

Tutorial 9 - Mathematics for CS

1. players pays \$1 to play a game where three fair dice are rolled. If three 6s are obtained, the player wins \$500. Otherwise the player wins nothing. What are the expected net winnings of this game? Would you want to play this game? Does your answer depend upon how many times you can play the game?

$$E(X - 1) = 499 \times \frac{1}{6^3} + (-1) \left(1 - \frac{1}{6^3}\right) = \frac{499}{216} - \frac{215}{216} = 71/54 = 1.31481481481$$

If we have infinite bets and money then $E(X)$ would tell us it is a good bet. However, if we don't have infinite money, the expected value fail to assess the risk factor (If we don't have money then we can't continue betting while expected value expect us to be able to continue). Calculating the standard deviation yeilds

$$\sigma = \sqrt{499^2 \times \frac{1}{6^3} + (-1)^2(1 - \frac{1}{6^3})} \approx 33.967304541$$

which is very high compared to the expected value.

A common approach is instead to calculate the expected growth of wealth after the bet. Suppose we have \$ M , then the expected growth of W

$$\begin{aligned} r &= e^{E(\ln(X+M) - \ln(M))} = e^{E(\ln(1 + \frac{X}{M}))} \\ &= e^{\ln(1 + \frac{499}{M}) \frac{1}{216} + \ln(1 - \frac{1}{M}) \frac{215}{216}} = \left(1 + \frac{499}{M}\right)^{\frac{1}{216}} \left(1 - \frac{1}{M}\right)^{\frac{215}{216}} \end{aligned}$$

Only for $M > 144$ is the rate of growth at least 1, hence we would only gain when $M \geq 145$. In other words, we would only bet is we have at least \$145 regardless of how many bets we have.

2. Suppose that the random variable X takes the values -2, 1, 4 and 6 with probability values 1/3, 1/6, 1/3 and 1/6 respectively.
 - (a) Find the expectation of X .

$$E(X) = -2 \times \frac{1}{3} + 1 \times \frac{1}{6} + 4 \times \frac{1}{3} + 6 \times \frac{1}{6} = 11/6$$

- (b) Find the variance of X using the formula

$$\text{Var}(X) = E((X - E(X))^2)$$

$$\text{Var}(X) = (-2 - 11/6)^2 \times \frac{1}{3} + (1 - 11/6)^2 \times \frac{1}{6} + (4 - 11/6)^2 \times \frac{1}{3} + (6 - 11/6)^2 \times \frac{1}{6} = \frac{341}{36}$$



- (c) Find the variance of X using the formula

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$\text{Var}(X) = (-2)^2 \times \frac{1}{3} + 1^2 \times \frac{1}{6} + 4^2 \times \frac{1}{3} + 6^2 \times \frac{1}{6} - (11/6)^2 = \frac{341}{36}$$

3. Suppose that you are organizing the game described in slide 7 of Lecture 9, where you charge players \$2 to roll two dice, and then you pay them the difference in the score.

- (a) What is the variance in your profit from each game? If you are playing a game in which you have positive expected winnings, would you prefer a small or a large variance in the winnings?

The expected loss is:

$$E(X) = 0 \times \frac{1}{6} + 1 \times \frac{5}{18} + 2 \times \frac{2}{9} + 3 \times \frac{1}{6} + 4 \times \frac{1}{9} + 5 \times \frac{1}{18} = \frac{35}{18} \approx 1.944$$

$$\text{Var}(X) = 0^2 \times \frac{1}{6} + 1^2 \times \frac{5}{18} + 2^2 \times \frac{2}{9} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{9} + 5^2 \times \frac{1}{18} = \frac{35}{6} \approx 5.83$$

If the expected value is the same, I would you prefer a small variance in the winnings.

- (b) If you fix the dice so that each die has a probability of 0.2 of scoring a 3 and equal probability of 0.16 of scoring the other five numbers, do your expected winnings increase beyond 6 cents per game? Is it a surprise?

$$P(\text{diff} = 0) = 5(0.16^2) + 0.2^2 = 0.168$$

$$P(\text{diff} = 1) = 2(3(0.16 \times 0.16) + 2(0.16 \times 0.2)) = 0.2816$$

$$P(\text{diff} = 2) = 2(2(0.16 \times 0.2) + 2(0.16 \times 0.16)) = 0.2304$$

$$P(\text{diff} = 3) = 2(2(0.16 \times 0.16) + 1(0.16 \times 0.2)) = 0.1664$$

$$P(\text{diff} = 4) = 2(2(0.16 \times 0.16) = 2(0.0256)) = 0.1024$$

$$P(\text{diff} = 5) = 2(0.16 \times 0.16) = 0.0512$$

Hence the expected loss is:

$$\begin{aligned} E(X') &= 0(0.168) + 1(0.2816) + 2(0.2304) + 3(0.1664) + 4(0.1024) + 5(0.0512) \\ &= 0 + 0.2816 + 0.4608 + 0.4992 + 0.4096 + 0.2560 = 1.9072 \end{aligned}$$

The expected earning is: $2 - 1.9072 = 0.0928$

The earning before was already close to 6 cents, hence it is not really a surprise that the earning increases above 6 cents when there is a bias towards the middle.

4. A random variable X has a probability density function $f(x) = A/\sqrt{x}$ for $3 \leq x \leq 4$.

- (a) What is the value of A ?

$$\int_3^4 f(x) dx = \int_3^4 A/\sqrt{x} dx = 1 \Rightarrow 2A\sqrt{x} \Big|_{x=3}^{x=4} = 1$$

$$\Rightarrow 2A(2 - \sqrt{3}) = 1 \Rightarrow A = \frac{1}{2(2 - \sqrt{3})} = \frac{2 + \sqrt{3}}{2}$$



(b) What is the cumulative distribution function of X?

$$F(x) = \int_3^x f(t) dt = \int_3^x A/\sqrt{t} dt = 1 \Rightarrow 2A\sqrt{t} \Big|_{t=3}^{t=x} = (2 + \sqrt{3})(\sqrt{x} - \sqrt{3})$$

(c) What is the expected value of X?

$$E(X) = \int_3^4 ax/\sqrt{x} dx = \int_3^4 a\sqrt{x} dx = 2A\sqrt{x}^3/3 \Big|_{x=3}^{x=4} \approx 3.48803387171$$

(d) What is the standard deviation of X?

$$\text{Var}(X) = \int_3^4 x^2 f(x) dx - E(X)^2 = \int_3^4 \frac{Ax^2}{\sqrt{x}} dx - E(X)^2 = \frac{2A}{5} \sqrt{x}^5 \Big|_{x=3}^{x=4} - E(X)^2 \approx 0.083361$$

$$\sigma = \sqrt{\text{Var}(X)} \approx 0.288724732191$$

(e) What is the median of X?

$$F(x) = 0.5 \Rightarrow (2 + \sqrt{3})(\sqrt{x} - \sqrt{3}) = 0.5 \Rightarrow \sqrt{x} = \frac{1}{2(2 + \sqrt{3})} + \sqrt{3} = \frac{2 + \sqrt{3}}{2}$$

$$\Rightarrow x = \frac{7 + 4\sqrt{3}}{4} \approx 3.4820507$$

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