Essential Mathematics

RMIT

Semester 2, 2017

Contents

Chapter 1

Sets, Functions and Inequalities

1.1 Sets

In mathematics, a *set* is a collection of distinct objects. For example, {1,2,3,4} is a set of numbers and {apples, bananas, figs, oranges} is a set of fruits. The following are **not** sets.

> {all young workers} is not a set as the objects are not clearly specified (some works may not be determined to be young or old);
>
> {1,2,3,4,3} is not a set as not all the objects are distinct or unique.

Sets are sometimes written using the set-builder notation in the examples below.

- $\{x \mid 3 < x < 6\}$ is the set of all numbers x greater than 3 and less than 6
- $\{x \mid x^2 - 2x + 4 = 0\}$ is the set of all numbers x that satisfies the equation shown

The objects in a set are called the *elements* of the set. For example, consider the set $A = \{1, 3, 5, 7\}$. The number 5 is an element of A. We say that 5 belongs to A. The number 2 is not an element of A, and we say that 2 does not belong to A.

The following are relational notations involving general sets (A and B, $\cdots$):

> $\in$: $x \in A$ means that x belongs to A
>
> $\notin$: $x \notin A$ means that x does not belong to A
>
> $\subset$: $A \subset B$ means A is contained in or equals to B
>
> $\subsetneq$: $A \subsetneq B$ means A is contained in but does not equal to B
>
> $\supset$: $A \supset B$ means A includes or equals to B
>
> $\supsetneq$: $A \supsetneq B$ means A includes but does not equal to B

The following examples are common sets operations (A, B, C denote sets):

The *union* of A and B:

$$A \cup B = \text{“}A \text{ and } B \text{ combined”}$$

The *union* of A, B and C:

$$A \cup B \cup C$$

The *intersection* of A and B:

$$A \cap B = \text{“the set of all elements common to both } A \text{ and } B\text{”}$$

The *intersection* of A, B and C:

$$A \cap B \cap C = \text{“all elements common to } A,\ B \text{ and } C\text{”}$$

The difference of A minus B:

$$A \setminus B = \text{the set of all elements of } A \text{ that do not belong to } B$$

The set of all real numbers, denoted $\mathbb{R}$, includes all the rational numbers (integer and fractions) and all the irrational numbers (irrational algebraic numbers such as $\sqrt{2}$ and $\sqrt{3}$, and transcendental numbers such as π and e).

Listed below are important subsets of $\mathbb{R}$ when dealing with functions.

$(a,\ b) = \{x \mid a < x < b\}$

$[a,\ b] = \{x \mid a \leq x \leq b\}$

$(a,\ b] = \{x \mid a < x \leq b\}$

$[a,\ b) = \{x \mid a \leq x < b\}$

$(a,\ \infty) = \{x \mid a < x < \infty\}$

$[a,\ \infty) = \{x \mid a \leq x < \infty\}$

$(-\infty,\ b) = \{x \mid \infty < x < b\}$

$(-\infty,\ b] = \{x \mid \infty < x \leq b\}$

$\mathbb{N} = \{1,\ 2,\ 3,\ \cdots\}$ – the set of all natural numbers

$\mathbb{Z} = \{\cdots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \cdots\}$ – the set of all integers

$\mathbb{R} = (-\infty,\ \infty) = \{x \mid -\infty < x < \infty\}$

Note: In some texts natural numbers may include 0

It is clear that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$.

1.1.1 Exercises

1. List the elements of each of the following sets:

   (a) $\{x | x \in \mathbb{N} \text{ and } x < 5\}$

   (b) $\{x | x \in \mathbb{N} \text{ and } 10 < x^2 < 90\}$

   (c) $\{x|x \in \mathbb{N} \text{ and } x^2 - 4 = 0\}$

2. Determine whether 3 belongs to each of the following sets:
   (a) $\{x|x > -2\} \cup \{x|x < 0\}$
   (b) $\{x|x^2 < 5\} \cap \{x|x^2 - 1 \text{ is an even integer}\}$
3. Describe all the subsets of each of the following sets:
   (a) $\{0, 1\}$
   (b) {dog, cat, budgie}
   (c) The set consisting of all pairs (x, y) where $x = 0$ or 1 and $y = 0$ or 1.
4. Describe the set of all real numbers for which the following holds:
   (a) $x^2 < 4$
   (b) $x(x - 1) > 0$
5. Use Julia to answer the following questions.
   (a) Find the union of sets A and B where A={1, 2, 3, 4} and B={4, 5, 6, 7}
   (b) Find the intersection of the sets A and B given in (a).
   (c) What are the common elements in the following two sets? Set A contains the positive even numbers less than or equal to 30 and set B contains multiples of 3 less than or equal to 30.

1.2 Functions

Example

Consider $f(x) = x^2$. The function f takes any real number and assigns its square:x^2.

1.2.1 Definition and Terminology

To each element of a given set X, a *function* assigns an element on another set Y. The set X is called the *domain* of the function. The elements of Y form a set called the *range* of the function. A function f is often called a *mapping*. A function with domain X and range Y is said to be a *mapping of X into Y or that f maps X into Y*

For our example, the *domain* of the function is the set of all real numbers $(x|-\infty < x < \infty)$.

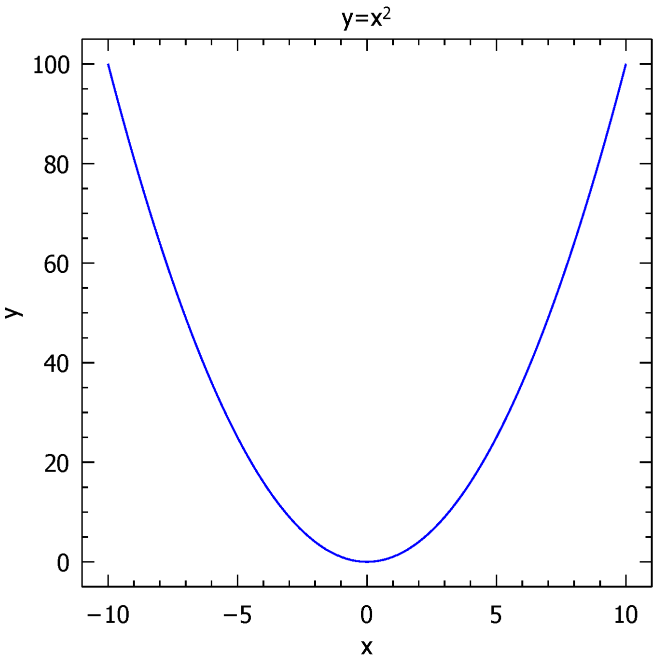


Figure 1.1: A simple function

Another way of thinking about a function is in terms of *input* and *output*. Inserting an input from the domain of the function we get a unique output in the range of the function. These inputs and outputs can be written as pairs in a set $x, f(x)$. Thus, for example, the **set of ordered pairs** $(1,1),(2,4),(3,9),(4,16)$ is a function with domain 1,2,3,4 and range 1,4,9,16. Ordered pairs can be written in a table and then they are sometimes referred to as **table functions**.

The **zeros** of a function are those values x for which $f(x) = 0$. For example, consider $f(x) = 2x - 6, \quad f(3) = 0$. Thus $x = 3$ is a zero of f.

Some special classes of function are given below.

Linear functions $y = mx+c$ where m and c are fixed real numbers (ie constants)

Polynomial functions $y = a_n x^n + ... + a_1 x + a_0$ where $a_n, ..., a_0$are fixed real numbers, called *coefficients*.

Rational functions $y = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials and q is not the zero polynomial.

Trigonometric functions $y = \sin x, \quad y = \cos x, \quad y = \tan x, ...$

Exponential functions $y = 2^x, \quad y = 3^x, ...$

Power functions $y = x^3, \quad y = x^{3/2}, ...$

Logarithmic functions $y = \log_{10} x, \quad y = \log_e x, ...$

1.2.2 Exercises

Refer to Tutorial 6 for these problems. They all require the use of Julia.

1. Locate the zeros of the polynomial function $x^3 - 6x^2 + 11x - 6$ by plotting the function.

2. Check that your answer is correct by using the Polynomials package to find the roots.

3. Given two polynomial functions $y_1 = x^4 - 6x^3 + 11x^2 - 6x$ and $y_2 = x^2 - 3x - 4$, plot these functions on the interval [0,3] Do either of the polynomials have any roots in this interval?

4. The rational function

$$z = \frac{x^4 - 6x^3 + 11x^2 - 6x}{x^2 - 3x + 4}c$$

   is defined over the domain [0,3]. Plot this function. Why are the roots the same as y_1 above?

5. Plot in the same graph the power function x^3 and the exponential function 3^x over the interval [0,4]. Which grows faster?

1.2.3 Linear Functions

We have already noted that a linear function can be written in the form $y = mx + c$ where m and c are constants. If we plot a linear function we observe that when $x = 0, \quad y = c$. In other words c is the y intercept, where the line cuts the y-axis. The slope of a line can be defined as 'the change in y over a change in x' and this is equal to m.

Example

The line $y = \frac{1}{2}x + 1$ is shown in figure 1.2 . Note the y intercept at $y = 1$. Note also that y increases by one unit for every two units increase in x. In other words the slope $= \frac{1}{2}$.

1.2.4 Exercises

1. Each of the following functions assigns the value y to x as given in the table. Find the domain and range of each function.

   (a) The function is given by

x	0	1	2	3
y	4	6	10	12

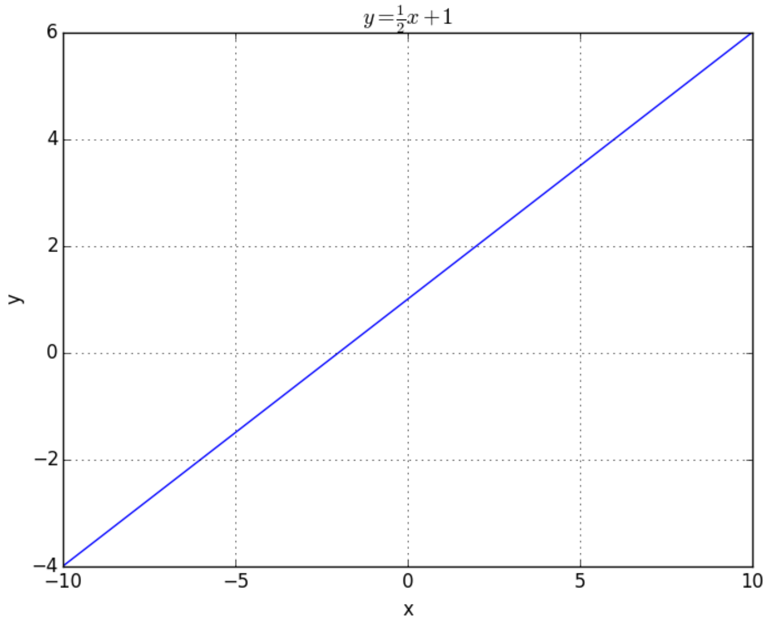


Figure 1.2: A linear function

(b) The function is given by

x	a	b	c	d
y	g	h	i	j

2. For each of the following real functions of a real variable given by a formula, state the domain and range.

   (a) $y = -2 - x^2$

   (b) $y = x^3 - x$

   (c) $y = \frac{1}{x-1}$

3. Find all real zeros of each of the following polynomials, and give the multiplicity of each.

   (a) $x - 3$

   (b) $x^2 - 4$

   (c) $(x-2)^4(x-5)^3$

4. Draw the graphs for each of the following functions.

   (a) $y = 2x + 3$

   (b) $y = 2x - 3$

   (c) $y = -2x + 1$

1.2.5 Some Basic Functions

Exponential Functions

An exponential function has the form

$$f(x) = b^x \quad (b > 0)$$

and has the following properties:

1. $b^x > 0$
2. $b^x \times b^y = b^{x+y}$
3. $\dfrac{b^x}{b^y} = b^{x-y}$
4. $b^0 = 1$
5. $b^{-x} = \dfrac{1}{b^x}$
6. $(ab)^x = a^x b^x, \quad (a > 0)$
7. $\left(b^x\right)^y = b^{xy}$
8. If $b > 1$, the function is increasing; if $0 < b < 1$, the function is decreasing.

Both increasing and decreasing functions are called monotone functions.

The most common exponential function is $f(x) = e^x$ where $e = 2.71828\ldots$.

NOTE. It will be helpful to know that $b^{1/2} = \sqrt{b}$, $b^{1/3} = \sqrt[3]{b}$, and in general, $b^{1/n} = \sqrt[n]{b}$.

Example 1.2.1 *Turn the following into a power of* 2.

$$\frac{\sqrt[3]{2^x}\ 4^x}{4^3\ 4^{x/3}}$$

.

Solution.

$$\begin{aligned}
\frac{\sqrt[3]{2^x}\ 4^x}{4^3\ 4^{x/3}} &= \frac{(2^x)^{1/3}\,(2^2)^x}{(2^2)^3\,(2^2)^{x/3}} \\
&= \frac{2^{x/3}2^{2x}}{2^6 2^{2x/3}} \\
&= x^{2x+x/3-2x/3-6} \\
&= 2^{6x/3-x/3-6} \\
&= 2^{5x/3-6}
\end{aligned}$$

1.2.6 Exercises

1. Simplify $\dfrac{3^x 2^{x+3}}{8}$. Answer: 6^x

Logarithmic Functions

The log function with base b is defined by

$$\log_b x = y \iff x = b^y.$$

From the definition, we have

$$\boxed{\log_b b^y = y \quad \text{and} \quad b^{\log_b x} = x.}$$

While any base b (> 0) can be used, in practice, $b = e \approx 2.718\ldots$ is most popular. $b = 10$ and $b = 2$ can also be seen occasionally.

We usually write $\log_e x$ as $\log x$ (or $\ln x$) and it is called the natural logarithmic function. In Julia, one may use $\log(x)$ for $\log_e x$ and $\log(b, x)$ for $\log_b x$.

Logarithmic Laws

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^y) = y \log_b x$

Example 1.2.2 *Express the following logarithms in terms of* $\ln 2$ *and* $\ln 3$.

1. $\log(2.25)$;
2. $\log(2\sqrt[3]{9})$.

Solution.

1.

$$\begin{aligned}\log(2.25) &= \log(\frac{9}{4}) \\ &= \log 9 - \log 4 \\ &= \log(3^2) - \log(2^2) \\ &= 2\log 3 - 2\log 2;\end{aligned}$$

2.

$$\begin{aligned}\log(2\sqrt[3]{9}) &= \log(2(3^2)^{1/3}) \\ &= \log(2 \times 3^{2/3}) \\ &= \log 2 + \frac{2}{3}\log 3.\end{aligned}$$

Example 1.2.3 *Simplify the following expressions:*

1. $e^{-2\ln x}$;
2. $\log_{10}(0.01 \times 10^x)$.

Solution.

1. $e^{-2\ln x} = \dfrac{1}{e^{2\ln x}} = \dfrac{1}{(e^{\log x})^2} = \dfrac{1}{x^2}$;
2. $\log_{10}(0.01 \times 10^x) = \log_{10}(10^{-2}10^x) = \log_{10} 10^{x-2} = x - 2$

Example 1.2.4 *Given that* $\log(x^2-1) - 2\log(x+1) = \ln y$, *find* y.

Solution. $\log(x^2-1) - 2\log(x+1) = \log y$

$$\Longrightarrow \quad \log(x^2-1) - \log(x+1)^2 = \log y \Longrightarrow \quad \log\frac{x^2-1}{(x+1)^2} = \log y$$

Since $\dfrac{x^2-1}{(x+1)^2} = \dfrac{(x+1)(x-1)}{(x+1)^2} = \dfrac{x-1}{x+1}$, we have

$$\log y = \log\frac{x-1}{x+1} \qquad \Longrightarrow \qquad y = \frac{x-1}{x+1}$$

Example 1.2.5 *Solve the equation* $\ln(y-1) = 2$ *for* y.

Solution.

$$\begin{aligned} \log(y-1) = 2 \quad &\Longrightarrow \quad e^{\log(y-1)} = e^2 \\ &\Longrightarrow \quad y - 1 = e^2 \\ &\Longrightarrow \quad y = e^2 + 1 \end{aligned}$$

1.2.7 Exercises

Answer

1. Simplify $e^{3\ln 6x}$; $216x^3$
2. Simplify $\log_2(4 \times 2^x)$; $2 + x$
3. Find x such that $(1 + e^{2x})^2 = 3$. $\frac{1}{2}\log(\sqrt{3}-1)$

   also $\log\sqrt{\sqrt{3}-1}$
4. Find the domain of the function $\log(1 - e^x)$. $\{x \mid x < 0\}$

Conversion (Change of Base)

Sometimes in a problem where the value of $\log_a x$ is available for some base a but we want to evaluate $\log_b x$ $(b \neq a)$. To be more precise, we want to express $y = \log_b x$ in terms of $\log_a x$.

To proceed we rewrite $y = \log_b x$ as:

$$b^y = x.$$

Now taking logarithm with base a and using the logarithmic laws we obtain

$$\log_a x = \log_a(b^y) = y\log_a b = (\log_b x)(\log_a b).$$

Dividing both sides by $\log_b a$, we end up with:

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

We will call this the log conversion formula.

Example 1.2.6 *Evaluate* $\log_7 23$ *using the natural logarithm function on your calculator.*

Solution.

By the log conversion formula,

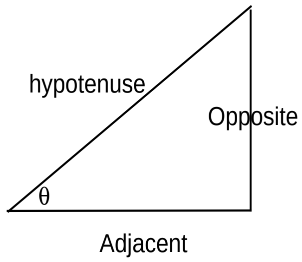
$$\log_7 23 = \frac{\log_e 23}{\log_e 7} = \frac{\ln 23}{\ln 7} \approx 1.61.$$

1.2.8 Exercises

1. Evaluate $\log_2 e^6$ using your calculator. Answer: $\log_2 e^6 \approx 8.656$

1.2.9 Trigonometric Functions

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



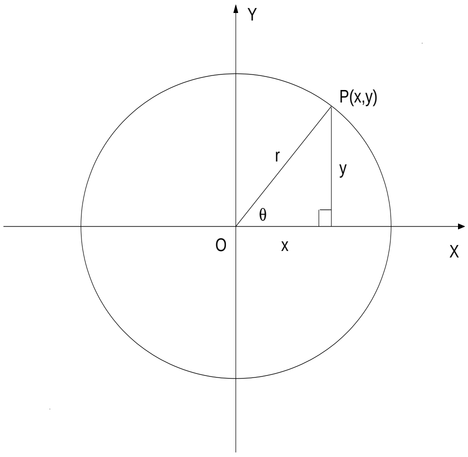
Note that the triangle must be right-angled and definitions of the three trigonometric functions only apply for $0° < \theta < 90°$ but are easily extended as follows:

Consider a circle centred at origin and a point (x, y) on it.

Note: θ is measured in anticlockwise direction from the positive x-axis. If $0 < \theta < 90°$, using the right angled triangle shown gives:

$$\begin{aligned} \sin\theta &= \frac{y}{r} \to y = r\sin\theta \\ \cos\theta &= \frac{x}{r} \to x = r\cos\theta. \end{aligned}$$

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$



Allowing θ to take any value (that is, letting (x, y) be any point on the circle) extends the definition of $\cos\theta$ and $\sin\theta$ to all angles. In particular, note that

$$\cos(\theta + 2\pi) = \cos\theta$$

and

$$\sin(\theta + 2\pi) = \sin\theta,$$

so both cosine and sine have period 2π.

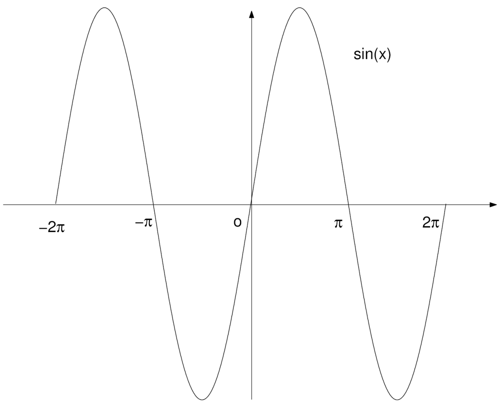


Figure 1.3: The graph of $\sin(x)$

A function $f(x)$ satisfying $f(x) = -f(-x)$ is called an odd function. If $f(x) = f(-x)$, it is called an even function.

$\sin(-\theta) = -\sin\theta \Rightarrow \sin(x)$ is an odd function.

$\cos(-\theta) = \cos\theta \Rightarrow cos(x)$ is an even function.

A function $f(x)$ satisfying $f(x+T) = f(x)$ for all $x \in (-\infty, \infty)$ is called a periodic function, where $T > 0$ is a constant called, in this case, a period.

It is easy to check that both $\sin(x)$ and $\cos(x)$ are periodic, with smallest period 2π.

One of the most important formulas is

$$(\cos\theta)^2 + (\sin\theta)^2 \quad = \quad 1$$

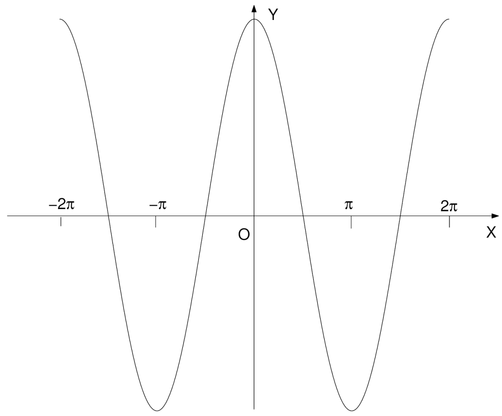


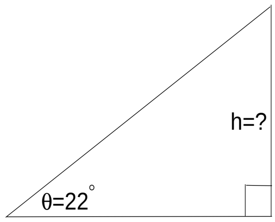
Figure 1.4: The graph of $\cos(x)$

Note: $(\sin\theta)^n$ is written as $\sin^n\theta$ if $n \neq -1$. $\sin\theta^n$ usually represents $\sin(\theta^n)$.

Example 1.2.7 *The angle of inclination of the top of a flagpole from a distance of* $30m$ *is* $22°$*. How high is the flagpole?*

Solution.

$$\begin{aligned} \tan 22° &= \frac{h}{30} \\ h &= 30\tan 22° \\ &\approx 12.1208(\mathrm{m}) \end{aligned}$$



1.2.10 Exercises

1. Are given functions even (E), odd (O) or neither (N)? Answer
   (a) $f(x) = x^4 + 1$ (E)
   (b) $f(x) = 2x - 1$ (N)
   (c) $f(t) = t|t|$ (O)
2. Simplify the following expressions:
   (a) $\frac{(a^3)(a^6)}{a^5}$ a^4
   (b) $\frac{b^x b^y}{b^{x-y}}$ b^{2y}
3. Given $\log a = 2$ and $\log b = 3$, evaluate the following without calculator:
   (a) $\log ab$ 5

(b) $\log \frac{a^3}{b^2}$ $\quad 0$

4. Simplify the following expressions:

   (a) $\log e^{-x}$ $\quad -x$

   (b) $e^{3\log y}$ $\quad y^3$

5. Solve the following equations for y:

   (a) $\log(y+3) = x$ $\quad y = e^x - 3$

   (b) $e^{-y} = x^4$ $\quad y = -\log(x^4)$

1.3 Inequalities

1.3.1 Introduction

The following are examples of inequalities:

$3 < 4$

$6 > 4$

All points (x, y) on the plane that is on or under the straight line $y = x$ satisfy the inequality $x \leq y$.

All points (x, y) that is strictly above the curve $y = x^2$ satisfy the inequality $y > x^2$.

1.3.2 Solving Inequalities

Sometimes it is not immediately clear what the set of numbers are that satisfy an inequality. For example, what is the set of numbers that satisfy the inequality $5x + 2 < 17$? The answer is $x|x < 3$ but how do we get that? We need some rules to manipulate inequalities into simpler forms. Without affecting the direction of an inequality we can do the following:

- Add or subtract a number from both sides
- Multiply or divide both sides by a **positive** number

For our example above subtracting 2 from both sides gives $5x < 15$. If we now divide both sides by 5 we get the result given.

We can also do the following provided we **change the direction of the inequality** :

- Multiply or divide both sides by a **negative** number
- Swap the left and right hand sides of the inequality

Example $-9 > -3x$.

Solution Multiply by -1 and change the direction of the inequality. This gives $9 < 3x$. We can now proceed to divide both sides by 3 to get $3 < x$. Swapping sides and changing the direction of the inequality gives $x > 3$.

1.3.3 Exercises

Solve for x.

1. $7x + 5 > 26$
2. $-2x - 2 < -10$

3. $4x - 7 < 11x + 7$

Consider $ax < 2a$. Dividing both sides by a gives $x < 2$ or does it? The rule says that we can divide both sides by a **positive** number without changing the direction of the inequality. But is a a positive number? We do not know. If a is negative the result would be $x > 2$. So all we can state is that if $b > 0$ then $x < 2$ but if $b < 0$ then $x > 2$.

Before considering some more difficult inequality problems we need to note the following Rule of Signs.

(a) $ab < 0 \implies a > 0$ and $b < 0$, or $a < 0$ and $b > 0$, $a, b \neq 0$

(b) $ab > 0 \implies a > 0$ and $b > 0$ or $a < 0$ and $b < 0$, $a, b \neq 0$

(c) $ab = 0 \implies a = 0$ or $b = 0$

(d) $\frac{a}{b} > 0$ Same as (a).

(e) $\frac{a}{b} < 0$ Same as (b).

(f) $\frac{a}{b} = 0 \implies a = 0$

Example 1.3.1 *Solve for x:*

$$\frac{x-3}{x+5} > 0$$

Solution Both numerator and denominator must have the same sign. Thus either (1) or (2) below must hold.

(1) $x - 3 > 0$ and $x + 5 > 0$

(2) $x - 3 < 0$ and $x + 5 < 0$

(1) implies

$$x > 3 \text{ and } x > -5$$

As both these inequalities must hold the solution to (1) is $x > 3$.

(2) implies

$$x < 3 \text{ and } x < -5$$

In this case the last inequality is the stricter so the solution to (2) is $x < -5$.

We now conclude that the solution is:

$$x < -5 \text{ or } x > 3$$