CS 4510 Automata and Complexity

Exam 1 Solutions

Prof. Jaeger

Name: ____________________ GTID: ____________________

- **When handing in this test, please go to a TA with your BuzzCard to verify the name on the test.**
- Do not open this exam until you are directed to do so. Read all the instructions first.
- **Write your name and GTID# (numbers, not letters) on the top of every page.** Your GTID# can be found on your BuzzCard.
- By submitting this exam, you agree that your actions during this exam conform with the Georgia Tech Honor Code.
- This exam is out of $N = 100$ points (with one extra credit point for writing your name). We reserve the right to decrease N if we misevaluated the difficulty of a question.
- Write your solutions in the space provided. If you run out of space, continue your answer on the back of the last page, and make a notation on the front of the sheet.
- There is a page of scratch paper at the end of the exam. We can provide additional scratch paper if you need it. The only outside material you may use on the exam is one (1) double-sided, hand-written, 8.5"x11" page of notes.
- Calculators are NOT permitted.
- You may use any of the theorems/facts/lemmas from the lecture notes, homeworks, or textbook without re-proving them unless explicitly stated otherwise.
- If you have a question, you may ask a TA. You should not communicate with anyone other than teaching staff during the exam.
- Do not spend too much time on any one problem! If you get stuck, move on and come back to that one later.
- Each problem is labeled with the number of points it is expected to be worth. These values may be shifted if any problems end up being more difficult than expected.
- Good luck!

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0. **Following Instructions [1 point extra credit]**

   Please write your name and GTID# (numbers, not letters) at the top of EACH page of this exam. Your GTID# can be found on your BuzzCard.

1. **Multiple Choices [32 points]** Fill ALL answers which apply.

   (a) [**8 points**] Which of the following statements are true. Choose ALL that apply.

   ● If D is a DFA and $D.q_0 \in D.F$, then $\varepsilon \in L(D)$.
   ● If R is a regular expression, then there exists an NFA N such that $L(R) = L(N)$.
   ○ If $L \in REG$, then L is finite.
   ● If L is finite, then $L \in REG$.
   ○ If $L \notin REG$ and $L' \notin REG$, then $L \cup L' \notin REG$

   (b) [**8 points**] Which of the following regular expressions DEFINITELY describe a non-empty language. (R is an arbitrary regular expression.) Choose ALL that apply.

   ● $111 \cup 1^*$
   ● R^*
   ○ $R \circ \emptyset$
   ○ $R \cup \emptyset$
   ○ $R \circ \varepsilon$
   ● $R \cup \varepsilon$

   (c) [**8 points**] Which of the following languages are regular? Choose ALL that apply.

   ● $L_1 = \{x \mid x \text{ contains more than two 1's}\}$
   ● $L_2 = \{\mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{Z}_{\geq 0}\} \cup \{x \mid \text{the length of } x \text{ is even}\}$
   ● $L_3 = \{x \mid x \text{ is a palindrome OR } x \text{ is not a palindrome}\}$
   ○ $L_4 = \{0^n10^n \mid n \in \mathbb{Z}_{\geq 0}\}$
   ● $L_5 = \{0^n1^n \mid n \in \mathbb{Z}_{\geq 0}, n < 15\}$
   ○ $L_6 = \overline{L_4}$
   ● $L_7 = \{0^n1^m \mid n, m \in \mathbb{Z}_{\geq 0}\}$

   (d) [**8 points**] You are using the pumping lemma to show that $\{0^n1^m \mid n \leq m,\ n, m \in \mathbb{Z}_{\geq 0}\}$ is not regular. You have assumed p is the pumping length. Which of the following choices for s allow you to successfully complete the proof, where s is the counterexample string which you show cannot be pumped? Choose ALL that apply.

   ○ 00000111111
   ● 0^p1^p
   ● 0^p1^{2p}
   ○ $0^{2p}1^p$
   ○ $0^{p-1}1^p$

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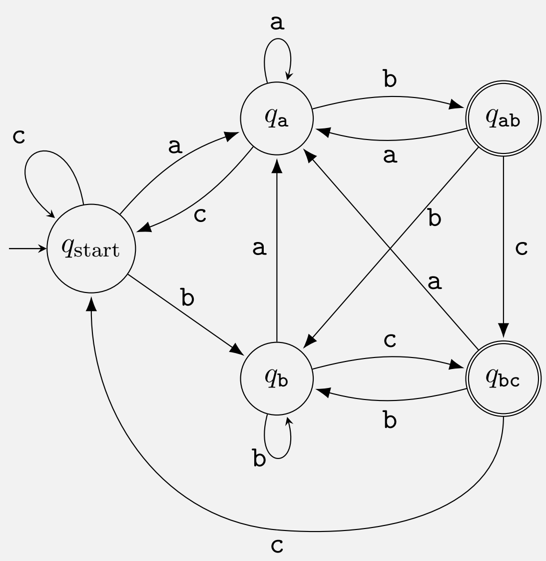
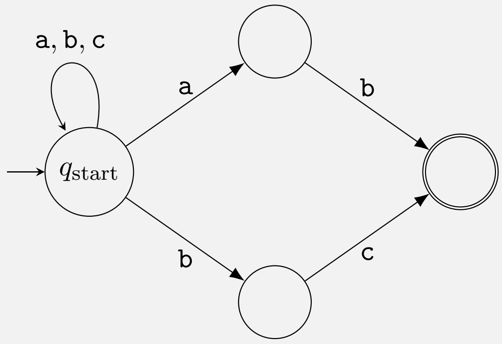
2. **[15 points]** Consider the language $L = \{x \mid x \text{ ends with } \mathtt{ab} \text{ or } \mathtt{bc}\}$ over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$. You will express this language in each of the following parts. Note that your NFA should be different from your DFA and must take advantage of nondeterminism.

   (a) **[5 points]** Draw the state diagram (circles and arrows) of a DFA accepting L.



   (b) **[5 points]** Draw the state diagram of a "simpler" NFA accepting L. If your NFA is the same as your DFA and does not take advantage of nondeterminism, it will result in no points being awarded.



   (c) **[5 points]** Construct the regular expression for L. Use **only** the symbols $\emptyset$, ε, a, b, c, $\circ$, $\cup$, *,), and (. You may leave $\circ$'s implicit.

$$(\mathtt{a} \cup \mathtt{b} \cup \mathtt{c})^*(\mathtt{ab} \cup \mathtt{bc})$$

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3. **[18 points]** If A and B are sets then we define the set difference operation by $A \smallsetminus B = \{x \mid x \in A \text{ and } x \notin B\}$.

(a) **[5 points]** Prove that the class of regular languages (REG) is closed under set difference using closure properties we've already proven.

> Suppose that A and B are two regular languages.
> Note that $A \setminus B = A \cap \overline{B}$. Since regular languages are closed under intersection and complement, we have that $A \cap \overline{B}$ is regular, and thus $A \setminus B$ is regular.
> Therefore the set difference of any two regular languages is regular, and we can say REG is closed under set difference.

(b) **[13 points]** Prove that the class of regular languages (REG) is closed under set difference WITHOUT using closure properties we've already proven.

> Suppose that languages A and B are regular. By definition of being regular, there must exist some DFA M and some DFA N such that $L(M) = A$ and $L(N) = B$. WLOG, assume that $M.\Sigma = N.\Sigma$.
> We now construct a new DFA D as follows:
>
> $$\begin{aligned} D.Q &= M.Q \times N.Q \\ D.\Sigma &= M.\Sigma \\ D.q_0 &= (M.q_0,\ N.q_0) \\ D.\delta((q,r),a) &= (M.\delta(q,a),\ N.\delta(r,a)) \\ D.F &= M.F \times \overline{N.F}. \end{aligned}$$
>
> Note that this machine runs M and N simultaneously and accepts exactly when M would accept and N would not.
> That is,
>
> $$\begin{aligned} L(D) &= \{x \mid x \in L(M) \text{ and } x \notin L(N)\} \\ &= \{x \mid x \in A \text{ and } x \notin B\} \\ &= A \setminus B. \end{aligned}$$
>
> By creating a DFA for it, the language $A \setminus B$ must be regular.
> Since for any regular languages, A, B, we have that their set difference is also regular, we get that REG is closed under set difference.

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4. **[20 points]** Recall that if $w = w_0 w_1 \dots w_n$, then we let $w^{\mathcal{R}}$ denote w written in reverse (i.e., $w^{\mathcal{R}} = w_n \dots w_1 w_0$). Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \dots, \mathtt{z}\}$ be the set of lowercase English letters. Now consider the following languages:

$$L_{\text{palindrome}} = \{w \in \Sigma^* \mid w = w^{\mathcal{R}}\}$$
$$L_{\text{flipped}} = \{vw \in \Sigma^* \mid v = w^{\mathcal{R}}\}.$$

Prove that L_{flipped} is not regular using the pumping lemma. Then prove that $L_{\text{palindrome}}$ is not regular using the fact that REG is closed under set difference (combined with some language we already know is regular).

Proof that L_{flipped} is not regular:

Assume for contradiction that language L_{flipped} is regular. Let $s = \mathtt{a}^p\mathtt{bba}^p$. We see that $s \in L_{\text{flipped}}$ and $|s| \geq p$, so by the pumping lemma there is a partition $s = xyz$, with $|xy| \leq p$ and $|y| \geq 1$ such that for all $i \in \mathbb{Z}_{\geq 0}$, the string xy^iz must also be in L_{flipped}. Note that since $|xy| \leq p$, we know that y must consist of only $\mathtt{a}$.
We now pump down to get xz, but in the string xz, there are two $\mathtt{b}$'s in the first half of the string and zero in the second half, so $xz \notin L_{\text{flipped}}$. Since for $i = 0$, we get a string that is not in the language, we have a contradiction. Therefore L_{flipped} is not regular.

Proof that $L_{\text{palindrome}}$ is not regular:

We observe that $L_{\text{palindrome}}$ is the set of all palindromes, whether even or odd length, whereas L_{flipped} is the set of even length palindromes.
Assume for contradiction that $L_{\text{palindrome}}$ is regular. Let $L_{\text{Odd}} = \{x \mid x \text{ is odd length}\}$. We know L_{Odd} is regular. Observe that $L_{\text{palindrome}} \setminus L_{\text{Odd}} = L_{\text{flipped}}$. Since regular languages are closed under set difference, this means that L_{flipped} must be regular, but since we just proved that it is not regular, we have a contradiction. Therefore $L_{\text{palindrome}}$ is not regular.

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5. **[15 points]** Consider the language $L = \{x \mid x \text{ does not contain } 111 \text{ as a substring}\}$. Prove that L is regular using the technique of your choice.

Solution:
Let $\Sigma = \{0, 1\}$. Consider the regular expression $R = \Sigma^* 111 \Sigma^*$. This is the language of all strings that do contain 111 as a substring, and by making a regular expression, R must be regular. By closure of regular languages under complement, $\overline{R}$ is also regular. Since $\overline{R} = L$, then L is regular.

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This page provides extra space if you need it. Clearly mark questions that use this space. If you finish early and want an additional fun challenge (for no points), figure out if REG is closed under the operation $L^{\mathcal{R}} = \{w \mid w^{\mathcal{R}} \in L\}$. What about non-regular languages? How does it relate to L_{flipped}?