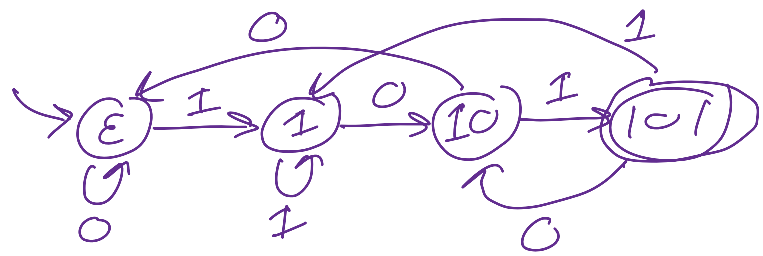
Announcements • HW 1 out, due next thurs **10 pm**

- "Live schedule" • Piazza post "notation"
- Corrected final date

DFA $\{w : w \text{ ends w/ } 101\}$



Not the same DFA

Or build $D = (Q, \Sigma, F, \delta, q_0)$ where

$Q = \{0,1\} \times \{0,1\} \times \{0,1\} = \{0,1\}^3$

$\Sigma = \{0,1\}$

$F = \{101\}$ $\qquad \delta(abc, x) = bcx$

$q_0 = 000$ $\quad$ // 100

Q

δ	0	1
ε	ε	1
1	10	⋮
10	⋮	⋮
101	⋮	⋮

ε

Broad Goal: Understand Computation

What are computational problems?

A "problem" is a function

$P : \Sigma^* \to S$.

often assume $\Sigma = \{0,1\}$

Similar idea.

Always assume $S = \{0,1\}$

Think of them $\{0,1\}^{n = \log_2 |S|}$.

Basic WLOG to assume $\{0,1\}$

Break down into problems "what is the ith bit of the output."

Consider $\Sigma = \{0,1,2,3\}$ $f : \Sigma \to \{0,1\}^2$

If $w = w_1 w_2 \cdots$ in Σ^*, let $f(w) = f(w_1) f(w_2) \ldots$

If $L \subseteq \{0,\ldots,3\}^*$, define $L' = \{f(w) : w \in L\}$.

L is regular iff L' is.

Easier direction $\Rightarrow$

L is regular so $L = \mathcal{L}(M)$ for some DFA M.

Let's build N for L'.

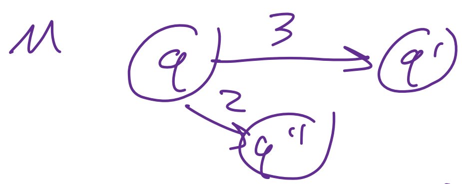
$N.Q = M.Q \times \{\varepsilon, 0, 1\}$

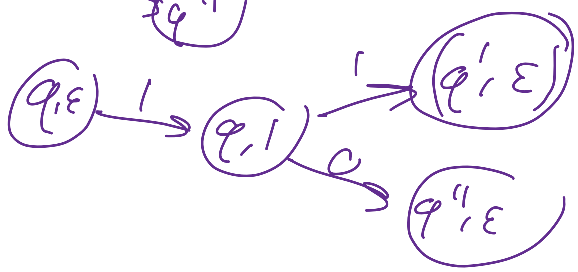
$N.F = M.F \times \{\varepsilon\}$

$N.q_0 = (M.q_0, \varepsilon)$

$N.\Sigma = \{0,1\}$

$$N.\delta((q,a),x) = \begin{cases} (q,x) & \text{if } a = \varepsilon \\ (M.\delta(q, \underbrace{ax}_{\text{interpret as } 0,\ldots,3 \text{ from binary}}), \varepsilon) & \end{cases}$$

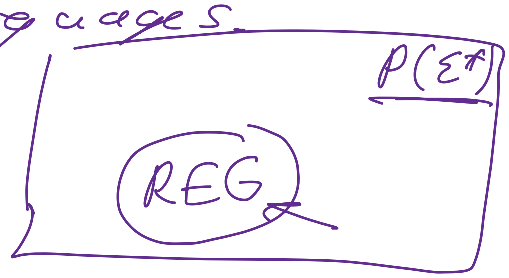




With $p : \Sigma^* \to \{0,1\}$ there's a natural correspondence with languages.

$$L \mapsto p(x) = \begin{cases} 1 & x \in L \\ 0 & x \notin L \end{cases}$$

$$p \mapsto L = \{x : p(x) = 1\}$$



Languages are sets. $P(\Sigma^*) = \{S : S \subseteq \Sigma^*\}$ is the set of all possible languages.

$\{0^n 1^n : n \in \mathbb{N}\}$

$\{1^{n^2} : n \in \mathbb{N}\}$

A ~~set~~ class S of languages is closed under a "rule" if the output of the rule is in S whenever the input(s) are.

Last time $\overline{} : P(\Sigma^*) \to P(\Sigma^*)$

$\wedge : P(\Sigma^*) \times P(\Sigma^*) \to P(\Sigma^*)$

Big Goal: REG is closed under $\cup, \circ, *$.

$L \circ L' = \{xy \mid x \in L, y \in L'\}$

$*$ Kleene star

$L^* = \{$finite strings made by concatenating elements of $L\}$

$= \{\varepsilon\} \cup L \cup (L \circ L) \cup (L \circ L \circ L) \cup \dots$

$= \underline{\{\varepsilon\}} \cup \bigcup_{i=1}^{\infty} L^i$

Fun: There is exactly two $L \in P(\{0,1\}^*)$ s.t. L^* is finite.

$\cup$ basically the same as $\cap$

$\{1\}^* = \{\varepsilon, 1, 11, 111, 1111, \ldots\}$

Let $L, L' \in REG$. Let D be for L and M for L'.

Build N.

$N.Q = D.Q \times M.Q$

$N.\delta((q,q'),x) = (D.\delta(q,x), M.\delta(q',x))$

$N.\Sigma = M.\Sigma = D.\Sigma$

$N.F = (D.F \times M.Q) \cup (D.Q \times M.F)$

$N.q_0 = (D.q_0, M.q_0)$

Alternate Proof

$L \cup L' = \overline{\overline{L} \cap \overline{L'}}$

and closure under $\cap$, $\overline{}$.

E.g. $\{w : w \text{ ends with } 101\} \leftarrow$

$= \{0,1\}^* \circ \underbrace{\{101\}}_{\{1\} \circ \{0\} \circ \{1\}} \leftarrow$

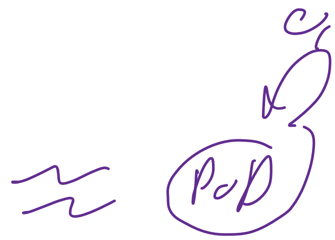
Approach for $\cup$, *

① Define stronger machine (NFA).
② Show NFAs and DFAs are equivalent.
③ Show closure w/ NFAs.

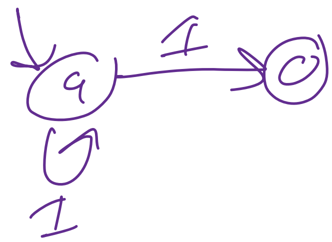
DFA $\longrightarrow$ NFA
↳ Nondeterministic.

Three new "powers"

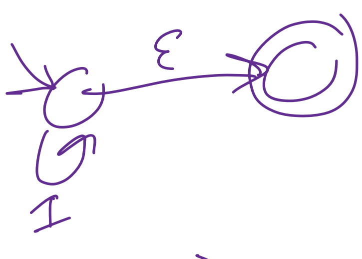
→ Missing transitions.



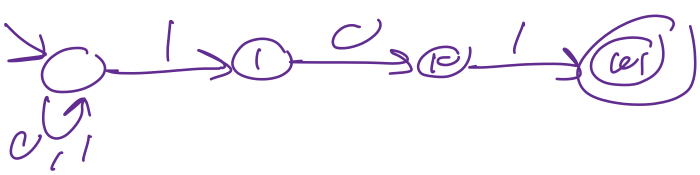
→ Multi-transitions
(Magically pick best one)



→ No-op transitions

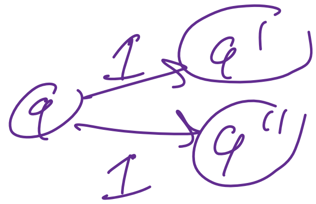


E.g. { end with 101 }



1010

Defn An NFA M is equivalent to a DFA except $M.\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to P(Q)$

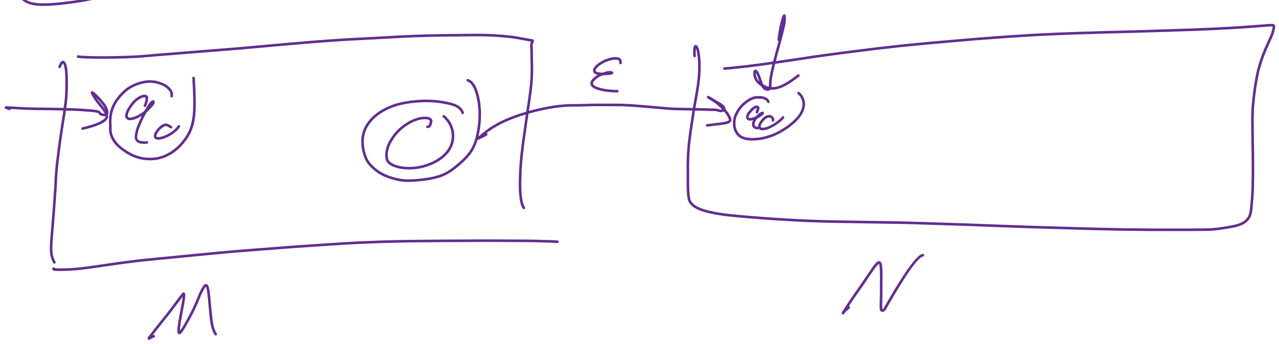


$\delta(q,1) = \{q', q''\}$.

Defn. An NFA M accepts w if there exist sequences $w_1 \dots w_n \in (\Sigma \cup \{\varepsilon\})^*$ s.t.
$q_1 \dots q_n$
$w_1 \circ \dots \circ w_n = w$
s.t.
$q_{i+1} \in \delta(q_i, w_{i+1})$

the character ε

Closure under $\circ$



More formally, let M, N be NFAs. We build D as follows

$D.Q = M.Q \cup N.Q$ (assume $M.Q \cap N.Q = \emptyset$)

$D.F = N.F$

$D.q_0 = M.q_0$

$D.\Sigma = M.\Sigma = N.\Sigma$

$$D.\delta(q,x) = \begin{cases} M.\delta(q,x) \cup \{N.q_0\} & \text{if } q \in M.F,\ x = \varepsilon \\ M.\delta(q,x) & \text{o.w. if } q \in M.Q \\ N.\delta(q,x) & \text{o.w. if } q \in N.Q \end{cases}$$

$\delta^*(q_0, \underbrace{w}_{\in \Sigma^*})$