Huynh Le An Khanh
Vietnamese-German University
Binh Duong, Vietnam

November 6, 2025

Tutorial 10 - Mathematics for CS

1. Consider the Air Conditioner Maintenance Example in Lecture 10.

		X = service time (hrs)			
		1	2	3	4
Y = number of air conditioner units	1	0.12	0.08	0.07	0.05
	2	0.08	0.15	0.21	0.13
	3	0.01	0.01	0.02	0.07

(a) Suppose that a location has only one air conditioner that needs servicing. What is the conditional probability mass function of the service time required? What are the conditional expectation and standard deviation of the service time?
Let $P(X = x, Y = y)$ be the probability that it takes x hours to service y AC. The probability that $Y = 1$: $P(Y = 1) = 0.12 + 0.08 + 0.07 + 0.05 = 0.32$
Then

$$
\begin{aligned}
P(X = 1|Y = 1) &= \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{0.12}{0.32} = 3/8 \\
P(X = 2|Y = 1) &= \frac{P(X = 2, Y = 1)}{P(Y = 1)} = \frac{0.08}{0.32} = 1/4 \\
P(X = 3|Y = 1) &= \frac{P(X = 3, Y = 1)}{P(Y = 1)} = \frac{0.07}{0.32} = 7/32 \\
P(X = 4|Y = 1) &= \frac{P(X = 4, Y = 1)}{P(Y = 1)} = \frac{0.05}{0.32} = 5/32
\end{aligned}
$$

Expectation knowing $Y = 1$:

$$E(X|Y = 1) = 1 \times 3/8 + 2 \times 1/4 + 3 \times 7/32 + 4 \times 5/32 = \frac{69}{32} = 2.15625$$

Variance knowing $Y = 1$:

$$\mathrm{Var}(X|Y = 1) = 1^2 \times 3/8 + 2^2 \times 1/4 + 3^2 \times 7/32 + 4^2 \times 5/32 - E_X(Y = 1)^2 = 1.1943359375$$

(b) Suppose that a location requires a service time of 2 hours. What is the conditional probability mass function of the number of air conditioner units serviced?
What are the conditional expectation and standard deviation of the number of air conditioner units serviced?

The probability that $X = 2$: $P(X = 2) = 0.08 + 0.15 + 0.01 = 0.24$
Then

$$P(Y = 1|X = 2) = \frac{P(X = 2, Y = 1)}{P(Y = 1)} = \frac{0.08}{0.24} = 1/3$$
$$P(Y = 2|X = 2) = \frac{P(X = 2, Y = 2)}{P(Y = 1)} = \frac{0.15}{0.24} = 5/8$$
$$P(Y = 3|X = 2) = \frac{P(X = 2, Y = 3)}{P(Y = 1)} = \frac{0.01}{0.24} = 1/24$$

Expectation knowing $X = 2$:

$$E(Y|X = 2) = 1 \times 1/3 + 2 \times 5/8 + 3 \times 1/24 = \frac{41}{24}$$

Variance knowing $X = 2$:

$$\mathrm{Var}(Y|X = 2) = 1^2 \times 1/3 + 2^2 \times 5/8 + 3^2 \times 1/24 - E_Y(X = 2)^2 \approx 0.289930556$$

2. Do the exercise on Slide 17 of Lecture 10.

   Let X be the random variable that gives the number of heads in the first and second tosses.
   Let Y be the random variable that gives the number of heads in the third toss.
   Let Z be the random variable that gives the number of head in the second and third tosses.
   Calculate the correlations of the random variables X, Y and Z in the Coin Tossing example. Interpret your results.

Calculate the variances:

$$\mathrm{Var}(X) = 0 \times 1/4 + 1 \times 1/2 + 2 \times 1/4 = 1$$
$$\mathrm{Var}(Y) = 0 \times 1/2 + 1 \times 1/2 = 1/2$$
$$\mathrm{Var}(Z) = 0 \times 1/4 + 1 \times 1/2 + 2 \times 1/4 = 1$$

Calculate the covariances:

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= \left(0 \times 0 \times \frac{1}{8} + 0 \times 1 \times \frac{1}{8} + 1 \times 0 \times \frac{1}{4} + 1 \times 1 \times \frac{1}{4} + 2 \times 0 \times \frac{1}{8} + 2 \times 1 \times \frac{1}{8}\right) - 1 \times 1/2 = 0 \\
\mathrm{Cov}(Y, Z) &= E(YZ) - E(Y)E(Z) \\
&= \left(0 \times 0 \times \frac{1}{4} + 1 \times 0 \times \frac{1}{4} + 1 \times 1 \times \frac{1}{4} + 2 \times 1 \times \frac{1}{4}\right) - 1 \times 1/2 = 0.25 \\
\mathrm{Cov}(X, Y) &= E(XZ) - E(X)E(Z) \\
&= \left(0 \times 0 \times \frac{1}{8} + 0 \times 1 \times \frac{1}{8} + 1 \times 1 \times \frac{2}{8} + 1 \times 2 \times \frac{1}{8} + 2 \times 1 \times \frac{1}{8} + 2 \times 2 \times \frac{1}{8}\right) - 1 \times 1 = 0.25
\end{aligned}$$

Calculate the correlations:

$$
\begin{aligned}
\operatorname{Corr}(X,Y) &= \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = 0 \\
\operatorname{Corr}(Y,Z) &= \frac{\operatorname{Cov}(Y,Z)}{\sqrt{\operatorname{Var}(Y)\operatorname{Var}(Z)}} = \frac{0.25}{\sqrt{1/2}} = \sqrt{2}/4 \approx 0.3535533906 \\
\operatorname{Corr}(X,Z) &= \frac{\operatorname{Cov}(X,Z)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Z)}} = \frac{0.25}{\sqrt{1}} = 1/4
\end{aligned}
$$

Interpretation: X,Y are independent; both Y, Z and X, Z are have possitive correlation, but Y, Z have greater correlation.

3. Two safety inspectors inspect a new building and assign it a "safety score" of 1, 2, 3, or 4. Suppose that the random variable X is the score assigned by the first inspector and the random variable Y is the score assigned by the second inspector, and that they have a joint probability mass function given below.

Y \ X	1	2	3	4
1	0.09	0.03	0.01	0.01
2	0.02	0.15	0.03	0.01
3	0.01	0.01	0.24	0.04
4	0.00	0.01	0.02	0.32

(a) What is the probability that both inspectors assign the same safety score?

$$P(X = Y) = 0.09 + 0.15 + 0.24 + 0.32 = 0.8$$

(b) What is the probability that the second inspector assigns a higher safety score than the first inspector?

$$P(X < Y) = 0.02 + 0.01 + 0.00 + 0.01 + 0.01 + 0.02 = 0.07$$

(c) What are the marginal probability mass function, expectation, and variance of the score assigned by the first inspector?

$$
\begin{aligned}
P(X = 1) &= 0.09 + 0.02 + 0.01 + 0.00 = 0.12 \\
P(X = 2) &= 0.03 + 0.15 + 0.01 + 0.01 = 0.2 \\
P(X = 3) &= 0.01 + 0.03 + 0.24 + 0.02 = 0.3 \\
P(X = 4) &= 0.01 + 0.01 + 0.04 + 0.32 = 0.38
\end{aligned}
$$

$$E(X) = 1 \times 0.12 + 2 \times 0.2 + 3 \times 0.3 + 4 \times 0.38 = 2.94$$

$$\operatorname{Var}(X) = E(X^2) - E(X)^2 = 1^2 \times 0.12 + 2^2 \times 0.2 + 3^2 \times 0.3 + 4^2 \times 0.38 - 2.94^2 = 1.0564$$

(d) What are the marginal probability mass function, expectation, and variance of the score assigned by the second inspector?

$$P(Y=1) = 0.09 + 0.03 + 0.01 + 0.01 = 0.14$$
$$P(Y=2) = 0.02 + 0.15 + 0.03 + 0.01 = 0.21$$
$$P(Y=3) = 0.01 + 0.01 + 0.24 + 0.04 = 0.3$$
$$P(Y=4) = 0.00 + 0.01 + 0.02 + 0.32 = 0.35$$

$$E(Y) = 1 \times 0.14 + 2 \times 0.21 + 3 \times 0.3 + 4 \times 0.35 = 2.86$$
$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = 1^2 \times 0.14 + 2^2 \times 0.21 + 3^2 \times 0.3 + 4^2 \times 0.35 - 2.86^2 = 1.1004$$

(e) Are the scores assigned by the two inspectors independent of each other? Would you expect them to be independent? How would you interpret the situation if they were independent?

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\ &= 0.09 + 2 \times (0.03 + 0.02) + 3 \times (0.01 + 0.01) + 4 \times (0.01 + 0.15 + 0.00) \\ &+ 6 \times (0.01 + 0.03) + 8 \times (0.01 + 0.01) + 9 \times (0.24) + 12 \times (0.02 + 0.04) \\ &\qquad + 16 \times 0.32 - 2.94 \times 2.86 = 9.29 - 8.4084 = 0.8816\end{aligned}$$

Correlation:

$$\mathrm{Corr}(X,Y) = \frac{0.8816}{\sqrt{1.0564 \times 1.1004}} \approx 0.817678$$

Since $\mathrm{Corr}(X,Y) \neq 0$, the scores are not independent. Since they are jugding the same metric (safety), the variables should be strongly dependent.
If the variables are independent, I would suspect that their definitions of "safety" are different.

(f) If the first inspector assigns a score of 3, what is the conditional probability mass function of the score assigned by the second inspector?

$$P(Y=1|X=3) = \frac{P(X=3,Y=1)}{P(X=3)} = \frac{0.09}{0.3} = 3/10$$
$$P(Y=2|X=3) = \frac{P(X=3,Y=2)}{P(X=3)} = \frac{0.03}{0.3} = 1/10$$
$$P(Y=3|X=3) = \frac{P(X=3,Y=3)}{P(X=3)} = \frac{0.24}{0.3} = 4/5$$
$$P(Y=4|X=3) = \frac{P(X=3,Y=4)}{P(X=3)} = \frac{0.02}{0.3} = 2/30$$

(g) What is the covariance of the scores assigned by the two inspectors?

$$\mathrm{Cov}(X,Y) = 0.8816$$

(h) What is the correlation between the scores assigned by the two inspectors? If you are responsible for training the safety inspectors to perform proper safety evaluations of buildings, what correlation value would you like there to be between the scores of two safety inspectors?

$$\mathrm{Corr}(X,Y) = \frac{0.8816}{\sqrt{1.0564 \times 1.1004}} \approx 0.817678$$

If I were responsible for training the safety inspectors to perform proper safety evaluations of buildings, I would like correlation value be between 0 and 1, ideally closer to 1, because that mean that they are scoring base on the same aspects.

Done by Huynh Le An Khanh on November 6, 2025.