Cart-Pole: Equations of Motion

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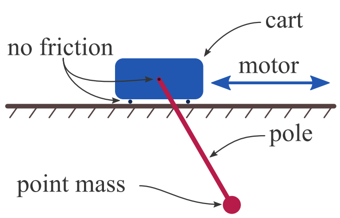
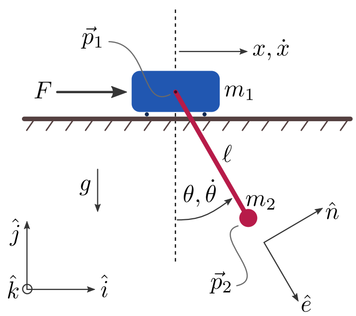
System:

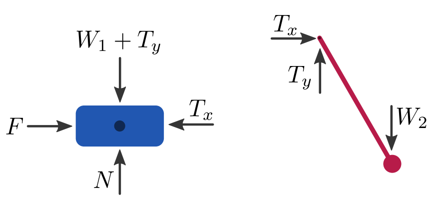


Symbols:

NOTATION
$\dot{x} = \frac{d}{dt}x(t)$



Free-body Diagrams:



Dynamics:

Eqn 1 - force balance on the cart:

$$(F - T_x)\,\hat{\imath} + (N - W_1 - T_y)\hat{\jmath} = m_1\,\ddot{\vec{p}}_1$$

Eqn 2 - force balance on the pole:

$$(T_x)\,\hat{\imath} + (T_y - W_2)\hat{\jmath} = m_2\,\ddot{\vec{p}}_2$$

Eqn 3 - torque balance on pole about pivot:

$$(\vec{p}_2 - \vec{p}_1) \times (-W_2\,\hat{\jmath}) = (\vec{p}_2 - \vec{p}_1) \times (m_2\,\ddot{\vec{p}}_2)$$

Kinematics:

$$\vec{p}_1 = x\,\hat{\imath} \qquad \vec{p}_2 = \vec{p}_1 + \ell\,\hat{e}$$
$$\dot{\vec{p}}_1 = \dot{x}\,\hat{\imath} \qquad \dot{\vec{p}}_2 = \dot{\vec{p}}_1 + \ell\,\dot{\hat{e}}$$
$$\ddot{\vec{p}}_1 = \ddot{x}\,\hat{\imath} \qquad \ddot{\vec{p}}_2 = \ddot{\vec{p}}_1 + \ell\,\ddot{\hat{e}}$$

Unit Vectors:

$$\begin{aligned}
\hat{e} &= \sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath} \\
\hat{n} &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \\
\dot{\hat{e}} &= \dot{\theta}\,\hat{n} \;=\; \dot{\theta}\cos\theta\,\hat{\imath} + \dot{\theta}\sin\theta\,\hat{\jmath} \\
\dot{\hat{n}} &= -\dot{\theta}\,\hat{e} \;=\; -\dot{\theta}\sin\theta\,\hat{\imath} + \dot{\theta}\cos\theta\,\hat{\jmath} \\
\ddot{\hat{e}} &= \ddot{\theta}\,\hat{n} + \dot{\theta}\,\dot{\hat{n}} \;=\; \ddot{\theta}\,\hat{n} - \dot{\theta}^2\,\hat{e} \\
\ddot{\hat{n}} &= -\ddot{\theta}\,\hat{e} + \dot{\theta}\,\dot{\hat{e}} \;=\; -\ddot{\theta}\,\hat{e} + \dot{\theta}^2\,\hat{n}
\end{aligned}$$

Algebra:

⟶ **Eqn 1 dot with horizontal direction:**

$$F - T_x = m_1\,(\hat{\imath}\cdot\ddot{\vec{p}}_1)$$
$$F - T_x = m_1\,\ddot{x}$$

⟶ **Eqn 2 dot with horizontal direction:**

$$T_x = m_2\,(\hat{\imath}\cdot\ddot{\vec{p}}_2)$$
$$T_x = m_2\left(\ddot{x} + \ell\,\hat{\imath}\cdot\left(\ddot{\theta}\,\hat{n} - \dot{\theta}^2\,\hat{e}\right)\right)$$
$$T_x = m_2\left(\ddot{x} + \ell\,\left(\ddot{\theta}\,\cos\theta - \dot{\theta}^2\,\sin\theta\right)\right)$$

⟶ **Eqn 3 dot with out of page direction:**

$$\hat{k}\cdot\left\{(\ell\hat{e})\times(-m_2\,g\,\hat{\jmath}) = (\ell\hat{e})\times\left(m_2\,\left(\ddot{x}\,\hat{\imath} + \ell\,\ddot{\hat{e}}\right)\right)\right\}$$
$$-m_2\,g\,\ell\sin\theta = m_2\ell\left(\ddot{x}\cos\theta + \ell\ddot{\theta}\right)$$
$$-g\,\sin\theta = \ddot{x}\cos\theta + \ell\,\ddot{\theta}$$

⟶ **Combine Eqn 1 and Eqn 2 to cancel tension:**

$$F - m_1\,\ddot{x} = m_2\,\left(\ddot{x} + \ell\,\ddot{\theta}\,\cos\theta - \,\ell\,\dot{\theta}^2\,\sin\theta\right)$$
$$F = (m_1 + m_2)\,\ddot{x} + m_2\,\ell\,\ddot{\theta}\,\cos\theta - \,m_2\,\ell\,\dot{\theta}^2\,\sin\theta$$

⟶ **Write equations of motion in matrix form:**

$$\boxed{\begin{pmatrix} \cos\theta & \ell \\ m_1 + m_2 & m_2\ell\cos\theta \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -g\sin\theta \\ F + m_2\,\ell\,\dot{\theta}^2\,\sin\theta \end{pmatrix}}$$