Motivic Cohomology
原相／母题／动机／主上同调

0 Introduction

Let X be a smooth separated scheme/$\mathbb{k}$. A. Beilinsun and S.Lichtenbaum conjectured around 1982–1987 that there should exist certain complexes $\mathcal{Z}(n)$, $n \in \mathbb{N}$ of schemes in Zariski topology on Sm/$\mathbb{k}$ which have the following properties:

1) $\mathcal{Z}(0)$ is the constant sheaf $\mathscr{Z}$.
2) $\mathcal{Z}(1)$ is the complex $0^*[-1]$:

$$\cdots \to 0 \to \underset{1}{\mathcal{O}^*} \to 0 \to \cdots.$$

3) For every field $F/\mathbb{k}$,

$$\mathbb{H}^n_{\mathrm{Zar}}(\mathscr{F}, \mathcal{Z}(n)) := H^n(\mathcal{Z}(n)(\operatorname{Spec}\mathscr{F})) = K_n^{\mathrm{Mil}}(\mathscr{F})$$

where $K_n^{\mathrm{Mil}}(\mathscr{F})$ is the n-th Milnor K-group of $K_0^{\mathrm{Mil}}(\mathscr{F}) = 2$, $K_1^{\mathrm{Mil}}(\mathscr{F}) = \mathscr{F}^\times$, $K_2^{\mathrm{Mil}}(F) = K_2(\mathscr{F})$.
4) $\mathbb{H}^{2n}_{\mathrm{Zar}}(X, \mathcal{Z}(n)) = CH^n(x)$, where

$$CH^n(x) = \mathcal{Z}\{\text{cycles of codim} = n\}/\text{ rat. equi.}$$

is the n-th Chow group.
5) There is a natural spectral sequere with

$$E_2^{p,q} = \mathbb{H}^p_{\mathrm{Zar}}(X, \mathcal{Z}(q)) \Rightarrow K_{2q-p}(X),$$

where $K_n(X) = \pi_{n+1}(BQ(\mathfrak{Vect}(X)), 0)$ is the Quillen K-theory. Tensoring with $\mathbb{Q}$, the spectral sequence degenerates and one has

$$\mathbb{H}^i_{\mathrm{Zar}}(X, \mathcal{Z}(n))_{\mathbb{Q}} = \mathrm{gr}_r^n K_{2n-i}(X)_{\mathbb{Q}},$$

where gr_r^n are quotient of the γ-filtration.

0.1 motivic cohomology

We denote by $H^{p,q} = \mathbb{H}^p_{\mathrm{Zar}}(X, \mathcal{Z}(q))$ the motivic cohomology of X. It satisfies the cancellation property:

$$H^{p,q}(X \times \mathbb{G}_m, \mathcal{Z}) = H^{p,q}(X, \mathcal{Z}) \oplus H^{p-1,q-1}(X, \mathcal{Z}).$$

It turns out that the group remains unchanged if we replace Zariski topology by Nisnevich topology. If one uses étale topology, the corresponding cohomologies are denoted by $H_L^{p,q}(X, \mathcal{Z})$, the Lichtenbanm motivic cohomologies. It admits the following comparison

$$H_l^{p,q}(X, \mathcal{Z}/n) = H^p_{\text{ét}}(X, \mathcal{Z}/\underset{\uparrow = \mu_n^{\otimes q}}{n(q)}).$$

f $\chi\mathbb{k} \nmid n$, The Beilinson–Lichtenbaum conjecture, proved by V.Voevodsky in 2011 , states that

$$H^{p,q}(X, \mathcal{Z}/n) \to H_L^{p,q}(X, \mathcal{Z}/n)$$

is an isomorphism if $p \le q$ and monomorplisin for $p = q + 1$. So we obtain

$$H^{p,q}(X, \mathcal{Z}/n) = H^p_{\text{ét}}(X, \mathcal{Z}/n(q))$$

if $p \le q$. If we take $X = \operatorname{Spec}(\mathbb{k})$, this is the Milnor Conjecture, proved by V. Voerosky in 1996:

$$H^{p,p}(\mathbb{k}, \mathcal{Z}/n) = K_p^{\mathrm{Mil}}(\mathbb{k})/n = H^p_{\text{ét}}(\mathbb{k}, \mathcal{Z}/n(p)).$$

$$H^{p,q}(\mathbb{k}, \mathcal{Z}/n) = \begin{cases} 0 & p > q \\ H^{p,p}(\mathbb{k}, \mathcal{Z}/n)\tau^{q-p} & p < q \end{cases}$$

where $\tau \in M_n(\mathbb{k}) = H^{0,1}(\mathbb{k}, \mathcal{Z}/n)$ is the primitive root.

Unlike finite coefficients, the $H^{p,q}(\mathbb{k}, \mathcal{Z})$ is quite hard to compute for small p. The open question is the Beilinson–Somle vanishing conjecture:

$$H^{p,q}(\mathbb{k}, \mathcal{Z}) = 0 \quad \text{if } p < 0.$$

If $\chi\mathbb{k} = 0$, this is known for number fields, function fields of genus 0 curves over number field and their inductive limits. If $\chi\mathbb{k} > 0$, this is known for finite fields and global fields.

The motivic cohomologies could be realized in a triangulated category $\mathfrak{DM}(\mathbb{k})$, such that

$$H^{p,q}(X, \mathcal{Z}) = H_{\mathfrak{DM}}(\mathcal{Z}(X), \mathcal{Z}(q)[p])$$

where $\mathcal{Z}(X)$ is the motive of X and $\mathcal{Z}(q)[p] = \mathbb{G}_m^{nq}[p-q]$. Moreover, we can define the motivic homology $H_{p,q}(X, \mathcal{Z})$ as

$$H_{p,q}(X, \mathcal{Z}) = H_{\mathfrak{DM}}(z(q)[p], \mathcal{Z}(X)).$$

From the aspect of motives, we can derive theorem of all (co)homologies which can be represented in $\mathfrak{DM}$. The main derives are the following:

1) If $E \to X$ is an $\mathbb{A}^n$-bundle, we have $\mathcal{Z}(E) = \mathcal{Z}(X)$ in $\mathfrak{DM}$.
2) If $\{U, V\}$ is an open covering of X, we have the Mayer-Vietoris sequence in $\mathfrak{DM}$.

$$\cdots \to \mathcal{Z}(U \cap V) \to \mathcal{Z}(U) \oplus \mathcal{Z}(V) \to \mathcal{Z}(X) \to \cdots.$$

3) If $Y \subseteq X$ is a closed embedding of codim c in $\mathrm{Sm}_{\mathbb{k}}$, we have a Gysin triangle

$$\mathcal{Z}(X\backslash Y) \to \mathcal{Z}(X) \to \underbrace{\mathcal{Z}(Y) \otimes \mathcal{Z}(c)[2c]}_{= \mathcal{Z}(Y)(C)[2c]} \to \cdots.$$

4) For any vector bundle $\mathscr{E}$ of rank n on X, we have the projective bundle formula:

$$\mathcal{Z}(\mathbb{P}(\mathscr{E})) = \bigoplus_{i=0}^{n} \mathcal{Z}(X)(i)[2i],$$

which defines the chern classes of $\mathscr{E}$.

5) For proper smooth x, the $\mathcal{Z}(x)$ has a strung dual $\mathcal{Z}(x)(-\dim x)[-2\dim x]$ in DM, which implies the Poincare duality

$$H^{p,q}(X,\mathcal{Z}) = H_{2\dim X-p,2\dim X-q}(X,\mathcal{Z}).$$

0.2 Outline Of The Lesson

The course consists of the following:

1) Intersection theory
2) Milnor K-thpory and cycle modules
3) The (effective and stable) (allegories of motives and basic tools.
4) Computation of $H^{n,n}$ and $H^{2n,n}$

1 Intersection Theory

1.1 〔Cycle〕 Let X be a finite type scheme over $\Bbbk$. Define

$$\mathcal{Z}_i(X) := \bigoplus_{\substack{C\subseteq X,\text{ irr. cld.}\\ \dim C=i}} \mathcal{Z}\cdot C, \quad \left(\mathcal{Z}(X) = \bigoplus_i \mathcal{Z}_i(X)\right),$$

$$\underbrace{\mathcal{K}_i(X)}_{\uparrow A\ set} := \{\mathcal{F}\text{ coherent on } X \dim\operatorname{supp}\mathcal{F}\leqslant i\}$$

1.2 〔Intersect Properly〕 Let $X\in \mathrm{Sm}_{\Bbbk}$, $U,V\subseteq X$ irr. closed and $W\subseteq U\cap V$ be an irr. component. If

$$\dim W = \dim U + \dim V - \dim X$$
$$\Leftrightarrow \operatorname{codim} W = \operatorname{codim} U + \operatorname{codim} V,$$

Say that U,V *intersection properly* at W.

1.3 Let $\Bbbk\leq A$ be a noetherian regular local ring. M,N are finitely generated A-mod and $\operatorname{len}(M\otimes N)<\infty$. Then

1) $\operatorname{len}\left(\operatorname{Tor}_i^A(M,N)\right)<\infty, \forall i\geqslant 0$
2) $\chi(M,N) := \sum_{i=0}^{\dim A}(-1)^i \operatorname{len}_A\left(\operatorname{Tor}_i^A(M,N)\right)\geqslant 0.$
3) $\dim M+\dim N\leq \dim A.$
4) $\dim M+\dim N<\dim A$ iff $\chi(M,N)=0.$

Proof. See J. P. Serre, *Local Algebra*. Q.E.D.

1.4 The 1.3 shows that, $\dim W \geqslant \dim V + \dim U - \dim X \Leftrightarrow \operatorname{codim} W\leq \operatorname{codim} U+\operatorname{codim} V$ in the context of Def 1.2 .

1.5 〔Intersection Multiplicity〕 Let $X, W\subseteq U\cap V$ as in Def 1.2. Define the *intersection multiplicity* $\operatorname{mult}_W(U,V)$ of U,V at W to be

$$\operatorname{mult}_W(U,V) := \chi^{\mathcal{O}_{X,W}}\left(\mathcal{O}_{X,W}/\mathfrak{p}_U, \mathcal{O}_{X,W}/\mathfrak{p}_V\right).$$

So $\operatorname{mult}_W(U,V)\geqslant 0$. $\operatorname{mult}_W(U,V)=0$ iff U,V do not intersect properly at W.

1.6 〔Intersection Product〕 Let $X\in\mathrm{Sm}_{\Bbbk}$, $U\in\mathcal{Z}_a(X)$, $V\in\mathcal{Z}_b(X)$, If U,V intersect properly at every component, define

$$U\cdot V = \sum_{\substack{W\subseteq U\cap V\\ \dim W=a+b-\dim X}} \underbrace{\operatorname{mult}_W(U,V)}_{\uparrow\in\mathcal{Z}_{a+b-\dim X}(X)}$$

Ex. X smooth projective surface, C,D divisors on $x\in C\cap D$), $\operatorname{mult}_X(C,D)=\operatorname{len}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f,g))$.

$C=\{f=0\}$, $D=\{g=0\}$ around x.

1.7 〔Sheaf Cycle〕 Suppose X is finite type/$\Bbbk$, $\mathcal{F}\in\mathcal{K}_a(X)$, define

$$\mathcal{Z}_a(\mathcal{F}) = \sum_{\dim\bar\eta=a}\operatorname{len}_{\mathcal{O}_{X,\eta}}(\mathcal{F}_\eta)\cdot \underbrace{\bar\eta}_{\uparrow\in\mathcal{Z}_a(X)}$$

1.8 〔Tor Sheaf〕 By [GTM52, III, Ex 6.8], every coherent $\mathcal{F}$ on $X\in\mathrm{Sm}_{\Bbbk}$ has a free resolution:

$$0\to\mathcal{E}_k\to\mathcal{E}_{k-1}\to\cdots\to\mathcal{E}_0\to\mathcal{F}\to 0,$$

where $\{\mathcal{E}_i\}$ are locally free of finite rank (vector bundle). So for any coherent $\mathcal{G}$, we define

$$\underbrace{\mathcal{T}\!or_i(\mathcal{F},\mathcal{G})}_{\uparrow A\ sheaf} := H_i(\mathcal{E}\otimes\mathcal{G}).$$

1.9 Let $X\in\mathrm{Sm}_{\Bbbk}$ and $\mathcal{F}\in\mathcal{K}_a(X)$, $\mathcal{G}\in\mathcal{K}_b(X)$ intersect properly, we have

$$\mathcal{Z}_a(\mathcal{F})\cdot\mathcal{Z}_b(\mathcal{G}) = \sum_{i=0}(-1)^i\mathcal{Z}_{a+b-\dim X}(\mathcal{T}\!or_i(\mathcal{F},\mathcal{G}))$$

Proof. The coefficients are defined locally, so we work locally. Let X be affine and count the coefficients of $\bar\xi$ where $\dim\bar\xi = a+b-\dim X$. We have to show:

$$\chi(\mathcal{F}_\xi,\mathcal{G}_\xi) = \sum_{\substack{\dim\bar\lambda=a,\dim\bar\eta=b\\ \xi\in\bar\lambda\cap\bar\eta}}^{\dim X}\operatorname{len}(\mathcal{F}_\lambda)\operatorname{len}(\mathcal{G}_\eta)\cdot\operatorname{mult}_{\bar\xi}(\bar\lambda,\bar\eta).$$

Both $\mathcal{F}$ and $\mathcal{G}$ are finitely generated modules over a noetherian ring, which admits a filtration

$$0=M_0\subseteq\cdots\subseteq M_d=\mathcal{F}\text{ or }\mathcal{G}$$

s.t. $M_i/M_{i-1}\cong\mathcal{O}_X/\text{a prime}$. By additivity of both sides(from property of Tor), we may assume $\mathcal{F}=\mathcal{O}_X/\mathfrak{P}$, $\mathcal{G}=\mathcal{O}_X/\mathfrak{Q}$, $\xi\in\bar\lambda\cap\bar\eta$, so LHS $=\operatorname{mult}_{\bar\xi}(\bar\lambda,\bar\eta)$.

If $\dim\bar\lambda=a$ and $\dim\bar\eta=b$, the equality follows by $\operatorname{len}(\mathcal{F}_\lambda)=\operatorname{len}(\mathcal{G}_\eta)=1$.

If $\dim\bar\lambda<a$ or $\dim\bar\eta<b$, the $\bar\lambda$ and $\bar\eta$ do not intersect properly at $\bar\xi$, LHS $=0=$ RHS.

Q.E.D.

1.10 The intersection product is commutative.

1.11 The intersection product is associative.

Proof. Suppose $\mathscr{F} \in \mathscr{K}_a(X), \mathscr{G} \in \mathscr{K}_b(X), \mathscr{H} \in \mathscr{K}_c(X)$ and they intersect properly.

Define a double complex $\mathcal{M}_{ij} = L^i \otimes \mathscr{G} \otimes M^j$, where $L^\bullet$ and $M^\bullet$ is resolution of $\mathscr{F}$ and $\mathscr{H}$. Then

$$'E^2_{p,q} = \mathscr{T}\!or_p\left(\mathscr{F}, \mathscr{T}\!or_q(\mathscr{G}, \mathscr{H})\right),$$
$$''E^2_{p,q} = \mathscr{T}\!or_p\left(\mathscr{T}\!or_q(\mathscr{F}, \mathscr{G}), \mathscr{H}\right).$$

Suppose we have some additive function l. we have

$$\sum_{\substack{p,q \\ \mathcal{Z}_a(\mathscr{F})\cdot(\mathcal{Z}_b(\mathscr{G})\mathcal{Z}_c(\mathscr{H}))}} (-1)^{p+q}\, l('E^2_{p,q}) = \sum_n (-1)^n\, l(H_n(\mathrm{Tot}(\mathcal{M}_{ij})))$$
$$= \sum_{\substack{p,q \\ (\mathcal{Z}_a(\mathscr{F})\mathcal{Z}_b(\mathscr{G}))\cdot\mathcal{Z}_c(\mathscr{H})}} (-1)^{p+q}\, l(''E^2_{p,q}).$$

Since len is addictive in ess Q.E.D.

1.12 〔**Exterior Product**〕 Suppose $X_1, X_2 \in \mathrm{Sm}_{\mathbb{k}}$, $\mathscr{F}_1 \in \mathscr{K}_{a_1}(X_1), \mathscr{F}_2 \in \mathscr{K}_{a_2}(X_2)$. Define

$$\mathcal{Z}_{a_1}(\mathscr{F}_1) \times \mathcal{Z}_{a_2}(\mathscr{F}_2) = \mathcal{Z}_{a_1+\dim X_2}(\pi_1^*\mathscr{F}_1) \cdot \mathcal{Z}_{a_2+\dim X_1}(\pi_2^*\mathscr{F}_2)$$

where $\pi_i : X_1 \times X_2 \to X_i$ is the projection. check it's well-define.

1.13 〔**Direct Image**〕 Suppose X, Y are finite type/$\mathbb{k}$ and $f : X \to Y$ is proper. For every irr. closed $C \subseteq X$ of dim a, define

$$f_*C = \begin{cases} [\mathbb{K}(C) : \mathbb{K}(f(C))] \cdot f(C) \in \mathcal{Z}_a(Y) & \dim f(C) = a; \\ 0 & \dim f(C) < a. \end{cases}$$

to be the *direct image* of C under f. Where $\mathbb{K}(C)$ is the field of rational functions of C.

1.14 Suppose X, Y are finite type/$\mathbb{k}$ of the same dimension n and that $f : X \to Y$ is proper. Then there exists an open set $U \subseteq Y$ s.t.

$$\dim(Y\backslash U) < n, \quad f : f^{-1}(U) \to U \text{ is finite.}$$

Proof. Generic point $\xi \in Y$ and $\dim \bar{\xi} = n$. We can find $\xi \in U$ s.t. $f|_U$ has finite fibers by [GTM52, II, Ex 3.7] [GTM52, III, 11.2] such f is finite. Q.E.D.

1.15 Let $f : X \to Y$ be a proper morphism between finite type schemes/$\mathbb{k}$ and $\mathscr{F} \in \mathscr{K}_a(x)$.

1) $f_*\mathscr{F} \in \mathscr{K}_a(Y)$ and $\mathrm{R}^i f_*\mathscr{F} \in \mathscr{K}_{a-1}(Y), i > 0$
2) $f_*\mathcal{Z}_a(\mathscr{F}) = \mathcal{Z}_a(f_*\mathscr{F})$.

Proof. 1) $\mathrm{R}^i f_*\mathscr{F}$ is coherent for all $i \geqslant 0$ by [GTM52, III, Thm 8.8]. We have $\mathrm{supp}\, \mathrm{R}^i f_*\mathscr{F} \subseteq \mathscr{F}$. If f is finite, f_* is exact so $\mathrm{R}^i f_*\mathscr{F} = 0, i > 0$. For general cases, we may suppose $\dim f(\mathrm{supp}\,\mathscr{F}) = a$ and set $W = \mathrm{supp}\,\mathscr{F}$. We have a diagram

$$\begin{array}{ccc} W & \xrightarrow{h} & f(W) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array},$$

where h is also proper. So by Lem 1.14, we get $V \subseteq f(W)$ s.t. $\dim(f(W)\backslash V) < a$ and $h|_V$ is finite.

Denote by $\mathscr{J}$ the ideal sheaf of W, we have

$$\mathscr{J}^s\mathscr{F} / \mathscr{J}^{s+a}\mathscr{F} = i_*i^*\, \mathscr{J}^s\mathscr{F} / \mathscr{J}^{s+1}\mathscr{F}.$$

By long exact sq., it suffices to de the cases $\mathscr{F} = i_*\mathscr{G}$ Then

$$(\mathrm{R}^k f_*)i_*\mathscr{G} = \mathrm{R}^k(f \circ i)_*\mathscr{G} = j_*\mathrm{R}^k h_*\mathscr{G}$$

Hence it suffices to consider h, But

$$(\mathrm{R}^h_*\mathscr{G})_V = \mathrm{R}^h(G|_{f^{-1}(V)}) = 0, \quad k > 0$$

so $\mathrm{supp}\, \mathrm{R}^k h_*\mathscr{G} \subseteq f(W)\backslash V$ if $k > 0$.

2) If f is finite, let us write down the coefficients of $\xi(\dim = a)$ of both sides, namely

$$l\left((f_*\mathscr{F})_\xi\right) = \sum_{\eta \in f^{-1}(\xi), \dim \bar{\eta} = a} l(F_\eta) \cdot [\mathbb{K}(\bar{\eta}) : \mathbb{K}(\overline{f(\eta)})].$$

By additivity, one reduces to the case when X is affine and $\mathscr{F} = \mathcal{O}_X / p$.

For general cases cause Lem 1.14 and the case f is finite. Q.E.D.

1.16 Suppose $f : X \to Y, Y \in \mathrm{Sm}_{\mathbb{k}}$ and X is closed in $Z \in \mathrm{Sm}_{\mathbb{k}}$. Denote by $j : X \to Z \times Y$ is the graph map. For any $C \in \mathcal{Z}_a(X), D \in \mathcal{Z}_b(Y)$ s.t. C and $f^{-1}(D)$ i.p., define

$$C \cdot_f D = j_*^{-1}(j(c) \cdot (Z \times D)) \in \mathcal{Z}_{a+b-\dim Y}(X)$$
$$f^*(D) = X \cdot_f D \quad (C \coloneqq X)$$

1.17 In the context above, for $\mathscr{F} \in \mathscr{K}_a(X), \mathscr{G} \in \mathscr{K}_b(Y)$, if $\mathscr{F}, f^*\mathscr{G}$ i.p., we have

$$\mathcal{Z}_a(\mathscr{F}) \cdot \mathcal{Z}_b(\mathscr{G}) = \sum_{i=0}^{\dim r} (-1)^i \mathcal{Z}_{a+b-\dim Y}\left(\mathrm{L}_i\left(\mathscr{F}_i \otimes f^*\right)\mathscr{G}\right).$$

1.18 Let $X \in \mathrm{Sm}_{\mathbb{k}}$, $\mathscr{F} \in \mathscr{K}_a(X), \mathscr{G} \in K_b(X)$ and $\mathscr{F}, \mathscr{G}$ i.p. Denote by $\Delta : X \to X \times X$ the diagond map, we have

$$\Delta^*\left(\mathcal{Z}_a(\mathscr{F}) \times \mathcal{Z}_b(\mathscr{G})\right) = \mathcal{Z}_a(\mathscr{F}) \cdot \mathcal{Z}_b(\mathscr{G}).$$

1.19 f^* is compatible with intersection product and $f^* \circ g^* = (g \circ f)^*$.

Proof. of Prop 1.17:

Denote by $\pi_2 : Z \times Y \to Y$ the second projection. By linearity,

$$\mathcal{Z}_a(\mathscr{F}) \cdot_f \mathcal{Z}_b(\mathscr{G}) = j_*^{-1}\left(\mathcal{Z}_a(j_*\mathscr{F}) \cdot \mathcal{Z}_{b+\dim Z}(\pi_2^*\mathscr{G})\right),$$

$j : X \to Z \times Y$.

Suppose $\mathscr{L}^\bullet \to \mathscr{G}$ is the locally free resolution of $\mathscr{G}$. Note that $\forall i \geqslant 0$, we have

$$j^*\left(j_*\mathscr{F} \otimes_{f^*} \mathscr{L}\right) = \mathscr{F} \otimes_{f^*} \mathscr{L}_i,$$

which induces an isomurphism

$$i_*\mathscr{F} \otimes \pi_2^*\mathscr{L} = j_*\left(F\mathscr{F} \otimes f^*\mathscr{L}\right).$$

Hence

$$\mathcal{T}or_i^{\mathcal{O}_{Z\times Y}}(j_*\mathscr{F}, \pi_2\mathscr{G}) = j_*\mathscr{L}(\mathscr{F} \otimes f^*)\mathscr{G},$$

So

$$j_*^{-1}\mathcal{Z}_{a+b-\dim Y}(\mathcal{T}or_i^{\mathcal{O}_{Z\times Y}}(j_*\mathscr{F}, \pi_2^*\mathscr{G})) = \mathcal{Z}_{a+b-\dim Y}(\mathscr{L}(\mathscr{F}\otimes f^*)\mathscr{G}).$$

So the statement follows. Q.E.D.

1.20 Let $\mathfrak{A}$ be an abelian category enough projective (resp, injecting) objects and F be a right (resp. left) exact functor from $\mathfrak{A}$. Suppose $\mathcal{C}$ is a homology complex in $\mathfrak{A}$. Then there is a double complex $\mathcal{M}_{ij}$ in $\mathfrak{A}$ s.t.

$$'E^2_{p,q} = \mathrm{L}_p\mathrm{F}H_q(\mathcal{C}) \quad \left(\text{resp. } 'E^2_{p,q} = \mathrm{R}^{-p}\mathrm{F}\left(H_q(\mathcal{C})\right)\right)$$

Proof. Use Cartan-Eilenberg(See YM, Methods of homological algebra, proposition II, page. 210) resolution $\mathcal{C}_{\bullet,\bullet} \to \mathcal{C}$ and consider the/double complex $\mathrm{F}\mathcal{C}_\bullet$. When F is right flat. Q.E.D.

1.21 Suppose $f : X \to Y$ is in $\mathrm{Sm}/\mathbb{k}$, $\mathscr{F} \in \mathscr{K}_a(X)$, $\mathscr{G} \in \mathscr{K}_b(Y)$ and all intersections needed are proper. ne have

$$z_a(\mathscr{F}) \cdot_f \mathcal{Z}_b(\mathscr{G}) = \mathcal{Z}_a(\mathscr{F}) \cdot_{f^*} \mathcal{Z}_b(\mathscr{G}).$$

Proof. We may assume X is affine. Let $\mathscr{L}^\bullet \to \mathscr{G}$ be a free resolution and apply lem 1.20 to $f^*\mathscr{L}^\bullet$ and $\mathscr{F} \otimes -$. We find a double complex s.t.

$$'E^2_{p,q} = \mathcal{T}or_p\left(\mathscr{F}, \mathscr{L}_q f^*\mathscr{G}\right), \quad ''E^2_{p,q} = \mathscr{L}_p\left(\mathscr{F} \otimes f^*\right)\mathscr{G}.$$

Q.E.D.

1.22 Let $X \subseteq Z$, and $Y, Z \in \mathrm{Sm}_{\mathbb{k}}$ and $f : X \to Y$ proper. Suppose $\mathscr{F} \in \mathscr{K}_a(x)$, $\mathscr{G} \in \mathscr{K}_b(Y)$, $\mathscr{F}$, $f^*\mathscr{G}$ i.p.. Then

$$f_*\left(\mathcal{Z}_a(\mathscr{F}) \cdot \mathcal{Z}_b(\mathscr{G})\right) = \left(f_*\mathcal{Z}_a(\mathscr{F})\right) \cdot \mathcal{Z}_b(\mathscr{G}).$$

Proof. Q.E.D.

Pick $\mathscr{L}^\bullet \to \mathscr{G}$ a resolution and Lem 1.20 to $\mathscr{F} \otimes f^*L^\bullet$ and f_*. We have a double complex $\mathcal{M}_{ij}$ s.t.

$$'E^2_{p,q} = \mathrm{R}^{-P}\mathscr{F}_*L_q\left(\mathscr{F} \otimes f^*\right)\mathscr{G}.$$

On the other hand,

$$H_q\left(M_{\bullet,n}\right) = \mathrm{R}^{-q}f_*\left(\mathscr{F} \otimes f^*L_n\right) = (\mathrm{R}^{-q}f_*\mathscr{F}) \otimes L_n.$$

so $''E_{p,q} = \mathcal{T}or_p(\mathrm{R}^{-q}f_*\mathscr{F}, \mathscr{G})$

2 Sheaves with Transfers

we fix an $S \in S_m/k$, called the base scheme Def 2. 1 Let X,Ytsm/s. We define the grows of finite correspondences

Ex2.1 For any $f : x \to 1$), the yroph $\int_f = (x, f(x)) \subseteq XXY$ is a finite colrespondence from X to Y, Ex.2.2 If $f : x \to y$ is finite and dimadimis, the guph C_f is also a fiwite rotrespondens fwh Y YX. Def2.3 Define anadditie categury (ors, whose objerts are the same as su/s und Cors (x, y) is defined in Def 2.1. Contraraviant additine functors

$$\text{Filors} \to \text{op}$$

are called presheaver with transfers. The corresponding category is denoted by psuss). we have a functor r : im/s $\to$ lors by Ex 2 .2. Ex2. 4 Fuery $x \in$ sim/s give an element. $\mathbb{Z}(x) \in \mathrm{Psh}(S)$ by $\mathbb{Z}(x)(y) = \mathrm{Cor}_s(y, x)$. ($\mathbb{Z}(s) = \mathbb{Z}$)

Ex2.5 The presheaves o and 0^* are in. psh (s). For awy $c \in \mathrm{cor}_s(x, y)$ and $f \in O(y)$ (resp. $O^*(y)$)

$$\begin{aligned} &C \xrightarrow{i} X_{\mathcal{L}\pi_1} \times Y \overline{\pi_2} Y \text{ define } O(c)(t) \\ &= \mathrm{Tr}_{(/x}\left((P_2 0 i)^{tx}(f)\right) \\ &(\text{resp. Nc/X } \left((\eta_2 \circ i)^t(t)\right) \end{aligned}$$

Def 2.6 Let us describe the composition in loos. Suppose $f \in \mathrm{Cor}(X/Y)$ and $g \in \mathrm{lor}(y, Z)$. xxz^{-113} Define $xx_5Y_5 \xrightarrow{P_{23}} y, y$

$$\mathrm{gl} = P_{13*}\left(P_{23}^*(g) \cdot P_{12}^*(t)\right)$$

(Check all intersection are proper). prop 2.7 The composition law is association. proof. Suppose $x \leftrightarrows y \xrightarrow{g}, z \underset{h}{\xrightarrow{h}} w$ are morplisms in tors. We have Cartesian squares

$$\begin{array}{ll} xyzw \to xzw & xyzw \to xyw \\ \frac{1}{2} \to x^2 z & \quad yzw \to yw \end{array}$$

2.1 Let $f : X \to Y$ be a proper morphism between finite type schemes/$\mathbb{k}$ and $\mathscr{F} \in \mathscr{K}_a(X)$.

1) $f_* \mathscr{F} \in \mathscr{K}_a(Y)$ and $R^i f_* \mathscr{F} \in \mathscr{K}_{a-1}(Y)$, $i > 0$.
2) $f_* \mathcal{Z}_a(\mathscr{F}) = \mathcal{Z}_a(f_* \mathscr{F})$.