
MSE, Bias², and Variance of the ML-Estimator of the Standard Deviation

SETUP:

- Assumptions: iid random sample (X_1, \dots, X_n) with $X_i \sim N(\mu, \sigma^2)$ for all $i = 1, \dots, n$.
- Fix values for the mean μ , the variance σ^2 , and the sample size n .
For instance: $\mu = 3$, $\sigma = 1.5$, and $n = \{2, 4, 6, \dots, 30\}$.

MONTE-CARLO ALGORITHM:

1. Simulate a realization from the iid random sample (X_1, \dots, X_n) .
2. Compute $s_n = \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2}$, where $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$

Repeat Steps 1-2 a large number of times, e.g., $R = 10,000$ times and save all estimates $s_{n,1}, \dots, s_{n,R}$. Then approximate the Mean Squared Error (MSE), the squared bias (Bias²), and the variance (Var) by

$$\begin{aligned} \text{MSE}(s_n) &\approx \frac{1}{R} \sum_{r=1}^R (s_{n,r} - \sigma)^2 \\ \text{Bias}^2(s_n) &\approx \left(\left(\frac{1}{R} \sum_{r=1}^R s_{n,r} \right) - \sigma \right)^2 \\ \text{Var}(s_n) &\approx \frac{1}{R} \sum_{r=1}^R \left(s_{n,r} - \left(\frac{1}{R} \sum_{r=1}^R s_{n,r} \right) \right)^2 \end{aligned}$$

PRESENTATION OF THE SIMULATION-RESULTS:

- Plot your results (y-axis: MSE, Bias², and Var; x-axis: n)
- Add the corresponding results for the unbiased estimator $\tilde{s}_n = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2}$