NUMERICAL SOLVER FOR MAC SCHEME STOKES EQUATION

A REPORT

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In this report, we consider Stokes equation:

$$
\begin{gathered}
-\Delta\vec{u}+\nabla p=\vec{F},\\
div\vec{u}=0,\\
\frac{\partial u}{\partial \mathbf{v}}=b, y=0; \frac{\partial u}{\partial \mathbf{v}}=t, y=1,\\
\frac{\partial v}{\partial \mathbf{v}}=l, x=0; \frac{\partial v}{\partial \mathbf{v}}=r, x=1,\\
u=0, x=0,1; v=0, y=0,1,
\end{gathered}
$$

with velocity $\vec{u}=(u,v)$, pressure p, external force $\vec{F}=(f,g)$ and outer normal direction $\mathbf{v}$. The domain given is $\Omega=(0,1)\times(0,1)$. We suggest the solution is

$$
\begin{cases}
u=(1-\cos(2\pi x))sin(2\pi y),\\
v=-(1-\cos(2\pi y))sin(2\pi x).
\end{cases}
$$

It follows that

$$
\begin{cases}
f(x,y)=-4\pi^2(2\cos(2\pi x)-1)\sin(2\pi y)+x^2,\\
g(x,y)=4\pi^2(2\cos(2\pi y)-1)\sin(2\pi x).
\end{cases}
$$

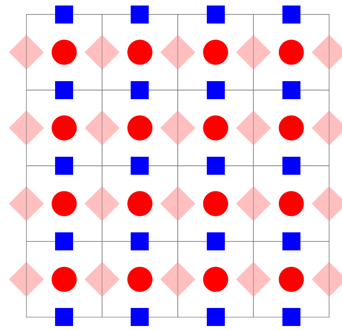
1 Numerical Scheme

We rewrite the Stokes equations into coordinate-wise:

$$-\Delta u+\partial_x p=f, \tag{1}$$
$$-\Delta v+\partial_y p=g, \tag{2}$$
$$-\partial_x u-\partial_y v=Div. \tag{3}$$

The MAC scheme is to discretize the x-coordinate momentum equation (1) at vertical edges, the y-coordinate momentum equation (2) at horizontal edges, and the continuity equation (3) at cell centers using central difference schemes. For grid of size N, let $h=1/N$. We denote by

$$
\begin{aligned}
f_{i,j}&=f(ih,(j-0.5)h), 1\le i\le N-1, 1\le j\le N; \quad g_{i,j}=g((i-0.5)h,jh), 1\le i\le N, 1\le j\le N-1;\\
b_i&=b(ih), t_i=t(ih), 1\le i\le N-1; \quad l_j=l(ih), r_j=r(jh), 1\le j\le N-1;\\
u_{i,j}&=u(ih,(j-0.5)h), 0\le i\le N, 1\le j\le N; \quad v_{i,j}=v((i-0.5)h,jh), 1\le i\le N, 0\le j\le N;\\
p_{i,j}&=p((i-0.5)h,(j-0.5)h), 1\le i\le N, 1\le j\le N.
\end{aligned}
$$



See the figure. Using central difference scheme we have the following equations:

$$
\begin{aligned}
&\frac{3u_{i,j}}{h^2} - \frac{u_{i-1,j}}{h^2} - \frac{u_{i+1,j}}{h^2} - \frac{u_{i,j+1}}{h^2} - (\frac{p_{i,j}}{h} - \frac{p_{i+1}}{h}) = f_{i,j} + \frac{b_i}{h}, \quad 1 \le i \le N-1, j = 1;\\
&\frac{4u_{i,j}}{h^2} - \frac{u_{i-1,j}}{h^2} - \frac{u_{i+1,j}}{h^2} - \frac{u_{i,j+1}}{h^2} - (\frac{p_{i,j}}{h} - \frac{p_{i+1}}{h}) = f_{i,j}, \quad 1 \le i \le N-1, 2 \le j \le N-1;\\
&\frac{3u_{i,j}}{h^2} - \frac{u_{i-1,j}}{h^2} - \frac{u_{i+1,j}}{h^2} - \frac{u_{i,j+1}}{h^2} - (\frac{p_{i,j}}{h} - \frac{p_{i+1}}{h}) = f_{i,j} + \frac{t_i}{h}, \quad 1 \le i \le N-1, j = N\\
&\qquad\qquad\qquad\qquad u_{0,j} = u_{N,j} = 0, \quad 1 \le j \le N;\\
&\frac{3v_{i,j}}{h^2} - \frac{v_{i+1,j}}{h^2} - \frac{v_{i,j-1}}{h^2} - \frac{v_{i,j+1}}{h^2} - (\frac{p_{i,j}}{h} - \frac{p_{i,j+1}}{h}) = g_{i,j} + \frac{l_j}{h}, \quad i = 1, 1 \le j \le N-1;\\
&\frac{4v_{i,j}}{h^2} - \frac{v_{i+1,j}}{h^2} - \frac{v_{i,j-1}}{h^2} - \frac{v_{i,j+1}}{h^2} - (\frac{p_{i,j}}{h} - \frac{p_{i,j+1}}{h}) = g_{i,j}, \quad 2 \le i \le N-1, 1 \le j \le N-1;\\
&\frac{3v_{i,j}}{h^2} - \frac{v_{i+1,j}}{h^2} - \frac{v_{i,j-1}}{h^2} - \frac{v_{i,j+1}}{h^2} - (\frac{p_{i,j}}{h} - \frac{p_{i,j+1}}{h}) = g_{i,j} + \frac{r_j}{h}, \quad i = N, 1 \le j \le N-1;\\
&\qquad\qquad\qquad\qquad v_{i,0} = vi, N = 0, \quad 1 \le i \le N;\\
&\qquad\qquad \frac{u_{i-1,j}}{h} - \frac{u_{i,j}}{h} + \frac{v_{i,j-1}}{h} - \frac{v_{i,j}}{h} = Div_{i,j}, \quad 1 \le i \le N, 1 \le j \le N.
\end{aligned}
$$

It can be written as matrix form

$$
\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F \\ Div \end{bmatrix}, \tag{4}
$$

where

$$
A = \begin{bmatrix}
\frac{1}{h^2}T_1 & -\frac{1}{h^2}I & & & & & & & & \\
-\frac{1}{h^2}I & \frac{1}{h^2}T_2 & -\frac{1}{h^2}I & & & & & & & \\
& \ddots & \ddots & \ddots & & & & & & \\
& & -\frac{1}{h^2}I & \frac{1}{h^2}T_2 & -\frac{1}{h^2}I & & & & & \\
& & & -\frac{1}{h^2}I & \frac{1}{h^2}T_1 & & & & & \\
& & & & & \frac{1}{h^2}T_3 & -\frac{1}{h^2}I & & & \\
& & & & & -\frac{1}{h^2}I & \frac{1}{h^2}T_3 & -\frac{1}{h^2}I & & \\
& & & & & & \ddots & \ddots & \ddots & \\
& & & & & & & -\frac{1}{h^2}I & \frac{1}{h^2}T_3 & -\frac{1}{h^2}I \\
& & & & & & & & -\frac{1}{h^2}I & \frac{1}{h^2}T_3
\end{bmatrix},
$$

$$
B = \begin{bmatrix}
-\frac{1}{h}D & & & \\
& \ddots & & \\
& & \ddots & \\
& & & -\frac{1}{h}D \\
-\frac{1}{h}I & \frac{1}{h}I & & \\
& \ddots & \ddots & \\
& & -\frac{1}{h}I & \frac{1}{h}I
\end{bmatrix},
$$

and submatrix

$$T_1 = \begin{bmatrix} 3 & -1 & & & \\ -1 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -1 \\ & & & -1 & 3 \end{bmatrix}, T_2 = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & 1 \\ & & & -1 & 4 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 3 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 3 \end{bmatrix}, D = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix},$$

and I is the $N-1$ order identity. Here

$$\begin{aligned} U &= (u_{1,1}, \dots, u_{N-1,1}, \dots, u_{N-1,N}, v_{1,1}, \dots, v_{1,N-1}, \dots, v_{N,N-1})^T, \\ P &= (p_{1,1}, \dots, p_{N,1}, \dots, p_{N,N})^T, \\ F &= (\tilde{f}_{1,1}, \dots, \tilde{f}_{N-1,1}, \dots, \tilde{f}_{N-1,N}, \tilde{g}_{1,1}, \dots, \tilde{g}_{1,N-1}, \dots, \tilde{g}_{N,N-1})^T \end{aligned},$$

$$\tilde{f}_{i,j} = \begin{cases} f_{i,j} + \dfrac{b_i}{h}, & j = 1 \\ f_{i,j}, & 2 \le j \le N \\ f_{i,j} + \dfrac{t_i}{h}, & j = N \end{cases}, \quad \tilde{g}_{i,j} = \begin{cases} g_{i,j} + \dfrac{l_j}{h}, & i = 1 \\ g_{i,j}, & 2 \le i \le N-1 \\ g_{i,j} + \dfrac{r_j}{h}, & i = N \end{cases}$$

We have the following theorem:

Theorem 1 *The spectral of $B^T A^{-1} B$ consist of 0s and 1s.*

Proof *The proof is left in appendix.*

2 Distributive Gauss Seidel

This section is based on 2020 fall's numerical linear algebra lecture and Long Chen's paper [1].

2.1 Algorithm Analysis

Since the matrix in (4) is not diagonally dominant, and the zeros block in the diagonal hampers the relaxation, we use distributive relaxation instead. To speak specifically, denote by

$$L = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}, \quad M = \begin{bmatrix} I & B \\ 0 & -B^T B \end{bmatrix}.$$

We have

$$LM = \begin{bmatrix} A & AB - BB^T B \\ B^T & B^T B \end{bmatrix} \approx \begin{bmatrix} A & 0 \\ B^T & B^T B \end{bmatrix} := \widetilde{LM}.$$

Let $A_p = B^T B$ and $\tilde{A}$, $\tilde{A_p}$ be approximation of A, A_p, respectivelly, then the DGS algorithm is given as following.

Algorithm 1: Distributive Gauss Seidel

1 Relax the momentum equation

$$U^{k+\frac{1}{2}} = U^k + \widetilde{A^{-1}}(F - AU^k - BP^k).$$

2 Relax the transformed mass equation

$$\delta = \widetilde{A_p^{-1}}(Div - BU^{k+\frac{1}{2}}).$$

3 Distribute the correction to the original varibales

$$\begin{aligned} U^{k+1} &= U^{k+\frac{1}{2}} + B\delta, \\ P^{k+1} &= P^k - A_p \delta. \end{aligned}$$

To implement it in Matlab, we can write the code by steps below.

(1) Using U^k as initial value, do one GS iteration for $AU = F - BP^k$ to get $U^{k+\frac{1}{2}}$. We notice that there is no need to compute A^{-1} and it is convenient to update one-by-one. For example, the update for velocity $u_{i,j}$ for $i = 1, \dots, N-1, j = 2, \dots, N-1$ can be simply written as

$$u_{i,j} = (h^2 * \tilde{f}_{i,j} - h * (p_{i+1,j} - p_{i,j}) + u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})/4,$$

and the coefficient 4 should be changed to 3 for u near the bottom or top edges accordingly. The fomula is similar for v.

(2) Compute divergence residual for internal grids and update the velocity and pressure. This is the same as in V-cycle lecture. Say

$$r = -\frac{u_{i,j} - u_{i-1,j}}{h} - \frac{v_{i,j} - v_{i,j-1}}{h} + Div_{i,j},$$

here Div is the divergence. Let $\delta = r * h/4$, we update velocity and pressure

$$\begin{aligned} u_{i,j} &= u_{i,j} + \delta \\ u_{i-1,j} &= u_{i_1,j} - \delta \\ v_{i,j} &= v_{i,j} + \delta \\ v_{i,j-1} &= v_{i,j-1} - \delta \\ p_{i,j} &= p_{i,j} + r \\ p_{i+1,j} &= p_{i+1,j} - r/4 \\ p_{i-1,j} &= p_{i-1,j} - r/4 \\ p_{i,j-1} &= p_{i,j-1} - r/4 \\ p_{i,j+1} &= p_{i,j+1} - r/4 \end{aligned}$$

(3) Compute divergence residual for grids on edges and update the velocity and pressure. This is similar to that above, except that we should replace 4 with 3 and do not change velocity on edges and pressure outside the domain.

(4) Compute divergence residual for grids on corners and update the velocity and pressure. This is almost the same except that we have to replace 4 with 2.

(5) After the update above, we get U^{k+1} and P^{k+1}.

The convergence analysis could be found in Chen's lecture [2]. We omit it here for simplicity.

2.2 V-cycle mutigrid

Consider the equation

$$\begin{bmatrix} A_h & B_h \\ B_h^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{U}_h \\ \tilde{P}_h \end{bmatrix} = \begin{bmatrix} \tilde{F}_h \\ \tilde{Div}_h \end{bmatrix}.$$

The V-cycle multigrid algorithm is described as followsing:

(1) On grid Ω^h, using U_h^0, P_h^0 as initial value, presmooth by DGS for n_1 times. Compute the residual $rF_h = F_h - A_h U_h^{n_1} - B_h P_h^{n_1}, rDiv_h = Div_h - B_h^T U_h^{n_1}$.

(2) Restrict the residual to the coarse grid.

(3) Using $U_{2h}^0, P_{2h}^0 = 0$ as initial values, presmooth

$$\begin{bmatrix} A_{2h} & B_{2h} \\ B_{2h}^T & 0 \end{bmatrix} \begin{bmatrix} U_{2h} \\ P_{2h} \end{bmatrix} = \begin{bmatrix} F_{2h} \\ Div_{2h} \end{bmatrix}.$$

by DGS for n_1 times. Compute the residual $rF_{2h} = F_{2h} - A_{2h} U_{2h}^{n_1} - B_{2h} P_{2h}^{n_1}, rDiv_{2h} = Div_{2h} - B_{2h}^T U_{2h}^{n_1}$.

(4) Restrict the residual to the coarse grid.

(5) Recursively use the steps above until grid Ω_{Lh}. Iterate DGS on Ω_{Lh} till converge. We get $P_{Lh} = B_{Lh}^T A_{Lh}^{-1} F_{Lh} - D_{Lh}, U_{Lh} = A_{Lh}^{-1}(F_{Lh} - B_{Lh} P_{Lh})$.

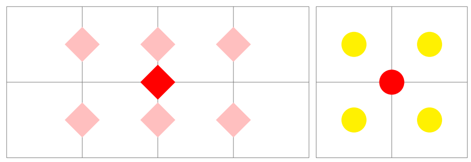
(6) Prolongate the correction to the fine grid. Using the corrected U, P as inital values, iterate with DGS for $n_2$times to get $U^{n_1+n_2}, P^{n_1+n_2}$.

(7) Continue this prolongation process until it comes to Ω_h. If the residual on Ω_h is smaller than toleration, the algorithm terminates. Otherwise restart from step 1 with U, P we've just got.

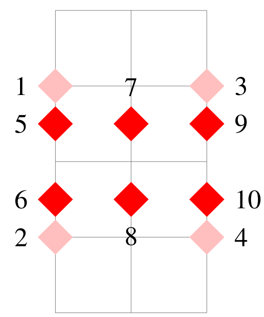
The restriction operators are

$$R^u_{h,2h} = \frac{1}{8}\begin{bmatrix}1 & 2 & 1\\ & * & \\ 1 & 2 & 1\end{bmatrix}, R^v_{h,2h} = \frac{1}{8}\begin{bmatrix}1 & & 1\\ 2 & * & 2\\ 1 & & 1\end{bmatrix}, R^p_{h,2h} = \frac{1}{4}\begin{bmatrix}1 & & 1\\ & * & \\ 1 & & 1\end{bmatrix}.$$

Using figures we can get it more easily.



The red point is the weighted average of the nearest points. For u, the weights of upper and lower point are twice of those on the corners. It is similar for v. For p it is just the average of the four nearest points. For the prolongation operators, we typically apply bilinear interpolation of neighboring coarse-grid unknowns in the staggerd grid. See the figure below.



We have $u_5 = \frac{3}{4}u_1 + \frac{1}{4}u_2, u_6 = \frac{1}{4}u_1 + \frac{3}{4}u_2, u_9 = \frac{3}{4}u_3 + \frac{1}{4}u_4, u_{10} = \frac{3}{4}u_4 + \frac{1}{4}u_3, u_7 = \frac{1}{2}(u_5 + u_9), u_8 = \frac{1}{2}(u_6 + u_{10})$. The transformation is similar for v. And for p, we just use the nearest point's value. Finally, let velocity on edges be 0.

2.3 Numerical Results

Using DGS smoothing with Vcycle multigrid method, choose $\frac{||r_h||_2}{||r_0||_2} \leq 10^{-8}$ as stop criteria. For initial value $U^0 = 0, P^0 = 0$ and different n_1, n_2, L, we have experimental results below.

We can see that, the CPU time cost is nearly $O(n^2)$ and of course V-cycle times decrease as n_1, n_2 increase. However, as the number of smoothing increases on each grid, the time consumption increases for one cycle. For this reason, it is better to use fewer smoothing operators for large N. Since our stop criteria is the same, the differences between errors are tiny. But for larger L, the bottom grid is croaser and the cost of accurate solution on this grid is lower, though it increases the number of prolongation and makes the solution more imprecise. Finally, because of the difference scheme's errors, when N is larger, the error between discrete solution and true solution is smaller.

3 Uzawa

3.1 Algorithm Analysis

Consider the equation

$$SP = \widetilde{Div},$$

where $S = B^T A^{-1} B$ and $\widetilde{Div} = B^T A^{-f} F - Div$. This is otained from equation (4) by eliminating u. One iteration for solve this equation is

$$P^{k+1} = P^k + \alpha(\widetilde{Div} - SP^k).$$

$L = N/2$				
N	n_1, n_2	V-cycle times	CPU time(s)	err
64	1,1	31	0.0642	0.0015
	2,2	22	0.0719	0.0015
	2,3	20	0.0809	0.0015
	3,3	18	0.0834	0.0015
	2,4	18	0.0757	0.0015
128	1,1	35	0.2451	3.7363e-04
	2,2	26	0.2853	3.7363e-04
	2,3	23	0.2743	3.7363e-04
	3,3	21	0.2973	3.7363e-04
	2,4	21	0.2971	3.7363e-04
256	1,1	40	1.0337	9.3398e-05
	2,2	30	1.3024	9.3398e-05
	2,3	26	1.1680	9.3398e-05
	3,3	24	1.1654	9.3398e-05
	2,4	24	1.3912	9.3398e-05
512	1,1	45	7.3908	2.3349e-05
	2,2	33	8.2950	2.3349e-05
	2,3	30	8.7525	2.3349e-05
	3,3	27	9.5431	2.3349e-05
	2,4	27	9.0193	2.3349e-05
1024	1,1	49	37.2065	5.8372e-06
	2,2	37	46.8557	5.8372e-06
	2,3	33	51.8100	5.8372e-06
	3,3	30	54.8474	5.8372e-06
	2,4	29	51.2733	5.8372e-06
2048	1,1	53	186.3297	1.4593e-06
	2,2	40	227.5852	1.4593e-06
	2,3	36	248.9021	1.4593e-06
	3,3	32	253.3137	1.4593e-06
	2,4	32	258.0706	1.4593e-06

Table 1: numerical results for DGS when $L = N/4$

$L = N/4$				
N	n_1, n_2	V-cycle times	CPU time(s)	err
64	1,1	30	0.0790	0.0015
	2,2	22	0.0724	0.0015
	2,3	20	0.0868	0.0015
	3,3	18	0.0783	0.0015
	2,4	18	0.0750	0.0015
128	1,1	36	0.2476	3.7363e-04
	2,2	26	0.2556	3.7363e-04
	2,3	23	0.2495	3.7363e-04
	3,3	21	0.2587	3.7363e-04
	2,4	21	0.3295	3.7363e-04
256	1,1	40	0.9557	9.3398e-05
	2,2	30	1.1379	9.3398e-05
	2,3	26	1.1035	9.3398e-05
	3,3	24	1.3206	9.3398e-05
	2,4	24	1.1006	9.3398e-05
512	1,1	45	6.3221	2.3349e-05
	2,2	33	7.6550	2.3349e-05
	2,3	29	7.7562	2.3349e-05
	3,3	27	8.1303	2.3349e-05
	2,4	27	8.7585	2.3349e-05

$L = N/4$				
N	n_1, n_2	V-cycle times	CPU time(s)	err
1024	1,1	50	34.4578	5.8372e-06
	2,2	37	44.1880	5.8372e-06
	2,3	32	46.8784	5.8372e-06
	3,3	29	52.0154	5.8372e-06
	2,4	29	46.6444	5.8372e-06
2048	1,1	54	167.5138	1.4593e-06
	2,2	40	215.7680	1.4593e-06
	2,3	35	226.3154	1.4593e-06
	3,3	32	249.8871	1.4593e-06
	2,4	32	254.8948	1.4593e-06

Table 2: numerical results for DGS when $L = N/2$

Algorithm 2: Uzawa

1 Solve $AU^{k+1} = F - BP^k$;
2 Update $P^{k+1} = P^k + \alpha(B^T U^{k+1})$;
3 Check if the error is smaller than toleration.

The reason for not using Jacobi or GS iteration is that we do not have explicit form of S which involves A^{-1}. Moreover, A^{-1} is possibly not even sparse. Substitute S into the iteration equation above we have $U^{k+1} = A^{-1}(F - BP^k)$. Thus we have Uzawa algorithm above. It can be proved (in fall's homework) that the optimal choice of $alpha$ is

$$\alpha_{opt} = \frac{2}{\lambda_{min}(S) + \lambda_{max}(S)},$$

and the corresponding convergence rate is

$$||P - P^k|| \leq \left(\frac{\kappa(S) - 1}{\kappa(S) + 1}\right)^k ||P - P^0||.$$

By Theorem 1, the optimal $alpha$ is 1.

3.2 Numerical Results

For $N = 64, 128, 256, 512$, we use Uzawa Iteration Method to solve the equation with $\alpha = 1$, stop criteria $||r_h||_2/||r_0||_2 \leq 1e - 8$ and compute the error. The solver for step 1 is Conjugate Gradient Method with relative error's toleration 1e-9. The results is as below.

N	iteration	CPU time(s)	err
64	2	0.0708	0.0015
128	3	0.3815	3.7363e-04
256	3	2.1382	9.3398e-05
512	3	23.2893	2.3349e-05

Table 3: numerical results for Uzawa

We can see that the time cost is nearly $O(n^3)$, which is much slower than DGS with V-cycle. The main time consumption is due to step 1, which solves a linear equation of size $2N(N-1)$. The update of P is easy and just similar to that of DGS.

4 Inexact Uzawa

4.1 Algorithm Analysis

Since the consumption of each iteration for solving $AU^{k+1} = F - BP^k$ is exhaustive, we use inexact solver and multigrid instead. The iteration for computing A^{-1} is called the inner iteration and the Uzawa iteration is called outer iteration. We have Inexact Uzawa Method as following.

Algorithm 3: Inexact Uzawa

Approximately solve $AU^{k+1} = F - BP^k$ and get $\tilde{U}^{k+1}$;
Update $P^{k+1} = P^k + \alpha(B^T\tilde{U}^{k+1})$;
Check if the error is smaller than toleration.

The linear algebraic proof of the convergence of Inexact Uzawa Method could be found in Cheng's paper [3].

4.2 Numerical Results

Let $N = 64, 128, \ldots, 2048$ we use Inexact Uzawa Iteration Method with V-cycle multigrid as preconditioner. Conjugate Gradient solver is used for $AU^{k+1} = F - BP^k$. For different α, n_1, n_2, L and CG's stop criteria, we have numerical results as below.

$L = N/2, \alpha = 1$, CG's $tol = 1e-2$				
N	n_1, n_2	V-cycle times	CPU time(s)	err
64	1,1	7	0.1667	0.0015
	2,2	3	0.1552	0.0015
	2,3	3	0.1577	0.0015
	3,3	2	0.1657	0.0015
	2,4	2	0.1682	0.0015
128	1,1	7	0.5049	3.7363e-04
	2,2	4	0.6934	3.7363e-04
	2,3	3	0.8086	3.7363e-04
	3,3	2	0.9011	3.7363e-04
	2,4	2	0.9229	3.7363e-04
256	1,1	7	1.7124	9.3398e-05
	2,2	5	3.0813	9.3398e-05
	2,3	4	3.3759	9.3398e-05
	3,3	4	3.7155	9.3398e-05
	2,4	4	3.5898	9.3398e-05
512	1,1	8	9.0273	2.3349e-05
	2,2	6	16.1597	2.3349e-05
	2,3	5	19.5823	2.3349e-05
	3,3	5	21.8906	2.3349e-05
	2,4	5	21.4553	2.3349e-05
1024	1,1	9	56.4277	5.8372e-06
	2,2	6	75.9721	5.8372e-06
	2,3	6	102.4055	5.8372e-06
	3,3	5	107.8853	5.8372e-06
	2,4	5	106.4592	5.8372e-06
2048	1,1	10	344.4211	1.4593e-06
	2,2	7	559.8833	1.4593e-06
	2,3	7	695.9285	1.4593e-06
	3,3	7	684.5836	1.4593e-06
	2,4	7	697.2869	1.4593e-06

Table 4: numerical results for Inexact Uzawa

It is seen that the main time consumption is due to CG iterations for each IU iteration. The time cost is about $O(n^2)$. As we decrease the the precision request for each iteration, the cost decreases significantly. Nevertheless, insufficiently precise iterations can lead to divergence. See the situation when $n_1 = n_2 = 1$. Besides, in most situations it is much better than Uzawa as solving $2n(n-1)$ order equation is typically tremendous work . We've found the time cost gets lowest when $n_1 = n_2 = 2$. We also do experiments when $N = 2048$ with other parameters. The results are as below.

$L = N/2, \alpha = 1, ite = 3$				
N	n_1, n_2	V-cycle times	CPU time(s)	err
64	1,1	-	-	-
	2,2	13	0.1218	0.0015
	2,3	12	0.1356	0.0015
	3,3	11	0.1533	0.0015
	2,4	11	0.1331	0.0015
128	1,1	-	-	-
	2,2	16	0.3051	3.7363e-04
	2,3	14	0.3757	3.7363e-04
	3,3	13	0.3540	3.7363e-04
	2,4	13	0.3647	3.7363e-04
256	1,1	-	-	-
	2,2	18	1.0464	9.3398e-05
	2,3	16	1.0476	9.3398e-05
	3,3	15	1.1236	9.3398e-05
	2,4	15	1.1627	9.3398e-05
512	1,1	-	-	-
	2,2	20	4.6836	2.3349e-05
	2,3	18	4.6569	2.3349e-05
	3,3	17	5.5668	2.3349e-05
	2,4	16	5.2103	2.3349e-05
1024	1,1	-	-	-
	2,2	22	23.1816	5.8372e-06
	2,3	20	29.0675	5.8372e-06
	3,3	19	30.5667	5.8372e-06
	2,4	18	26.1623	5.8372e-06
2048	1,1	-	-	-
	2,2	24	109.4843	1.4593e-06
	2,3	22	135.8847	1.4593e-06
	3,3	21	147.7990	1.4593e-06
	2,4	21	144.7638	1.4593e-06

Table 5: numerical results for Inexact Uzawa

$N = 2048, n_1 = n_2 = 2, ite = 3$				
L	α	V-cycle times	CPU time(s)	err
$N/2$	0.6	42	206.1259	1.4593e-06
	0.8	29	139.2482	1.4593e-06
	1.0	24	109.4843	1.4593e-06
	1.2	31	152.5146	1.4593e-06
$N/4$	0.6	-	-	-
	0.8	-	-	-
	1.0	-	-	-
	1.2	-	-	-
$N/8$	0.6	-	-	-
	0.8	-	-	-
	1.0	-	-	-
	1.2	-	-	-

Table 6: numerical results for Inexact Uzawa

$N = 2048, n_1 = n_2 = 2, ite = 10$				
L	α	V-cycle times	CPU time(s)	err
$N/2$	0.6	13	128.4066	1.4593e-06
	0.8	13	129.4319	1.4593e-06
	1.0	14	140.1027	1.4593e-06
	1.2	14	137.7923	1.4593e-06
$N/4$	0.6	13	122.5156	1.4593e-06
	0.8	14	143.7004	1.4593e-06
	1.0	14	137.3772	1.4593e-06
	1.2	14	164.6454	1.4593e-06
$N/8$	0.6	21	207.4238	1.4593e-06
	0.8	15	150.8334	1.4593e-06
	1.0	14	136.6724	1.4593e-06
	1.2	40	389.6522	1.4593e-06

Table 7: numerical results for Inexact Uzawa

Theoptimal α in theory is experimentally better than others as shown in the above results except two situations. For smaller L, the bottom grids are coaser thus more precise solution is needed for each IU iteration. As we can see, moderately higher precision of equation solving can significantly decrease the number of outer cycles. Additionally, the time cost is less than that of DGS.

Appendix A

We prove theorem 1 in appendix A.

Proof (Theorem 1) *Write A, B as*

$$A = \begin{bmatrix} \frac{1}{h^2}A_1 & \\ & \frac{1}{h^2}A_2 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{h}B_1 \\ \frac{1}{h}B_2 \end{bmatrix}.$$

We have $B^T A^{-1} B = B_1^T A_1^{-1} B_1 + B_2^T A_2^{-1} B_2$. Let

$$C = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}_{(N-1)\times N}$$

then we have

$$CC^T = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & & \\ & & & -1 & 2 \end{bmatrix}_{(N-1)\times(N-1)} \qquad C^T C = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}_{N\times N}.$$

Suppose $C = UDV^T$ is the SVD with $U = (u_{ij})_{(N-1)\times(N-1)}, V = (v_{ij})_{N\times N}$ and D is the diagonal matrix of size $(N-1)\times N$ with singular value $\lambda_i, i = 1, \dots, N-1$. We denote $U = [\alpha_1, \dots, \alpha_{N-1}], V_0 = [\beta_1, \dots, \beta_N]$, then $\lambda_i \alpha_i = C\beta_i, \lambda_i \beta_i = C^T \alpha_i, 1 \le i \le N-1$. Furthermore, α_i, β_i are eigenvectors of $CC^T, C^T C$, respectively. Since $C^T C$'s rank is no more than $N-1$, we can WLOG suppose $\lambda_N = 0$ and substitute it to the equation we have $\beta_N = \frac{1}{N} 1^{N\times 1}$. Now we have

$$B_1 B_1^T = CC^T \otimes I_N,$$
$$B_2 B_2^T = I_N \otimes T_{N-1},$$

Here $\otimes$ is Kronecker product and

$$T_{N-1} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(N-1)\times(N-1)}$$

Thus for $1 \leq i \leq N-1, 1 \leq j \leq N$, *we have*

$$B_1 B_1^T(\beta_j \otimes \alpha_i) = \lambda_i^2(\beta_j \otimes \alpha_i)$$
$$B_2 B_2^T(\alpha_i \otimes \beta_j) = \lambda_i^2(\alpha_i \otimes \beta_j)$$

Therefore, there are SVDs of $B_1 = U_1 D_1 V_1^T, B_2 = U_2 D_2 V_2^T$, *respectively, such that*

$$U_1 = V \otimes U$$
$$U_2 = U \otimes V$$

Substitute them into the equations above we get

$$V_1 = V_2 = V \otimes V$$

and $D_1 = D_2 = diag(\lambda_i^2 I_N)_{N-1\times N}$,$U_1^T B_1 B_1^T U_1 = U_2^T B_2 B_2^T U_2 = diag(\lambda_i I_N), i = 1, \dots, N-1$. *Moreover,* $U_1^T(T_N \otimes I_{N-1})U_1 = diag(D^T D)$,$U_2^T(I_{N-1} \otimes C^T C)U_2 = diag(D^T D)$. *Here*

$$\begin{aligned} B^T A^{-1} B &= B_1^T A_1^{-1} B_1 + B_2^T A_2^{-1} B_2 \\ &= V_1 D_1^T U_1^T A_1^{-1} U_1 D_1 V_1^T + V_2 D_2^T U_2^T A_2^{-1} U_2 D_2 V_2^T \\ &= V_1 D_1^T (U_1^T A_1 U_1)^{-1} D_1 V_1^T + V_2 D_2^T (U_2^T A_2 U_2)^{-1} D_2 V_2^T \end{aligned}.$$

We could write A_1, A_2 *as*

$$A_1 = B_1 B_1^T + (T_N \otimes I_{N-1}), A_2 = B_2 B_2^T + diag(C^T C)$$

Now we can induce that

$$D_1^T(U_1^T A_1 U_1)^{-1} D_1 = diag(\frac{\lambda_1^2}{\lambda_1^2+\lambda_1^2}, \dots, \frac{\lambda_1^2}{\lambda_1^2+\lambda_{N-1}^2}, 1, \dots, \frac{\lambda_{N-1}^2}{\lambda_{N-1}^2+\lambda_1^2}, \dots, \frac{\lambda_{N-1}^2}{\lambda_{N-1}^2+\lambda_{N-1}^2}, 1, 0, \dots, 0, 0),$$

$$D_2^T(U_2^T A_2 U_2)^{-1} D_2 = diag(\frac{\lambda_1^2}{\lambda_1^2+\lambda_1^2}, \dots, \frac{\lambda_{N-1}^2}{\lambda_{N-1}^2+\lambda_1^2}, 1, \dots, \frac{\lambda_1^2}{\lambda_{N-1}^2+\lambda_1^2}, \dots, \frac{\lambda_{N-1}^2}{\lambda_{N-1}^2+\lambda_{N-1}^2}, 1, 0, \dots, 0, 0).$$

Thus

$$D_1^T(U_1^T A_1 U_1)^{-1} D_1 + D_2^T(U_2^T A_2 U_2)^{-1} D_2 = diag(I_{N^2-1}, 0)$$

and the proof is done.

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