Solution to Week 1 Exercises

The following Exercises correspond to those in the notes.

Sets

Exercise 1.1.1

1. List the elements of each of the following sets:

   (a) $\{x|x \in \mathbb{N} \text{ and } x < 5\}$

   This is the set of all natural numbers that are less than 5. Thus,

   $$\{x|x \in \mathbb{N} \text{ and } x < 5\} = \{1, 2, 3, 4\}$$

   (b) $\{x|x \in \mathbb{N} \text{ and } 10 < x^2 < 90\}$

   $10 < x^2 < 90 \iff \sqrt{10} < x < \sqrt{90}$ or $-\sqrt{90} < x < \sqrt{10}$.

   $$3 < \sqrt{10} < 4, \quad 9 < \sqrt{90} < 10 \quad -4 < -\sqrt{10} < -3, \quad -10 < -\sqrt{90} < -9$$

   This means that the set consists of integers that are either in interval $(-10, -4)$ or in interval $(4, 10)$.

   $\Longrightarrow$

   $$\{x|x \in \mathbb{N} \text{ and } 10 < x^2 < 90\} = \{4, 5, 6, 7, 8, 9\}$$

   (c) $\{x|x \in \mathbb{N} \text{ and } x^2 - 4 = 0\}$

   The set of numbers satisfying $x^2 - 4 = 0$ are $\{-2, 2\}$. Thus,

   $$\{x|x \in \mathbb{N} \text{ and } x^2 - 4 = 0\} = \{2\}$$

2. Determine whether 3 belongs to each of the following sets:

   (a) $\{x|x > -2\} \cup \{x|x < 0\}$

   Yes. 3 is in the first set, and hence is in the union.

   (b) $\{x|x^2 < 5\} \cap \{x|x^2 - 1 \text{ is an even integer}\}$

   No. $3^2 = 9 > 5$ hence $3 \notin \{x|x^2 < 5\}$. Thus, 3 is not in the intersection.

3. Describe all the subsets of each of the following sets:

(a) $\{0, 1\}$

ϕ (the empty set), {0}, {1}, {0, 1}

(b) {dog, cat, budgie}

ϕ (the empty set), {dog}, {cat}, {budgie}, {dog, cat}, {dog, budgie}, {cat, budgie}, {dog, cat, budgie}

(c) The set consisting of all pairs (x, y) where $x = 0$ or 1 and $y = 0$ or 1.

ϕ (the empty set),

{(0,0)}, {(0,1)}, {(1,0)}, {(1,1)}

{(0,0),(0,1)},{(0, 0), (1,0)},{(0,0),(1,1)}, {(0,1),(1,0)},{(0,1),(1,1)},{(1,0),(1,1)}

{(0,1),(1,0),(1,1)}, {(0,0),(1,0),(1,1)},{(0,0),(0,1),(1,1)},{(0,0),(0,1),(1,0)}

{(0,0),(0,1),(1,0),(1,1)}

4. Describe the set of all real numbers for which the following holds:

(a) $x^2 < 4$

$-2 < x < 2$

(b) $x(x - 1) > 0$

x and $x - 1$ are either both positive or both negative.

$$1 < x < \infty \quad \text{or} \quad -\infty < x < 0$$

5. Solution is given in Week 1 Exercise Sol.ipynb.

Functions

Exercise 1.2.2

Solutions are given in Week 1 Exercise Sol.ipynb.

Exercise 1.2.4

1. Each of the following functions assigns the value y to x as given in the table. Find the domain and range of each function.

(a) The function is given by

x	0	1	2	3
y	4	6	10	12

Domain: $\{0, 1, 2, 3\}$

Range: $\{4, 6, 10, 12\}$

(b) The function is given by

x	a	b	c	d
y	g	h	i	j

Domain: $\{a, b, c, d\}$

Range: $\begin{cases} \{g, h, i, j\}, & \text{if } g, h, i, j \text{ are distinct} \\ \{g\} \cup \{h\} \cup \{i\} \cup \{j\}, & \text{if } g, h, i, j \text{ are not distinct} \end{cases}$

(Example: If $g = h = 0$ and $i = j = 1$, for example, then Range$= \{0, 1\}$ instead of $\{0, 0, 1, 1\}$.)

2. For each of the following real functions of a real variable given by a formula, state the domain and range.

   (a) $y = -2 - x^2$

   Domain: $(-\infty, \infty)$; Range: $[-2, \infty)$

   (b) $y = x^3 - x$

   Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

   (c) $y = \frac{1}{x-1}$

   Domain: $(-\infty, 1) \cup (1, \infty)$; Range: $(-\infty, 0) \cup (0, \infty)$

3. Find all real zeros of each of the following polynomials, and give the multiplicity of each.

   (a) $x - 3$

   $x = 3$ multiplicity 1

   (b) $x^2 - 4$

   $x = -2$ multiplicity 1, and $x = 2$ multiplicity 1

   (c) $(x - 2)^4(x - 5)^3$

   $x = 2$ multiplicity 4, and $x = 5$ multiplicity 3

4. Draw the graphs for each of the following functions.

   See Week 1 Exercise Sol.ipynb

Exercise 1.2.6

1. Simplify $\dfrac{3^x 2^{x+3}}{8}$.

$$\frac{3^x 2^{x+3}}{8} = \frac{3^x 2^x 2^3}{8} = \frac{(3 \times 2)^x \times 8}{8} = 6^x$$

Exercises 3.2.7

1. Simplify $e^{3 \log 6x}$

$$e^{3 \log 6x} = e^{\log (6x)^3} = (6x)^3 = 6^3 x^3 = 216x^3$$

2. Simplify $\log_2(4 \times 2^x)$;

   $\log_2(4 \times 2^x) = \log_2 2^2 + \log_2 2^x = 2 + x$

3. Find x such that $(1 + e^{2x})^2 = 3$.

$$\begin{aligned}
&(1+e^{2x})^2 = 3\\
&2\log(1+e^{2x}) = \log 3\\
&\log(1+e^{2x}) = \frac{1}{2}\log 3\\
&\log(1+e^{2x}) = \log 3^{\frac{1}{2}}\\
&1+e^{2x} = 3^{\frac{1}{2}}\\
&e^{2x} = 3^{\frac{1}{2}} - 1\\
&2x = \log(3^{\frac{1}{2}} - 1)\\
&x = \frac{1}{2}\log(3^{\frac{1}{2}} - 1)\\
&x = \log\sqrt{\sqrt{3} - 1}
\end{aligned}$$

4. Find the domain of the function $\log(1 - e^x)$

   The function is well-defined for (and only for) $1 - e^x > 0$. This implies $e^x < 1$ or $x < \log 1 = 0$

   Domain: $\{x \mid x < 0\}$

Exercise 1.2.8

1. Evaluate $\log_2 e^6$ using your calculator.

   $\log_2 e^6 = 8.6562$

Exercises 3.2.10

1.

2. Are given functions even (E), odd (O) or neither (N)?

   (a) $f(x) = x^4 + 1$

   Since $f(-x) = (-x)^4 + 1 = (-1)^4 x^4 + 1 = x^4 + 1 = f(x) \implies$ Answer(E)

   (b) $f(x) = 2x - 1$

   Since $f(-1) = 2(-1) - 1 = -2 - 1 = -3$, we see that

   $f(1) = 2 - 1 = 1 \neq f(-1) \implies f(x)$ is not even,

   and that $-f(1) = -(2-1) = -1 \neq f(-1) \implies f(x)$ is not odd.

   Hence answer: (N)

(c) $f(t) = t|t|$

Since $f(-t) = -t|-t| = -t|t| = -f(t)$, $f(t)$ is odd. $\Longrightarrow$ Answer: (O)

3. Simplify the following expressions:

(a) $\dfrac{(a^3)(a^6)}{a^5}$

$\dfrac{(a^3)(a^6)}{a^5} = \dfrac{a^{3+6}}{a^5} = a^{9-5} = a^4$

(b) $\dfrac{b^x b^y}{b^{x-y}}$

$\dfrac{b^x b^y}{b^{x-y}} = \dfrac{b^{x+y}}{b^{x-y}} = b^{(x+y)-(x-y)} = b^{2y}$

4. Given $\log a = 2$ and $\log b = 3$, evaluate the following without calculator:

(a) $\log ab$ 5

$\log ab = \log a + \log b = 2 + 3 = 5$

(b) $\log \frac{a^3}{b^2}$

$\log \frac{a^3}{b^2} = \log a^3 - \log b^2 = 3 \log a - 2 \log b = 6 - 6 = 0$

5. Simplify the following expressions:

(a) $\log e^{-x}$

$\log e^{-x} = -x \log e = -x$

(b) $e^{3 \log y}$

$e^{3 \log y} = e^{\log y^3} = y^3$

6. Solve the following equations for y:

(a) $\log(y + 3) = x$

$\log(y+3) = x \;\Rightarrow\; e^{\log(y+3)} = e^x \;\Rightarrow\; y + 3 = e^x \;\Rightarrow\; y = e^x - 3$

(b) $e^{-y} = x^4$

$e^{-y} = x^4 \;\Rightarrow\; \log(e^{-y}) = \log(x^4) \;\Rightarrow\; -y = 4 \log x \;\Rightarrow\; y = -4 \log(x)$

Inequalities

Exercise 1.3.3

Solve each of the following inequalities for x.

1. $7x + 5 > 26$

Solution.

$$7x + 5 > 26 \Leftrightarrow 7x > 21 \Leftrightarrow x > 3$$

2. $-2x - 2 < -10$

Solution.

$$-2x - 2 < -10 \Leftrightarrow -2x < -10 + 2 = -8 \Leftrightarrow x > 4$$

(dividing by a negative number changes the direction of the inequality)

3. $4x - 7 < 11x + 7$

Solution.

$$4x - 7 < 11x + 7 \Leftrightarrow -7 - 7 < 11x - 4x \Leftrightarrow -14 < 7x \Leftrightarrow x > -2$$