CS 4510 Automata and Complexity 3/05/2022

Solutions for Worksheet 5: Pumping Lemma for CFG

1. Show that L is not context free.
   L = $\{0^n \quad |$ where n is prime.$\}$

   Solution:

   Assuming L is context free,
   there is a pumping length p such that any string $s \in L$ of length $\geq p$ can be written as $s = uvxyz$, where $vy \neq \varepsilon$, $|vxy| \leq p$, and for all $i \geq 0, uv^i xy^i z \in L$.

   Let $s = uvxyz = 0^p$, where p is prime.

   If v is 0^a and y is 0^b $(1 \leq a + b \leq p)$, $uv^{1+p}xy^{1+p}z$ will be $0^{p+p(a+b)} = 0^{p(1+a+b)} \notin L$, where $p(1 + a + b)$ is not prime.

   Therefore, L is not context free by contradiction.

2. Show that L is not context free.
L = $\{w \quad | \text{ where } w \in \{0,1\}^*$, w is a palindrome with an equal # of 0's and 1's.$\}$

Solution:

Assuming L is context free,
there is a pumping length p such that any string $s \in L$ of length $\geq p$ can be written as $s = uvxyz$, where $vy \neq \varepsilon$, $|vxy| \leq p$, and for all $i \geq 0, uv^ixy^iz \in L$.

Let $s = uvxyz = 0^p1^{2p}0^p$.

Case 1: v or y contains multiple types of characters.
Then, uv^2xy^2z will contain characters out of order. Therefore, $uv^2xy^2z \notin L$.

Case 2: v and y contain the same character.
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^p1^{2p}0^p$, uv^2xy^2z will be $0^{p+a}1^{2p}0^p \notin L$, where # of 0's and 1's are different.
If vy is 1^a $(1 \leq a \leq p)$, and it is in $0^p1^{2p}0^p$, uv^2xy^2z will be $0^p1^{2p+a}0^p \notin L$, where # of 0's and 1's are different.
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^p1^{2p}0^p$, uv^2xy^2z will be $0^p1^{2p}0^{p+a} \notin L$, where # of 0's and 1's are different.

Case 3: v and y contain different characters.
If v is 0^a and y is 1^b $(2 \leq a+b \leq p$ and $a, b \geq 1)$, and it is in $0^p1^{2p}0^p$, uv^2xy^2z will be $0^{p+a}1^{2p+b}0^p \notin L$, which is not palindrome. (# of 0's and 1's can be the same here.)
If v is 1^a and y is 0^b $(2 \leq a+b \leq p$ and $a, b \geq 1)$, and it is in $0^p1^{2p}0^p$, uv^2xy^2z will be $0^p1^{2p+a}0^{p+b} \notin L$, which is not palindrome. (# of 0's and 1's can be the same here.)

Therefore, L is not context free by contradiction.

3. Show that L is not context free.
L = $\{w \quad |$ where $w \in \{0,1\}^*$, w has length 2 mod 3, and
the characters at position $\ulcorner \frac{n}{3} \urcorner$ and $\ulcorner \frac{2n}{3} \urcorner$ are 0's.$\}$

Solution:

Assuming L is context free,
there is a pumping length p such that any string $s \in L$ of length $\geq p$ can be written as $s = uvxyz$, where $vy \neq \varepsilon$, $|vxy| \leq p$, and for all $i \geq 0, uv^ixy^iz \in L$.

Let $s = uvxyz = 1^p01^p01^p$.

Case 1: v or y contains either of the 0's.
Then, uxz contains less than two 0's which does not satisfy the second requirement. (At least two 0's are required.) Therefore, $uxz \notin L$.

Case 2: v and y contain only the 1's.
To satisfy the second requirement: *the characters at position $\ulcorner \frac{n}{3} \urcorner$ and $\ulcorner \frac{2n}{3} \urcorner$ are 0's,*
the three strings before/between/after 0's must have the same length.
However, pumping up/down can only change the length of two out of three strings.

Therefore, L is not context free by contradiction.

4. Show that L is not context free.
   L $= \{w\overline{w} \mid$ where $w \in \{0,1\}^*$, $\overline{w}$ is a complement of $w.\}$

   For example, if $w = 000011$, then $\overline{w} = 111100$.
   The choice of string s is very important to this question.
   Convince yourself that $s = 0^p1^p1^p0^p$ will not work.

Solution:

Assuming L is context free,
there is a pumping length p such that any string $s \in L$ of length $\geq p$ can be written as $s = uvxyz$, where $vy \neq \varepsilon$, $|vxy| \leq p$, and for all $i \geq 0, uv^ixy^iz \in L$.

Let $s = uvxyz = 0^p1^p0^p1^p0^p1^p$.

Case 1: v or y contains multiple types of characters.
Then, uv^2xy^2z will contain characters out of order. Therefore, $uv^2xy^2z \notin L$.

Case 2: v and y contain the same character.
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^p1^p0^p1^p0^p1^p$, uv^2xy^2z will be $0^{p+a}1^p0^p1^p0^p1^p \notin L$.
If vy is 1^a $(1 \leq a \leq p)$, and it is in $0^p1^p0^p1^p0^p1^p$, uv^2xy^2z will be $0^p1^{p+a}0^p1^p0^p1^p \notin L$.
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^p1^p0^p1^p0^p1^p$, uv^2xy^2z will be $0^p1^p0^{p+a}1^p0^p1^p \notin L$.
The following solution process is also the same...

Case 3: v and y contain different characters.
If v is 0^a and y is 1^b $(2 \leq a+b \leq p$ and $a, b \geq 1)$, and it is in $0^p1^p0^p1^p0^p1^p$, uv^2xy^2z will be $0^{p+a}1^{p+b}0^p1^p0^p1^p \notin L$.
If v is 1^a and y is 0^b $(2 \leq a+b \leq p$ and $a, b \geq 1)$, and it is in $0^p1^p0^p1^p0^p1^p$, uv^2xy^2z will be $0^p1^{p+a}0^{p+b}1^p0^p1^p \notin L$.
The following solution process is also the same...

Therefore, L is not context free by contradiction.

5. Show that L is not context free.
   L = $\{0^i 1^j \quad | \text{ where } i \geq 0,\ i^2 = j.\}$

Solution:

Assuming L is context free,
there is a pumping length p such that any string $s \in L$ of length $\geq p$ can be written as $s = uvxyz$, where $vy \neq \varepsilon$, $|vxy| \leq p$, and for all $i \geq 0, uv^ixy^iz \in L$.

Let $s = uvxyz = 0^p 1^{p^2}$.

Case 1: v or y contains multiple types of characters.
Then, uv^2xy^2z will contain characters out of order. Therefore, $uv^2xy^2z \notin L$.

Case 2: v and y contain the same character.
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^p 1^{p^2}$, uv^2xy^2z will be $0^{p+a}1^{p^2} \notin L$.
If vy is 1^a $(1 \leq a \leq p)$, and it is in $0^p 1^{p^2}$, uv^2xy^2z will be $0^p 1^{p^2+a} \notin L$.

Case 3: v and y contain different characters.
If v is 0^a and y is 1^b $(2 \leq a + b \leq p$ and $a, b \geq 1)$, and it is in $0^p 1^{p^2}$, uv^2xy^2z will be $0^{p+a}1^{p^2+b} \notin L$.

For the above, assume $(p+a)^2 = p^2 + b$

$p^2 + 2ap + a^2 = p^2 + b \equiv 2ap + a^2 = b$

Because $p > b, 2ap + a^2 \neq b$ and $(p+a)^2 \neq p^2 + b$

Therefore, L is not context free by contradiction.

6. Show that L is not context free.
L = $\{0^i 1^j 2^k \quad | \text{ where } i, j, k \geq 0, i \times j = k.\}$

Solution:

Assuming L is context free,
there is a pumping length p such that any string $s \in L$ of length $\geq p$ can be written as $s = uvxyz$, where $vy \neq \varepsilon$, $|vxy| \leq p$, and for all $i \geq 0, uv^i xy^i z \in L$.

Let $s = uvxyz = 0^p 1^p 2^{p^2}$.

Case 1: v **or** y **contains multiple types of characters.**
Then, uv^2xy^2z will contain characters out of order. Therefore, $uv^2xy^2z \notin L$.

Case 2: v **and** y **contain the same character.**
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^p 1^p 2^{p^2}$, uv^2xy^2z will be $0^{p+a} 1^p 2^{p^2} \notin L$.
If vy is 1^a $(1 \leq a \leq p)$, and it is in $0^p 1^p 2^{p^2}$, uv^2xy^2z will be $0^p 1^{p+a} 2^{p^2} \notin L$.
If vy is 2^a $(1 \leq a \leq p)$, and it is in $0^p 1^p 2^{p^2}$, uv^2xy^2z will be $0^p 1^p 2^{p^2+a} \notin L$.

Case 3: v **and** y **contain different characters.**
If v is 0^a and y is 1^b $(2 \leq a + b \leq p$ and $a, b \geq 1)$, and it is in $0^p 1^p 2^{p^2}$, uv^2xy^2z will be $0^{p+a} 1^{p+b} 2^{p^2} \notin L$.

For the above, assume $(p + a)(p + b) = p^2$

$p^2 + (a + b)p + ab = p^2 \equiv (a + b)p + ab = 0$

Because $a, b, p > 0$, $(a + b)p + ab \neq 0$ and $(p + a)(p + b) \neq p^2$

If v is 1^a and y is 2^b $(2 \leq a + b \leq p$ and $a, b \geq 1)$, and it is in $0^p 1^p 2^{p^2}$, uv^2xy^2z will be $0^p 1^{p+a} 2^{p^2+b} \notin L$.

For the above, assume $p(p + a) = p^2 + b$

$p^2 + ap = p^2 + b \equiv ap = b$

Because $p > b, ap \neq b$ and $p(p + a) \neq p^2 + b$

Therefore, L is not context free by contradiction.

7. Show that L is not context free.
L $= \{0^i 1^j 2^k 3^r \quad |$ where $i, j, k \geq 0, i + j = k, i = r$ or $j = r$ or both.$\}$

Solution:

Assuming L is context free,
there is a pumping length p such that any string $s \in L$ of length $\geq p$ can be written as $s = uvxyz$, where $vy \neq \varepsilon$, $|vxy| \leq p$, and for all $i \geq 0, uv^ixy^iz \in L$.

Let $s = uvxyz = 0^{p+1}1^p2^{2p+1}3^p$.

Case 1: v or y contains multiple types of characters.
Then, uv^2xy^2z will contain characters out of order. Therefore, $uv^2xy^2z \notin L$.

Case 2: v and y contain the same character.
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^{p+1}1^p2^{2p+1}3^p$, uv^3xy^3z will be $0^{p+1+2a}1^p2^{2p+1}3^p \notin L$.
If vy is 1^a $(1 \leq a \leq p)$, and it is in $0^{p+1}1^p2^{2p+1}3^p$, uv^3xy^3z will be $0^{p+1}1^{p+2a}2^{2p+1}3^p \notin L$.
If vy is 2^a $(1 \leq a \leq p)$, and it is in $0^{p+1}1^p2^{2p+1}3^p$, uv^3xy^3z will be $0^{p+1}1^p2^{2p+1+2a}3^p \notin L$.
If vy is 3^a $(1 \leq a \leq p)$, and it is in $0^{p+1}1^p2^{2p+1}3^p$, uv^3xy^3z will be $0^{p+1}1^p2^{2p+1}3^{p+2a} \notin L$.

Case 3: v and y contain different characters.
If v is 0^a and y is 1^b $(2 \leq a + b \leq p$ and $a, b \geq 1)$, and it is in $0^{p+1}1^p2^{2p+1}3^p$, uv^3xy^3z will be $0^{p+1+2a}1^{p+2b}2^{2p+1}3^p \notin L$.
If v is 1^a and y is 2^b $(2 \leq a + b \leq p$ and $a, b \geq 1)$, and it is in $0^{p+1}1^p2^{2p+1}3^p$, uv^3xy^3z will be $0^{p+1}1^{p+2a}2^{2p+1+2b}3^p \notin L$.
If v is 2^a and y is 3^b $(2 \leq a + b \leq p$ and $a, b \geq 1)$, and it is in $0^{p+1}1^p2^{2p+1}3^p$, uv^3xy^3z will be $0^{p+1}1^p2^{2p+1+2a}3^{p+2b} \notin L$.

Therefore, L is not context free by contradiction.

8. Show that L is not context free.
L = $\{ww^Rw \mid \text{where } w \in \{0,1\}^*.\}$

The choice of string s is very important to this question.
Convince yourself that $s = 0^p0^p0^p$ and $s = 0^p110^p0^p1$ will not work.

Solution:

Assuming L is context free,
there is a pumping length p such that any string $s \in L$ of length $\geq p$ can be written as $s = uvxyz$, where $vy \neq \varepsilon$, $|vxy| \leq p$, and for all $i \geq 0, uv^ixy^iz \in L$.

Let $s = uvxyz = 0^p1^p1^p0^p0^p1^p = 0^p1^{2p}0^{2p}1^p$.

Case 1: v or y contains multiple types of characters.
Then, uv^2xy^2z will contain characters out of order. Therefore, $uv^2xy^2z \notin L$.

Case 2: v and y contain the same character.
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^p1^{2p}0^{2p}1^p$, uv^2xy^2z will be $0^{p+a}1^{2p}0^{2p}1^p \notin L$.

The length of uv^2xy^2z is $6p + a$, which means that w is at least $2p$ in length.

Counting from the beginning,

we can say that w has $(p + a)$ 0's and some 1's.

Because there are two more w's remaining, we need $(2p + 2a)$ 0's.

Because there are $2p$ 0's left, this case does not make sense.

If vy is 1^a $(1 \leq a \leq p)$, and it is in $0^p1^{2p}0^{2p}1^p$, uv^2xy^2z will be $0^p1^{2p+a}0^{2p}1^p \notin L$.
Follow the above approach.
If vy is 1^a $(1 \leq a \leq p)$, and it is in $0^p1^{2p}0^{2p}1^p$, uv^2xy^2z will be $0^p1^{2p}0^{2p+a}1^p \notin L$.
Follow the above approach.
If vy is 0^a $(1 \leq a \leq p)$, and it is in $0^p1^{2p}0^{2p}1^p$, uv^2xy^2z will be $0^p1^{2p}0^{2p}1^{p+a} \notin L$.
Follow the above approach.

Continued...

Case 3: v and y contain different characters.
If v is 0^a and y is 1^b ($2 \leq a+b \leq p$ and $a, b \geq 1$), and it is in $0^p1^{2p}0^{2p}1^p$, uv^2xy^2z will be $0^{p+a}1^{2p+b}0^{2p}1^p \notin L$.
Follow the above approach.
If v is 1^a and y is 0^b ($2 \leq a+b \leq p$ and $a, b \geq 1$), and it is in $0^p1^{2p}0^{2p}1^p$, uv^2xy^2z will be $0^p1^{2p+a}0^{2p+b}1^p \notin L$.
Follow the above approach.
If v is 0^a and y is 1^b ($2 \leq a+b \leq p$ and $a, b \geq 1$), and it is in $0^p1^{2p}0^{2p}1^p$, uv^2xy^2z will be $0^p1^{2p}0^{2p+a}1^{p+b} \notin L$.
Follow the above approach.

Therefore, L is not context free by contradiction.