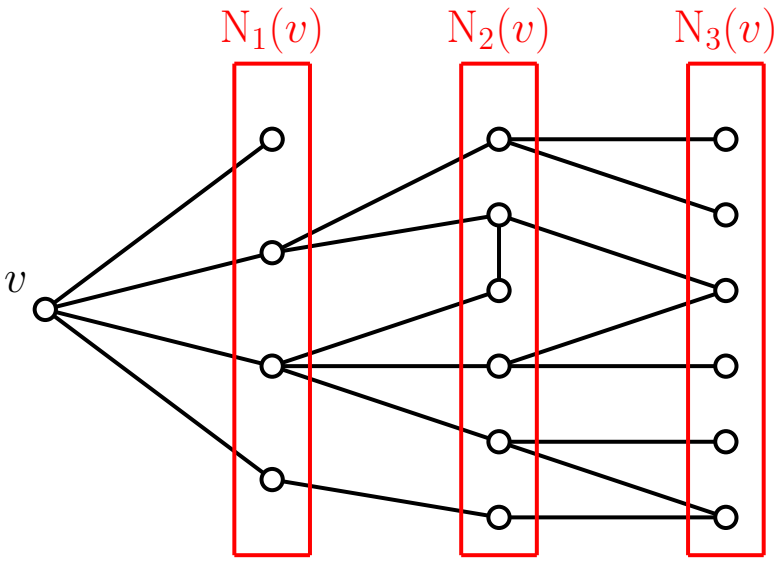


Figure 2.1: $|N_k(v)|$ for $k = 2$

For every $v \in V$ the following is true:

$$\sum_{i=0}^{k} |N_i(v)| = |\{u \in V \mid \delta(v, u) \leq k\}| \leq |V| = n \tag{2.1}$$

However, if we look at the sum:

$$\begin{aligned}
\sum_{v \in V} \sum_{i=0}^{k} |N_i(v)| &= \sum_{v \in V} (1 + |N_1(v)| + \sum_{i=2}^{k} |N_i(v)|) = \\
&= n + \sum_{v \in V} (|N_1(v)|) + \sum_{v \in V} \sum_{i=2}^{k} |N_i(v)| = \\
&= n + \sum_{v \in V} (\deg(v)) + \sum_{v \in V} \sum_{i=2}^{k} \sum_{u \in N_{i-1}(v)} (\deg(u) - 1) = \quad \text{(for } k = 2\text{)} \\
&= n + \sum_{v \in V} (\deg(v)) + \sum_{v \in V} \sum_{u \in N_1(v)} (\deg(u) - 1) = \\
&= n + \sum_{v \in V} (\deg(v)) + \sum_{u \in V} ((|N_1(u)|)(\deg(u) - 1)) = \\
&= n + \sum_{v \in V} (\deg(v)) + \sum_{u \in V} ((\deg(u))(\deg(u) - 1)) = \\
&= n + \sum_{u \in V} ((\deg(u))(\deg(u))) \geq n + n(\text{avg}\deg)^2 = \\
&= n + n(n^{1/2})^2 = n(n+1) > n^2
\end{aligned}$$

where $\text{avg}\deg = \frac{2m}{n}$ is the average degree of a vertex.

This contradicts Equation 2.1. □

It is conjectured by Erdös [2] and others that Theorem 2.3 is tight. Namely, it is conjectured that for any $k \geq 1$, that there are graphs with $\Omega(n^{1+1/k})$ edges and girth greater than $2k$. The conjecture is known to be true for $k = 1, 2, 3, 5$.

Observation 2.4. *For any graph of girth $2k+1$ its bipartite subgraphs have girth $2k+2$ or greater as simple cycles in bipartite graphs have even length and subgraphs can't have smaller girth than original graph.*

Since graphs with girth greater than 1 do not have loops, they have a bipartite subgraph with at least half the edges, so there are graphs with at least $\Omega(n^{1+1/k})$ edges and girth greater than $2k+1$.

Lemma 2.5. *An $(\alpha, \beta)-$distance oracle must use at least m bits of storage for girth $k > \alpha + \beta + 1$ graphs, where m is a number of edges.*

Proof. Let G be a girth $k > \alpha + \beta + 1$ unweighted graph with n vertices and m edges. Let H be any subgraph of G and $\delta_H(u, v)$ be a distance between vertices u, v in H.

Consider any edge (u, v) from G. If (u, v) is in H then $\delta_H(u, v) = 1$, but otherwise $\delta_H(u, v) \geq k - 1$, since the girth of H is not less than k. The oracle will report distance between u and v at most $\alpha + \beta$, if edge (u, v) in H, but not less than $k - 1$ otherwise.

Consequently, the generated structure must be different for subgraphs of G with different edge sets. There are 2^m subgraphs of G with different edge sets, so for at least one of them the generated oracle must take m bits of space. □

Observation 2.6. *An $(\alpha, \beta)-$distance oracle must take at least $\Omega(n^{1+1/k})$ memory, where $k = \lceil (\alpha + \beta + 1)/2 \rceil$, if the girth conjecture is true.*

Observation 2.7. *This points to the space lower bound of $\Omega(n^{3/2})$ for $(3, 0)$ and $(2, 1)-$distance oracles for both weighted and unweighted graphs. The bound for $(3, 0)-$distance oracle is tight up to a factor of $\Theta(k \log n)$[1] due to oracle by Thorup and Zwick [4], described in Chapter 3. Additionally any oracle with $\alpha + \beta < 3$ must take at least $\lceil \frac{n(n-1)}{4} \rceil$ bits of storage for some graphs as there exists girth 4 graph with that many edges.*

Conjecture 2.8 ([8])**.** *Consider a data structure that preprocesses sets $S_1, \ldots, S_n \subseteq [X]$, and answers queries of the form "does S_i intersect S_j?". Let $X = lg^c n$ for a large enough constant c. If the query takes constant time, the space must be $\tilde{\Omega}(n^2)$.*

Theorem 2.9 ([8])**.** *A distance oracle for undirected, unweighted graphs with $m = \tilde{O}(n)$ edges, which can distinguish between distances of 2 and 4 in constant time requires $\tilde{O}(n^2)$ space assuming Conjecture 2.8*

[1] $\log n$ factor is here because machine word is assumed to have $\Theta(\log n)$ bits.

Proof. Build a bipartite graph, with n vertices on the left, numbered from 1 to n and X on the right, numbered 1 to X. The number of vertices is $n + X = O(n)$. Connect left vertex i to the elements of S_i on the right. The number of edges is no more than $nX = \tilde{O}(n)$. Two left vertices are at distance 2 if the corresponding sets intersect, and distance at least 4 otherwise. Thus, the distance oracle can solve set intersection queries. □

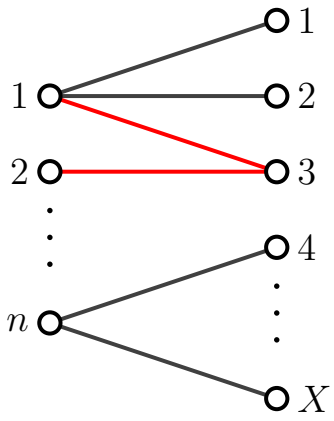


Figure 2.2: Example graph from Theorem 2.9. S_1 and S_2 intersect.

Observation 2.10. *For any graph we can insert $c-1$ vertices on each edge, splitting the edge into c edges, so that each pair of the original vertices will have their distance multiplied by c, where c is a positive integer. The number of vertices in the new graph will be $n + (c-1)m = O(n+m)$, number of edges will be $cm = O(m)$.*

Consequently any oracle that can distinguish between distances $2c$ and $4c$ in constant time must use $\tilde{\Omega}(n'^2) = \tilde{\Omega}(n^2)$ space, where $n = \tilde{O}(cn')$, so $n' = \tilde{O}(n)$.

An $(\alpha, \beta)-$distance oracle must take $\tilde{O}(n^2)$ space if $\alpha < 2$, assuming Conjecture 2.8.

2.2 Preprocessing Time Lower Bound

Stretch-2 or constant surplus distance oracle is smallest stretch oracle that possibly could calculate all distance estimates in less time than boolean matrix multiplication. This is because any algorithm that computes all-pairs approximate distances with stretch less than 2 (or surplus only) could be used to compute boolean matrix multiplication [15].

For directed graphs any constant stretch could be used to compute boolean matrix multiplication (also in [15]).

$m^{5/3-o(1)}$ time is required for a stretch $(2 + o(1))$ oracle based on 3-SUM Conjecture [16].

Chapter 3

(3,0)-Distance Oracle

3.1 Oracle by Thorup and Zwick

We present the slightly modified $(2k-1)-$*approximate* distance oracle of Thorup and Zwick [4].

Theorem 3.1. *Let $G = (V, E)$ be an undirected weighted graph with non-negative weights and $|V| = n$ vertices and $|E| = m$ edges. Let $\delta(u, v)$ denote the distance between vertices $u, v \in V$. Let $k \geq 1$ be an integer. The graph can be preprocessed in $O(nm + kn^2)$ expected time[1]. in order to obtain data structure of size $O(kn^{1+1/k})$ such that any distance approximation $\hat{\delta}(u, v)$ satisfying $\delta(u, v) \leq \hat{\delta}(u, v) \leq (2k - 1)\delta(u, v)$ can be retrieved in $O(k)$ time.*

Definition 3.2. A distance between sets $X, Y \subseteq V$ denoted $\delta(X, Y)$ is equal to the distance between closest vertices in those sets. More formally $\delta(X, Y) = \liminf\{\delta(u, v) \mid u \in X, v \in Y\}$. A distance between vertex v and set $X \subseteq V$ denoted $\delta(v, X)$ or $\delta(X, v)$ is equal to $\delta(\{v\}, X)$ or $\delta(X, \{v\})$ respectively.

Definition 3.3. A ball $B_X(v)$ for a given set $X \subseteq V$ and a vertex $v \in V$ is a set of vertices $\{u \in V \mid \delta(v, u) < \delta(v, X)\}$. We call $\delta(v, X)$ the radius of ball $B_X(v)$.

Definition 3.4. A cluster $C_X(v)$ is, informally an inverse of ball: $u \in C_X(v) \iff v \in B_X(u)$ or in other words $C_X(v) = \{u \in V \mid \delta(u, v) < \delta(u, X)\}$.

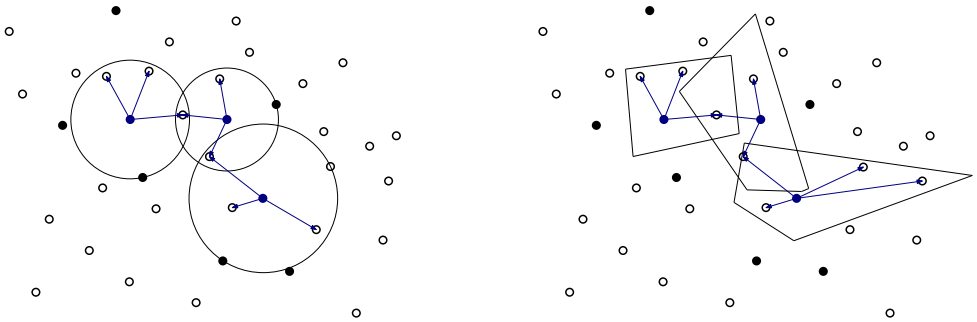
Let $V = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_k = \emptyset$ be a non-increasing sequence of sets.

Definition 3.5. A portal $p_{A_i}(v) \in A_i$ is a vertex that satisfies $\delta(v, p_{A_i}(v)) = \delta(v, A_i)$. Ties are broken arbitrarily.

Definition 3.6. A bunch $\mathrm{bun}(v) = \bigcup_{i=0}^{k-1}\{u \in B_{A_{i+1}}(v) \cap A_i\}$ or more verbosely $\mathrm{bun}(v) = \bigcup_{i=0}^{k-1}\{u \in A_i \setminus A_{i+1} \mid \delta(v, u) < \delta(v, A_{i+1})\}$

[1] By skipping All-Pairs Shortest Path preprocessing and preprocessing the graph directly it is possible to create the oracle in $O(n^2)$ expected time [6] or $O(mn^{1/k})$ deterministic time [5]

Definition 3.7. A clump $\text{clum}(v) = \bigcup_{i=0}^{k-1}\{u \in C_{A_{i+1}}(v) \cap A_i\}$ or more verbosely $\text{clum}(v) = \bigcup_{i=0}^{k-1}\{u \in A_i \setminus A_{i+1} \mid \delta(u,v) < \delta(u, A_{i+1})\}$



(a) Balls (b) Clusters

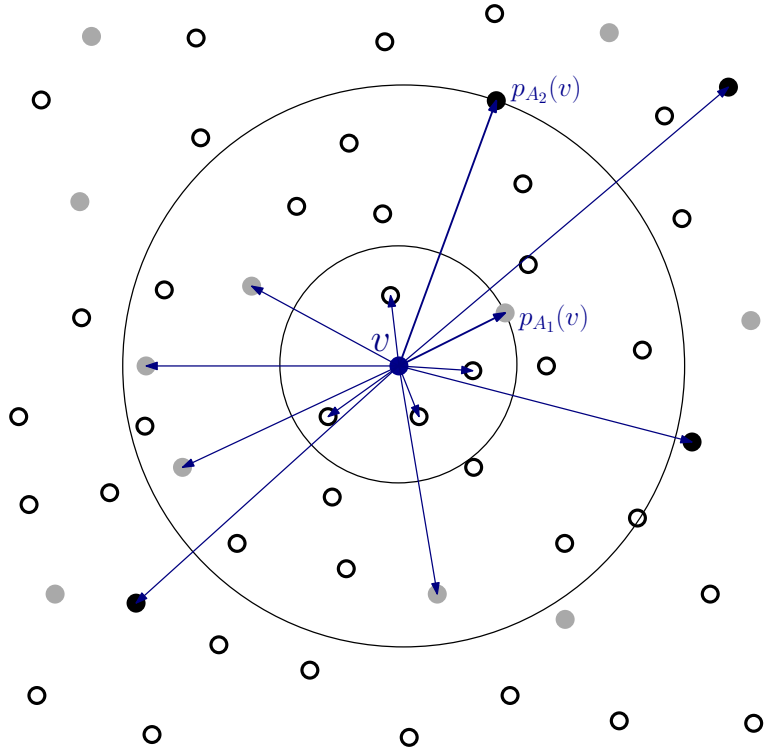


Figure 3.2: Bunch for $k = 3$

algorithm **basicprepro**$_k(G)$
calculate $\delta(u, v)$ for every $u, v \in V$
$A_0 \leftarrow V; A_k \leftarrow \emptyset$
for $i \leftarrow 1$ to $k-1$
  let A_i be an uniform sample from A_{i-1} of size $\lceil n^{-1/k}|A_{i-1}| \rceil$
for every $v \in V$
  for $i \leftarrow 0$ to $k-1$
    calculate $\delta(v, A_i)$ and $p_{A_i}(v)$
  calculate bun(v)
return structure that for every $v, w \in V$, $0 \leq i \leq k-1$ can retrieve $p_{A_i}(v)$,
  $\delta(v, u)$ for $u \in$ bun(v) and can determine if $w \in$ bun(v), all in constant time

algorithm **basicdist**$_k(u, v)$
$w \leftarrow p_0(u); i \leftarrow 0$
while $w \notin$ bun(v)
  $i \leftarrow i + 1$
  $(u, v) \leftarrow (v, u)$
  $w \leftarrow p_{A_i}(u)$
return $\delta(u, w) + \delta(w, v)$

As $n^{-1/k} > 0$ the sample size is greater than 0, so $A_i \neq \emptyset$, for $0 \leq i < k$.

3.2 Correctness

The algorithm **basicdist**$_k(u, v)$ always terminates in at most $k-1$ steps, since $p_{k-1}(u) \in A_{k-1} \subseteq$ bun(v) for every $u, v \in V$. This means query time is $O(k)$.

Lemma 3.8. *$\delta(u, p_{A_i}(u))$ is at most $i\delta(u, v)$ for as long as the* **basicdist**$_k(u, v)$ *did not terminate in $i-1$ steps.*

Proof. The proof is by induction: If the algorithm terminates immediately then $\delta(u, p_0(u)) = \delta(u, u) = 0$. If the algorithm did not terminate in $i-1$ steps then $p_{i-1}(v) \notin$ bun(u), so $\delta(u, p_{i-1}(v)) \geq \delta(u, A_i) = \delta(u, p_{A_i}(u))$ and $\delta(p_{i-1}(v), v) \leq (i-1)\delta(u, v)$, this combined gives

$$\delta(u, p_{A_i}(u)) \leq \delta(u, p_{i-1}(v)) \leq \delta(p_{i-1}(v), v) + \delta(v, u) \leq i\delta(u, v)$$

□

The distance returned by **basicdist**$_k(u, v)$ is at most $(2k-1)\delta(u, v)$ as with the triangle inequality we have

$$\begin{aligned}\delta(u, p_{A_i}(u)) + \delta(p_{A_i}(u), v) &\leq 2\delta(u, p_{A_i}(u)) + \delta(u, v) = \\ &= (2i+1)\delta(u, v) \leq (2k-1)\delta(u, v)\end{aligned}$$

3.3 Size of the Structure

For each vertex $v \in V$ we store:

- $p_{A_i}(v)$ for $i = 0 \dots k-1$ and its corresponding distance $\delta(v, p_{A_i}(v)) = \delta(v, A_i)$
- the hash table for the bunch $\text{bun}(v)$ holding $\delta(v,u)$ for every $u \in \text{bun}(v)$ (the hash table is also able to determine whatever $u \in \text{bun}(v)$)

In order to store balls and distances in constant memory per entry and constant retrieve time we will use hash tables. They can be computed in linear expected time [17] or they can be computed deterministically is $O(s \log s \log n)$ time [18], where s in number of entries to be stored.

The total size of the data structure is therefore $O(kn + \sum_{v \in V} |\text{bun}(v)|)$.

Lemma 3.9. *Let $X \subseteq V$ and S be an uniform sample from X of size $\lceil p|X| \rceil$. Then for $v \in V$ the expected size of $B_S(v) \cap X$ is less than $1/p$.*

Proof. For we show that expected size of $B_S(v) \cap X$ is *stochastically dominated* by a random variable with parameter p. Let $w_1, w_2 \dots w_l$ be the elements of X arranged in any non-decreasing order of distance from v. If $w_j \in S$ then $w_j, w_{j+1}, \dots, w_l \notin B_S(v)$ as those elements are not closer to v than S.

If we look at $\Pr[w_j \in S] \geq p$ we see that

$$\Pr[w_j \in B_S(v)] \leq \Pr[\{w_1, w_2, \dots, w_j\} \cap S = \emptyset] \leq (1-p)^j$$

$$\mathbf{E}[B_S(v) \cap X] = \sum_{j=1}^{l} \Pr[w_j \in B_S(v)] \leq \sum_{j=1}^{l} (1-p)^j < p^{-1}$$

This concludes the proof. □

Theorem 3.10. *Let $X \subseteq V$, S be an uniform sample from X of size $q\lceil p|X| \rceil$, where q is an positive integer. Then for $v \in V$ the expected value of $|B_S(v) \cap X|^q$ is less than $1/p^q$.*

Proof. By Lemma 3.9 we can take q uniform samples $S_1, S_2, \dots, S_q$ of size $\lceil p|X| \rceil$, and the expected value of $|B_{S_i}(v) \cap X|$, for $v \in V$, $0 < i \leq q$ will be less than $1/p$. However for a given $v \in V$ we can choose the same sequence $w_1, w_2, \dots, w_l$ for all samples S_i, which means that it is independent from the samples. Additionally the samples are independent from each other, so the expected values of the upper bounds on $|B_{S_i}(v) \cap X|$ are independent, which gives:

$$\mathbf{E}[|B_S(v) \cap X|^q] \leq \mathbf{E}[|B_{(\cup_{i=1}^{q} S_i)}(v) \cap X|^q] \leq \mathbf{E}[\Pi_{i=1}^{q} |B_{S_i}(v) \cap X|] < 1/p^q$$

We can think of $B_{\cup_{i=1}^{q} S_i}(v)$ as an uniform sample S' from X of size $\leq q\lceil p|X| \rceil$. We can then select uniformly elements one by one that are not yet in the sample and

add them to sample until we got a sample of size $|S|$. The sample constructed that way will be uniform, and in each step $B_{S'}(v) \cap X$ will not get any new elements. □

Theorem 3.10 shows that balls $B_S(v)$, $v \in V$, S uniform sample from V of size $\lceil pn \rceil$ are unlikely to grow much bigger than $1/p$.

Note that this does not apply to clusters $C_S(v)$, which in turn are unlikely to intersect significantly more than $4n/p^2$ vertices, as every cluster intersection means that both vertices that have their clusters intersecting belong to the same ball. We have n balls with expected square of their size (amount of different pairs of vertices in them) being lower than $\sim 1/(p/2)^2$ (= if pn is even)

Theorem 3.11. *Expected size of bun(v), $v \in V$ is at most $kn^{1/k} + k$*

Proof. The size of $A_{k-1} \setminus A_k$ is less than $n^{1/k} + k$, as $A_k = \emptyset$ and by induction

$$\begin{aligned}(n^{-1/k})^{i-1}|A_{k-i}| + i - 1 &\leq (n^{-1/k})^{i-1}(n^{-1/k}|A_{k-(i+1)}| + 1) + i - 1 \leq \\ &\leq (n^{-1/k})^{i}|A_{k-(i+1)}| + i \quad (\text{for } \ 1 \leq i \leq k-1)\end{aligned}$$

so $|A_{k-1} \setminus A_k| = |A_{k-1}| = (n^{-1/k})^0|A_{k-1}| \leq (n^{-1/k})^{k-1}|A_0| + k - 1 = n^{1/k} + k - 1$

For $0 \leq i < k-1$ we show that expected size of $\text{bun}(v) \cap A_i$ for $0 \leq i < k-1$ is by Lemma 3.9 at most $n^{1/k}$. Indeed, A_{i+1} is an uniform sample from A_i of size $\left\lceil p|A_i| \right\rceil$ for $p = n^{-1/k}$. Since no element of $B_{A_j}(v)$ is in A_i for $0 \leq j \leq i$ and no element of A_j is in $B_{A_{i+1}}(v)$ for $i+1 \leq j \leq k-1$, sets $B_{A_{i+1}}(v) \cap A_i$ for $0 \leq i \leq k-1$ are disjoint, so $\text{bun}(v) \cap A_i = B_{A_{i+1}}(v) \cap A_i \leq 1/p = n^{1/k}$.

This together with $V = \cup_{i=0}^{k-1}(A_i \setminus A_{i+1})$, as $V = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_k = \emptyset$ and $|A_{k-1} \setminus A_k| < n^{1/k} + k$ completes the proof. □

The expected size of the structure is $O(kn + n(kn^{1/k} + k)) = O(kn^{1+1/k})$. We can get data structure of deterministic size $O(kn^{1+1/k})$ by re-running the algorithm until data structure produced is small enough. By Markov's inequality the expected number of repetitions required is constant, so this will not affect the expected running time of the algorithm.

3.4 Preprocessing Memory and Time

The calculation of all distances takes $O(nm+n \log n)$ time and $O(n^2)$ memory when using Dijkstra's algorithm with Fibonacci heaps. (see Cormen et al. [12, Chapter 21]) The time could be further improved to $O(nm)$ by using Thorup's algorithm for undirected graphs with floating point edge weights [13, 14].

The creation of A_i sets takes $O(kn)$ memory and time.

The rest of the **basicprepro**$_k(G)$ takes $O(kn^2)$ expected time and $O(kn^{1+1/k})$ deterministic memory (program is re-run if memory usage becomes too high) as all calculated values are stored in the oracle.

Definition 3.12. Auxiliary memory is the memory used by program without without counting the input, which can be read, but not written to.

basicprepro$_k(G)$ takes $O(nm + kn^2)$ expected time and $O(n^2)$ memory in total. However this approach is very inefficient due to calculation of all distances. By cleverly calculating only needed distances, the running time could be reduced to $O(kmn^{1/k})$ [7], or $O(n^2)$ expected time for unweighted graphs [6]. The auxiliary memory in both cases is $O(kn^{1+1/k})$ – the same as the oracle size. By using deterministic A_i sets creation methods, we can achieve the $\tilde{O}(kmn^{1/k})$ deterministic time and $O(kn^{1+1/k})$ auxiliary memory [5].

Chapter 4

(2,1)-Distance Oracle

4.1 Oracle by Baswana, Goyal, and Sen

We present the $(2,1)-$*approximate* distance oracle of Baswana, Goyal, and Sen [7], modified by us to work on weighted graphs.

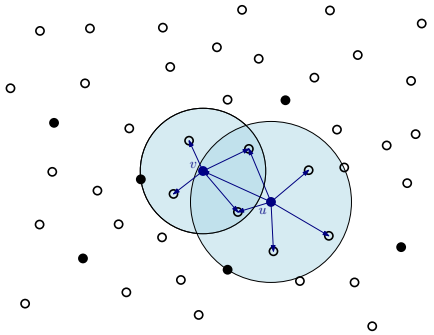
Theorem 4.1. *Let $G = (V, E)$ be an undirected weighted graph with non-negative weights and $|V| = n$ vertices and $|E| = m$ edges. Let $\delta(u, v)$ denote the distance between vertices $u, v \in V$. The graph can be preprocessed in $O(n^{7/3} \log n + mn^{2/3} \log n)$ expected time in order to obtain data structure of size $O(n^{5/3} \log n)$ such that any distance approximation $\hat{\delta}(u, v)$ satisfying $\delta(u, v) \leq \hat{\delta}(u, v) \leq 2\delta(u, v) + h$, where h is the weight of the heaviest edge, can be retrieved in $O(1)$ time.*

```
algorithm prepro(G)
A_0 ← V; A_2 ← ∅
A_1 ← sample(A_0, p)
for every u ∈ A_1
    for every v ∈ V
        calculate δ(u, v)
for every v ∈ V
    calculate p_{A_1}(v), B_{A_1}(v) and C_{A_1}(v)
    for every u ∈ B_{A_1}(v)
        calculate δ(v, u)
    for every pair (u, w) ∈ C_{A_1}(v) × C_{A_1}(v)
        O(u, w) ← v
```

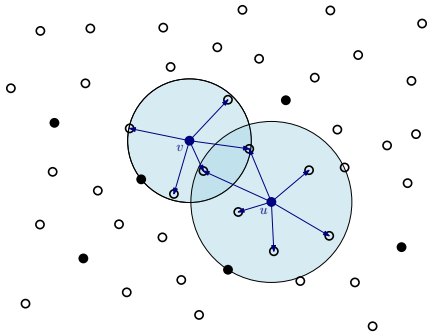
algorithm **dist**(u, v)
if $u \in B_{A_1}(v)$ or $v \in B_{A_1}(u)$
    return $\delta(u, v)$
else if $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$
    return $\delta(u, \mathcal{O}(u, v)) + \delta(\mathcal{O}(u, v), v)$
else
    return minimum of $\delta(u, p_{A_1}(u)) + \delta(p_{A_1}(u), v)$ and $\delta(u, p_{A_1}(v)) + \delta(p_{A_1}(v), v)$

The biggest difference between this oracle and the $(3, 0)-$distance oracle is that here we keep track of vertices $u, v \in V$ such that their balls intersect, and store the vertex from their intersection in $\mathcal{O}$. In order to do that we had to modify the construction of the A_i sets in order to ensure that not too many balls will intersect with each other.

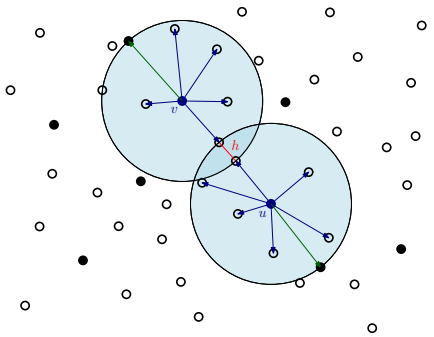
4.2 Correctness



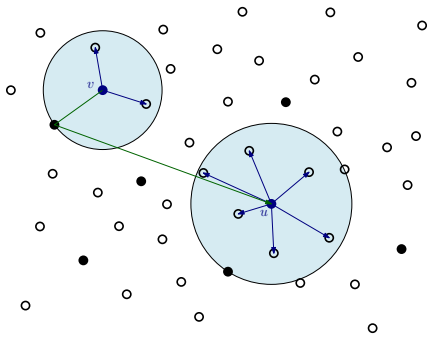
(a) Case 1: $u \in B_{A_1}(v)$ or $v \in B_{A_1}(u)$



(b) Case 2: $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$



(c) Case 3: $(2, h)-$approximation case. The distance between vertices is smaller than the sum of their ball's radii, but they do not share a vertex.



(d) Case 3: stretch-2 case. The reported distance is stretch-2 approximation or even more accurate, if the balls are far away from each other.

The worst case happens when the balls have equal radius, portals are in the opposite direction to destination and balls are as close to each other as possible without sharing a vertex.

Lemma 4.2. *The distance returned by* $\mathbf{dist}(u, v)$ *is not less than* $\delta(u, v)$ *and at most* $2\delta(u, v) + h$*, where* h *is the weight of the heaviest edge.*

Proof. We will go through each of cases

1. $u \in B_{A_1}(v)$ or $v \in B_{A_1}(u)$:
   Our structure stores the $B_{A_1}(w)$, $\delta(w, q)$ for $w \in V$, $q \in B_{A_1}(w)$. Exact distance is returned.

2. $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$:
   Since last case did not occur, $u \notin B_{A_1}(v)$ and $v \notin B_{A_1}(u)$ we have $\delta(u, v) \geq \delta(u, A_1)$ and $\delta(u, v) \geq \delta(v, A_1)$. Let $w \in B_{A_1}(u) \cap B_{A_1}(v)$, we see that $\delta(u, w) < \delta(u, A_1)$, $\delta(v, w) < \delta(v, A_1)$, so $\delta(u, w) + \delta(w, v) < \delta(u, A_1) + \delta(v, A_1) \leq \delta(u, v) + \delta(u, v)$. The $\delta(u, w)$ and $\delta(w, v)$ are stored in our structure.

   Note that for $u, v, w \in V$ we have

   $$w \in B_{A_1}(u) \cap B_{A_1}(v) \implies u, v \in C_{A_1}(w) \implies \mathcal{O}(u, v) \text{ is defined}$$

   $$\mathcal{O}(u, v) = w \implies u, v \in C_{A_1}(w) \implies w \in B_{A_1}(u) \cap B_{A_1}(v)$$

   so we can retrieve $w \in B_{A_1}(u) \cap B_{A_1}(v)$ from $\mathcal{O}(u, v)$ or tell that $B_{A_1}(u) \cap B_{A_1}(v) = \emptyset$ otherwise.

   For unweighted graphs we can remember exact $\delta(u, v)$ by choosing $w = \mathcal{O}(u, v)$ such that it is on the shortest path from u to v. Such w exists in unweighted graphs as the shortest path is the one with the least vertices and first $\delta(u, A_1)$ vertices, last $\delta(v, A_1)$ vertices of a path from u to v will be in corresponding balls. This also allows us to skip the first case as

   $u \in B_{A_1}(v) \implies v \in B_{A_1}(v)$ for $v \in V$.

3. None of the above:
   Let $x = \delta(B_{A_1}(u), B_{A_1}(v))$, $a = \delta(u, A_1)$, $b = \delta(v, A_1)$. Since $B_{A_1}(u) \cap B_{A_1}(v) = \emptyset$ we have $x > 0$. Moreover $a + b - h \leq \delta(u, v) < a + b + x$ as any vertices $w \in B_{A_1}(u)$, $q \in B_{A_1}(v)$ realizing $\delta(w, q) = x$ are also satisfying $a \leq \delta(u, q)$, $b - h \leq \delta(q, v)$, $\delta(u, w) < a$ and $\delta(v, q) < b$.

   Without the loss of generality we assume that $a \leq b$. We then have

   $$\delta(u, v) \leq \delta(u, p_{A_1}(v)) + \delta(p_{A_1}(v), v) \leq 2\delta(u, p_{A_1}(v)) + \delta(u, v) =$$

   $$= 2a + \delta(u, v) \leq (a + b - h) + h + \delta(u, v) \leq 2\delta(u, v) + h$$

   $p_{A_1}(v) \in A_1$ so distances from $p_{A_1}(v)$ to all other vertices are stored in our structure.

Since every case returns $(2,h)-$approximation of the distance, the lemma holds. □

Observation 4.3. *Distance estimate error is bounded from above by* $2\delta(u,p_{A_1}(u))$*, as case* 1 *distances are exact, case* 2 *and* 3 *report exact distance between* w *and* v *plus* $\delta(u,w)$*, where* $w \in V$ *satisfies* $\delta(u,w) < \delta(u,p_{A_1}(u))$*. Analogically the distance estimate error is bounded from above by* $2\delta(v,p_{A_1}(v))$*.*

This remark points to the interesting property of this oracle − for a vertex u we report exact distances to every vertex closer than $\delta(u,p_{A_1}(u))$ (and we know then that the distance reported is exact), and we report almost stretch-1 distances to every vertex v satisfying $\delta(u,p_{A_1}(u)) << \delta(u,v)$ which is equivalent to $\delta(u,p_{A_1}(u)) << \delta(p_{A_1}(u),v)$ (and both of those values are stored in the oracle).

Observation 4.4. *This oracle is a stretch*-2 *oracle for all cases except for part of case 3. For distance estimate* $\hat{\delta}(u,v)$*,* $u,v \in V$ *to be more than* $2\delta(u,v)$*, the following must be true: distance is reported at case* 3*,* $\delta(p_{A_1}(u),v) < 3\delta(u,p_{A_1}(u))$ *and* $\delta(p_{A_1}(v),u) < 3\delta(v,p_{A_1}(v))$ *Those conditions could be easily verified in constant time during oracle query.*

We can make even stronger argument, this distance oracle is not a stretch-2 distance oracle only if the balls of the queried vertices have no common vertex, but they are connected by a single edge. This however can not be verified by this oracle.

4.3 Improved Sampling Algorithm

algorithm **sample**(X,p)
$R \leftarrow \emptyset; X' \leftarrow X$
while $X' \neq \emptyset$
  Let S be an uniform sample from X' of size $\lceil p|X| \rceil$ or X' if $\lceil p|X| \rceil > |X'|$
  $R \leftarrow R \cup S$
  For every $v \in X'$
    calculate $C_R(v) \cap X$
  $X' \leftarrow \{v \in X \mid |C_R(v) \cap X| > 4/p\}$
return R

Observation 4.5. *The algorithm* **sample**(X,p) *ensures that all* $C_R(v)$ *clusters have at most* $4/p$ *vertices, so all* $C_{A_1}(v)$ *clusters have at most* $4/p$ *vertices. This ensures that* $\mathcal{O}$ *has at most* $16n/p^2$ *defined values. Recall that* $\mathcal{O}$ *is a hash table that for each pair of vertices* $(u,v) \in V \times V$ *satisfying* $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$ *stores the vertex* $w \in B_{A_1}(u) \cap B_{A_1}(v)$*.*

Lemma 4.6. *In each step of sample* **sample**(X, p)*, the value of* $|X'|$ *is at least halved with probability at least* $1/2$*.*

Proof. Let X_i be the set X' at the beginning of i-th step. Note that $X_i \supseteq X_{i+1}$ as R never shrinks so clusters C_R never grow larger. If $|X_i| < \lceil p|X| \rceil$ then $|X_{i+1}| = 0$ and lemma holds. In other case after the set R is augmented by sample S ($R \supseteq S$) we note that

$$\sum_{v \in X_i} |C_R(v) \cap X| = \sum_{v \in X} |B_R(v) \cap X_i| \leq \sum_{v \in X} |B_S(v) \cap X_i|$$

by Lemma 3.9

$$\mathbf{E}[\sum_{v \in X} |B_S(v) \cap X_i|] < |X|(p|X|/|X_i|)^{-1} = |X_i|/p$$

By Markov's inequality with probability $1/2$ we have $\sum_{v \in X_i} |C_R(v) \cap X| < 2|X_i|/p$, and since in that case

$$|X_{i+1}|4/p \leq \sum_{v \in X_i} |C_R(v) \cap X| < 2|X_i|/p$$

so $2|X_{i+1}| < |X_i|$ with probability at least $1/2$. □

So the expected number of steps is at most $2 \log n$.

By combining the above lemma with Lemma 3.9 we can prove the following lemma

Lemma 4.7. *Given an undirected weighted graph with non-negative weights, a set* X*,* $X \subseteq V$ *and number* $0 < p < 1$*, we can compute a sample set* R *of expected size* $O(p|X| \log |X|)$ *such that for every* $v \in V$ $C_R(v) \cap X$ *is of size at most* $4/p$ *and expected size of* $B_R(v) \cap X$ *is less than* $1/p$*.*

4.4 Efficient Ball and Cluster Computation

Lemma 4.8. *Given an undirected weighted graph with non-negative weights, number* $0 < p < 1$ *and set* A_1 *generated by algorithm* **sample**(V, p)*, we can calculate* $B_{A_1}(v)$*,* $C_{A_1}(v)$ *and* $\delta(v, u)$ *for every* $v \in V$*,* $u \in B_{A_1}(v)$ *in* $O(m/p + n/p \log n)$ *time.*

Proof. We start by calculating the $\delta(v, A_1)$ for every $v \in V$. We can do this by adding to the graph a source vertex s and connecting it to every $u \in A_1$ by edge of weight 0, and then running the Dijkstra's algorithm from there. It is not hard to see that this will calculate $\delta(v, A_1)$ for every $v \in V$ in $O(m + n \log n)$ time.

Now for every $v \in V$ we know the $\delta(v, A_1)$, so we can run modified Dijkstra's algorithm to calculate $B_{A_1}(v)$ and $\delta(v, u)$ for $u \in B_{A_1}(v)$. We modify Dijkstra's

algorithm to ignore edges that lead to too far away vertices − vertices $u \in V$ such that $\delta(v,u) \geq \delta(v,A_1)$. Such modified Dijkstra's will work in $O(\xi(B_{A_1}(v))+|B_{A_1}(v)|$ log $|B_{A_1}(v)|)$ time, where $\xi(X)$, $X \subseteq V$ is the number of edges incident to vertices of X.

Observation 4.9. *Note that when $B_{A_1}(v)$ is calculated for every $v \in V$, we can easily calculate $C_{A_1}(v)$ as $u \in B_{A_1}(v) \iff v \in C_{A_1}(u)$, $u \in V$. This will not change the time complexity as $\sum_{v\in V} |B_{A_1}(v)| = \sum_{v\in V} |C_{A_1}(v)|$.*

Thorup claims that his algorithm can be modified in a similar way (at least for calculating clusters) [4], achieving working time of $O(\xi(B_{A_1}(v)) + |B_{A_1}(v)|) = O(\xi(B_{A_1}(v)))$.

We will now calculate an upper bound on $\sum_{v\in V} \xi(B_{A_1}(v))$:

$$\begin{aligned}\sum_{v\in V} \xi(B_{A_1}(v)) &= \sum_{v\in V} \sum_{u\in B_{A_1}(v)} \xi(u) = \\ &= \sum_{u\in V} \sum_{v\in C_{A_1}(u)} \xi(u) = \\ &= \sum_{u\in V} |C_{A_1}(u)|\xi(u) \leq \sum_{u\in V} 4\xi(u)/p \leq 8m/p\end{aligned}$$

So the total working time of the above is

$$O(m + n\log n + 8m/p + \sum_{v\in V} |B_{A_1}(v)| \log |B_{A_1}(v)|) = O(m/p + n/p \log n)$$

. □

The above weighted graphs Ball and Cluster computation requires the clusters to be bounded by their size, without it the bound on computation time will increase.

Observation 4.10. *If A_1 would be a set containing uniform sample from V of size $\lceil p|V| \rceil$ instead, then the expected size of $\sum_{v\in V} \xi(B_{A_1}(v))$ would be bounded by Lemma 3.9 and since $B_X(v) \leq B_Y(v)$ for $X \subseteq Y \subseteq V$, $v \in V$*

$$\sum_{u\in V} |C_{A_1}(u)|\xi(u) \leq \sum_{u\in V} |C_{A_1}(u)|n = n\sum_{u\in V} |C_{A_1}(u)| = n\sum_{v\in V} |B_{A_1}(v)| \leq n^2/p$$

Which changes the computation time from deterministic $O(m/p + n/p$ log $n)$ to expected $O(n^2/p)$.

For undirected graphs however, computation is much simpler and will work for **sample**(V,p)

Lemma 4.11. *Given an undirected unweighted graph with no multiple edges, number $0 < p < 1$ and uniform sample S from V of size $\lceil p|V| \rceil$ we can calculate $B_S(v)$ and $C_S(v)$ for every $v \in V$ in $O(m + n/p^2)$ expected time.*

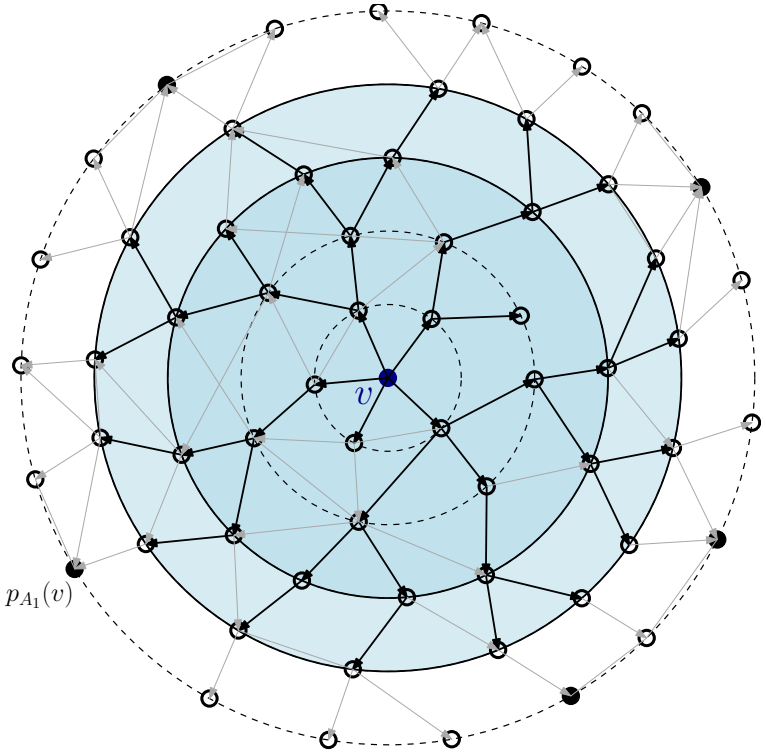


Figure 4.2: Ball in the unweighted graph. Black edges belong to the BFS tree and full black vertices belong to A_1.

Proof. First we compute $\delta(v, S)$ for every $v \in V$. This can be done in $O(m)$ time by a single BFS traversal. We do that by adding a source vertex s that is connected to every $u \in S$ and starting BFS in that vertex. Indeed, if $\delta(v, S) = x$ then $\delta(v, u) = x - 1$ for some $u \in S$ and $\delta(v, w) \geq x - 1$ for every $w \in S$, so $\delta(v, s) = \delta(v, S) + 1$.

Now for every $v \in V$ we compute ball $B_S(v)$ using BFS, but we immediately halt the search when we reach a vertex $u \in V$ such that $\delta(v, S) - 1 \leq \delta(v, u)$. That way we are guaranteed to visit every $w \in V$ such that $\delta(v, w) \leq \delta(v, S) - 1$, but we will not process any edges from vertices $q \in V$ such that $\delta(v, q) \geq \delta(v, S) - 1$ (Unless the edge was processed by a vertex visited earlier).

Let us count the expected number of edges processed. Let $u \in V$ be any vertex and $v_1, v_2, \ldots, v_n \in V$ be vertices of V arranged in BFS non-decreasing order of distance from u. Each edge will be processed at most twice, from both (or one) of vertices it is connecting. We will count only the first time − when the edge is connecting to the vertex that was not visited (or when edge is a loop). BFS can process the edges of v_i when both of the below conditions are satisfied (required, but not sufficient):

1. ψ_1^i: There is no vertex in the set $\{v_j | 0 < j < i\}$ which is selected in the sample S.

2. ψ_2^i: There is no vertex in the set $\{v_j | i \leq j \leq n\}$ that is in S and there is edge

between it and v_i.

The chance for the ψ_1^i is no more than $(1-p)^{i-1}$ as the chance for $v_j \in S$ is $\lceil p|V| \rceil / |V| \geq p$.

If ψ_1^i is true then chance for ψ_2^i is no more than:

$$(1 - \Pr[v_j \in S \text{ for } i < j \leq n \mid \psi_1^i])^{\xi'(v_i)}$$

where $\xi'(v_i)$ is number of edges connecting v_i with any v_j where $i \leq j \leq n$.

We also have $\Pr[v_j \in S \text{ for } i < j \leq n \mid \psi_1^i] > p$ as ψ_1^i for $i < j \leq n$ increase the chance for $v_j \in S$ that was originally $\lceil p|V| \rceil / |V| \geq p$. That gives $\Pr[\psi_2^i \mid \psi_1^i] \leq (1-p)^{\xi'(v_i)}$.

Now let us look at the expected number of edges processed at least once – it is no more than

$$\begin{aligned}
\sum_{i=1}^{n} \Pr[\psi_1^i] \Pr[\psi_2^i \mid \psi_1^i] \xi'(v_i) &\leq \sum_{i=1}^{n} ((1-p)^{i-1}(1-p)^{\xi'(v_i)} \xi'(v_i)) \leq \\
&\leq \sum_{i=1}^{n} ((1-p)^{i-1} \sum_{j=1}^{\xi'(v_i)} (1-p)^{j-1}) \leq \\
&\leq \sum_{i=1}^{n} ((1-p)^{i-1} 1/p) \leq \\
&\leq 1/p \sum_{i=1}^{n} ((1-p)^{i-1}) \leq 1/p^2
\end{aligned}$$

As computing $\delta(v, S)$ takes $O(m+n)$ time and for every vertex we process the expected $1/p^2$ edges the total expected time is $O(m + n/p^2)$. □

4.5 Preprocessing Memory and Time

We first compute the **sample**(V, p) which consists of while loop with expected 2 log n iterations, in each one we compute uniform sample S in time $O(n)$, and then $B_R(v)$, $C_R(v)$ for $v \in V$, where R contains uniform sample from V of size $\lceil p|V| \rceil$. This can be done by Lemma 4.8 and by Observation 4.10 for weighted graphs or by Lemma 4.11 for unweighted graphs. We can compute the $B_R(v)$ in $O(m + n^2/p)$ expected time for weighted graphs or $O(m + n/p^2)$ expected time for unweighted graphs. The calculation of clusters and their sizes can be done by Observation 4.9 without affecting the complexity. The total expected time is $O(m \log n + n^2/p \log n)$ for weighted graphs or $O(m \log n + n/p^2 \log n)$ for unweighted graphs. The balls and clusters have expected size of $O(1/p)$ so they will not take more than $O(n/p \log n)$ expected auxiliary memory.

We can compute $\delta(u, v)$, $p_{A_1}(v)$ for every $u \in A_1$, $v \in V$ by running Dijkstra's algorithm for each of the $u \in A_1$. However this would take $O(|A_1|(m + n \log n))$ time.

We can do this faster, in $O(m|A_1|) = O(mnp \log n)$, by using Thorup's algorithm for undirected graphs with floating point edge weights [13, 14]. For unweighted graphs a simple BFS will be enough. Finding $p_{A_1}(v)$ can be done afterwards by checking every distance calculated. The expected auxiliary memory is $O(n|A_1|) = O(n^2 p \log n)$.

Next we will compute balls and clusters using Lemma 4.8 in $O(m/p + n/p \log n)$ expected time and $O(n/p)$ deterministic auxiliary memory.

And at the end we will compute $\mathcal{O}$ in $O(n/p^2)$ memory as cluster size is bounded by $4/p$ (Observation 4.5). The time could be either expected $O(n/p^2)$ or deterministic $O(n/p^2 \log n)$ depending on hash table construction method used.

This will total to $O(m \log n + n^2/p \log n + mnp \log n + m/p + n/p \log n + n/p^2) = O(n^2/p \log n + mnp \log n + n/p^2)$ expected time for weighted graphs or $O(n/p^2 \log n + mnp \log n)$ expected time for unweighted graphs.

The expected auxiliary memory is $O(n/p \log n + n^2 p \log n + n/p + n/p^2) = O(n^2 p \log n + n/p^2)$.

For $p = n^{-1/3}$ and $m \geq n$ the expected auxiliary memory is the lowest $- O(n^{5/3} \log n)$. The expected time is then $O(n^{7/3} \log n + mn^{2/3} \log n)$ for weighted graphs and $O(mn^{2/3} \log n)$ for unweighted graphs. For $m = O(n^2)$ the expected time is $O(n^{8/3} \log n)$ in both cases.

By Markov's inequality we can get deterministic auxiliary memory $O(n^{5/3} \log n)$ in expected constant number of **prepro**(G) executions.

Observation 4.12. *As the oracle uses only data structures that were generated during preprocessing the auxiliary preprocessing memory is an upper bound on the oracle memory.*

Chapter 5

Removing the Log Factors

5.1 Oracle by Patrascu and Roditty

We present the slightly modified $(2,1)$ – *approximate* distance oracle of Patrascu and Roditty [8].

algorithm **preproPatRod**(G)
let R be an uniform sample from V of size $\lceil n^{-1/3}n \rceil$
let S be an uniform sample from V of size $\lceil n^{-2/3}n \rceil$
let $A_1 \leftarrow R \cup \bigcup_{u \in S} B_R(u)$
for every $u \in A_1$
    for every $v \in V$
        calculate $\delta(u,v)$
for every $v \in V$
    calculate $p_{A_1}(v)$, $B_{A_1}(v)$ and $C_{A_1}(v)$
    for every $u \in B_{A_1}(v)$
        calculate $\delta(v,u)$
    for every pair $(u,w) \in C_{A_1}(v) \times C_{A_1}(v)$
        $\mathcal{O}(u,w) \leftarrow v$

The only difference between this oracle and oracle from Chapter 4 is that this oracle constructs A_1 by adding $B_R(u)$, $u \in S$ to the set R, instead of adding $O(\log n)$ sets of size R to the set R, which reduces the memory complexity by $\log n$. Here we get a weaker bound on $C_{A_1}(v)$, $v \in V$, which fortunately is claimed by Knudsen to not increase preprocessing time for sparse graphs too much [11, Chapter 4, Lemma 10].

Lemma 5.1. *There are at most $O(n^{5/3})$ expected number pairs of vertices $(u,v) \in V \times V$ such that $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$, if A_1 was constructed by* **preproPatRod**(G).

Proof. We will prove the lemma by showing that for $u \in V$ expected number of balls intersecting with $B_{A_1}(u)$ is less than $n^{2/3}$.

The algorithm generates first the sample R. Let $u_1, u_2, \ldots, u_n$ be the elements of V arranged in any non-decreasing order of distance from u.

Now let $w_1, w_2, \ldots, w_n$ be the elements of V arranged in any order such that if $i \leq j$ then for every l such that $w_j \in C_R(u_l)$ there is such k that $w_i \in C_R(u_k)$ we have $k \leq l$. In other words we order the vertices by the first cluster (the one with the smallest i) $C_R(u_i)$, $i \in \{1, \ldots, n\}$ that it belongs to.

Now the algorithm generates the sample S. What is the expected size of the set $L = \{w_1, w_2, \ldots, w_l\}$ such that $L \cap S = \emptyset$ and L is largest possible?

$$\mathbf{E}[|L|] = \sum_{i=1}^{n} \Pr[w_i \in L] = \sum_{i=1}^{n} \Pr[\{w_1, w_2, \ldots, w_i\} \cap S = \emptyset] \leq$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{i} Pr[w_j \notin S] \leq \sum_{i=1}^{n} (1 - n^{-2/3})^i < n^{2/3}$$

For $v \in V$ we claim that $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset \implies v \in L$.

We prove the claim by contradiction: Suppose that $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$ and $v \notin L$

We have $v = w_k$ for some $l + 1 \leq k \leq n$. Note that w_{l+1} exists as otherwise $L = V$, $v \in L$ and $w_{l+1} \in S$, as L is largest possible

Let u_i $1 \leq i \leq n$ be such vertex, that $v \in C_R(u_i)$ and i is smallest possible. If no such u_i exists then $\delta(v, R) = 0$ as otherwise $v \in C_R(v)$, however $\delta(v, R) = 0 \implies B_R(v) = B_{A_1}(v) = \emptyset$, contradiction.

Now let u_j $1 \leq j \leq n$ be such vertex, that $w_{l+1} \in C_R(u_j)$ and j is smallest possible. If no such u_j exists then since u_i exists we have $k < l + 1$, so $v = w_k \in L$.

Since $k \geq l + 1$ we have $i \geq j$, so $\delta(u, u_i) \geq \delta(u, u_j)$, but $w_{l+1} \in C_R(u_j)$, so $u_j \in B_R(w_{l+1})$ and $w_{l+1} \in S$ which means that $u_j \in A_1$, so $u_j \notin B_{A_1}(u)$ and $u_i \notin B_{A_1}(u)$. However u_i is the closest vertex to u such that $u_i \in B_R(v)$ and since $B_R(v) \supseteq B_{A_1}(v)$, we get a contradiction $B_{A_1}(u) \cap B_{A_1}(v) = \emptyset$. □

Observation 5.2. *The above proof also gives an estimation on expected value of* $\sum_{v \in V} |C_{A_1}(v)|^2$. *Indeed when we look at set* L_u *(*L *generated for a given* u*) we see that it contains all elements from clusters* $C_R(v)$*, where* $v \in B_{A_1}(u)$*. This points to:*

$$\sum_{v \in V} |C_{A_1}(v)|^2 = \sum_{v \in V} \sum_{u \in C_{A_1}(v)} |C_{A_1}(v)| = \sum_{u \in V} \sum_{v \in B_{A_1}(u)} |C_{A_1}(v)| \leq$$

$$\leq \sum_{u \in V} \sum_{v \in B_{A_1}(u)} L_u = \sum_{u \in V} |B_{A_1}(u)| L_u \leq \sum_{u \in V} |B_R(u)| L_u$$

This implies

$$\mathbf{E}[\sum_{v\in V}|C_{A_1}(v)|^2] \leq \mathbf{E}[\sum_{u\in V}|B_R(u)|L_u] = \sum_{u\in V}\mathbf{E}[|B_R(u)|L_u] \leq nn^{1/3}n^{2/3} = n^2$$

as our upper bounds on $\mathbf{E}[|B_R(u)|]$ *and* $\mathbf{E}[|L_u|]$ *are independent − we generate the* $B_R(u)$ *and* $w_1, w_2, \ldots, w_n$ *(for a given* u*) based on sample* R *and graph* G *and then we generate random sample* S *that gives us* L_u *together with its upper bound, while* $B_R(u)$ *remains unaffected*

The calculation of the hash table $\mathcal{O}$ is possible $O(n^2)$ by the above, however it can be improved to $O(mn^{2/3})$ [11].

The generation of R and S takes $O(n)$, the computation $B_R(u), u \in S$ takes $O(m+|S|/p^2) = O(m+n)$ expected time as stated by Lemma 4.11. The computation of A_1 then happens in $O(n)$ time. The expected auxiliary memory is $O(n)$. We see that the generation A_1 is very fast and the expected size of A_1 is $O(n^{2/3})$ as $|R| = \lceil n^{2/3} \rceil$, $|S| = \lceil n^{1/3} \rceil$ and expected size of $B_R(u), u \in S$ is less than $n^{1/3}$ by Lemma 3.9.

The calculation of $\delta(u, v)$, $p_{A_1}(v)$ for every $u \in A_1$, $v \in V$ takes only expected $O(mn^{2/3})$ time and $O(n^{5/3})$ expected auxiliary memory. This is an improvement by the factor of $\log n$.

The expected value of $\sum_{v\in V}|C_{A_1}(v)|^2$ is n^2, so the computation of $\mathcal{O}$ takes $O(n^2)$ expected time. The expected auxiliary memory remains at $O(n^{5/3})$ thanks to Lemma 5.1.

The total expected time is $O(n^2 + mn^{2/3})$ (or $O(mn^{2/3})$ by claim from [11]), expected auxiliary memory $O(n^{5/3})$.

5.2 Further Improvements

With small modifications, the presented oracles are also able to retrieve the paths realizing the approximation in constant time per edge.

If we additionally remember the distances between vertices, that have their balls connected by a single edge, we can construct $(2, 0)-$distance oracle for weighted graphs with $O(n^{4/3}m^{1/3})$ expected memory in a similar way [8].

The oracle preprocessing time could be further improved to $O(n^2)$ by applying spanners − subgraphs of a graph that have less edges than original but the distances between vertices are preserved up to some error (usually additive or multiplicative). This however comes with the drawbacks as either the distance approximation is worse − stretch $(2, 3)$ [7] or the oracle becomes more complicated [10, 11]. However using spanners will only improve the preprocessing time for graphs with $\omega(n^{4/3})$ edges, as currently known construction of good enough spanner takes $\Theta(n^2)$ time.

The oracle can be generalized similar to oracle from Chapter 3 to produce $(2k-2,1)-$*approximate* distance oracle of size $O(n^{1+2/(2k-1)})$ [9].

There are also more efficient algorithms for special classes of graphs, like planar graphs.

Chapter 6

Results

6.1 Perfect Hashing Functions Generation

Definition 6.1. A minimal perfect hash function is a function that given a set of keys S maps them bijectively to into the set $\{1, 2, \ldots, |S|\}$.

In the following we will call minimal perfect hash functions simply a perfect hash functions.

The hash table used in the programs is PTHash [19], minimal perfect hash algorithm. The implementation of PTHash is written in C++ and available at `https://github.com/jermp/pthash`

The construction time of PTHash is superlinear, fortunately this will not affect the time complexity as all algorithm use $\Omega(n^2)$ preprocessing time, but only $O(n^{5/3})$ write operations.

6.2 Programs Compared

The first program marked as **BFS** compared is an oracle with $A_1 = V$, every exact distance is calculated by BFS algorithm and stored in memory. This is our control sample.

The second program **Basic** is an oracle with A_1 being an uniform sample from V of size $\lceil (\rceil p|V|) = \lceil (\rceil n^{2/3})$. It is the most basic $(2,1)-$distance oracle without any expansion of the A_1 that would ensure small number of ball intersections. This oracle have expected memory of $\Omega(n^{7/3})$ for some graphs.

The third program **BasGoySen** is Chapter 4. Sample A_1 is expanded by vertices with big clusters until none remains.

The fourth and the last program **PatRod** is Section 5.1 oracle. Sample A_1 is

expanded by random balls.

As the programs are very similar, we run the same tests with the same seed for all random sampling (different tests have different seeds). This ensures that the first sample will be the same for all programs, removing some noise from our comparison. However we have made no attempt to modify the PTHash code, so the perfect hash functions generated are different with each run.

6.3 Random Graphs Comparison

The oracles were tested on random graphs with $n = 2000 \pm 100$ in order to avoid any distortions from dense graphs with particular number of vertices, such as program behaving differently on semi-clique graphs with even or odd number of vertices. The number of edges was randomly chosen from $n^a(n-1)/10$ to $n^a(n-1)/2$ for a parameter $0 \leq a \leq 1$. Edges themselves were an uniform sample from all possible edges (no loops and multiple edges).

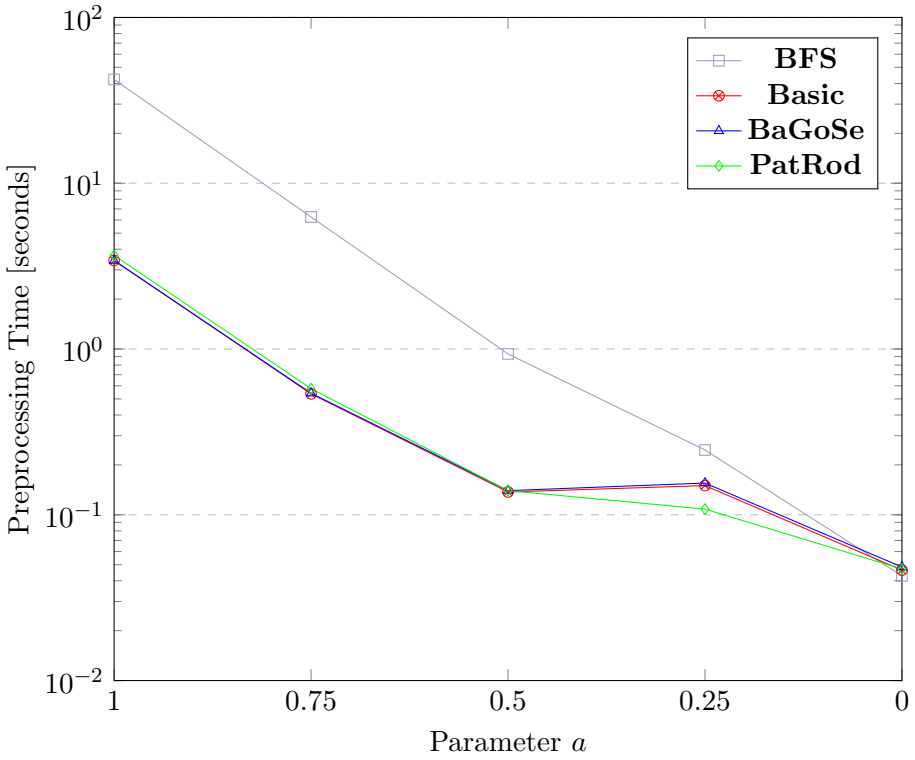


Figure 6.1: Preprocessing Time for $n = 2000 \pm 100$, Random Graph, arithmetic average of 10 tests

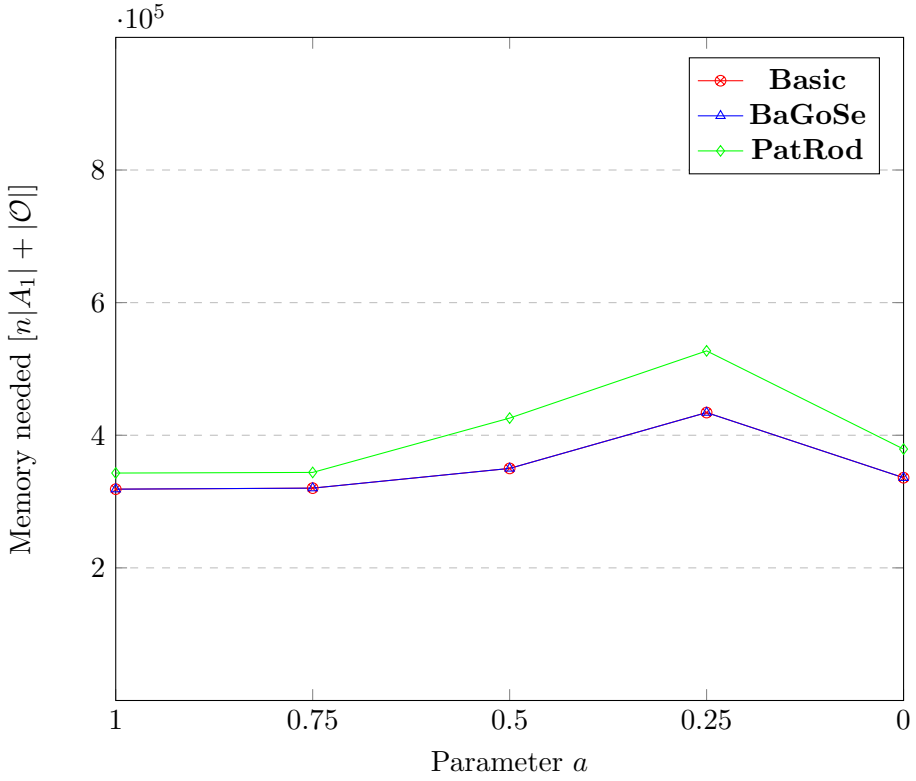


Figure 6.2: Memory needed for $n = 2000 \pm 100$, Random Graph, arithmetic average of 10 tests, calculated as number of distances that need to be stored. Note that memory for the perfect hash functions and $p_{A_1}(v)$ for $v \in V$ was not included. If the $\mathcal{O}$ stores distances instead of vertices we do not need to store balls.

The following quantities were measured:

- Preprocessing Time – Time taken (in seconds) from the moment after input was loaded into memory to the moment oracle is ready to answer queries.
- Average Query Time – Time taken (in seconds) to report $\hat{\delta}(u, v)$, for every $u, v \in V$ divided by $|V|^2$.
- Size of the Set A_1 – self explanatory.
- $n|A_1|$ – Memory needed (in machine words) to store $\delta(u, v)$, for every $u \in A_1$, $v \in V$.
- Sum of Ball sizes – $\sum_{v \in V} |B_{A_1}(v)| = \sum_{v \in V} |C_{A_1}(v)|$ – Memory needed (in machine words) to store $\delta(v, u)$, for every $v \in V$, $u \in B_{A_1}(v)$.
- $|\mathcal{O}|$ – Count of pairs of vertices $(u, v) \in V \times V$ such that their balls intersect $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$.
- Space PHFs B_{A_1} – Memory needed (in bits) to store the perfect hash function for every $B_{A_1}(v)$, $v \in V$ that allows to access any ball element in constant

time, while not using additional memory. The hash function generator seems to struggle with low key numbers, hence large number of bits for empty balls. This could be fixed by combining all functions into one, if needed.

- Space PHFs $\mathcal{O}$ − Memory needed (in bits) to store the perfect hash function for table $\mathcal{O}$.

- Average Relative Error − Arithmetic average of errors calculated as $(\hat{\delta}(u,v) - \delta(u,v))/\delta(u,v)$. If $\delta(u,v) = 0$ or there is no path between u and v, all oracles recognize it flawlessly and hence the error is 0.

- Worst Error percentage − The number of times the reported distance was $2\delta(u,v) + 1$ divided by number of all distances reported.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	42.2736	3.4243	3.4170	3.6729
Avg. Query Time	1.3815e-07	1.4916e-07	1.5012e-07	1.5001e-07
$\|A_1\|$	1993.3	158.9	158.9	171.1
$n\|A_1\|$	3.9756e+06	316860	316860	341175
$\sum_{v\in V} \|B_{A_1}(v)\|$	0	1834.4	1834.4	1822.2
$\|\mathcal{O}\|$	0	1834.4	1834.4	1822.2
Space PHFs B_{A_1}	6.4035e+06	6.4041e+06	6.4043e+06	6.4041e+06
Space PHFs $\mathcal{O}$	3216	13638.4	13644.8	13558.4
Average Error	0	0.6674	0.6674	0.6584
Worst Error (%)	0	10.4053	10.4053	10.2423

Table 6.1: $n = 2000 \pm 100$, $a = 1.0$, Random Graph, arithmetic average of 10 tests

The very dense random graphs − $a = 1.0$ with at least 1/5 of all possible edges have all or almost all balls and clusters of size ≤ 1. The preprocessing time is the largest here especially for the naive **BFS** approach. The oracle report the approximate with over 10% of answers being reported with bigger stretch than 2. The average reported distance is $\sim 66\%$ greater than real distance. The errors are surprisingly large considering that most if not all distances are not larger than 2.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	6.2580	0.5368	0.5411	0.5764
Avg. Query Time	1.3829e-07	1.5242e-07	1.5335e-07	1.5128e-07
$\|A_1\|$	1998.3	159.2	159.2	171.1
$n\|A_1\|$	3.9954e+06	318240	318240	342014
$\sum_{v\in V} \|B_{A_1}(v)\|$	0	1892.2	1892.2	1857.9
$\|\mathcal{O}\|$	0	2010.2	2010.2	1921.9
Space PHFs B_{A_1}	6.4194e+06	6.4206e+06	6.4207e+06	6.4208e+06
Space PHFs $\mathcal{O}$	3216	13188.8	14188.8	13902.4
Average Error	0	0.4681	0.4629	0.4629
Worst Error (%)	0	6.1077	6.0448	6.0448

Table 6.2: $n = 2000 \pm 100$, $a = 0.75$, Random Graph, arithmetic average of 10 tests

The slightly less dense random graphs $-$ $a = 0.75$ have some of the balls intersecting, much better preprocessing time by about the same factor 6.7 for all programs. And smaller errors both average and percent of worst case by about factor of 2/3.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.9339	0.1373	0.1399	0.1399
Avg. Query Time	1.3671e-07	1.9346e-07	1.9343e-07	1.7201e-07
$\|A_1\|$	1995.9	159	159	205.1
$n\|A_1\|$	3.9849e+06	317416	317416	409890
$\sum_{v\in V} \|B_{A_1}(v)\|$	0	6785.7	6785.7	4650.3
$\|\mathcal{O}\|$	0	32557.1	32557.1	16005.1
Space PHFs B_{A_1}	6.4119e+06	6.4399e+06	6.4396e+06	6.4208e+06
Space PHFs $\mathcal{O}$	3196.8	131400	130677	13902.4
Average Error	0	0.3012	0.3012	0.2796
Worst Error (%)	0	0.9458	0.9458	0.9842

Table 6.3: $n = 2000 \pm 100$, $a = 0.5$, Random Graph, arithmetic average of 10 tests

For random graphs with about $n^{3/2}$ edges $-$ $a = 0.5$ the **BFS** speeds up again by a factor $\sim$ 6.7, however other programs speed up about 4 times as the balls here starts to intersect in bigger numbers. As balls grow in size the Worst Error Percentage drops rapidly, while the average error drops again by factor $\sim 2/3$.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.2460	0.1500	0.1554	0.1082
Avg. Query Time	1.4172e-07	2.6281e-07	2.6099e-07	1.9993e-07
$\|A_1\|$	1972.2	157.8	157.8	234.7
$n\|A_1\|$	3.8929e+06	311392	311392	464122
$\sum_{v\in V} \|B_{A_1}(v)\|$	0	12177.8	12177.8	8636.1
$\|\mathcal{O}\|$	0	122816	122816	63118.9
Space PHFs B_{A_1}	6.3360e+06	6.3941e+06	6.3942e+06	6.3736e+06
Space PHFs $\mathcal{O}$	3216	301478	299878	167926
Average Error	0	0.1751	0.1751	0.1393
Worst Error (%)	0	0.2024	0.2024	0.1562

Table 6.4: $n = 2000 \pm 100$, $a = 0.25$, Random Graph, arithmetic average of 10 tests

For $a = 0.25$ the random graphs have a lot of balls intersecting, which causes the **Basic** and **BaGoSe** to work slower despite the better theoretical complexity for **BaGoSe**. Here the **PatRod** is faster then the rest of algorithms, despite being similar on other tests. The Average Error drops as usual.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.0428	0.0464	0.0484	0.0471
Avg. Query Time	1.1239e-07	1.2350e-07	1.2445e-07	1.2298e-07
$\|A_1\|$	2008.9	159.7	159.7	181.5
$n\|A_1\|$	4.0401e+06	321056	321056	364936
$\sum_{v\in V} \|B_{A_1}(v)\|$	0	3521.7	3521.7	3432.8
$\|\mathcal{O}\|$	0	14910.9	14910.9	14202.8
Space PHFs B_{A_1}	6.4539e+06	6.4597e+06	6.4604e+06	6.3736e+06
Space PHFs $\mathcal{O}$	3216	24480	24467.2	167926
Average Error	0	7.1658e-05	7.1658e-05	6.5491e-05
Worst Error (%)	0	7.4778e-07	7.4778e-07	7.4778e-07

Table 6.5: $n = 2000 \pm 100$, $a = 0$, Random Graph, arithmetic average of 10 tests

The last test serries are the tests with $m < n$, so the graph is disconnected which massively improves the preprocessing time and the approximation errors are rather low, as all oracles can tell if the path between two vertices exists or not.

The oracles behave rather similarly, however oracle **PatRod** does seem to work faster on graphs where balls are usually larger.

6.4 Star-Like Graphs

It rarely happens that some clusters become sizeable on random graphs. This causes **Basic** and **BaGoSe** to have almost always identical set A_1.

Consider a vertex u that has a large cluster $C_{A_1}(u)$ − a cluster of size greater than q/p, for a large enough constant q. u is contained in a lot of balls, however balls are unlikely to be large as stated by Theorem 3.10, so radius of ball u is smaller than most of the radii of balls from $C_{A_1}(u)$. Balls from the C_{A_1} despite their usually larger radii, rarely contain each other as their size is unlikely to be large. This hints that graph around u should be more dense than around most of vertices from its cluster.

In order to generate graphs where large clusters are likely, we decided to split the vertices into two sets − the dense core and sparse corona. To simplify analysis the graph will be bipartite with edges only occurring between core and corona, and the core will have size $1/p$, so the sample S of size pn will have reasonable chance of missing the core, resulting in $\sum_{v\in V} |C_S(v)|^2 = \Omega(n^2/p)$ preprocessing time and $\Omega(n^2)$ memory if our star graph have a lot of edges.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.2417	2.0036	0.0570	0.0618
Avg. Query Time	1.3576e-07	4.7731e-07	1.4782e-07	1.5623e-07
$\|A_1\|$	1995.8	159.05	171.3	181.45
$n\|A_1\|$	3.9855e+06	317550	342000	362224
$\sum_{v\in V} \|B_{A_1}(v)\|$	0	5542.25	1824.5	1914.5
$\|\mathcal{O}\|$	0	1.2664e+06	1824.5	8401.6
Space PHFs B_{A_1}	6.4116e+06	6.4123e+06	6.4121e+06	6.4122e+06
Space PHFs $\mathcal{O}$	3212.8	4.2605e+06	13580.8	46821.6
Average Error	0	0.1530	0.1196	0.1270
Worst Error (%)	0	0.0354	0	0.0205

Table 6.6: $n = 2000 \pm 100$, $a = 0.3$, Star-Like, arithmetic average of 20 tests

On Star-Like graphs the **Basic** algorithm performs much worse than even simple **BFS**. While the **BaGoSe** slightly outperforms the **PatRod** as the set of vertices with clusters with size > 1 is rather small and those vertices are usually added to the **BaGoSe** sample.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.0972	0.3266	0.0440	0.0622
Avg. Query Time	1.2391e-07	3.0933e-07	1.2971e-07	1.6602e-07
$\|A_1\|$	1984.05	158.4	170.1	177.2
$n\|A_1\|$	3.9389e+06	314404	337636	351724
$\sum_{v\in V} \|B_{A_1}(v)\|$	0	3316.45	1836	2154.65
$\|\mathcal{O}\|$	0	215708	2685.1	25307.5
Space PHFs B_{A_1}	6.3737e+06	6.3744e+06	6.3743e+06	6.3746e+06
Space PHFs $\mathcal{O}$	3216	1.0054e+06	18024.8	131357
Average Error	0	0.0960	0.0095	0.0258
Worst Error (%)	0	3.1012e-05	2.1535e-06	0.0128

Table 6.7: $n = 2000 \pm 100$, $a = 0.15$, Star-Like, arithmetic average of 10 tests

6.5 Improving the Star-Like Graphs Core

The Star-Like graphs from the previous section have one important flaw − when a vertex from the core is added to the sample most of the balls it is contained in will have radius equal to 1 and thus size equal to 1. This however can be fixed by slightly modifying the construction.

We present the graph for which **sample**$(V, n^{-1/3})$ requires $\Omega(\log n)$ expected number of steps.

We first split the graph into $np/(2q)$ disjoint parts $G_1, G_2, \ldots, G_{np/(2q)}$, where $p = n^{-1/3}$ and $q \geq 8$ is a constant. Each of the parts is constructed as follows:

- Let $D_1, D_2, \ldots, D_{p^{-1/3}}$ be the disjoint sets of vertices of size $1/pp^{1/3}$. We will call them the (dense) core.
- Let $S_1, S_2, \ldots, S_{p^{-1/3}}$ be the disjoint sets of vertices of size $q/pp^{1/3}$. We will call them the (sparse) corona.
- Now for every $i, j \in \{1, 2, \ldots p^{-1/3}\}$ we will connect all vertices of D_i to w_1, all vertices of S_j to $w_{p^{1/3}+i}$ and every vertex w_k to w_{k+1} for $1 \leq k < p^{-1/3} + i$, where $w_1, w_2, \ldots, w_{p^{-1/3}+i}$ is a set of vertices $J_{(i,j)}$ called joint of size at most $2p^{-1/3}$.

All sets $D_i, S_j, J_{(i,j)}$ are disjoint and their combined size totals to at most $(q+3)/p < 2q/p$.

We now estimate the lower bound on expected number of **sample**(V, p) steps. For simplicity we will consider only the clusters of core vertices.

For a part to contain no large clusters ($> 4/p$) one of the following conditions must be satisfied:

1. ψ_1 there is a vertex u that is in the sample and belongs to $D_1 \cup_{j=1}^{p^{-1/3}} J_{(1,j)}$.

2. ψ_2 there is less than $4/qp^{-1/3}$ sets $S_j \cup_{i=1}^{p^{-1/3}} J_{(i,j)}$ that do not have any of the vertices in the sample.

if neither of the above holds then all vertices from sets S_j from the ψ_2 condition contain D_1 resulting in clusters from D_1 being $\geq 4/p + p^{-1/3} + 1$.

What is the expected number $\mathbf{E}[t]$ of vertices from a given G_k, $1 \leq k \leq np/(2q)$ that will get added to the sample? For the ψ_1 to occur the set of size $3/p^{2/3}$ must be hit. However the chance for next sampled vertex to hit this set is less than: (assuming no ψ_2 have occurred)

$$\frac{3/p^{2/3}}{l/p^{2/3}}$$

where $1 \leq l \leq p^{-1/3}$ is the largest integer such that $\cup_{i=1}^{l}(D_1 \cup_{j=1}^{p^{-1/3}} J_{(1,j)})$ does not contain any sampled vertex. For the ψ_2 to occur t must be not less than $1/2p^{-1/3}$ as $q \geq 8$.

For every x we will estimate the upper bound $\phi(x)$ on the number of core sets $D_1, D_2, \ldots D_{\phi(x)}$ where every vertex is a large cluster, needed for $\mathbf{E}[t]$ to be at least x on condition that $t \leq 2x$. If $2x < 1/2p^{-1/3}$ then ψ_2 does not hold, so we will assume that ψ_2 is false.

Note that for the sets $D_1, D_2, \ldots D_{\phi(x)}$ to be large it is sufficient that there is no vertices from $\cup_{i=1}^{\phi(x)}(D_i \cup_{j=1}^{p^{-1/3}} J_{(i,j)})$ in the sample, and ψ_2 is false.

The estimate will be calculated by induction Suppose that $\phi(x-0.5) = a$, then $\phi(x) = 4a$, as adding vertex to the sample have at most $a3/(a3 + (4-1)a) \leq 1/2$ chance of hitting the $D_1, D_2, \ldots, D_a$ or their joints, which results in $\mathbf{E}[t] \geq x - 0.5$ on condition $t \leq 2x - 1$ and when those sets were not hit the $\mathbf{E}[t] \geq x - 0.5 + 1$ on condition $t \leq 2x$ as we have at least a core sets with large clusters. This combined gives $\mathbf{E}[t] \geq x$ on condition $t \leq 2x$.

$\phi(1) = 1$ even when $t \leq 1 < 2$ as we need at least one additional vertex in sample to satisfy ψ_1. The largest x such that $\phi(x) \leq p^{-1/3}$ is $\lfloor(\rfloor \log_{16} p^{-1/3}) + 1$.

This concludes that expected size of the sample from $\mathbf{sample}(V, p)$ is not less than $np/(2q)\log_2 p^{-1/3}/4 = np/q(\log_2 n)/72 = \Omega(np\frac{\log n}{q})$. As in each step the sample from $\mathbf{sample}(V, p)$ grows by $\lceil(\rceil pn)$ we have $\Omega(np\log n)$ expected steps for small constant q (like $q = 8$).

The **BaGoSe** memory and expected preprocessing time log factors therefore are tight − there are graphs where **BaGoSe** takes $\Theta(n^{5/3} \log n)$ memory and $\Theta(mn^{2/3} \log n)$ expected preprocessing time.

6.6 Conclusion

The both oracles – **BaGoSe** and **PatRod** – are fast, achieving even subquadratic expected preprocessing time on sparse graphs. The distance approximation errors seems to be low on sparse graphs, as expected due to similar construction of $(2,0)-$distance oracle with $O(n^{4/3}m^{1/3})$ expected memory used [8].

The **PatRod** seems to perform the best on difficult graphs as expected, however the **Basic** outperforms the **PatRod** memory wise on most graphs (dense or not). As calculating balls and clusters is rather fast **BaGoSe** performs about as fast as **Basic** and takes the same memory on most graphs, while only increasing the A_1 sample when worst case scenario is more likely to happen.

This points to the possibility of combining those programs: we take the sample as in **Basic**, we can calculate the $\sum_{v\in V}|C_{A_1}(v)|^2$ in $O(m+n/p^2)$ expected time for unweighted graphs. If this value is too large (say larger than n/p^2) then we expand the A_1 sample like in **PatRod**.

This combined program will behave like **PatRod** on difficult graphs with negligible preprocessing time increase, while behaving **Basic** on easier graphs, where **PatRod** unlikely to be better especially memory wise.

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