Huynh Le An Khanh
Vietnamese-German University
Binh Duong, Vietnam

November 7, 2025

Tutorial 11 - Mathematics for CS

1. Suppose that the random variables X, Y and Z are independent with $E(X) = 2$, $\mathrm{Var}(X) = 4$, $E(Y) = -3$, $\mathrm{Var}(Y) = 2$, $E(Z) = 8$ and $Var(Z) = 7$. Calculate the expectation and variance of the following random variables.

   (a) $3X + 7$
   $$E(3X + 7) = 3E(X) + 7 = 3(2) + 7 = 13$$
   $$\mathrm{Var}(3X + 7) = 3^2\mathrm{Var}(X) = 9(4) = 36$$

   (b) $4X - 3Y$
   $$E(4X - 3Y) = 4E(X) - 3E(Y) = 4(2) - 3(-3) = 8 + 9 = 17$$
   Since X and Y are independent, $\mathrm{Corr}(X, Z) = 0$ and hence
   $$\mathrm{Var}(4X - 3Y) = 4^2\mathrm{Var}(X) + (-3)^2\mathrm{Var}(Y) = 16(4) + 9(2) = 64 + 18 = 82$$

   (c) $5X - 9Z + 8$
   $$E(5X - 9Z + 8) = 5E(X) - 9E(Z) + 8 = 5(2) - 9(8) + 8 = 10 - 72 + 8 = -54$$
   Since X and Z are independent, $\mathrm{Corr}(X, Z) = 0$ and hence
   $$\mathrm{Var}(5X - 9Z + 8) = 5^2\mathrm{Var}(X) + (-9)^2\mathrm{Var}(Z) = 25(4) + 81(7) = 100 + 567 = 667$$

   (d) $X + 2Y + 3Z$
   $$E(X + 2Y + 3Z) = E(X) + 2E(Y) + 3E(Z) = 2 + 2(-3) + 3(8) = 2 - 6 + 24 = 20$$
   Since X, Z and Y, Z are independent pairs, $X + 2Y$, $3Z$ are independent, and since X and Y are independent,
   $$\begin{aligned}\mathrm{Var}(X + 2Y + 3Z) = \mathrm{Var}(X + 2Y) + 3^2\mathrm{Var}(Z) &= \mathrm{Var}(X) + 2^2\mathrm{Var}(Y) + 3^2\mathrm{Var}(Z)\\ &= 4 + 4(2) + 9(7) = 4 + 8 + 63 = 75\end{aligned}$$

2. Recall that for any function $g(X)$ of a random variable X,
   $$E(g(X)) = \int g(x)f(x)\,dx$$
   where $f(x)$ is the probability density function of X. Use this result to show that
   $$E(aX + b) = aE(X) + b \tag{1}$$
   and
   $$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X) \tag{2}$$

$$E(aX+b) = \int (ax+b)f(x)\,dx = a\int xf(x)\,dx + b\int f(x)\,dx = aE(X)+b$$

$$\begin{aligned}
\mathrm{Var}(aX+b) &= E((aX+b)^2) - (E(aX+b))^2 \\
&= E(a^2X^2 + 2abX + b^2) - (aE(X)+b)^2 \\
&= a^2E(X^2) + 2abE(X) + b^2 - (a^2E(X)^2 + 2abE(X) + b^2) \\
&= a^2(E(X^2) - E(X)^2) = a^2\mathrm{Var}(X)
\end{aligned}$$

3. Suppose that components are manufactured such that their heights are independent of each other with $\mu = 65.9$ and $\sigma = 0.32$.

   (a) What are the mean and the standard deviation of the average height of five components?
   Let $\bar{X}$ be the sample mean of 5 independent components.

   $$E(\bar{X}) = \mu = 65.9$$
   $$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{0.32^2}{5} = 0.02048$$
   $$\sigma(\bar{X}) = \sqrt{\mathrm{Var}(\bar{X})} = \sqrt{0.02048} \approx 0.1431083506$$

   (b) If eight components are stacked on top of each other, what are the mean and the standard deviation of the total height?
   Let $T = X_1 + X_2 + \cdots + X_8$

   $$E(T) = 8\mu = 8(65.9) = 527.2$$
   $$\mathrm{Var}(T) = 8\sigma^2 = 8(0.32^2) = 0.8192$$
   $$\sigma(T) = \sqrt{\mathrm{Var}(T)} = \sqrt{0.8192} \approx 0.9050966799$$

4. If \$$x$ is invested in mutual fund A, the annual return has an expectation of \$$0.1x$ and a standard deviation of \$$0.02x$. If \$$x$ is invested in mutual fund B, the annual return has an expectation of \$$0.1x$ and a standard deviation of \$$0.03x$. Suppose that the returns on the two funds are independent of each other and that I have \$1000 to invest.

   (a) What are the expectation and variance of my annual return if I invest all my money in fund A?

   $$E(R_A) = 0.1(1000) = 100$$
   $$\mathrm{Var}(R_A) = (0.02 \times 1000)^2 = 400$$

   (b) What are the expectation and variance of my annual return if I invest all my money in fund B?

   $$E(R_B) = 0.1(1000) = 100$$
   $$\mathrm{Var}(R_B) = (0.03 \times 1000)^2 = 900$$

   (c) What are the expectation and variance of my total annual return if I invest half of my money in fund A and half in fund B?

   $$E(R_T) = 0.1(500) + 0.1(500) = 100$$
   $$\mathrm{Var}(R_T) = (0.02 \times 500)^2 + (0.03 \times 500)^2 = 100 + 225 = 325$$
   $$\sigma(R_T) = \sqrt{325} = 18.03$$

(d) Suppose I invest \$x in fund A and the rest of my money in fund B. What value of x minimizes the variance of my total annual return? Explain why your answers illustrate the importance of diversity in an investment strategy.

Let total return be

$$E(R_T) = 0.1x + 0.1(1000 - x) = 0.1(1000) = 100$$

$$\begin{aligned}\operatorname{Var}(R_T) = (0.02x)^2 + (0.03(1000-x))^2 &= \frac{x^2}{1/0.0004} + \frac{(1000-x)^2}{1/0.0009} \\ &\overset{AM-GM}{\geq} \frac{(x + 1000 - x)^2}{1/0.0004 + 1/0.0009} = \frac{3600}{13}\end{aligned}$$

Equality happens iff

$$\frac{x}{1/0.0004} = \frac{1000-x}{1/0.0009} \Rightarrow 0.0004x = 0.0009(1000-x) \Rightarrow x = 0.9/0.0013 \approx 692.3076923077$$

This shows the importance of diversification to reduces total risk (variance).

5. Suppose that the random variable X has a probability density function $f(x) = 2x$ for $0 \leq x \leq 1$. Find the probability density function and the expectation of the random variable Y in the following cases.

(a) $Y = X^3$

Note: Since Y is monotonic for all of these cases, $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

Let $x = y^{1/3}$, then $\frac{dx}{dy} = \frac{1}{3}y^{-2/3}$

$$f_Y(y) = f_X(y^{1/3})\left|\frac{dx}{dy}\right| = 2y^{1/3}\frac{1}{3}y^{-2/3} = \frac{2}{3}y^{-1/3}, \quad 0 \leq y \leq 1$$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y\frac{2}{3}y^{-1/3}\,dy = \frac{2}{3}\int_0^1 y^{2/3}\,dy = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5}$$

(b) $Y = \sqrt{X}$

Let $x = y^2$, then $\frac{dx}{dy} = 2y$

$$f_Y(y) = f_X(y^2)|2y| = 2y^2(2y) = 4y^3, \quad 0 \leq y \leq 1$$

$$E(Y) = \int_0^1 y(4y^3)\,dy = 4\int_0^1 y^4\,dy = \frac{4}{5}$$

(c) $Y = \frac{1}{1+x}$

Let $x = \frac{1-y}{y}$, then $\frac{dx}{dy} = -\frac{1}{y^2}$

$$f_Y(y) = f_X\left(\frac{1-y}{y}\right)\left|\frac{dx}{dy}\right| = 2\left(\frac{1-y}{y}\right)\frac{1}{y^2} = \frac{2(1-y)}{y^3}, \quad \frac{1}{2} \leq y \leq 1$$

$$\begin{aligned}E(Y) = \int_{1/2}^1 y\frac{2(1-y)}{y^3}\,dy = 2\int_{1/2}^1 \frac{1-y}{y^2}\,dy = 2\left(\int_{1/2}^1 \frac{1}{y^2} - \frac{1}{y}\,dy\right) \\ = 2\left(-y^{-1} - \ln y\right)\Big|_{y=1/2}^{y=1} = 2(-1-(-2)-\ln 2) = 2 - 2\ln 2\end{aligned}$$

(d) $Y = 2^X$

Let $x = \log_2 y$, then $\frac{dx}{dy} = \frac{1}{y \ln 2}$

$$f_Y(y) = f_X\left(\log_2 y\right)\left|\frac{dx}{dy}\right| = f_X(\log_2 y)\frac{1}{y \ln 2} = \frac{2 \log_2 y}{y \ln 2}, \quad 1 \le y \le 2$$

$$\begin{aligned} E(Y) = \int_1^2 y \frac{2 \log_2 y}{y \ln 2}\, dy = \frac{2}{\ln 2}\int_1^2 \log_2 y \, dy = \frac{2}{(\ln 2)^2}\ (y \ln y - y)\big|_{y=1}^{y=2} \\ = \frac{2}{(\ln 2)^2}(2 \ln 2 - 1) \end{aligned}$$

Done by Huynh Le An Khanh on November 7, 2025.