Chapter 1

Introduction to Categories

1.1 Categories

Definition 1.1.1 (Category)**.** A *category* $\mathscr{C}$ consists of a collection of mathematical structures, called the objects, denoted by $\text{Ob}(\mathscr{C})$, and a class of morphisms (or arrows), denoted by $\text{Hom}(\mathscr{C})$, between the objects. The morphisms must obey the following axioms:

i. If A, B, C in $\mathscr{C}$, $f : A \to B$, and $g : B \to C$, then there is a morphism $g \circ f : A \to C$ (composition).

ii. For each X in $\mathscr{C}$, there exists an identity function. That is, there exists a function $\text{id}_X : X \to X$ such that for every morphism $f : A \to X$ and every morphism $g : X \to A$, we have $\text{id}_X \circ f = f$ and $g \circ id_X = g$.

iii. The composition is associative. That is, whenever we have $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Example 1.1.1.1 (**Set**)**.** The category of sets, denoted by **Set**, is a category where the objects are sets and the morphism are functions.

Example 1.1.1.2 (**Cat**)**.** **Cat** is a category where the objects are small categories (categories where $\text{Ob}(\mathscr{C})$ and $\text{Hom}(\mathscr{C})$ are actual sets, not classes) and the morphisms are functors.

Example 1.1.1.3 (The opposite category)**.** Every category $\mathscr{C}$ has an *opposite category* $\mathscr{C}^{op}$. The objects in this category are the exact same as those in $\mathscr{C}$, but the morphisms (arrows) point the other direction. In other words, $\text{Hom}_{\mathscr{C}^{op}}(X, Y) = \text{Hom}_{\mathscr{C}}(Y, X)$. Note that in category theory, morphisms do not need to be actual functions.

1.2 Functors

Definition 1.2.1 (Functor)**.** A *functor* is a mapping between categories. Specifically, if $\mathscr{C}$ and $\mathscr{D}$ are categories, then a functor F from $\mathscr{C}$ to $\mathscr{D}$ does the following:

i. maps an object X in $\mathscr{C}$ with $F(X)$ in $\mathscr{D}$ (for every X).

ii. for every morphism f in $\mathrm{Hom}(X,Y)$, the functor maps f to $F(f)$ with $F(f)$ in $\mathrm{Hom}(F(X),F(Y))$, for every X,Y in $\mathscr{C}$ and $F(X),F(Y)$ in $\mathscr{D}$.

such that the following axioms hold:

i. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every X in $\mathscr{C}$.

ii. $F(g \circ f) = F(g) \circ F(f)$ for every morphism f, g.

Example 1.2.1.1 (Power set)**.** The power set $P : \mathbf{Set} \to \mathbf{Set}$ is a functor. A functor which maps a category to itself is called an *endofunctor*.

Example 1.2.1.2 (Forgetful functor)**.** The functor $F : \mathbf{Grp} \to \mathbf{Set}$ defined by sending a group to its underlying set and sending a group homomorphism to its underlying set function is functor.

Example 1.2.1.3 (Opposite functor)**.** Just like opposite categories, each functor admits an *opposite functor*. If $F : \mathscr{C} \to \mathscr{D}$, then $F^{op} : \mathscr{C}^{op} \to \mathscr{D}^{op}$.

Example 1.2.1.4 (Contravariant functor)**.** Let $\mathbf{Vect}_{\Bbbk}$ be the category of vector spaces over a field $\Bbbk$, where the morphisms are linear transformations. Let $F : \mathbf{Vect}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$ be the functor that sends a vector space V to its dual space $V^* = \mathrm{Hom}(V, \Bbbk)$. It is clear what this functor does to objects, but what about morphisms? What is the result of applying F to linear transformations? Suppose we had vector spaces V and W and linear transformations $f : V \to W$, $\psi : V \to \Bbbk$:

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "\psi"'] & & W \arrow[dl, dashed, "?"] \\ & \Bbbk & \end{tikzcd}$$

How are we supposed to take ψ and f to create a map $F(f)$ from W to $\Bbbk$? From the diagram, this looks impossible. But it also hints at a solution. If we instead start with a linear transform $\phi : W \to \Bbbk$, then we can define $\psi = \phi \circ f$. We can thus define $F(f) := f^*$, called a *pullback*, which takes $\phi : W \to \Bbbk$ to $\phi \circ f : V \to \Bbbk$. In other words, $f^*(\phi) = \phi \circ f$. Notice that this endofunctor reverses the arrows of the morphisms. If F send V to W, then F sends $\mathrm{Hom}(W, \Bbbk)$ to $\mathrm{Hom}(V, \Bbbk)$. This is called a *contravariant functor*.

Definition 1.2.2. A functor is *full* if it is "surjective", *faithful* if it is "injective", and *fully faithful* if it is "bijective".

Theorem 1.2.3. *Let X, Y be objects of a category $\mathscr{C}$ and $F : \mathscr{C} \to \mathscr{D}$ a fully faithful functor. If $F(X) \cong F(Y)$, then $X \cong Y$.*

Proof. Let $h : F(X) \to F(Y)$ be an isomorphism with inverse h^{-1}. Since F is fully faithful, we must be able to find a unique morphism $f : X \to Y$ satisfying $F(f) = h$. Similarly, we can also find a $g : Y \to X$ such that $F(g) = h^{-1}$. Then:

$$\mathrm{id}_{F(X)} = h^{-1} \circ h = F(g) \circ F(f) = F(fg).$$

But by defintion of a functor, $\mathrm{id}_{F(X)} = F(\mathrm{id}_X)$, so we have that

$$F(fg) = F(\mathrm{id}_X).$$

Since F is fully faithful, $fg = \mathrm{id}_X$. A similar argument shows that $gf = \mathrm{id}_Y$, establishing the isomorphism. □

1.3 Natural Transformations

Definition 1.3.1 (Natural Transformation)**.** Let F and G both be functors from a category $\mathscr{C}$ to a category $\mathscr{D}$. A *natural transformation* $\eta : F \Rightarrow G$ is a family of morphisms that satisfies the following:

i. For all X in $\mathscr{C}$, there is a morphism $\eta_X : F(X) \to G(X)$, called the component of η at X.

ii. For all morphisms $f : X \to Y$ in $\mathrm{Hom}(\mathscr{C})$, $G(f) \circ \eta_X = \eta_Y \circ F(f)$. Equivalently, this condition is satisfied if and only if the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\eta_Y"'] & G(Y)
\end{tikzcd}$$

A diagram *commutes* if taking any path from point A to point B gives equivalent results.

Example 1.3.1.1. Of course, the identity transformation is a natural transformation.

$$\begin{tikzcd}
F(X) \arrow[r, "\mathrm{id}_X"] \arrow[d, "F(f)"'] & F(X) \arrow[d, "F(f)"] \\
F(Y) \arrow[r, "\mathrm{id}_Y"'] & F(Y)
\end{tikzcd}$$

Remark. The word "natural" or "canonical" is used in mathematics to describe relationships that arise natural from construction. Intuitively, we can think of it as if two people were to define a relationship independently, they end up with the same thing. For example, the canonical map from a set S to its power set $\mathcal{P}(S)$ is defined by $a \mapsto \{a\}$, and the Jordan Canonical Form is named that way because there is only one (up to isomorphism) JCF for each similarity class of linear transformations.

Example 1.3.1.2. Let V be a finite dimensional vector space. Then there is a natural transformation from V to V^{**}, the double dual of V. If V and W be finite dimensional vector spaces, and $T : V \to W$ is a linear transformation, we can define the natural transformations like so:

$$\eta_V(v) = \mathrm{eval}_v$$
$$\eta_W(w) = \mathrm{eval}_w$$

where eval_v takes an element $f : V \to \Bbbk$ from V^* and applies v to it. Equationally, this translates to $\mathrm{eval}_v = f(v)$. Now consider the following diagram:

$$\begin{tikzcd} V \arrow[r, "\mathrm{eval}_v"] \arrow[d, "T"'] & V^{**} \arrow[d, "T^{**}"] \\ W \arrow[r, "\mathrm{eval}_w"'] & W^{**} \end{tikzcd}$$

Is it commutative? Yes, it is! However, the reader is encouraged to check it themselves.

Remark. Although there is also an isomorphism $V \cong V^*$, the isomorphism is not "natural" in any way. To even define a map from V to V^*, we have to choose a basis; and you know that when you need to choose a basis your life is already going downhill. For every different basis we pick, we get a different isomorphism. In fact, it can be shown that there is *no* "natural" isomorphism between V and V^*.

Definition 1.3.2 (Representable functors)**.** Let $\mathscr{C}$ be a locally small category, and for each object A in $\mathscr{C}$ let $\mathrm{Hom}(A, -)$ be the functor that maps X to the set $\mathrm{Hom}(A, X)$. A functor $F : \mathscr{C} \to \mathbf{Set}$ is said to be *representable* if there is a natural transformation from F to $\mathrm{Hom}(A, -)$ for some object A in $\mathscr{C}$. A *representation* is a pair (A, Φ) such that $\Phi : \mathrm{Hom}(A, -) \to F$ is a natural transformation.

Example 1.3.2.1. Let $F : \mathbf{Grp} \to \mathbf{Set}$ be the functor that sends a group to its underlying set. Then $(\mathbb{Z}, 1)$ is a representation of F.

Proof. We would like to establish the natural transformation $\Phi : \mathrm{Hom}(\mathbb{Z}, -) \to F$. Let G be an object in $\mathbf{Grp}$, then for each g in G, there is a unique homomorphism satisfying $1 \mapsto g$. This sets up a bijection between $F(G)$ and $\mathrm{Hom}(\mathbb{Z}, G)$. Since we can do this for every group, let us define this mapping to Φ_G, the component of Φ at G. Now let us verify Φ is natural. Let G, H be groups and f be a (group) homomorphism.

$$\begin{tikzcd} F(G) \arrow[r, "\Phi_G"] \arrow[d, "F(f)"'] & \mathrm{Hom}(\mathbb{Z}, G) \arrow[d, "\mathrm{Hom}(\mathbb{Z}, f)"] \\ F(H) \arrow[r, "\Phi_H"'] & \mathrm{Hom}(\mathbb{Z}, H) \end{tikzcd}$$

Recall that if g in $\mathrm{Hom}(A, f)$, then $g \mapsto f \circ g$, so this diagram does indeed commute. □

We can do similar things to other forgetful functors (try it yourself for **Ring**).

Chapter 2

Yoneda's Lemma

2.1 The Yoneda Embedding

The Yoneda embedding is a special case of the Yoneda lemma, but we'll work through this first before moving on to the general version.

Corollary 2.1.0.1 (The Yoneda Embedding). *Let $\mathscr{C}$ be a locally small category (recall that locally small means that* $\mathrm{Hom}(\mathscr{C})$ *are actual sets). Then there is a natural functor F from $\mathscr{C}$ to* $\mathbf{Set}$.

Proof. Let X be an object in $\mathscr{C}$ and define $H_A(X) = \mathrm{Hom}(A, X)$, for all A in $\mathscr{C}$, the set of all morphisms to X. Now for any morphism $f : X \to Y$, we want to map it to some morphism $H_A(f) : H_A(X) \to H_A(Y)$. Note that objects of $H_A(X)$ are morphisms themselves, $u : A \to X$ for some object A in $\mathscr{C}$. Thus we can simply define $H_A(f) = f \circ u$. □

Theorem 2.1.1 (Yoneda's Lemma). *Let $\mathscr{C}$ be a locally small category, $F : \mathscr{C} \to$ **Set** a functor, and $H^A = \mathrm{Hom}(A, -)$. Then*

$$\mathbf{Set}^{\mathscr{C}}(H^A, F) \cong F(A).$$

That is, there is a one-to-one correspondence between the natural tranformations from H^A to F and the elements of $F(A)$.

Proof. Let X be an object in $\mathscr{C}$ and a in X. Define the natural transformation η by declaring each component $\eta_X : \mathrm{Hom}(A, X) \to F(X)$ to be the map $g \mapsto (F(g))(a)$. □

Corollary 2.1.1.1. *Let X, Y be objects in a category $\mathscr{C}$. Then the Yoneda embedding $\mathscr{Y} : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}}$ given by*

$$\mathrm{Hom}(X, Y) \mapsto \mathbf{Set}^{\mathscr{C}}(\mathrm{Hom}(X, -), \mathrm{Hom}(Y, -))$$

is full and faithful.

Corollary 2.1.1.2. *Let X, Y be objects in a category $\mathscr{C}$. Then*

$$X \cong Y \text{ if and only if } \operatorname{Hom}(X, -) \cong \operatorname{Hom}(Y, -).$$

Corollary 2.1.1.3 (Cayley's Theorem)**.** *Let G be a group. Then G is isomorphic to a subgroup of the symmetric group acting on G.*

Proof. For every group G, we can view it as a category $\mathscr{G}$ consisting of a single object $\bullet$ where the (iso)morphisms are the group elements. Let $F : \mathscr{G} \to \mathbf{Set}$ be the functor that maps $\bullet$ to the underlying set X and morphisms g to a function $x \mapsto gx$. In particular, $F = \operatorname{Hom}(\bullet, -)$. According to Yoneda, we have:

$$\mathbf{Set}^{\mathscr{G}}(\operatorname{Hom}(\bullet, -), \operatorname{Hom}(\bullet, -)) \cong \operatorname{Hom}(\bullet, \bullet).$$

Notice that the right hand side is actually G itself. As for the left hand side, let us consider the components $\eta_g : G \to G$ in $\mathbf{Set}^{\mathscr{G}}(\operatorname{Hom}(\bullet, \bullet), \operatorname{Hom}(\bullet, \bullet))$. These, however, were just the maps $x \mapsto gx$ we defined earlier, which are the automorphisms of G. Thus the left hand side is a subgroup of the symmetric group acting ong G, and the right hand side is G, and we are done. $\square$