CS 4510 Automata and Complexity 1/24/2022

Worksheet 1: Regular Language Closure

Worksheets are provided for your study purposes only. They are not graded. The answers to prime-numbered questions will be uploaded to Canvas on Friday, Jan 29th. For the rest of questions, however, you are encouraged to discuss on Canvas and try to solve them collaboratively. You are also welcomed to discuss the problem during office hours.

In this worksheet, you will be tasked with proving regular languages are closed under certain given operations. That is, to show applying the given operation to any regular language(s) will result in another regular language. As a reminder, your proof should be general enough so that it works for any regular languages that the operation is being applied upon.

1. **Separated Repetition**

   In class, we learned that regular languages are closed under Kleene star (repetition). However, what if there must be a separator between each repetition?

   a. Let L be a language and let **COMMA-SEPARATED**(L) be the operation that repeats strings in L for at least once and inserts a comma (",") between each repetition. For example, if L contains exactly 2 strings "cat" and "neko", **COMMA-SEPARATED**(L) will contain "cat", "neko", "cat,neko", "cat,cat", "neko,cat", "neko,neko", "cat,cat,cat,cat,cat,neko", etc. Show that regular languages are closed under **COMMA-SEPARATED**.

   Note: L's alphabet may contain commas (",").

   b. Let L be a language and let **ANY-SEPARATED**(L) be the operation that repeats strings in L for at least once and inserts any same character (drawn from a finite and predetermined alphabet) between each repetition. For example, if L contains only string "cat", **ANY-SEPARATED**(L) will contain (but not limited to) "cat", "cat,cat", "cat;cat", "catOcat", "catOcatOcat", "catacatacatacat", but not "catAcatBcat" because the separator has to be the same character. Show that regular languages are closed under **ANY-SEPARATED**.

   Note: the separator of choice may be in the alphabet of L.

2. **Set Operations**

   We learned that regular languages are closed under complement and union, but what about some other set operations?

   For example purposes, let L_1 be a language that contains exactly 2 strings "a" and "b". Let L_2 be a language that contains exactly 2 strings "b" and "c".

   a. Show that the intersection of two regular languages forms a regular language. For example, the intersection of L_1 and L_2 will contain exactly one string "b".

b. Show that the set difference of two regular languages forms a regular language. For example, the set difference $L_1 - L_2$ will contain exactly one string "a".

c. Show that the symmetric difference of two regular languages forms a regular language. For example, the symmetric difference of L_1 and L_2 will contain exactly two strings "a" and "c".

3. **SUBSTRING**

   Let **SUBSTRING**(L) be the set of all of the substrings of strings in L. For example, if "cat" is in L, then "", "c", "a", "t", "ca", "at", "cat" will all be in **SUBSTRING**(L). Show that regular languages are closed under **SUBSTRING**.

4. **SKIP-EVEN**

   Let **SKIP-EVEN**(L) consists of strings of L where each character at an even index has been omitted. For example, if $L = \{\text{cat}, \text{hamburger}\}$, **SKIP-EVEN**($L$) $= \{\text{ct}, \text{hmugr}\}$. Show that regular languages are closed under **SKIP-EVEN**.

5. **DOUBLE**

   Let $\Sigma = \{0\}$ (i.e. the alphabet only contains the character "0") and consider the following operation defined on languages over Σ.

   $$\textbf{DOUBLE}(L) = \{w \in \Sigma^* \mid \text{There exists } x \in L \text{ such that } |w| = 2|x|\}$$

   For example, if $L = \{00, 000\}$, **DOUBLE**(L) $= \{0000, 000000\}$. Show that regular languages are closed under **DOUBLE**.

6. **REVERSE**

   Let **REVERSE**(L) consists of reversed strings of L. For example, if $L = \{\text{cat}, \text{neko}\}$, **REVERSE**($L$) $= \{\text{tac}, \text{oken}\}$. Show that regular languages are closed under **REVERSE**.

7. **SLICE**

   Let **SLICE**(L) consists of strings of L where the first and last characters are removed. For example, if $L = \{\text{cat}, \text{neko}\}$, **SLICE**($L$) $= \{\text{a}, \text{ek}\}$. Show that regular languages are closed under **SLICE**.

8. **SKIP-ONCE**

   Let **SKIP-ONCE**(L) consists of strings of L where exactly one character is removed. For example, if $L = \{\text{cat}, \text{four}\}$, then **SKIP-ONCE**($L$) $= \{\text{at}, \text{ct}, \text{ca}, \text{our}, \text{fur}, \text{for}, \text{fou}\}$.

9*. **Union of Infinitely Many Regular Languages** (Out of scope for this worksheet)

   In class, we learned that the union of two regular languages is regular. That easily generalizes to union of any finitely many regular languages being regular. However, what about infinitely many regular languages? Prove or disprove that the union of infinitely many regular languages is regular.

   Hint: $\{a^n b^n \mid n \textit{ is natural number}\}$ *is not a regular language*