Experimental Comparison of (2,1)-Distance Oracles

(Eksperymentalne porównanie
wyroczni odległości (2,1))

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Abstract

Let $G = (V, E)$ be an undirected unweighted graph with $|V| = n$ vertices and $|E| = m$ edges. Let $\delta(u, v)$ denote the distance between vertices $u, v \in V$. An $(\alpha, \beta) -$ *approximate* distance oracle for G is a data structure that supports the following distance queries between pair of vertices in G: Given two vertices u, v it can in constant time compute a distance estimate $\hat{\delta}(u, v)$ that satisfies

$$\delta(u, v) \leq \hat{\delta}(u, v) \leq \alpha\delta(u,v) + \beta$$

What is remarkable in this structure is that it can take significantly less than quadratic space, but still answer distance queries in constant time, despite quadratic space structure required to retrieve exact distances. The $(3, 0)$ distance oracle is the most precise oracle that have $\beta = 0$ and takes only $O(n^{2-\epsilon})$ memory $-$ $O(n^{3/2})$ to be exact. By using the additive error $-$ $\beta > 0$ we are able to decrease α to 2 preserving the $O(n^{2-\epsilon})$ memory $-$ $(2, 1)$ distance oracle takes only $O(n^{5/3})$ memory. Unfortunately the Oracle with additive error no longer works on weighted graphs, only on unweighted. In this paper we will compare $(2, 1)$ distance oracles on different graph sets, looking at their memory use, query time and average error. We will also compare the preprocessing time.

Niech $G = (V, E)$ będzie nieskierowanym, nieważonym grafem, gdzie $n = |V|$ jest liczbą wierzchołków, a $m = |E|$ liczbą krawędzi. Oznaczmy jako $\delta(u, v)$ odległość pomiędzy wierzchołkami $u, v \in V$. Wyrocznią odległości (α, β)-przybliżoną grafu G, gdzie $\alpha \geq 1$, $\beta \geq 0$ nazwiemy strukturę danych, która umożliwia następujące zapytania o odległości pomiędzy parami wierzchołków G: Mając wierzchołki u, v możemy w stałym czasie policzyć przybliżoną odległość $\hat{\delta}(u, v)$, która spełnia

$$\delta(u, v) \leq \hat{\delta}(u, v) \leq \alpha\delta(u,v) + \beta$$

To co jest w tej strukturze niezwykłe to to, że może ona zajmować znacznie mniej niż kwadratowo wiele pamięci ale odpowiedzi na zapytania, których wszystkich możliwych jest kwadratowo wiele, pozostają w czasie stałym.
Wyrocznia odległości $(3, 0)$ jest najdokładniejszą wyrocznią, która ma $\beta = 0$ i dodatkowo zajmuje tylko $O(n^{2-\epsilon})$ pamięci $-$ dokładniej $O(n^{3/2})$. Jednak używając błędu addytywnego $-$ $\beta > 0$ jesteśmy w stanie zmniejszyć α do 2 zachowując pamięć $O(n^{2-\epsilon})$ $-$ wyrocznia odległości $(2, 1)$ zajmuje jedynie $O(n^{5/3})$ pamięci. Niestety wyrocznia z błędem addytywnym nie działa na grafy ważone, jedynie na nieważone. W poniższej pracy porównamy na różnych zestawach grafów kilka wyroczni odległości (2,1) pod względem ich zajmowanej pamięci, szybkości działania i średniego błędu. Porównamy również czas jaki zajmuje stworzenie wyroczni.

Contents

Chapter 1

Introduction

In chapter 2 we will discuss why $(3,0)$ and $(2,1)$ distance oracles have minimum possible error allowing for subquadratic space structure. Next, in chapter 3 we will present the basic $(3,0)$ distance oracle by Thorup and Zwick [3] chapter 4 will be dedicated to how the $(3,0)$ multiplicative error barrier was breached by adding the additive error $-$ $(2,1)$ distance oracle, sacrificing weighted graphs and some memory in the process. The next two chapters will be dedicated to the different variations of the $(2,1)$ distance oracle, in the chapter 5 we will discuss the theoretical differences, while in the chapter 6 we will show the results of the tests on different graph sets.

Definition 1.1. Surplus t estimate for $t > 0$ is any estimate $\hat{\psi}$ of a value ψ that satisfies $\psi \leq \hat{\psi} \leq \psi + t$ It is also called the additive error estimate.

Definition 1.2. Stretch t estimate for $t > 1$ is any estimate $\hat{\psi}$ of a value ψ that satisfies $\psi \leq \hat{\psi} \leq t\psi$. It is also called the multiplicative error estimate.

Definition 1.3. (α, β)-approximation for $\alpha \geq 1, \beta \geq 0$ is any estimate $\hat{\psi}$ of a value ψ that satisfies $\psi \leq \hat{\psi} \leq \alpha\psi + \beta$. (α, β)-approximation have additive and multiplicative errors combined, when $\alpha > 1, \beta > 0$.

Usually when $\beta > 0$ the graph is assumed to be unweighted or β depends on the largest edge.

The $(\alpha, \beta) -$ *approximate* distance oracle or in short (α, β) distance oracle returns stretch (α, β) estimates.

Definition 1.4. Single-Source Shortest Path or SSSP Problem consists of finding the distances between a given vertex v and all other vertices in the graph.

Definition 1.5. All-Pairs Shortest Path or APSP Problem consists of finding the distances between all pairs of vertices in the graph.

Definition 1.6. All-Pairs *Approximate* Shortest Path or APASP Problem consists of finding approximate distances between all pairs of vertices in the graph.

Chapter 2

Theoretical Limitations

2.1 Space Lower Bound

Definition 2.1. A simple cycle is a path that have distinct edges and only the first and the last of its vertices are equal.

Definition 2.2. The girth of the graph is the length of it's shortest simple cycle.

Theorem 2.3. *A graph with at least* $\frac{1}{2} * n^{1+1/k}$ *edges have girth not greater than* $2k$

Proof. We will prove the lemma for $k = 2$. Full proof can be found in Alon *et al*. [1].
Suppose that there exist graph G with $\frac{1}{2} * n^{1+1/k}$ or more edges, that have girth at least $2k + 1$. In other words G does not have a simple cycle of length $2k$ or less.
The vertex u have degree $deg(u) = a$ when number of edges incident to u plus number of edges connecting u with itself (loops) amounts to a [1].
Let $nei_a(v) = \{u \in V \mid \delta(v, u) = a\}$
For $2 \leq i \leq k$ we show that

$$|nei_i(v)| = \sum_{u \in nei_{i-1}(v)} (deg(u) - 1)$$

as
1. Clearly every vertex u from set $nei_{i-1}(v)$ have exactly one edge (u, l) such that $\delta(v, l) \leq i - 1$
and
2. Clearly there is no two distinct edges (u, l), (w, l) satisfying $u, w \in nei_{i-1}(v)$, $l \in V$

[1] Every loop is also incident to its vertex

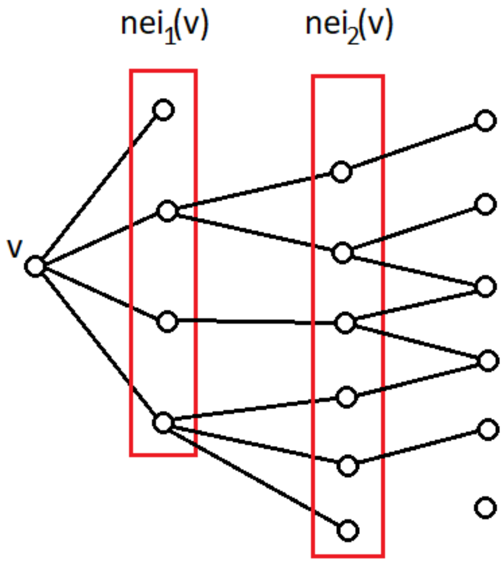


Figure 2.1: Placeholder for $|nei_2(v)|$

We note that $u \in nei_i(v) \iff v \in nei_i(u)$, so for $2 \le i \le k$ we have:

$$\sum_{v \in V} |nei_i(v)| = \sum_{v \in V} \sum_{u \in nei_{i-1}(v)} (deg(u) - 1) =$$

$$= \sum_{u \in V} \sum_{v \in nei_{i-1}(u)} (deg(u) - 1)$$

$$= \sum_{u \in V} ((|nei_{i-1}(u)|) * (deg(u) - 1))$$

For every $v \in V$ the following is true:

$$\sum_{i=0}^{k} |nei_i(v)| = |\{u \in V \mid \delta(v, u) \le k\}| \le |V| = n \tag{2.1}$$

However, if we look at the sum:

$$\sum_{v \in V} \sum_{i=0}^{k} |nei_i(v)| = \sum_{v \in V} (1 + |nei_1(v)| + \sum_{i=2}^{k} |nei_i(v)|) =$$

$$= n + \sum_{v \in V} (|nei_1(v)|) + \sum_{v \in V} \sum_{i=2}^{k} |nei_i(v)| =$$

$$= n + \sum_{v \in V} (deg(v)) + \sum_{v \in V} \sum_{i=2}^{k} \sum_{u \in nei_{i-1}(v)} (deg(u) - 1) = \quad (\text{for } k = 2)$$

$$= n + \sum_{v \in V} (deg(v)) + \sum_{v \in V} \sum_{u \in nei_1(v)} (deg(u) - 1) =$$

$$= n + \sum_{v \in V} (deg(v)) + \sum_{u \in V} ((|nei_1(u)|) * (deg(u) - 1)) =$$

$$= n + \sum_{v \in V} (deg(v)) + \sum_{u \in V} ((deg(u)) * (deg(u) - 1)) =$$

$$= n + \sum_{u \in V} ((deg(u)) * (deg(u))) \geq n + n * (avg\ deg)^2 =$$

$$= n + n * (n^{1/2})^2 = n * (n + 1) > n^2$$

where $avg\ deg = \frac{2m}{n}$ is the average degree of a vertex.

This contradicts 2.1 □

It is conjectured by Erdös [2] and others that the Theorem 2.3 is tight. Namely, it is conjectured that for any $k \geq 1$, that there are graphs with $\Omega(n^{1+1/k})$ edges and girth greater than $2k$. The conjecture is known to be true for $k = 1, 2, 3, 5$

Observation 2.4. For any graph of girth $2k + 1$ it's bipartisan subgraphs have girth $2k+2$ or greater as simple cycles in bipartisan graphs have even length and subgraphs can't have smaller girth.

Since graphs with girth greater than 1 do not have loops, they have a bipartisan subgraph with at least half the edges, so there are graphs with at least $\Omega(n^{1+1/k})$ edges and girth greater than $2k + 1$.

Lemma 2.5. *An (α, β) distance oracle must use at least m bits of storage for girth $k > \alpha + \beta + 1$ graphs, where m is a number of edges.*

Proof. Let G be a girth $k > \alpha + \beta + 1$ graph with n vertices and m edges. Let H be any subgraph of G and $\delta_H(u, v)$ be a distance between vertices u, v in H.

Consider any edge (u, v) from G. If (u, v) is in H then $\delta_H(u, v) = 1$, but otherwise $\delta_H(u, v) \geq k - 1$, since the girth of H is not less than k. The oracle will report distance between u and v at most $1 * \alpha + \beta$, if edge (u, v) in H, but not less than $k - 1$ otherwise.

Consequently, the generated structure must be different for subgraphs of G with different edge sets. There is 2^m subgraphs of G with different edge sets, so for at least one of them the generated oracle must take m bits of space. □

Observation 2.6. An (α, β) distance oracle must take at least $\Omega(n^{1+1/k})$ memory, where $k = \lceil (\alpha + \beta + 1)/2 \rceil$, if the 1963 Erdös girth conjecture is true.

Observation 2.7. This points to the space lower bound of $\Omega(n^{3/2})$ for $(3, 0)$ and $(2, 1)$ distance oracles. The bound for $(3, 0)$ distance oracle is tight up to $\Theta(k\ log\ n)$ due to Oracle by Thorup and Zwick [3], described in this paper. Additionally any oracle with $\alpha + \beta < 3$ must take at least $\lceil \frac{n*(n-1)}{4} \rceil$ bits of storage for some graphs as there exists girth 4 graph with that many edges.

Conjecture 2.8. *[7] Consider a data structure that preprocesses sets $S_1, \ldots, S_n \subseteq [X]$, and answers queries of the form "does S_i intersect S_j?". Let $X = lg^c n$ for a large enough constant c. If the query takes constant time, the space must be $\Omega(n^2/$ poly log $n)$.*

Theorem 2.9. *[7] A distance oracle for undirected, unweighted graphs with $m = O(n$ poly log $n)$ edges, which can distinguish between distances of 2 and 4 in constant time requires $O(n^2$ poly log $n)$ space assuming conjecture 2.8*

Proof. Build a bipartite graph, with n vertices on the left, numbered from 1 to n and X on the right, numbered 1 to X. The number of vertices is $n + X = O(n)$. Connect left vertex i to the elements of S_i on the right. The number of edges is no more than $nX = O(n*$ poly log $n)$. Two left vertices are at distance 2 if the corresponding sets intersect, and distance at least 4 otherwise. Thus, the distance oracle can solve set intersection queries. □

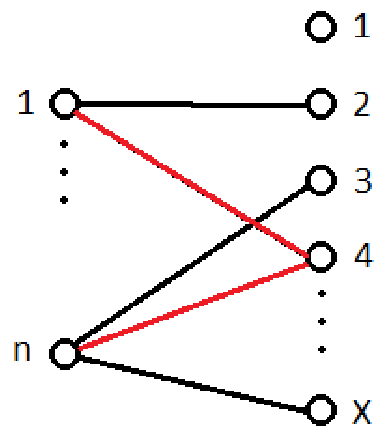


Figure 2.2: Placeholder for Theorem 2.9

Observation 2.10. For any graph we can insert $c-1$ vertices on each edge, splitting the edge into c edges, so that each pair of the original vertices will have their distance multiplied by c, where c is a positive integer. The number of vertices in the new graph will be $n + (c-1)m = O(n+m)$, number of edges will be $cm = O(m)$.

In consequence any Oracle that can distinguish between distances $2c$ and $4c$ in constant time must use $\Omega(n'^2/$ poly log $n') = \Omega(n^2/$ poly log $n)$ space, where $n = O(c * (n' *$ poly log $n'))$, so $n' = O(n/$ poly log $n)$.

An (α, β) distance oracle must take $O(n^2/$ poly log $n)$ space if $\alpha < 2$, assuming conjecture 2.8

2.2 Preprocessing Time Lower Bound

The stretch 2 or constant surplus distance oracle is smallest stretch oracle that possibly could calculate all distance estimates in less time than boolean matrix multiplication. This is because any algorithm that computes all-pairs approximate distances with stretch less than 2 (or surplus only) could be used to compute boolean matrix multiplication [14].

For directed graphs any constant stretch could be used to compute boolean matrix multiplication (also in [14])

$m^{5/3-o(1)}$ time is required for a stretch $(2 + o(1))$ oracle based on 3-SUM conjecture[15]

Chapter 3

(3,0) Distance Oracle

We present the slightly modified $(2k-1)-$ *approximate* distance oracle of Thorup and Zwick[3]

Theorem 3.1. *Let $G = (V, E)$ be an undirected weighted graph with non-negative weights and $|V| = n$ vertices and $|E| = m$ edges. Let $\delta(u,v)$ denote the distance between vertices $u, v \in V$. Let $k \geq 1$ be an integer. The graph can be preprocessed in $O(nm+kn^2)$ expected time[1] in order to obtain data structure of size $O(kn^{1+1/k})$ such that any distance approximation $\hat{\delta}(u,v)$ satisfying $\delta(u,v) \leq \hat{\delta}(u,v) \leq (2k-1)\delta(u,v)$ can be retrieved in $O(k)$ time.*

Definition 3.2. A distance between sets $X, Y \subseteq V$ denoted $\delta(X,Y)$ is equal to the distance between closest vertices in those sets.
More precisely $\delta(X,Y) = \liminf\{\delta(u,v) \mid u \in X, v \in Y\}$
A distance between vertex v and set $X \subseteq V$ denoted $\delta(v,X)$ or $\delta(X,v)$ is equal to $\delta(\{v\},X)$ or $\delta(X,\{v\})$ respectively.

Definition 3.3. A ball $B_X(v)$ for a given set $X \subseteq V$ and a vertex $v \in V$ is a set of vertices $\{u \in V \mid \delta(v,u) < \delta(v,X)\}$. We call $\delta(v,X)$ a radius of ball $B_X(v)$

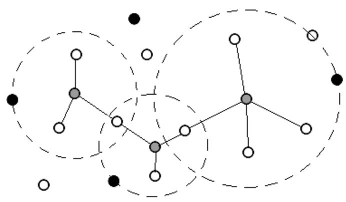
Definition 3.4. A cluster $C_X(v)$ is an inverse of ball $u \in C_X(v) \iff v \in B_X(u)$ or in other words $C_X(v) = \{u \in V \mid \delta(u,v) < \delta(u,X)\}$

Let $V = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_k = \emptyset$ be a non increasing sequence of sets.

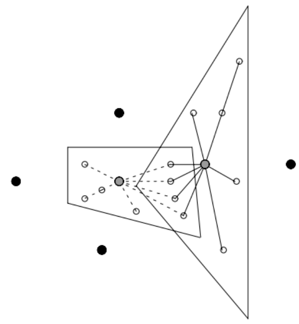
Definition 3.5. A portal $p_{A_i}(v) \in A_i$ is a vertex that satisfies $\delta(v, p_{A_i}(v)) = \delta(v, A_i)$. Ties are broken arbitrarily.

Definition 3.6. A bunch $\mathrm{bun}(v) = \bigcup_{i=0}^{k-1}\{u \in B_{A_{i+1}}(v) \cap A_i\}$
or more verbosely $\mathrm{bun}(v) = \bigcup_{i=0}^{k-1}\{u \in A_i \setminus A_{i+1} \mid \delta(v,u) < \delta(v, A_{i+1})\}$

Definition 3.7. A clump $\mathrm{clum}(v) = \bigcup_{i=0}^{k-1}\{u \in C_{A_{i+1}}(v) \cap A_i\}$
or more verbosely $\mathrm{clum}(v) = \bigcup_{i=0}^{k-1}\{u \in A_i \setminus A_{i+1} \mid \delta(u,v) < \delta(u, A_{i+1})\}$



(a) Placeholder for ball



(b) Placeholder for cluster

algorithm $\mathbf{basicprepro}_k(G)$
calculate $\delta(u,v)$ for every $u,v \in V$
$A_0 \leftarrow V; A_k \leftarrow \emptyset$
for $i \leftarrow 1$ to $k-1$
  let A_i be an uniform sample from A_{i-1} of size $\lceil n^{-1/k} * |A_{i-1}| \rceil$
for every $v \in V$
  for $i \leftarrow 0$ to $k-1$
    calculate $\delta(v, A_i)$ and $p_{A_i}(v)$
  calculate $\text{bun}(v)$
return structure that for every $v, w \in V$, $0 \le i \le k-1$ can retrieve $p_{A_i}(v)$,
  $\delta(v,u)$ for $u \in \text{bun}(v)$ and can determine if $w \in \text{bun}(v)$, all in constant time

algorithm $\mathbf{basicdist}_k(u,v)$
$w \leftarrow p_0(u); i \leftarrow 0$
while $w \notin \text{bun}(v)$
  $i \leftarrow i+1$
  $(u,v) \leftarrow (v,u)$
  $w \leftarrow p_{A_i}(u)$
return $\delta(u,w) + \delta(w,v)$

As $n^{-1/k} > 0$ the sample size is greater than 0, so $A_i \neq \emptyset$, for $0 \le i < k$.

3.1 Correctness

The algorithm $\mathbf{basicdist}_k(u,v)$ always terminates in at most $k-1$ steps, since $p_{k-1}(u) \in A_{k-1} \subseteq \text{bun}(v)$ for every $u, v \in V$. This means query time $O(k)$.

Lemma 3.8. *$\delta(u, p_{A_i}(u))$ is at most $i * \delta(u,v)$ for as long as the* $\mathbf{basicdist}_k(u,v)$ *did not terminate in $i-1$ steps.*

[1] by skipping All-Pairs Shortest Path preprocessing and preprocessing the graph directly it is possible to create the oracle in $O(n^2)$ expected time [4] or $O(mn^{1/k})$ deterministic time [5]

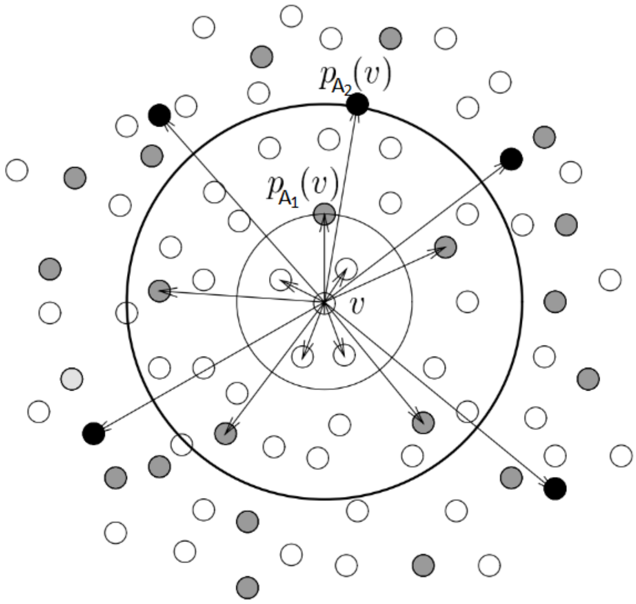


Figure 3.2: Placeholder for bunch

Proof. The proof is by induction:
If the program terminates immediately then $\delta(u, p_0(u)) = \delta(u, u) = 0$.
If the program did not terminate in $i-1$ steps then
$p_{i-1}(v) \notin \text{bun}(u)$, so $\delta(u, p_{i-1}(v)) \geq \delta(u, A_i) = \delta(u, p_{A_i}(u))$ and
$\delta(p_{i-1}(v), v) \leq (i-1)\delta(u, v)$, this combined gives
$\delta(u, p_{A_i}(u)) \leq \delta(u, p_{i-1}(v)) \leq \delta(p_{i-1}(v), v) + \delta(v, u) \leq i * \delta(u, v)$ □

The distance returned by $\mathbf{basicdist}_k(u, v)$ is at most $(2k-1)\delta(u, v)$ as with the triangle inequality we have
$\delta(u, p_{A_i}(u)) + \delta(p_{A_i}(u), v) \leq 2*\delta(u, p_{A_i}(u)) + \delta(u, v) = (2i+1)\delta(u, v) \leq (2k-1)\delta(u, v)$

3.2 Size of the Structure

For each vertex $v \in V$ we store:

- $p_{A_i}(v)$ for $i = 0 \ldots k-1$ and its corresponding distance $\delta(v, p_{A_i}(v)) = \delta(v, A_i)$
- the hash table for the bunch $\text{bun}(v)$ holding $\delta(v, u)$ for every $u \in \text{bun}(v)$ (the hash table is also able to determine whatever $u \in \text{bun}(v)$)

In order to store balls and distances in constant memory per entry and constant retrieve time we will use hash tables. They can be computed in linear expected time[16] or they can be computed deterministically is $O(s \log s \log n)$ time.[17], where s in number of entries to be stored.

The total size of the data structure is $O(kn + \sum_{v \in V} |\text{bun}(v)|)$

Lemma 3.9. *Let $X \subseteq V$, S be an uniform sample from X of size $\lceil p * |X| \rceil$. Then for $v \in V$ the expected size of $B_S(v) \cap X$ is less than $1/p$.*

Proof. For we show that expected size of $B_S(v) \cap X$ is *stochastically dominated* by a random variable with parameter p. Let $w_1, w_2 \ldots w_l$ be the elements of X arranged in any non-decreasing order of distance from v. If $w_j \in S$ then $w_j, w_{j+1}, \ldots, w_l \notin B_S(v)$ as those elements are not closer to v than S.

If we look at $\Pr[w_j \in S] \geq p$ we see that

$$\Pr[w_j \in B_S(v)] \leq \Pr[\{w_1, w_2, \ldots, w_j\} \cap S = \emptyset] \leq (1-p)^j$$

$$\mathbf{E}[B_S(v) \cap X] = \sum_{j=1}^{l} \Pr[w_j \in B_S(v)] \leq \sum_{j=1}^{l} (1-p)^j < p^{-1}$$

□

Theorem 3.10. *Let $X \subseteq V$, S be an uniform sample from X of size $q * \lceil p * |X| \rceil$, where q is an positive integer. Then for $v \in V$ the expected value of $|B_S(v) \cap X|^q$ is less than $1/p^q$.*

Proof. By Lemma 3.9 we can take q uniform samples $S_1, S_2, \ldots, S_q$ of size $\lceil p * |X| \rceil$, and the expected value of $|B_{S_i}(v) \cap X|$, for $v \in V$, $0 < i \leq q$ will be less than $1/p$. However for a given $v \in V$ the order $w_1, w_2, \ldots, w_l$ could be constant for all samples S_i, which means that it is independent from the samples. Additionally the samples are independent from each other, so the expected values of the upper bounds on $|B_{S_i}(v) \cap X|$ are independent, which gives:

$$\mathbf{E}[|B_S(v) \cap X|^q] \leq \mathbf{E}[|B_{(\cup_{i=1}^q S_i)}(v) \cap X|^q] \leq \mathbf{E}[\Pi_{i=1}^q |B_{S_i}(v) \cap X|] < 1/p^q$$

We can think of $B_{\cup_{i=1}^q S_i}(v)$ as an uniform sample S' from X of size $\leq q * \lceil p * |X| \rceil$. We can then select uniformly elements one by one that are not yet in the sample and add them to sample until we got a sample of size $|S|$. The sample constructed that way will be uniform, and in each step $B_{S'}(v) \cap X$ will not get any new elements. □

Theorem 3.10 shows that balls $B_S(v)$, $v \in V$, S uniform sample from V of size $\lceil (\rceil p * n)$ are unlikely to grow much bigger then $1/p$.
Note that this does not applies to clusters $C_S(v)$, which in turn are unlikely to intersect with each other much more times than $4n/p^2$, as every cluster intersection

means that both vertices that have their clusters intersecting belong to the same ball. We have n balls with expected square of their size (amount of different pairs of vertices in them) being lower than $\sim 1/(p/2)^2$ (= if $p * n$ is even)

Theorem 3.11. *expected size of bun(v), $v \in V$ is at most $kn^{1/k} + k$*

Proof. The size of $A_{k-1} \setminus A_k$ is less than $n^{1/k} + k$, as $A_k = \emptyset$ and by induction

$$(n^{-1/k})^{i-1} * |A_{k-i}| + i - 1 \leq (n^{-1/k})^{i-1} * (n^{-1/k} * |A_{k-(i+1)}| + 1) + i - 1 \leq$$

$$\leq (n^{-1/k})^{i} * |A_{k-(i+1)}| + i \quad \text{(for } 1 \leq i \leq k-1)$$

so $|A_{k-1} \setminus A_k| = |A_{k-1}| = (n^{-1/k})^0 * |A_{k-1}| \leq (n^{-1/k})^{k-1} * |A_0| + k - 1 = n^{1/k} + k - 1$

For $0 \leq i < k-1$ we show that expected size of bun$(v) \cap A_i$ for $0 \leq i < k-1$ is by Lemma 3.9 at most $n^{1/k}$. Indeed, A_{i+1} is an uniform sample from A_i of size $\lceil p * |A_i| \rceil$ for $p = n^{-1/k}$. Since no element of $B_{A_j}(v)$ is in A_i for $0 \leq j \leq i$ and no element of A_j is in $B_{A_{i+1}}(v)$ for $i+1 \leq j \leq k-1$, sets $B_{A_{i+1}}(v) \cap A_i$ for $0 \leq i \leq k-1$ are disjoint, so bun$(v) \cap A_i = B_{A_{i+1}}(v) \cap A_i \leq 1/p = n^{1/k}$.

This together with $V = \cup_{i=0}^{k-1}(A_i \setminus A_{i+1})$, as $V = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_k = \emptyset$ and $|A_{k-1} \setminus A_k| < n^{1/k} + k$ completes the proof. □

The expected size of the structure is $O(kn + n * (kn^{1/k} + k)) = O(kn^{1+1/k})$ We can get data structure of deterministic size $O(kn^{1+1/k})$ by re-running the algorithm until data structure produced is small enough. By Markov's inequality the expected number of repetitions required is constant, so this will not affect the expected running time of the algorithm.

3.3 Preprocessing Memory and Time

The calculation of all distances takes $O(nm+n \log n)$ time and $O(n^2)$ memory when using Dijkstra algorithm with Fibonacci heaps. (see Cormen et al. [[11], Chapter 21]) The time could be further improved to $O(nm)$ by using efficient Thorup algorithm for undirected graphs with floating point edge weights[12][13]

The creation of A_i sets takes $O(kn)$ memory and time.

The rest of the **basicprepro**$_k(G)$ takes $O(kn^2)$ expected time and only $O(kn^{1+1/k})$ deterministic memory (program is re-run if memory gets too high) as all calculated values are stored in Oracle.

Definition 3.12. auxiliary memory is the memory used by program without without counting the input, which can be read, but not written to.

basicprepro$_k(G)$ takes $O(nm + kn^2)$ expected time and $O(n^2)$ memory in total. However this approach is very inefficient due to calculation of all distances.

By cleverly calculating only needed distances, the running time could be reduced to $O(n^2)$ expected time and $O(kn^{1+1/k})$ auxiliary memory[4]. By using deterministic A_i sets creation methods, we can achieve the $O(mn^{1/k})$ deterministic time and $O(kn^{1+1/k})$ auxiliary memory[5]

Chapter 4

(2,1) Distance Oracle

We present the $(2,1)-$ *approximate* distance oracle of Surender Baswana, Vishrut Goyal, and Sandeep Sen.[6], modified to work on weighted graphs.

Theorem 4.1. *Let $G = (V,\ E)$ be an undirected weighted graph with non-negative weights and $|V| = n$ vertices and $|E| = m$ edges. Let $\delta(u,v)$ denote the distance between vertices $u, v \in V$. The graph can be preprocessed in $O(n^{7/3}\ \log n + mn^{2/3}$ $\log n)$ expected time in order to obtain data structure of size $O(n^{5/3}\ \log n)$ such that any distance approximation $\hat{\delta}(u,v)$ satisfying $\delta(u,v) \le \hat{\delta}(u,v) \le 2\delta(u,v) + h$, where h is the weight of the heaviest edge, can be retrieved in $O(1)$ time.*

```
algorithm prepro(G)
A_0 ← V; A_2 ← ∅
A_1 ← sample(A_0, p)
for every u ∈ A_1
    for every v ∈ V
        calculate δ(u, v)
for every v ∈ V
    calculate p_{A_1}(v), B_{A_1}(v) and C_{A_1}(v)
    for every u ∈ B_{A_1}(v)
        calculate δ(v, u)
    for every pair (u, w) ∈ C_{A_1}(v) × C_{A_1}(v)
        O(u, w) ← v
```

```
algorithm dist(u, v)
if u ∈ B_{A_1}(v) or v ∈ B_{A_1}(u)
     return δ(u, v)
else if B_{A_1}(u) ∩ B_{A_1}(v) ≠ ∅
    return δ(u, O(u, v)) + δ(O(u, v), v)
else
    return minimum of δ(u, p_{A_1}(u)) + δ(p_{A_1}(u), v) and δ(u, p_{A_1}(v)) + δ(p_{A_1}(v), v)
```

The biggest difference between this oracle and the $(3, 0)$ distance oracle is that here we keep track whatever the balls of the any two vertices intersect. In order to do that we had to modify the construction of the A_i sets in order to ensure that not too many balls will intersect with each other.

4.1 Correctness

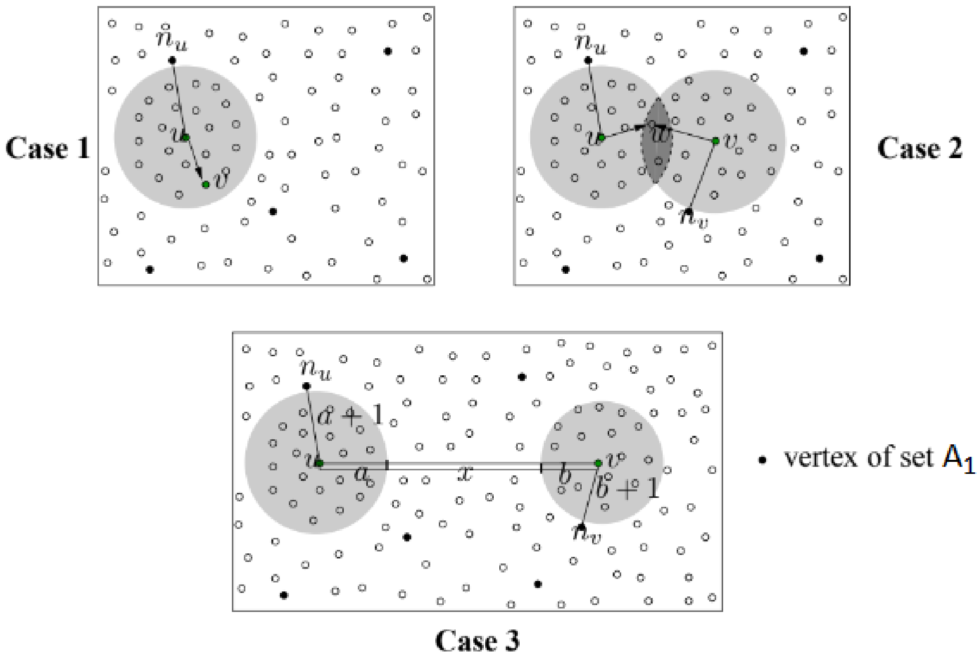


Figure 4.1: Placeholder for **dist**(u, v)

Lemma 4.2. *The distance returned by* **dist**(u, v) *is not less than* $\delta(u, v)$ *and at most* $2 * \delta(u, v) + h$.

Proof. We will go through each of cases

1. <u>$u \in B_{A_1}(v)$ or $v \in B_{A_1}(u)$:</u>
   Our structure stores the $B_{A_1}(w)$, $\delta(w, q)$ for $w \in V$, $q \in B_{A_1}(w)$. Exact distance is returned.

2. $\underline{B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset}$:
   Since last case did not occur, $u \notin B_{A_1}(v)$ and $v \notin B_{A_1}(u)$ we have $\delta(u,v) \geq \delta(u,A_1)$ and $\delta(u,v) \geq \delta(v,A_1)$. Let $w \in B_{A_1}(u) \cap B_{A_1}(v)$, we see that $\delta(u,w) < \delta(u,A_1)$, $\delta(v,w) < \delta(v,A_1)$, so $\delta(u,w) + \delta(w,v) < \delta(u,A_1) + \delta(v,A_1) \leq \delta(u,v) + \delta(u,v)$ $\delta(u,w)$, $\delta(w,v)$ are stored in our structure.
   Note that for $u,v,w \in V$ we have

$$w \in B_{A_1}(u) \cap B_{A_1}(v) \implies u,v \in C_{A_1}(w) \implies \mathcal{O}(u,v) \text{ is defined}$$

$$\mathcal{O}(u,v) = w \implies u,v \in C_{A_1}(w) \implies w \in B_{A_1}(u) \cap B_{A_1}(v)$$

   so we can retrieve $w \in B_{A_1}(u) \cap B_{A_1}(v)$ from $\mathcal{O}(u,v)$ or tell that $B_{A_1}(u) \cap B_{A_1}(v) = \emptyset$ otherwise.

   For unweighted graphs we can remember exact $\delta(u,v)$ by choosing $w = \mathcal{O}(u,v)$ such that it is on the shortest path from u to v. Such w exists in unweighted graphs as the shortest path is the one with the least vertices and first $\delta(u,A_1)$ vertices, last $\delta(v,A_1)$ vertices of a path from u to v will be in corresponding balls. This also allows us to skip the first case as
   $u \in B_{A_1}(v) \implies v \in B_{A_1}(v)$ for $v \in V$.

3. <u>None of the above:</u>
   Let $x = \delta(B_{A_1}(u), B_{A_1}(v))$, $a = \delta(u,A_1)$, $b = \delta(v,A_1)$. Since $B_{A_1}(u) \cap B_{A_1}(v) = \emptyset$ we have $x > 0$. Moreover $a + b - h \leq \delta(u,v) < a + b + x$ as any vertices $w \in B_{A_1}(u)$, $q \in B_{A_1}(v)$ realizing $\delta(w,q) = x$ are also satisfying $a \leq \delta(u,q)$, $b - h \leq \delta(q,v)$, $\delta(u,w) < a$ and $\delta(v,q) < b$.
   Without the loss of generality we assume that $a \leq b$. We then have

$$\delta(u,v) \leq \delta(u,p_{A_1}(v)) + \delta(p_{A_1}(v),v) \leq 2 * \delta(u,p_{A_1}(v)) + \delta(u,v) =$$

$$= 2a + \delta(u,v) \leq (a+b-h) + h + \delta(u,v) \leq 2 * \delta(u,v) + h$$

   $p_{A_1}(v) \in A_1$ so distances from $p_{A_1}(v)$ to all other vertices are stored in our structure.

□

Observation 4.3. This Oracle is a stretch 2 Oracle for all cases except for part of case 3. For distance estimate $\hat{\delta}(u,v)$, $u,v \in V$ to be more than $2 * \delta(u,v)$ the following must be true: distance is reported at case 3 and $\delta(u,v) < 2 * \delta(u,p_{A_1}(u))$ as our error is at most $2 * \delta(u,p_{A_1}(u))$, so both of the below must be true:
$\delta(p_{A_1}(u),v) < 3 * \delta(u,p_{A_1}(u))$ and $\delta(p_{A_1}(v),u) < 3 * \delta(v,p_{A_1}(v))$
Those conditions could be easily verified during Oracle query.

We can make even stronger argument, the Oracle is not a stretch 2 only if the balls of the queried vertices have no common vertex, but they are connected by a single edge. This however can not be verified by this Oracle.

4.2 Improved Sampling Algorithm

```
algorithm sample(X, p)
R ← ∅; X' ← X
while X' ≠ ∅
    Let S be an uniform sample from X' of size ⌈p * |X|⌉ or X' if ⌈p * |X|⌉ > |X'|
    R ← R ∪ S
    For every v ∈ X'
        calculate C_R(v) ∩ X
    X' ← {v ∈ X | |C_R(v) ∩ X| > 4/p}
return R
```

Observation 4.4. The algorithm **sample**(X, p) ensures that all $C_R(v)$ clusters have at most $4/p$ vertices, so all $C_{A_1}(v)$ clusters have at most $4/p$ vertices. This ensures that $\mathcal{O}$ have at most $16n/p^2$ defined values.

Lemma 4.5. *In each step of sample* **sample**(X, p)*, the* $|X'|$ *is at least halved with probability at least* $1/2$*.*

Proof. Let X_i be the set X' at the beginning of i-th step. Note that $X_i \supseteq X_{i+1}$ as R never shrinks so clusters C_R never grow larger. If $|X_i| < \lceil p * |X| \rceil$ then $|X_{i+1}| = 0$ and lemma holds. In other case after the set R is augmented by sample S ($R \supseteq S$) we note that

$$\sum_{v \in X_i} |C_R(v) \cap X| = \sum_{v \in X} |B_R(v) \cap X_i| \leq \sum_{v \in X} |B_S(v) \cap X_i|$$

by Lemma 3.9

$$\mathbf{E}[\sum_{v \in X} |B_S(v) \cap X_i|] < |X| * (p * |X|/|X_i|)^{-1} = |X_i|/p$$

By Markov inequality with probability $1/2$ we have $\sum_{v \in X_i} |C_R(v) \cap X| < 2|X_i|/p$, and since in that case

$$|X_{i+1}| * 4/p \leq \sum_{v \in X_i} |C_R(v) \cap X| < 2|X_i|/p$$

so $2 * |X_{i+1}| < |X_i|$ with probability at least $1/2$.

□

So the expected number of steps is at most 2 $\log n$

By combining above Lemma with Lemma 3.9 we can prove the following Lemma

Lemma 4.6. *Given an undirected weighted graph with non-negative weights, a set* X*,* $X \subseteq V$ *and number* $0 < p < 1$*, we can compute a sample set* R *of expected size* $O(p * |X| \ log\ |X|)$ *such that for every* $v \in V$ $C_R(v) \cap X$ *is of size at most* $4/p$ *and expected size of* $B_R(v) \cap X$ *is less than* $1/p$*.*

4.3 Efficient Ball and Cluster Computation

Lemma 4.7. *Given an undirected weighted graph with non-negative weights, number $0 < p < 1$ and set A_1 generated by algorithm* **sample**(V, p)*, we can calculate $B_{A_1}(v)$, $C_{A_1}(v)$ and $\delta(v, u)$ for every $v \in V$, $u \in B_{A_1}(v)$ in $O(m/p + n/p \log n)$ time.*

Proof. We start by calculating the $\delta(v, A_1)$ for every $v \in V$. We can do this by adding to the graph a source vertex s and connecting it to every $u \in A_1$ by edge of weight 0, and then running the Dijkstra algorithm from there. It is not hard to see that this will calculate $\delta(v, A_1)$ for every $v \in V$ in $O(m + n \log n)$ time.

Now for every $v \in V$ we know the $\delta(v, A_1)$, so we can run modified Dijkstra algorithm to calculate $B_{A_1}(v)$ and $\delta(v, u)$ for $u \in B_{A_1}(v)$. We modify Dijkstra algorithm to ignore edges that lead to too far away vertices $-$ vertices $u \in V$ such that $\delta(v, u) \geq \delta(v, A_1)$. Such modified Dijkstra will work in $O(\xi(B_{A_1}(v)) + |B_{A_1}(v)| \log |B_{A_1}(v)|)$ time, where $\xi(X)$, $X \subseteq V$ is the number of edges incident to vertices of X.

Observation 4.8. Note that when $B_{A_1}(v)$ is calculated for every $v \in V$, we can easily calculate $C_{A_1}(v)$ as $u \in B_{A_1}(v) \iff v \in C_{A_1}(u)$, $u \in V$. This will not change the time complexity as $\sum_{v \in V} |B_{A_1}(v)| = \sum_{v \in V} |C_{A_1}(v)|$

The Mikkel Thorup claims that his algorithm can be modified in a similar way (at least for calculating clusters) [3], achieving working time of $O(\xi(B_{A_1}(v)) + |B_{A_1}(v)|) = O(\xi(B_{A_1}(v)))$.

Lets calculate an upper bound on $\sum_{v \in V} \xi(B_{A_1}(v))$:

$$\sum_{v \in V} \xi(B_{A_1}(v)) = \sum_{v \in V} \sum_{u \in B_{A_1}(v)} \xi(u) =$$

$$= \sum_{u \in V} \sum_{v \in C_{A_1}(u)} \xi(u) =$$

$$= \sum_{u \in V} |C_{A_1}(u)| \xi(u) \leq \sum_{u \in V} 4\xi(u)/p \leq 8m/p$$

So the total working time of the above is
$O(m + n \log n + 8m/p + \sum_{v \in V} |B_{A_1}(v)| \log |B_{A_1}(v)|) =$
$= O(m/p + n/p \log n)$

□

The above weighted graphs Ball and Cluster computation requires the clusters to be bounded by their size, without it the bound on computation time will increase.

Observation 4.9. If A_1 would be a set containing uniform sample from V of size $\lceil p * |V| \rceil$ instead, then the expected size of $\sum_{v \in V} \xi(B_{A_1}(v))$ would be bounded by Lemma 3.9 and since $B_X(v) \leq B_Y(v)$ for $X \subseteq Y \subseteq V$, $v \in V$

$$\sum_{u \in V} |C_{A_1}(u)|\xi(u) \leq \sum_{u \in V} |C_{A_1}(u)| * n = n * \sum_{u \in V} |C_{A_1}(u)| = n * \sum_{v \in V} |B_{A_1}(v)| \leq n^2/p$$

Which changes the computation time from deterministic $O(m/p + n/p \log n)$ to expected $O(n^2/p)$.

For undirected graphs however, computation is much simpler and will work for **sample**(V, p)

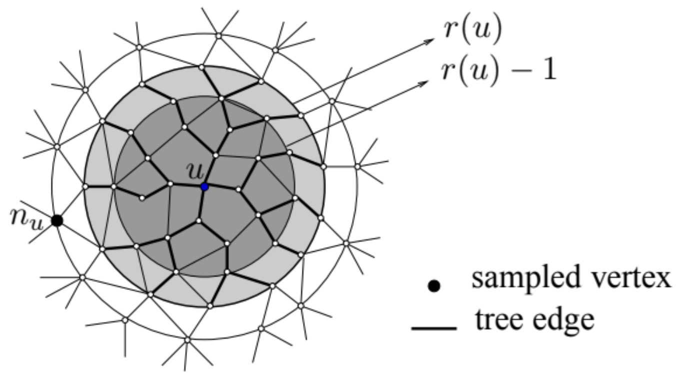


Figure 4.2: Placeholder for unweighted ball computation

Lemma 4.10. *Given an undirected* <u>*unweighted*</u> *graph with no multiple edges, number* $0 < p < 1$ *and uniform sample* S *from* V *of size* $\lceil p * |V| \rceil$ *we can calculate* $B_S(v)$ *and* $C_S(v)$ *for every* $v \in V$ *in* $O(m + n/p^2)$ *expected time.*

Proof. First we compute $\delta(v, S)$ for every $v \in V$. This can be done in $O(m)$ time by a single BFS traversal. We do that by adding a source vertex s that is connected to every $u \in S$ and starting BFS in that vertex. Indeed, if $\delta(v, S) = x$ then $\delta(v, u) = x - 1$ for some $u \in S$ and $\delta(v, w) \geq x - 1$ for every $w \in S$, so $\delta(v, s) = \delta(v, S) + 1$

Now for every $v \in V$ we compute ball $B_S(v)$ using BFS, we immediately halt the edge processing when we reach a vertex $u \in V$ such that $\delta(v, S) - 1 \leq \delta(v, u)$. That way we are guaranteed to visit every $w \in V$ such that $\delta(v, w) \leq \delta(v, S) - 1$, but we will not process any edges from vertices $q \in V$ such that $\delta(v, q) \geq \delta(v, S) - 1$ (Unless the edge was processed by a vertex visited earlier).

Lets count the expected number of edges processed. Let $u \in V$ be any vertex and $v_1, v_2, \ldots, v_n \in V$ be vertices of V arranged in BFS non-decreasing order of distance from u. Each edge will be processed at most twice, from both (or one) of vertices it is connecting. We will count only the first time $-$ when the edge is connecting to the vertex that was not visited (or when edge is a loop). The BFS can process the edges of v_i when both of the below conditions are satisfied (required, but not sufficient):

1. ψ_1^i: There is no vertex in the set $\{v_j | 0 < j < i\}$ which is selected in the sample S.

2. ψ_2^i: There is no vertex in the set $\{v_j | i \le j \le n\}$ that is in S and there is edge between it and v_i.

The chance for the ψ_1^i is no more than $(1-p)^{i-1}$ as the chance for $v_j \in S$ is $\lceil p * |V| \rceil / |V| \ge p$.

If ψ_1^i is true then chance for ψ_2^i is no more than $(1 - \Pr[v_j \in S \text{ for } i < j \le n \mid \psi_1^i])^{\xi'(v_i)}$, where $\xi'(v_i)$ is number of edges connecting v_i with any v_j where $i \le j \le n$.
$\Pr[v_j \in S \text{ for } i < j \le n \mid \psi_1^i] > p$ as ψ_1^i for $i < j \le n$ increase the chance for $v_j \in S$ that was originally $\lceil p * |V| \rceil / |V| \ge p$. That gives $\Pr[\psi_2^i \mid \psi_1^i] \le (1-p)^{\xi'(v_i)}$
Now lets look at the expected number of edges processed at least once − it is no more than

$$\sum_{i=1}^{n} \Pr[\psi_1^i] * \Pr[\psi_2^i \mid \psi_1^i] * \xi'(v_i) \le \sum_{i=1}^{n} ((1-p)^{i-1}(1-p)^{\xi'(v_i)}\xi'(v_i)) \le$$

$$\le \sum_{i=1}^{n}((1-p)^{i-1}\sum_{j=1}^{\xi'(v_i)}(1-p)^{j-1}) \le \sum_{i=1}^{n}((1-p)^{i-1} * 1/p) \le 1/p * \sum_{i=1}^{n}((1-p)^{i-1}) \le 1/p^2$$

□

4.4 Preprocessing Memory and Time

We first compute the **sample**(V, p) which consists of while loop with expected 2 log n iterations, in each one we compute uniform sample S in time $O(n)$, and then $B_R(v)$, $C_R(v)$ for $v \in V$, where R contains uniform sample from V of size $\lceil p * |V| \rceil$. This can be done by Lemma 4.7 and by Observation 4.9 for weighted graphs or by Lemma 4.10 for unweighted graphs. We can compute the $B_R(v)$ in $O(m + n^2/p)$ expected time for weighted graphs or $O(m + n/p^2)$ expected time for unweighted graphs. The calculation of clusters and their sizes can be done by Observation 4.8 without affecting the complexity. The total expected time is $O(m \log n + n^2/p \log n)$ for weighted graphs or $O(m \log n + n/p^2 \log n)$ for unweighted graphs. The balls and clusters have expected size of $O(1/p)$ so they will not take more than $O(n/p \log n)$ expected auxiliary memory.

We can compute $\delta(u, v)$, $p_{A_1}(v)$ for every $u \in A_1$, $v \in V$ by running Dijkstra algorithm for each of the $u \in A_1$. However will take $O(|A_1| * (m + n \log n))$ time. We can do this faster, in $O(m|A_1|) = O(mnp \log n)$, by using efficient Thorup algorithm for undirected graphs with floating point edge weights [12][13]. For unweighted graphs a simple BFS will be enough. Finding $p_{A_1}(v)$ can be done afterwards by checking every distance calculated. The expected auxiliary memory is $O(n|A_1|) = O(n^2p\ \log n)$.

Next we will compute balls and clusters using Lemma 4.7 in $O(m/p + n/p$ log $n)$ expected time and $O(n/p)$ deterministic auxiliary memory.

And at the end we will compute $\mathcal{O}$ at constant time per entry $-$ $O(n/p^2)$ time and memory as cluster size is bounded by $4/p$ (Observation 4.4). The time could be either expected $O(n/p^2)$ or deterministic $O(n/p^2 \log n)$ depending on hash table construction method used.

This will total to $O(m \log n + n^2/p \log n + mnp \log n + m/p + n/p \log n + n/p^2) = O(n^2/p \log n + mnp \log n + n/p^2)$ expected time for weighted graphs or $O(n/p^2$ log $n + mnp \log n)$ expected time for unweighted graphs.

The expected auxiliary memory is $O(n/p \log n + n^2p\ \log n + n/p + n/p^2) = O(n^2p$ log$n + n/p^2)$.

For $p = n^{-1/3}$ and $m \geq n$ the expected auxiliary memory is the lowest $-$ $O(n^{5/3}$ log $n)$. The expected time is then $O(n^{7/3} \log n + mn^{2/3} \log n)$ for weighted graphs and $O(mn^{2/3} \log n)$ for unweighted graphs. For $m = O(n^2)$ the expected time is $O(n^{8/3} \log n)$ in both cases.

By Markov inequality we can get deterministic auxiliary memory $O(n^{5/3} \log n)$ in expected constant number of **prepro**(G) executions.

Observation 4.11. As Oracle uses only data structures that were generated during preprocessing the auxiliary preprocessing memory is an upper bound on the oracle memory.

Chapter 5

Theoretical Comparison of (2,1) Distance Oracles

5.1 Oracle by Mihai Patrascu and Liam Roditty

We present the slightly modified $(2, 1) -$ *approximate* distance oracle of Mihai Patrascu and Liam Roditty [7]

```
algorithm preproPatRod(G)
let R be an uniform sample from V of size ⌈n^{-1/3} * n⌉
let S be an uniform sample from V of size ⌈n^{-2/3} * n⌉
let A_1 ← R ∪ ⋃_{u∈S} B_R(u)
for every u ∈ A_1
    for every v ∈ V
        calculate δ(u, v)
for every v ∈ V
    calculate p_{A_1}(v), B_{A_1}(v) and C_{A_1}(v)
    for every u ∈ B_{A_1}(v)
        calculate δ(v, u)
    for every pair (u, w) ∈ C_{A_1}(v) × C_{A_1}(v)
        O(u, w) ← v
```

The only difference between this Oracle and Oracle from chapter 4 is that this oracle constructs A_1 by adding $B_R(u)$, $u \in S$ to the set R, instead of adding $\tilde{\log}\, n$ sets of size R to the set R, which reduces the memory complexity by $\log n$. However we get a weaker bound on $C_{A_1}(v)$, $v \in V$ resulting in increased preprocessing time for sparse graphs.

Lemma 5.1. *There is at most $O(n^{5/3})$ expected pairs of vertices $(u, v) \in V \times V$ such that $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$, if A_1 was constructed by* **preproPatRod**(G).

Proof. We will proof the Lemma by showing that for $u \in V$ expected number of balls intersecting with $B_{A_1}(u)$ is less than $n^{2/3}$.

The program generates first the sample R. Let $u_1, u_2, \dots, u_n$ be the elements of V arranged in any non-decreasing order of distance from u.

Now let $w_1, w_2, \dots, w_n$ be the elements of V arranged in any order such that if $i \leq j$ then for every l such that $w_j \in C_R(u_l)$ there is such k that $w_i \in C_R(u_k)$ we have $k \leq l$. In other words we order the vertices by the first cluster (the one with the smallest i) $C_R(u_i)$, $i \in \{1, \dots, n\}$ that it belongs to.

Now the program generates the sample S. What is the expected size of the set $L = \{w_1, w_2, \dots, w_l\}$ such that $L \cap S = \emptyset$ and L is largest possible?

$$\mathbf{E}[|L|] = \sum_{i=1}^{n} \Pr[w_i \in L] = \sum_{i=1}^{n} \Pr[\{w_1, w_2, \dots, w_i\} \cap S = \emptyset] \leq$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{i} Pr[w_j \notin S] \leq \sum_{i=1}^{n} (1 - n^{-2/3})^i < n^{2/3}$$

For $v \in V$ we claim that $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset \implies v \in L$.

We prove the claim by contradiction: Suppose that $B_{A_1}(u) \cap B_{A_1}(v) \neq \emptyset$ and $v \notin L$

We have $v = w_k$ for some $l + 1 \leq k \leq n$. Note that w_{l+1} exists as otherwise $L = V$, $v \in L$ and $w_{l+1} \in S$, as L is largest possible

Let u_i $1 \leq i \leq n$ be such vertex, that $v \in C_R(u_i)$ and i is smallest possible. If no such u_i exists then $\delta(v, R) = 0$ as otherwise $v \in C_R(v)$, however $\delta(v, R) = 0 \implies B_R(v) = B_{A_1}(v) = \emptyset$, contradiction.

Now let u_j $1 \leq j \leq n$ be such vertex, that $w_{l+1} \in C_R(u_j)$ and j is smallest possible. If no such u_j exists then since u_i exists we have $k < l + 1$, so $v = w_k \in L$.

Since $k \geq l + 1$ we have $i \geq j$, so $\delta(u, u_i) \geq \delta(u, u_j)$, but $w_{l+1} \in C_R(u_j)$, so $u_j \in B_R(w_{l+1})$ and $w_{l+1} \in S$ which means that $u_j \in A_1$, so $u_j \notin B_{A_1}(u)$ and $u_i \notin B_{A_1}(u)$. However u_i is the closest vertex to u such that $u_i \in B_R(v)$ and since $B_R(v) \supseteq B_{A_1}(v)$, we get a contradiction $B_{A_1}(u) \cap B_{A_1}(v) = \emptyset$. □

Observation 5.2. The above proof also gives an estimation on expected value of $\sum_{v \in V} |C_{A_1}(v)|^2$. Indeed when we look at set L_u (L generated for a given u) we see that it contains all elements from clusters $C_R(v)$, where $v \in B_{A_1}(u)$. This points to:

$$\sum_{v \in V} |C_{A_1}(v)|^2 = \sum_{v \in V} \sum_{u \in C_{A_1}(v)} |C_{A_1}(v)| = \sum_{u \in V} \sum_{v \in B_{A_1}(u)} |C_{A_1}(v)| \leq$$

$$\leq \sum_{u \in V} \sum_{v \in B_{A_1}(u)} L_u = \sum_{u \in V} |B_{A_1}(u)| * L_u \leq \sum_{u \in V} |B_R(u)| * L_u$$

$$\mathbf{E}[\sum_{v\in V} |C_{A_1}(v)|^2] \leq \mathbf{E}[\sum_{u\in V} |B_R(u)| * L_u] = \sum_{u\in V} \mathbf{E}[|B_R(u)| * L_u] \leq n * n^{1/3} * n^{2/3} = n^2$$

as our upper bounds on $\mathbf{E}[|B_R(u)|]$ and $\mathbf{E}[L_u]$ are independent $-$ we generate the $B_R(u)$ and $w_1, w_2, \ldots, w_n$ (for a given u) based on sample R and graph G and then we generate random sample S that gives us L_u together with its upper bound, while $B_R(u)$ remains unaffected

It is claimed that the calculation of the hash table $\mathcal{O}$ is possible in $O(mn^{2/3})$ [10]

The generation of R and S takes $O(n)$, the computation $B_R(u), u \in S$ takes $O(m+|S|/p^2) = O(m+n)$ expected time as stated by Lemma 4.10. The computation of A_1 then happens in $O(n)$ time. The expected auxiliary memory is $O(n)$. We see that the generation A_1 is very fast and the expected size of A_1 is $O(n^{2/3})$ as $|R| = \lceil n^{2/3} \rceil$, $|S| = \lceil n^{1/3} \rceil$ and expected size of $B_R(u), u \in S$ is less than $n^{1/3}$ by Lemma 3.9.

The calculation of $\delta(u, v)$, $p_{A_1}(v)$ for every $u \in A_1$, $v \in V$ takes only expected $O(mn^{2/3})$ time and $O(n^{5/3})$ expected auxiliary memory. This is an improvement by the factor of $\log n$.

The expected value of $\sum_{v\in V} |C_{A_1}(v)|^2$ is n^2, so the computation of $\mathcal{O}$ takes $O(n^2)$ expected time. The expected auxiliary memory remains at $O(n^{5/3})$ thanks to Lemma 5.1.

The total expected time is $O(n^2 + mn^{2/3})$, expected auxiliary memory $O(n^{5/3})$.

5.2 Further Improvements

The Oracle preprocessing time could be further improved by applying spanners $-$ subgraphs of a graph that have less edges than original but the distances between nodes are preserved up to some error (usually additive or multiplicative). This however comes with the drawbacks as either the distance approximation is worse $-$ stretch $(2, 3)$ [6] or the oracle becomes more complicated [9] [10]

The Oracle can be generalized similar to Oracle from Chapter 3 to produce $(2k-2, 1) -$ *approximate* distance oracle of size $O(n^{1+2/(2k-1)})$ [8]

The are also more efficient algorithms for the special classes of graphs $-$ sparse graphs, planar graphs.

Chapter 6

Results

6.1 Perfect Hashing Functions Generation

Definition 6.1. A minimal perfect hash function is a function that given a set of keys S maps them bijectively to into the set $\{1, 2, \ldots, |S|\}$.

In the following we will call minimal perfect hash functions simply a perfect hash functions.

The hash table used in the programs is PTHash [18], minimal perfect hash algorithm The implementation of PTHash is written in C++ and available at `https://github.com/jermp/pthash`

The construction time of PTHash is superlinear, fortunately this will not affect the time complexity as all algorithm use $\Omega(n^2)$ preprocessing time, but only $O(n^{5/3})$ write operations. However, the practical construction time should improve as PTHash is very efficient.

6.2 Programs Compared

The first program marked as **BFS** compared is an Oracle with $A_1 = V$, every exact distance is calculated by BFS algorithm and stored in memory. This is our control sample.

The second program **Basic** is an Oracle with A_1 being an uniform sample from V of size $\lceil(\rceil p * |V|) = \lceil(\rceil n^{2/3})$. Its the most basic $(2, 1)$ Distance Oracle without any expansion of the A_1 that would ensure small number of ball intersections. This Oracle have expected memory of $\Omega(n^{7/3})$ for some graphs.

The third program **BasGoySen** is Chapter 4. Sample A_1 is expanded by vertices with big clusters until none remains.

The fourth and the last program **PatRod** is Chapter 5 section 5.1 Oracle. Sample A_1 is expanded by random balls.

As the programs are very similar, we run the same tests with the same seed for all random sampling (different tests have different seeds). This ensures that the first sample will be the same for all programs, removing some noise from our comparison. However we have made no attempt to modify the PTHash code, so the perfect hash functions generated are different with each run.

6.3 Random Graphs Comparison

The Oracles were tested on random graphs with $n = 2000 \pm 100$ in order to avoid any distortions from dense graphs with particular number of edges. The number of edges was randomly chosen from $n^a(n-1)/10$ to $n^a(n-1)/2$ for a parameter $0 \leq a \leq 1$. Edges themselves were an uniform sample from all possible edges (no loops and multiple edges).

The following quantities were measured:

- Preprocessing Time − Time taken (in seconds) from the moment after input was loaded into memory to the moment oracle is ready to answer queries.
- Average Query Time − Time taken (in seconds) to report $\hat{\delta}(u,v)$, for every $u, v \in V$ divided by $|V|^2$.
- Size of the Set A_1 − self explanatory.
- $n{*}|A_1|$ − Memory needed (in machine words) to store $\delta(u,v)$, for every $u \in A_1$, $v \in V$.
- Sum of Ball sizes − $\sum_{v \in V} |B_{A_1}(v)| = \sum_{v \in V} |C_{A_1}(v)|$ − Memory needed (in machine words) to store $\delta(v,u)$, for every $v \in V$, $u \in B_{A_1}(v)$.
- Space PHFs B_{A_1} − Memory needed (in bits) to store the perfect hash function for every $B_{A_1}(v)$, $v \in V$ that allows to access any ball element in constant time, while not using additional memory. The hash function generator seems to struggle with low key numbers, hence large number of bits for empty balls. This could be fixed by combining all functions into one, if needed.
- Space PHFs $\mathcal{O}$ − Memory needed (in bits) to store the perfect hash function for table $\mathcal{O}$.
- Average Relative Error − Arithmetic average of errors calculated as $(\hat{\delta(u,v)} - \delta(u,v))/\delta(u,v)$. If $\delta(u,v) = 0$ or there is no path between u and v, all oracles recognize it flawlessly and hence the error is 0.

- Worst Error percentage – The number of times the reported distance was $2 * \delta(u, v) + 1$ divided by number of all distances reported.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	42.2736	3.4243	3.4170	3.6729
Avg. Query Time	1.3815e-07	1.4916e-07	1.5012e-07	1.5001e-07
$\lvert A_1 \rvert$	1993.3	158.9	158.9	171.1
$n * \lvert A_1 \rvert$	3.9756e+06	316860	316860	341175
$\sum_{v \in V} \lvert B_{A_1}(v) \rvert$	0	1834.4	1834.4	1822.2
$\lvert \mathcal{O} \rvert$	0	1834.4	1834.4	1822.2
Space PHFs B_{A_1}	6.4035e+06	6.4041e+06	6.4043e+06	6.4041e+06
Space PHFs $\mathcal{O}$	3216	13638.4	13644.8	13558.4
Average Error	0	0.6674	0.6674	0.6584
Worst Error (%)	0	10.4053	10.4053	10.2423
Total Space				

Table 6.1: $n = 2000 \pm 100$, $a = 1.0$, Random Graph, arithmetic average of 10 tests

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	6.2580	0.5368	0.5411	0.5764
Avg. Query Time	1.3829e-07	1.5242e-07	1.5335e-07	1.5128e-07
$\lvert A_1 \rvert$	1998.3	159.2	159.2	171.1
$n * \lvert A_1 \rvert$	3.9954e+06	318240	318240	342014
$\sum_{v \in V} \lvert B_{A_1}(v) \rvert$	0	1892.2	1892.2	1857.9
$\lvert \mathcal{O} \rvert$	0	2010.2	2010.2	1921.9
Space PHFs B_{A_1}	6.4194e+06	6.4206e+06	6.4207e+06	6.4208e+06
Space PHFs $\mathcal{O}$	3216	13188.8	14188.8	13902.4
Average Error	0	0.4681	0.4629	0.4629
Worst Error (%)	0	6.1077	6.0448	6.0448
Total Space				

Table 6.2: $n = 2000 \pm 100$, $a = 0.75$, Random Graph, arithmetic average of 10 tests

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.9339	0.1373	0.1399	0.1399
Avg. Query Time	1.3671e-07	1.9346e-07	1.9343e-07	1.7201e-07
$\lvert A_1 \rvert$	1995.9	159	159	205.1
$n * \lvert A_1 \rvert$	3.9849e+06	317416	317416	409890
$\sum_{v \in V} \lvert B_{A_1}(v) \rvert$	0	6785.7	6785.7	4650.3
$\lvert \mathcal{O} \rvert$	0	32557.1	32557.1	16005.1
Space PHFs B_{A_1}	6.4119e+06	6.4399e+06	6.4396e+06	6.4208e+06
Space PHFs $\mathcal{O}$	3196.8	131400	130677	13902.4
Average Error	0	0.3012	0.3012	0.2796
Worst Error (%)	0	0.9458	0.9458	0.9842
Total Space				

Table 6.3: $n = 2000 \pm 100$, $a = 0.5$, Random Graph, arithmetic average of 10 tests

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.2460	0.1500	0.1554	0.1082
Avg. Query Time	1.4172e-07	2.6281e-07	2.6099e-07	1.9993e-07
$\lvert A_1 \rvert$	1972.2	157.8	157.8	234.7
$n * \lvert A_1 \rvert$	3.8929e+06	311392	311392	464122
$\sum_{v \in V} \lvert B_{A_1}(v) \rvert$	0	12177.8	12177.8	8636.1
$\lvert \mathcal{O} \rvert$	0	122816	122816	63118.9
Space PHFs B_{A_1}	6.3360e+06	6.3941e+06	6.3942e+06	6.3736e+06
Space PHFs $\mathcal{O}$	3216	301478	299878	167926
Average Error	0	0.1751	0.1751	0.1393
Worst Error (%)	0	0.2024	0.2024	0.1562
Total Space				

Table 6.4: $n = 2000 \pm 100$, $a = 0.25$, Random Graph, arithmetic average of 10 tests

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.0428	0.0464	0.0484	0.0471
Avg. Query Time	1.1239e-07	1.2350e-07	1.2445e-07	1.2298e-07
$\lvert A_1 \rvert$	2008.9	159.7	159.7	181.5
$n * \lvert A_1 \rvert$	4.0401e+06	321056	321056	464122
$\sum_{v \in V} \lvert B_{A_1}(v) \rvert$	0	3521.7	3521.7	8636.1
$\lvert \mathcal{O} \rvert$	0	14910.9	14910.9	63118.9
Space PHFs B_{A_1}	6.4539e+06	6.4597e+06	6.4604e+06	6.3736e+06
Space PHFs $\mathcal{O}$	3216	24480	24467.2	167926
Average Error	0	7.1658e-05	0.1751	0.1393
Worst Error (%)	0	7.4778e-07	0.2024	0.1562
Total Space				

Table 6.5: $n = 2000 \pm 100$, $a = 0$, Random Graph, arithmetic average of 10 tests

6.4 Star Like Graphs

It rarely happens that some clusters become sizeable on random graphs. This causes **Basic** and **BaGoSe** to have almost always identical set A_1.

Consider a vertex u that has a large cluster $C_{A_1}(u)$ $-$ a cluster of size greater than q/p, for a large enough constant q. u is contained in a lot of balls, however balls are unlikely to be large as stated by Theorem 3.10, so radius of ball u is smaller than most of the radii of balls from $C_{A_1}(u)$. Balls from the C_{A_1} despite their usually larger radii, rarely contain each other as their size is unlikely to be large. This hints that graph around u should be more dense than around most of vertices from its cluster.

In order to generate graphs where large clusters are likely, we decided to split the vertices into two sets $-$ the dense core and sparse corona. To simplify analysis the graph will be bipartite with edges only occurring between core and corona, and the core will have size $1/p$, so the sample S of size $p * n$ will have reasonable chance of missing the core, resulting in $\sum_{v \in V} |C_S(v)|^2 = \Omega(n^2/p)$ if our star graph have a lot of edges.

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.2417	2.0036	0.0570	0.0618
Avg. Query Time	1.3576e-07	4.7731e-07	1.4782e-07	1.5623e-07
$\|A_1\|$	1995.8	159.05	171.3	181.45
$n * \|A_1\|$	3.9855e+06	317550	342000	362224
$\sum_{v \in V} \|B_{A_1}(v)\|$	0	5542.25	1824.5	1914.5
$\|\mathcal{O}\|$	0	1.2664e+06	1824.5	8401.6
Space PHFs B_{A_1}	6.4116e+06	6.4123e+06	6.4121e+06	6.4122e+06
Space PHFs $\mathcal{O}$	3212.8	4.2605e+06	13580.8	46821.6
Average Error	0	0.1530	0.1196	0.1270
Worst Error (%)	0	0.0354	0	0.0205
Total Space				

Table 6.6: $n = 2000 \pm 100$, $a = 0.3$, Star Like, arithmetic average of 20 tests

	BFS	**Basic**	**BaGoSe**	**PatRod**
Preprocess Time	0.0972	0.3266	0.0440	0.0622
Avg. Query Time	1.2391e-07	3.0933e-07	1.2971e-07	1.6602e-07
$\|A_1\|$	1984.05	158.4	170.1	177.2
$n * \|A_1\|$	3.9389e+06	314404	337636	351724
$\sum_{v \in V} \|B_{A_1}(v)\|$	0	3316.45	1836	2154.65
$\|\mathcal{O}\|$	0	215708	2685.1	25307.5
Space PHFs B_{A_1}	6.3737e+06	6.3744e+06	6.3743e+06	6.3746e+06
Space PHFs $\mathcal{O}$	3216	1.0054e+06	18024.8	131357
Average Error	0	0.0960	0.0095	0.0258
Worst Error (%)	0	3.1012e-05	2.1535e-06	0.0128
Total Space				

Table 6.7: $n = 2000 \pm 100$, $a = 0.15$, Star Like, arithmetic average of 10 tests

6.5 Open Problems

Does algorithm **sample**(X, p) from chapter 4 have $O(1)$ expected number of while loop iterations, if the graph is unweighted?

It is argued by Lemma 4.5 that in each step balls contributing to the large clusters are likely to be shrunk, when one of the vertices with large cluster will be added to the sample, halving the balls contribution on average. However for unweighted graphs the radius of the contributing ball would be shrunk by 1, which might drastically reduce the ball size. Intuitively, increasing the radii of every ball by 1 transforms the $(2, 1)$ distance oracle into $(2, 0)$ distance oracle increasing the memory consumption to $\Omega(n^2/poly \log n)$ on some graphs. Decreasing the radii of ball could have opposite effects, as most of the vertices of a ball is usually located at the border of it.

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