Introduction to Probability

Tutorial 11

1. Suppose that the random variables X, Y and Z are independent with $E(X) = 2$, Var(X) $= 4$, $E(Y) = -3$, Var(Y) $= 2$, $E(Z) = 8$ and Var(Z) $= 7$. Calculate the expectation and variance of the following random variables.

   (a) $3X + 7$ (b) $4X - 3Y$ (c) $5X - 9Z + 8$ (d) $X + 2Y + 3Z$

2. Recall that for any function $g(X)$ of a random variable X,

$$E(g(X)) = \int g(x)\, f(x)\, dx,$$

   where $f(x)$ is the probability density function of X. Use this result to show that

$$E(aX + b) = aE(X) + b$$

   and

$$\mathrm{Var(aX + b)} = \mathrm{a^2\, Var(X)}.$$

3. Suppose that components are manufactured such that their heights are independent of each other with $\mu = 65.9$ and $\sigma = 0.32$.

   (a) What are the mean and the standard deviation of the average height of five components?

   (b) If eight components are stacked on top of each other, what are the mean and the standard deviation of the total height?

4. If \$x is invested in mutual fund A, the annual return has an expectation of $\$0.1x$ and a standard deviation of $\$0.02x$. If \$x is invested in mutual fund B, the annual return has an expectation of $\$0.1x$ and a standard deviation of $\$0.03x$. Suppose that the returns on the two funds are independent of each other and that I have \$1000 to invest.

   (a) What are the expectation and variance of my annual return if I invest all my money in fund A?

   (b) What are the expectation and variance of my annual return if I invest all my money in fund B?

   (c) What are the expectation and variance of my total annual return if I invest half of my money in fund A and half in fund B?

   (d) Suppose I invest \$x in fund A and the rest of my money in fund B. What value of x minimizes the variance of my total annual return?
   Explain why your answers illustrate the importance of diversity in an investment strategy.

5. Suppose that the random variable X has a probability density function

$$f(x) = 2x$$

for $0 \leq x \leq 1$. Find the probability density function and the expectation of the random variable Y in the following cases.

(a) $Y = X^3$ (b) $Y = \sqrt{X}$ (c) $Y = \dfrac{1}{1+X}$ (d) $Y = 2^X$