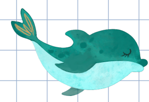
Announcements

- HW due thursday
- Exam in 2 weeks



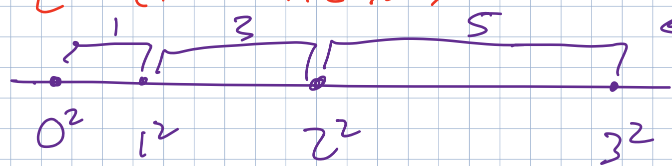
Today: Non-regularity Toolbox

1. Show infinite states would be needed (Myhill-Nerode Thm.)
2. Pumping lemma
3. Closure properties

(1) To prove L non-regular, pick an infinite set S, then show for every $x \neq y \in S$ there exist z s.t. $xz \in L$ and $yz \notin L$ (or vice versa).

E.g. Last time $L = \{0^n 1^n : n \in \mathbb{N}\}$. Picked $S = \{0^i : i \in \mathbb{N}\}$. Then $x = 0^i$ $y = 0^j$ pick $z = 1^i$.

E.g. 2: $L = \{1^{n^2} : n \in \mathbb{N}\}$



← these keep increasing! ($\approx$ super-constant derivative)

Distance $\Delta_n = (n+1)^2 - (n^2) = 2n+1$

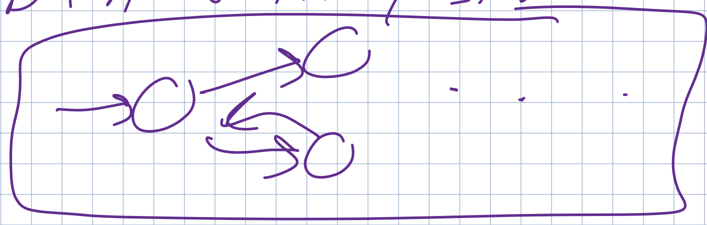
Pick $S = \{1^{n^2} : n \in \mathbb{N}\}$. Given $x = 1^{i^2}$ and $y = 1^{j^2}$ where $i < j$ wlog, pick $z = 1^{\Delta_i}$. Then

- $xz = 1^{i^2 + \Delta_i} = 1^{(i+1)^2} \in L$.
- $yz = 1^{j^2 + \Delta_i} \notin L$

$$j^2 < \underbrace{j^2 + \Delta_i}_{\text{between two consecutive squares}} < j^2 + \Delta_j = (j+1)^2$$

$\Rightarrow$ Not squared.

Consider a DFA with p states.



Consider running this on a loooong string

$x = x_1 x_2 \; ---- \; x_n$

$q_0 \; q_1 \; \ldots$

<u>Claim</u> (if $|x| > p$) then state must repeat at some point

$x_1 x_2 \ldots x_n \; --- \; x_m \; --- \; x_{|x|}$

$q_n \qquad q_m = q_n \qquad q_{|x|} \in F$

Can take this any number of times! → loop in the graph.

<u>Pumping lemma</u>: If A regular language, then there is a number p (the pumping length) where if $s \in A$ with $|s| \geq p$ then we can divide $s = xyz$ s.t.

(i) $xy^iz \in A$ for all $i \in \mathbb{N}$.

(ii) $|y| > 0$

(iii) $|xy| \leq p$

<u>Pf.</u> Picture above. Let D be a DFA for A.

Pick p = # of states in D. Given s, pick y to be the first loop we find. □

Contrapositive

Suppose for all $p>0$, there exists $s \in A$ with $|s| \geq p$ s.t. for all divisions $s = xyz$ with $|y|>0$ and $|xy| \leq p$, there exists i s.t. $xy^iz \notin A$. Then A is not regular.

E.g. $A = \{0^n1^n \mid n \in \mathbb{N}\}$ is not regular.

Pf. Let p be given. Then we pick

$$s = 0^p1^p \qquad \underbrace{0^{p/2}1^{p/2}}$$

Let division $xyz = s$ be given with $|y|>0$ and $|xy| \leq p$.

Then let $i=2$, then $xy^iz \notin A$ because $xy^2z = 0^{p+|y|}1^p$ and $|y|>0$. So by PL contrapositive, A is not regular.

If $s = \overbrace{\underbrace{0}_{x} \ldots \underbrace{0}_{y}}^{p}\ \overbrace{\underbrace{1 \ldots 1}_{z}}^{p}$

$y = 0^I$

$xy^2z = 0^{p+I}1^p$

If $s = \overbrace{0 \ldots 0}^{p/2}\ \overbrace{1 \ldots 1}^{p/2}$

Case 1: $y = 0^I$

$xy^2z = 0^{p/2+I}1^{p/2}$

Case 2: $y = 0^I1^J$

$xy^2z = 0^{p/2}1^J0^I1^{p/2}$

Case 3: $y = 1^I$

$xy^2z = 0^{p/2}1^{p/2+I}$

Must analyze separately.

$i=1$ never works because $xy^1z = xyz = s \in A$.

$$\underbrace{0 - 0}_{x}\underbrace{0 \cdot 0 \; 1 \, . 1}_{y}\underbrace{1 - 1}_{z}$$

$xy^2z = 0^{p/2} 1^J 0^I 1^{p/2}$

If $s = 0^{p/2} 1^{p/2}$, then for any $xyz = s$ we have $xy^2z \notin L$ (with $|y| > 0$, $|xy| < p$)

<u>Case 1</u>: If y has an uneven number of zeros and one's so will xy^2z.

<u>Case 2</u>: If y has an even number, xy^2z will have 0's and 1's in the wrong order

e.g. $\underbrace{0 \ldots 0}_{x}\underbrace{0 \ldots 01 \ldots 1}_{y}\underbrace{0 \ldots 01 \ldots}_{y}\underbrace{1 \ldots 1}_{z}$

<u>E.g. 2</u> $\{1^{n^2} : n \in \mathbb{Z}\}$ is not regular.

Let p be given. Pick $s = 1^{p^2}$, then consider any division $xyz = s$ with $|y| > 0$ and $|xy| < p$. Pick $i = 2$.

So $xy^2z = 1^{p^2 + |y|}$

$$p^2 \underset{|y|>0}{<} p^2 + |y| \underset{|xy|<p}{<} p^2 + p < p^2 + 2p = (p+1)^2$$

alternate way to write this

$$p^2 + p = (p+1)p < (p+1)^2$$

Practice:

1. $\{ww \mid w \in \{0,1\}^*\}$
2. $\{0^m 1^n \mid m > n\}$
3. $\{wv \mid v$ is w with bits flipped$\}$

1

s	i
$0^p 1^p 0^p 1^p$	2

$0^p 1^p 0^p$ ← bad, not in language

$0^{2p} 1^{2p}$ ← won't work

2 $s = 0^{p-1} 1^?$ does not work

$y = 1$ would make you fail.

$s = 0^p 1^{p-1}$

$i = 2$ fails

$i = 0$ works!

"Pumping Down"

This is incorrect, y=1 would not be an issue because you could pick something like i=p-1, so there would be p 1's which is more than the p-1 m's.

However, y=0 would be an issue. Choosing i>1 would just give more 0's. That would not result in anything outside the language. Choosing i=0 would not work because we would still have p-2 0's, which may be more than the number of 1's.

For the solution which worked, it was important that there was only one more 0 than 1, so it was guaranteed that removing even a single 0 would give us something out of the language.

If we chose something like s=0^p-1 1^p-2, then our choice of i would need to depend on the given split xyz=s. If s has more 0's than 1's we can pick i=0 for the same reason. If y contained 1's we would need to pump up.