Motivic Cohomology
原相／母题／动机／主上同调

0 Introduction

1 Intersection Theory

2 Sheaves with Transfers

We fix an $S \in \mathrm{Sm}/k$, called the *base scheme*.

2.1 Let $X, Y \in \mathrm{Sm}/\mathbb{k}$. We define the groups of finite correspondences:

$$\mathrm{Cor}_S(X, Y) = \mathcal{Z}\{C \subseteq X \times_S Y\}C \to X \text{ finite}.$$

For any $f : X \to Y$, the graph $\Gamma_f = (x, f(x)) \subseteq X \times_S Y$ is a finite colrespondence from X to Y.

2.2 If $f : X \to Y$ is finite and $\dim X = \dim Y$, the guph Γ_f is also a fiwite correspondence from $Y \to X$.

2.3 Define anadditie category $\mathfrak{Cor}_S$, whose objerts are the same as Sm/S und $\mathrm{Cor}_S(X, Y)$ is defined in Def 2.1. Contraraviant additine functors

$$F_i \mathfrak{Cor}_S^{\mathrm{op}} \to \mathfrak{Ab}.$$

are called presheaver with transfers. The corresponding category is denoted by $\mathfrak{PSh}(S)$. we have a functor $r\colon \mathrm{Sm}/S \to \mathfrak{Cor}_S$ by 2.2

Ex2. 4 Fuery $x \in \mathrm{sim}/s$ give an element. $\mathbb{Z}(x) \in \mathrm{Psh}(S)$ by $\mathbb{Z}(x)(y) = \mathrm{Cor}_s(y, x)$. $(\mathbb{Z}(s) = \mathbb{Z})$

Ex2.5 The presheaves o and 0^* are in. psh (s). For awy $c \in \mathrm{cor}_s(x, y)$ and $f \in O(y)$ (resp. $O^*(y)$)

$$\begin{aligned} &C \xrightarrow{i} X_{\mathcal{L}\pi_1} \times Y \overrightarrow{\pi_2} Y \text{ define } O(c)(t) \\ &= \mathrm{Tr}_{(/x}\left((P_2 0 i)^{tx}(f)\right) \\ &(\text{resp. } \mathrm{Nc}/X \left((\eta_2 \circ i)^t(t)\right) \end{aligned}$$

Def 2.6 Let us describe the composition in loos. Suppose $f \in \mathrm{Cor}(X/Y)$ and $g \in \mathrm{lor}(y, Z)$. xxz^{-113} Define $xx_5 Y_5 \xrightarrow{P_{23}} y, y$

$$\mathrm{gl} = P_{13*}\left(P_{23}^*(g) \cdot P_{12}^*(t)\right)$$

(Check all intersection are proper).

2.4〔**2.7**〕The composition law is associative.

Proof. Suppose $Z \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{W}$ are morplisms in tors. We have Cartesian squares

$$\begin{tikzcd} XYZW \arrow[r] \arrow[d] & XZW \arrow[d] \\ XYZ \arrow[r] & XZ \end{tikzcd}$$

Q.E.D.

2.5 Let $f\colon X \to Y$ be a proper morphism between finite type schemes/$\mathbb{k}$ and $\mathscr{F} \in \mathscr{K}_a(X)$.

i) $f_*\mathscr{F} \in \mathscr{K}_a(Y)$ and $R^i f_*\mathscr{F} \in \mathscr{K}_{a-1}(Y), i > 0$.

ii) $f_*\mathcal{Z}_a(\mathscr{F}) = \mathcal{Z}_a(f_*\mathscr{F})$.

2.6〔**2.8**〕

2.31〔**Cosimplicial Object**〕For and $n \in \mathbb{N}$, define

$$\mathbb{A}^n \cong \Delta^n = \mathrm{Spec}^k[x_0, \cdots, x_n]/(\sum x_i - 1).$$

2.32〔**Associated Complexes**〕For $X \in \mathfrak{PSh}$

3 Milnor $\mathcal{K}$-Theory

3.1〔**Milnor $\mathcal{K}$-theory**〕For any field $\mathbb{F}$, define $\forall x \in \mathbb{F}^\times$,

$$\mathcal{K}_\bullet^{\mathrm{Mil}}(\mathbb{F}) = T(\mathbb{F}^\times)/x \otimes (1 - x)$$

to be the Milnor $\mathcal{K}$-theory of $\mathbb{F}$, which is a graded algebra, where $T(X)$ is

For example, $\mathcal{K}_0^{\mathrm{Mil}}(\mathbb{F}) = \mathcal{Z}$, $\mathcal{K}_1^{\mathrm{Mil}}(\mathbb{F}) = \mathbb{F}^\times$.

3.2

- $[x][y] + [y][x] = 0$;
- $[x][x] = [x][-1]$.

Proof. 1) $[x][-x] = [x]\left[\frac{1-x}{1-x^{-1}}\right]$

$$\begin{aligned} &= [x][1-x] + \left[x^{-1}\right]\left[1 - x^{-1}\right] \\ &= 0 \end{aligned}$$

$$\begin{aligned} S_0[x][y] + [y][x] =& [x][-x] + [x][y] + [y][x] \\ &+ [y][-y] \\ =& [x](-xy) + [y][-xy] \\ =& (xy)[-xy] = 0 \end{aligned}$$

2) $[x][x] = [x][-1] + [x][-x] = [x][-1].D$ Q.E.D.

3.3〔**L**〕t $\mathbb{k}$ be a field and V be a normalized discrete valuation on $\mathbb{k}$. Let $\mathbb{k}(V) = \mathcal{O}_V/m_V$ be

Prop 3.3 Let k be a field and v be a normalized discrete valuation on k. Let $R(v) = 0r/m_v$ be the residue field 1 . Then $\exists$! homomorphism Sit. $\forall u_1, \ldots, u_{n-1} \in k_n^m(k) - k_{n-1}^m(k(v))$ ∂_v ([x] [u_1] and $x \in K^x$ whee $\bar{u}_i$ is the (lass of u_i in $R(v)^x$.
Prof. The uniqueness is clear. For exist. pence, choosing a uniformizer π, define a graded ring orphism