Huynh Le An Khanh
Vietnamese-German University
Binh Duong, Vietnam

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Tutorial 15 - Mathematics for CS

1. Suppose that $X \sim N(3.2, 6.5)$, $Y \sim N(-2.1, 3.5)$ and $Z \sim N(12, 7.5)$ are independent random variables. Find the probability that

   (a) $X + Y \geq 0$
   Since $X + Y \sim N(3.2 - 2.1, 6.5 + 3.5) = N(1.1, 10)$

   $$P(X + Y \geq 0) = 1 - P(X + Y < 0) = 1 - \Phi\left(\frac{0 - 1.1}{\sqrt{10}}\right) \approx 0.636023785195$$

   (b) $X + Y - 2Z \leq -20$
   Since $X + Y - 2Z \sim N(3.2 - 2.1 - 2 \times 12, 6.5 + 3.5 + 2^2(7.5)) = N(-22.9, 40.5)$

   $$P(X + Y - 2Z \leq -20) = \Phi\left(\frac{-20 - (-22.9)}{\sqrt{40.5}}\right) \approx 0.675693915837$$

   (c) $3X + 5Y \geq 1$
   Since $3X + 5Y \sim N(3(3.2) + 5(-2.1), 3^2(6.5) + 5^2(3.5)) = N(-0.9, 146)$

   $$P(3X + 5Y \geq 1) = 1 - P(3X + 5Y < 1) = 1 - \Phi\left(\frac{1 - (-0.9)}{\sqrt{146}}\right) \approx 0.437525835439$$

   (d) $4X - 4Y + 2Z \leq 25$
   Since $4X - 4Y + 2Z \sim N(4(3.2) - 4(-2.1) + 2(12), 4^2(6.5) + 4^2(3.5) + 2^2(7.5)) = N(45.2, 190)$

   $$P(4X - 4Y + 2Z \leq 25) = \Phi\left(\frac{25 - 45.2}{\sqrt{190}}\right) = 0.0713974528747$$

   (e) $|X + 6Y + Z| \geq 2$
   Since $X + 6Y + Z \sim N(3.2 + 6(-2.1) + 12, 6.5 + 6^2(3.5) + 7.5) = N(2.6, 140)$

   $$P(|X + 6Y + Z| \geq 2) = 1 - \Phi\left(\frac{2 - 2.6}{\sqrt{140}}\right) + \Phi\left(\frac{-2 - 2.6}{\sqrt{140}}\right) \approx 0.868944192685$$

   (f) $|2X - Y - 6| \leq 1$
   Since $2X - Y - 6 \sim N(2(3.2) - (-2.1) - 6, 4(6.5) + (3.5)) = N(2.5, 29.5)$

   $$\begin{aligned} P(|2X - Y - 6| \leq 1) &= P(2X - Y - 6 \leq 1) - P(2X - Y - 6 \leq -1) \\ &= \Phi\left(\frac{1 - 2.5}{\sqrt{29.5}}\right) - \Phi\left(\frac{-1 - 2.5}{\sqrt{29.5}}\right) \approx 0.131550514394 \end{aligned}$$

2. Let $X_1, \ldots, X_{15}$ be independent identically distributed $N(4.5, 0.88)$ random variables, with an average $\bar{X}$.

(a) Calculate $P(4.2 \leq \bar{X} \leq 4.9)$
Since $\bar{X} \sim N(15(4.5), 0.88/\sqrt{15}) = N(67.5, 0.88/\sqrt{15})$

$$P(4.2 \leq \bar{X} \leq 4.9) = \Phi(\frac{4.9 - 4.5}{0.88/\sqrt{15}}) - \Phi(\frac{4.2 - 4.5}{0.88/\sqrt{15}}) \approx 0.867334239354$$

(b) Find the value of c for which $P(4.5 - c \leq \bar{X} \leq 4.5 + c) = 0.99$

$$\begin{aligned} 0.99 = P(4.5 - c \leq \bar{X} \leq 4.5 + c) &= \Phi(\frac{c}{0.88/\sqrt{15}}) - \Phi(\frac{-c}{0.88/\sqrt{15}}) \\ &= 2\Phi(\frac{c}{0.88/\sqrt{15}}) - 1 \Rightarrow c = 0.88\Phi^{-1}(0.995)/\sqrt{15} \approx 0.585267114392 \end{aligned}$$

3. A piece of wire is cut and the length of the wire has a normal distribution with a mean 7.2 m and a standard deviation 0.11 m. If the piece of wire is then cut exactly in half, what are the mean and the standard deviation of the lengths of the two pieces?
Each piece has mean $7.2/2 = 3.6$ m, $\sigma = 0.11/2 = 0.055$ m.

4. Suppose that a fair coin is tossed n times. Estimate the probability that the proportion of heads obtained lies between 0.49 and 0.51 for $n = 100, 200, 500, 1000$ and 2000.

   $\bar{X} \sim B(n,p)/n$, $\mu = 0.5n/n = 0.5$, $\sigma^2 = n(0.5)(0.5)/n^2 = 0.25n/n^2 = 0.25/n$ Approximating with the normal distribution $Y \sim N\left(0.5, \frac{0.25}{n}\right)$.

$$P(0.49 \leq \bar{X} \leq 0.51) \approx \Phi\left(\frac{0.51 - 0.5}{\sqrt{0.25/n}}\right) - \Phi\left(\frac{0.49 - 0.5}{\sqrt{0.25/n}}\right)$$

   For each n:

$$\begin{aligned} n = 100: &\quad Z = \pm 0.2 \Rightarrow P \approx 0.158519418878 \\ n = 200: &\quad Z = \pm 0.283 \Rightarrow P \approx 0.22270258921 \\ n = 500: &\quad Z = \pm 0.447 \Rightarrow P \approx 0.345279153981 \\ n = 1000: &\quad Z = \pm 0.632 \Rightarrow P \approx 0.472910743134 \\ n = 2000: &\quad Z = \pm 0.894 \Rightarrow P \approx 0.628906630477 \end{aligned}$$

5. Suppose that a fair die is rolled 1000 times.

   (a) Estimate the probability that the number of 6s is between 150 and 180.
   $X \sim B(1000, \frac{1}{6})$, $\mu = \frac{1000}{6}$, $\sigma = \sqrt{1000(\frac{1}{6})(\frac{5}{6})}$
   Approximating with the normal distribution $Y \sim N\left(\frac{1000}{6}, 1000(\frac{1}{6})(\frac{5}{6})\right)$

$$P(150 \leq X \leq 180) \approx \Phi\left(\frac{180.5 - \mu}{\sigma}\right) - \Phi\left(\frac{149.5 - \mu}{\sigma}\right) \approx 0.931843503324$$

   (b) What is the smallest value of n for which there is a probability of at least 99% of obtaining at least 50 6s in n rolls of a fair die?
   Approximating with the normal distribution $Y \sim N\left(\frac{n}{6}, \frac{5n}{36}\right)$

$$P(X \geq 50) \approx 1 - \Phi\left(\frac{49.5 - np}{\sqrt{np(1-p)}}\right) = 0.99 \Rightarrow \frac{49.5 - n/6}{\sqrt{5n/36}} \approx \Phi^{-1}(0.01) \Rightarrow n \approx 367.51512$$

6. A multiple-choice test consists of a series of questions, each with four possible answers.

   (a) If there are 60 questions, estimate the probability that a student who guesses blindly at each question will get at least 30 questions right.
   $X \sim B(60, 0.25)$, $\mu = 15$, $\sigma = \sqrt{60(0.25)(0.75)}$
   Approximating with the normal distribution $Y \sim N(\mu, \sigma^2)$

   $$P(X \geq 30) \Rightarrow 1 - \Phi(\frac{29.5 - 15}{3.354}) \approx 4.8605708347 \times 10^{-10}$$

   (b) How many questions are needed in order to be 99% confident that a student who guesses blindly at each question scores no more than 35% on the test?
   Suppose there are n questions.

   $$0.99 = P(X \leq 0.35n) \approx \Phi\left(\frac{0.35n - 0.25n}{\sqrt{n(0.25)(0.75)}}\right)$$

   $$\Rightarrow \frac{0.35n - 0.25n}{\sqrt{n(0.25)(0.75)}} \approx \Phi^{-1}(0.99) \Rightarrow n = 51$$

7. The lifetimes of batteries are independent with an exponential distribution with a mean of 84 days. Consider a random selection of 350 batteries. What is the probability that at least 55 of the batteries have lifetimes between 60 and 100 days?

   $$p = P(60 \leq D \leq 100) = e^{-60/84} - e^{-100/84}$$

   $X \sim B(350, p)$, $\mu = 350p$, $\sigma = \sqrt{350p(1-p)}$
   Approximating with the normal distribution $Y \sim B(350, p)$

   $$P(Y \geq 55) \approx 1 - \Phi\left(\frac{54.5 - \mu}{\sigma}\right) = 0.978575571147$$

Done by Huynh Le An Khanh on November 12, 2025.