MATH2267 Week 7

Limits, Derivatives and Applications

Semester 2, 2018

MATH2267 Week 7

Overview

Part 1: Limits
Introduction
Evaluation

Part 2: Derivatives
Introduction
Calculation

Part 3: Applications
Optimization (one variable)

Part 1. Limits

1. The concept.
2. Find limit by substitution, cancellation, graph, print values.
3. Limit of polynomials and rational function (without computer).
4. Limit at ∞.

Part 1: Limits

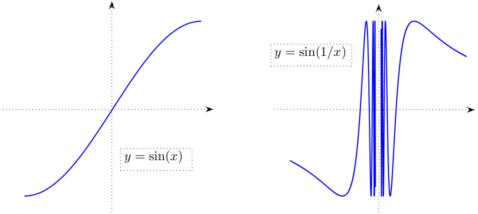
1.1. Introduction

The limit of $f(x)$ at $x = a$ is the trend of the $f(x)$ values when x approaches (not equal) to a. This limit is denoted by $\lim_{x\to a} f(x)$.

For example, at $x = 0$,

$f(x) = \sin(x)$ has limit 0 (or $\lim_{x\to 0} \sin(x) = 0$),

$g(x) = \sin(1/x)$ has no limit ($\lim_{x\to 0} \sin(1/x)$ doesn't exist.)



Part 1: Limits

1.1. Introduction

- If $\lim_{x \to a} f(x) = v$, we say $f(x)$ converges to v when $x \to a$.
- If $\lim_{x \to a} f(x)$ doesn't exist, we say $f(x)$ diverges when $x \to a$.

Methods for Finding Limits

There are many methods for finding limits. Some simplest ones:

Substitution

Cancellation

Graphic

Print out nearby values

Rules of limit

1. Substitution. If $f(x)$ is continuous, then $\lim_{x\to a} f(x) = f(a)$

1. $f(x)$ is continuous if its graph is not broken.
2. Polynomials, $\sin(x)$, $cos(x)$, e^x are continuous on $\mathbb{R}$.
3. $\log(x)$ and x^a (a is real) are continuous on $(0, +\infty)$.
4. $x^{1/3}$, $x^{1/5}$, $x^{1/7}, \cdots$ are continuous on $(-\infty, +\infty)$.
5. $x^{1/2}$, $x^{1/4}$, $x^{1/6}, \cdots$ are continuous on $(0, \infty)$.

Example. Find $\lim_{x\to 5} x^2 + 1$.
Solution. Polynomial are continuous. So, use substitution to obtain

$$\lim_{x\to 5} x^2 + 1 = 5^2 + 1 = 26$$

2. Simplification.

Example.

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}\\ &= \lim_{x\to 1} x+1\\ &= 1+1=2 \qquad \text{(use substitution)}\end{aligned}$$

Example.

$$\begin{aligned}\lim_{x\to 2}\frac{(x^2-4)^2}{\sqrt[3]{x-2}} &= \lim_{x\to 2}\frac{(x-2)^2(x+2)^2}{(x-2)^{1/3}}\\ &= \lim_{x\to 2}(x-2)^{5/3}(x+2)^2\\ &= 0 \qquad \text{(use substitution)}\end{aligned}$$

3. Graphic Method (Using Julia or otherwise)

Example. $f(x) = \dfrac{x^2-4}{x-2}$ undefined at $x = 2$. Graph shows $f(x)$ approaches 4 as $x \to 2$ from both side of 2.

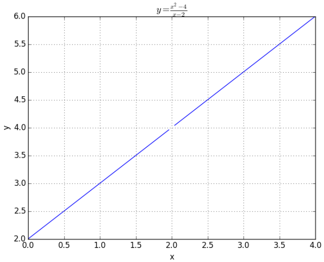
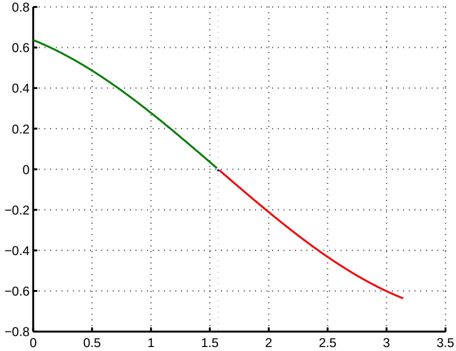


Figure: Graph of $\frac{x^2-4}{x-2}$

Part 1: Limits

1.2. Finding limits

Example. $\lim_{x \to \pi/2} \dfrac{\sin(x) - 1}{(x - \pi/2)^2} = 0$ can be obtained from thee graph of $(\sin(x) - 1)/(x - \pi/2)^2$ near $x = \pi/2$.



4. Print out Nearby Values

Example. Find $\lim_{x\to 0} \dfrac{e^x - 1}{x}$.

Solution. Choose two sequences of x values (put in vector form):

1. one approaches $x = 0$ from left: here choose $\mathbf{x}_1 = [-1, -1/10, -1/100, -1/1000, -1/10000]$
2. the other approaches $x = 0$ from right: say $\mathbf{x}_2 = [2, 1/5, 1/50, 1/500, 1/5000]$

In Julia, do the following

```
In [1]:  x1=[-1 -1/10 -1/100 -1/1000 -1/10000]
         x2=[2 1/5 1/50 1/500 1/5000]

Out[1]:  1x5 Array{Float64,2}:
          2.0  0.2  0.02  0.002  0.0002

In [2]:  (exp(x1)-1)./x1

Out[2]:  1x5 Array{Float64,2}:
          0.632121  0.951626  0.995017  0.9995  0.99995

In [3]:  (exp(x2)-1)./x2

Out[3]:  1x5 Array{Float64,2}:
          3.19453  1.10701  1.01007  1.001  1.0001

In [ ]:
```

We see when $x \to 0$, $f(x) \to 1$. Thus, $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$

Limit Rules

Let $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then, the following limit laws hold:

- $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$, ($c$ is constant)
- $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$,
- $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$,
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$.

Limit Rules (for $\lim_{x\to a}[f(x)/g(x)]$)

(1) $\begin{cases} \lim_{x\to a} f(x) = 0 \\ \lim_{x\to a} g(x) = v \neq 0 \end{cases} \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = 0$

(2) $\begin{cases} \lim_{x\to a} f(x) = v \ (v \text{ finite}) \\ \lim_{x\to a} g(x) = \infty \end{cases} \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = 0$

(3) $\begin{cases} \lim_{x\to a} f(x) = \infty \\ \lim_{x\to a} g(x) = v \end{cases} \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \begin{cases} \infty, & v > 0 \\ -\infty, & v < 0 \end{cases}$

(4) $\begin{cases} \lim_{x\to a} f(x) = \infty \text{ or } v \ (\neq 0) \\ \lim_{x\to a} g(x) = 0 \end{cases} \Rightarrow \lim_{x\to a} \left| \frac{f(x)}{g(x)} \right| = \infty$

(5) When $\begin{cases} \lim_{x\to a} f(x) = \infty \\ \lim_{x\to a} g(x) = \infty \end{cases}$ OR $\begin{cases} \lim_{x\to a} f(x) = 0 \\ \lim_{x\to a} g(x) = 0 \end{cases}$

$\lim_{x\to a}[f(x)/g(x)]$ is indeterminate limit of type $\frac{\infty}{\infty}$ or $\frac{0}{0}$, resp.

(5′) Other indeterminate types: $\infty - \infty$, $0 \cdot \infty$, 1^∞, 0^0, ∞^0, $\cdots$

Limit at Infinity

Sometimes we need to consider limits when $x \to \infty$ and $x \to -\infty$.

Note: Here ∞ means $+\infty$.

- $\lim_{x\to\infty} \frac{1}{x} = 0$
- $\lim_{x\to\infty} 3x = \infty$
- $\lim_{x\to\infty} \frac{1}{x} - 2x = -\infty$
- $\lim_{x\to\infty} \frac{\sin x}{3x} = 0$
- $\lim_{x\to\infty} \frac{x^4+3x+2}{x^2+20x} = \frac{x^2+3\frac{1}{x}+\frac{2}{x^2}}{1+\frac{20}{x}} = \infty$
- $\lim_{x\to\infty} \frac{3x^2+2}{2x^2+10} = \lim_{x\to\infty} \frac{3+\frac{2}{x^2}}{2+\frac{10}{x^2}} = \frac{3}{2}$
- $\lim_{x\to\infty} \frac{-x^3+2}{2x^2+10} = \lim_{x\to\infty} \frac{-x+\frac{2}{x^2}}{2+\frac{10}{x^2}} = -\infty$

Part 2

Derivatives

Skills and knowledge to develop:

1. Physical significance of derivative (- rate of change).
2. Geometric significance of derivative (- slope of tangent).
3. Derivative formulas for commonly used functions.
4. Derivative rules.
5. Apply derivative formulas and rules to find derivatives.
6. Solving problems involving rate of change.

Physical explanation – Rate of Change

Let $s(t)$ be the distance traveled at time t. What is the instant speed at time t?
Consider the average speed from time t to $t + h$:

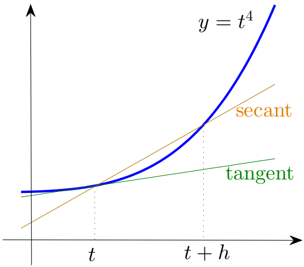
$$\frac{s(t+h) - s(t)}{h},$$

when $h \to 0$ the limit of the average speed over $[t, t+h]$

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

is the instant speed at time t.

Geometrical explanation – Slope of tangent



$\dfrac{y(t+h) - y(t)}{h}$ = slope of the secant.

Secant approaches to the tangent when $h \to 0$.

Thus, $y'(t) = \lim_{h \to 0} \dfrac{y(t+h) - y(t)}{h}$ is the slope of the tangent.

Part 2: Derivatives

2.1. Introduction

We define the derivative of a function $f(x)$ at x as the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and denote by $f'(x)$ or $\frac{df(x)}{dx}$.

Example

The distance traveled at time t is $y(t) = t^2$. What is the instant speed at a specific time t?

Solution.

$$\frac{y(t+h) - y(t)}{h} = \frac{(t+h)^2 - t^2}{h} = \frac{2th + h^2}{h} = 2t + h$$

Instant speed at t:

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} = \lim_{h \to 0} 2t + h = 2t$$

Example

Calculating derivatives from the first principles:

Example. Find $f'(x)$ for $f(x) = \sqrt{x}$.
Solution.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})h} \\ &= \lim_{h\to 0} \frac{(x+h)x}{(\sqrt{x+h}+\sqrt{x})h} = \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} \\ &= \frac{1}{2\sqrt{x}} \quad \text{(which exists when } x \neq 0) \end{aligned}$$

Part 2: Derivatives

2.2. Finding derivatives

Basic Derivatives:

1. $(c)' = 0$ (derivative of const. =0)
2. $(x^n)' = nx^{n-1}$ (n is real, e.g. $n = 1/2$)
3. $(e^x)' = e^x$
4. $[\log(x)]' = 1/x$
5. $[\sin(x)]' = \cos(x)$
6. $[\cos(x)]' = -\sin(x)$
7. $[\tan(x)]' = 1/\cos^2(x)$

Higher Derivatives

$f''(x) = (f'(x))'$ — second derivative

$f'''(x) = (f''(x))'$ — third derivative

$f^{(n)}(x) = (f^{(n-1)}(x))'$ — nth derivative

Rules of Derivatives:

1. $[af(x)]' = af'(x)$ a is constant
2. $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
3. $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$ Product Rule
4. $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ Quotient Rule
5. $[f(g(x))]' = f'(g(x))g'(x)$ Chain Rule

Many derivatives can be calculated by combining the basic derivatives and the derivative rules.

Part 2: Derivatives

2.2. Finding derivatives

Example.

1. Find the derivative of $y = 6x^4 - 3x^3 + x - 15$.

   Solution.

$$\begin{aligned} y'(x) &= 6(x^4)' - 3(x^3)' + (x)' - (15)' \\ &= 24x^3 - 9x^2 + 1 \end{aligned}$$

Derivatives of more complicated functions can be obtained using WolframAlpha:

Access WolframAlpha (from any browser):

```
http://www.wolframalpha.com
```

To find $[\sin(x^2)]'$, for example, type in the following then click "=":

```
derivative sin(x^2)
```

Part 3

Applications

1. Optimization (one variable).
2. Optimality conditions.
3. Examples.

Example. The cost to produce x units of some product is

$$C(x) = 0.01x^2 + 4x \ (\$).$$

The product is sold at price 100 $ per unit. Determine the sales volume at which profit reaches its maximum.

Solution.

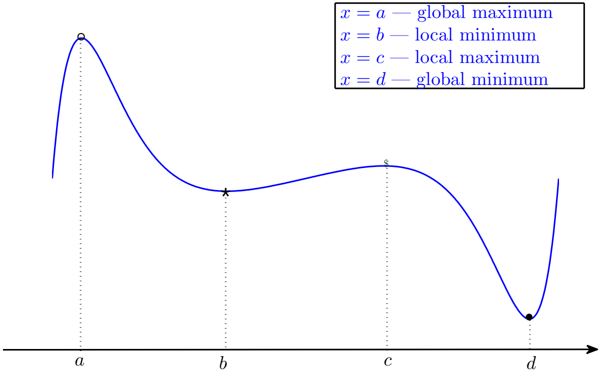
Profit = Income − Cost

The profit $P(x)$ by selling x units is

$$P(x) = 100x - 0.01x^2 - 4x = 96x - 0.01x^2$$

We will see that when selling 4800 units, the profit reaches the maximum of 230,400 $.

Local/global minimum/maximum.



Let $f(x)$ be defined on and differentiable in open interval (a, b). Then, local minima and local maxima only occur at critical points (i.e. solutions of $f'(x) = 0$)

Finding local minima and maxima

1. Finding all critical points (i.e. find solutions of $f'(x) = 0$)
2. Second Derivative Test:

   For each critical point x_0,

   if $f''(x_0) < 0$, there is a local maximum at x_0
   if $f''(x_0) > 0$, there is a local minimum at x_0
   if $f''(x_0) = 0$, the test fails (other method has to be used)

Example. The cost to produce x units of some product is

$$C(x) = 0.01x^2 + 4x \ (\$).$$

The product is sold at price 100 \$ per unit. Determine the sales volume at which profit reaches its maximum.
Solution. As we have seen, the profit is

$$P(x) = 100x - 0.01x^2 - 4x = 96x - 0.01x^2$$

$P'(x) = 96 - 0.02x$
$P'(x) = 0 \ \Rightarrow \ 0.02x = 96 \ \Rightarrow \ x = 96/0.02 = 4800$ (units)
$P''(x) = -0.02 < 0$ ($P''(4800) < 0$) $\ \Rightarrow \ x = 4800$ is maximum
Profit maximized when sales volume$=$ 4800 units
Maximum profit$=$ $P(4800) = 230,400$ \$

Example. A company produces and sells 1000 units per month at 2000 \$ /unit. For every \$ reduced from the selling price, the company can sell 1 extra unit per month. Determine the price at which the company has a maximum revenue and calculate this maximum value.

Solution.

Let x \$ be deducted from base price 2000 \$. New price: $2000 - x$

Sales volume: $1000 + x$ units

Revenue:
$R(x) = (1000 + x)(2000 - x) = 2000000 + 1000x - x^2$
$R'(x) = 1000 - 2x = 0 \Rightarrow x = 500$
$R''(x) = -2 < 0 \Rightarrow x = 500$ is maximum

Thus, $x = 1500$ is maximum point, $R(500) = 2250000$ \$ is maximum revenue

Next Week

- Function of One Variable
  - Antiderivative
  - Indefinite Integral
  - Definite Integral
- Function of Two Variables
  - Partial Derivatives
  - Applications