Essential Mathematics

Tutorial 4

Semester 2 2017

Before you begin, have a look over **matricesintro.ipynb**.

1. Let

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & -3 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}.$$

Find:

(a) $5\mathbf{A}$

Solution:

$$5\mathbf{A} = 5\begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 10 & -15 & -5 \\ -5 & 10 & -15 \end{bmatrix}$$

(b) $\mathbf{B}+\mathbf{C}$

Solution:

$$\begin{aligned} \mathbf{B}+\mathbf{C} &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 & 2 \\ 1 & 4 & -6 \\ 3 & 2 & -2 \end{bmatrix} \end{aligned}$$

(c) $(\mathbf{B}+\mathbf{C})^T$

Solution:

$$\begin{aligned}(\mathbf{B}+\mathbf{C})' &= \begin{bmatrix} 3 & -2 & 2 \\ 1 & 4 & -6 \\ 3 & 2 & -2 \end{bmatrix}' \\ &= \begin{bmatrix} 3 & 1 & 3 \\ -2 & 4 & 2 \\ 2 & -6 & -2 \end{bmatrix}\end{aligned}$$

(d) $\mathbf{B}^T + \mathbf{C}^T$

Solution:

$$\begin{aligned}\mathbf{B}^T + \mathbf{C}^T &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}' + \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}' \\ &= \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ -1 & -3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & 1 \\ 3 & -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 & 3 \\ -2 & 4 & 2 \\ 2 & -6 & -2 \end{bmatrix}\end{aligned}$$

(e) Is $(BC)^T = B^T C^T$ true?

Solution:

$$
\begin{aligned}
BC &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -4 & 7 \\ -1 & 2 & -6 \\ 5 & -1 & 4 \end{bmatrix} \\
(BC)' &= \begin{bmatrix} -1 & -1 & 5 \\ -4 & 2 & -1 \\ 7 & -6 & 4 \end{bmatrix} \\
B'C' &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}' \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}' \\
&= \begin{bmatrix} 9 & -6 & -2 \\ -1 & -1 & 0 \\ -2 & -5 & -3 \end{bmatrix} \\
&\neq (BC)^T
\end{aligned}
$$

(f) Is $(BC)^T = C^T B^T$ true?

Solution:

$$
\begin{aligned}
C'B' &= \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}' \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}' \\
&= \begin{bmatrix} 9 & -6 & -2 \\ -1 & -1 & 0 \\ -2 & -5 & -3 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -1 & 5 \\ -4 & 2 & -1 \\ 7 & -6 & 4 \end{bmatrix}
\end{aligned}
$$

Thus, $(BC)' = C'B'$.

2. For the matrix B given in Question 1, complete the following EROs successively:
   (a) Perform the following 2 EROs on B to obtain matrix B_1
   - Row 1 is added to Row 2
   - (-2) time of Row 1 is added to Row 3

   (b) Perform the following ERO on B_1 to obtain B_2:
   - (-3) times of Row 2 is added to Row 3

   (c) Perform the following ERO on B_2 to obtain B_3:

- Row 3 is divided by 13

(d) Check if all elements under the diagonal are 0's and all diagonal elements are 1's.

Solution: Performing the required EROs (a)-(d), we obtain

$$
\begin{array}{lll}
B: & \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix} & \\
B_1: & \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -4 \\ 0 & 3 & 1 \end{bmatrix} & \begin{array}{r} R_2 + R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \\
B_2: & \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 13 \end{bmatrix} & R_3 - 3R_2 \to R_3 \\
B_3: & \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} & R_3/13 \to R_3
\end{array}
$$

3. Consider the following system of equations:

$$
\begin{array}{rrrl}
-5x & -4y & +7z & = 0 \\
6x & +5y & -4z & = 2 \\
4x & +3y & -5z & = -1
\end{array}
$$

.

(a) Express this in matrix form (i.e. write the augmented matrix).

Solution:

$$
[A\,|\,b] = \begin{bmatrix} -5 & -4 & 7 & 0 \\ 6 & 5 & -4 & 2 \\ 4 & 3 & -5 & -1 \end{bmatrix}
$$

(b) Reduce this matrix to reduced row echelon form.

Solution:

$$
\begin{aligned}
A &= \begin{bmatrix} -5 & -4 & 7 & 0 \\ 6 & 5 & -4 & 2 \\ 4 & 3 & -5 & -1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 1 & 3 & 2 \\ 6 & 5 & -4 & 2 \\ 4 & 3 & -5 & -1 \end{bmatrix} \quad R_1 + R_2 \to R_1 \\
&\to \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & -22 & -10 \\ 0 & -1 & -17 & -9 \end{bmatrix} \quad \begin{array}{r} R_2 - 6R_1 \to R_2 \\ R_3 - 4R_1 \to R_3 \end{array} \\
&\to \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 22 & 10 \\ 0 & 0 & 5 & 1 \end{bmatrix} \quad \begin{array}{r} -R_2 \to R_2 \\ R_3 - R_2 \to R_3 \end{array} \\
&\to \begin{bmatrix} 1 & 1 & 0 & 7/5 \\ 0 & 1 & 0 & 28/5 \\ 0 & 0 & 1 & 1/5 \end{bmatrix} \quad \begin{array}{r} R_1 - (3/5)R_3 \to R_1 \\ R_2 - (22/5)R_3 \to R_2 \\ (1/5)R_3 \to R_3 \end{array} \\
&\to \begin{bmatrix} 1 & 0 & 0 & -21/5 \\ 0 & 1 & 0 & 28/5 \\ 0 & 0 & 1 & 1/5 \end{bmatrix} \quad R_1 - R_2 \to R_1
\end{aligned}
$$

(c) Using Julia, and without using row operations, find x, y and z.

Solution:

```
A=[-5 -4 7; 6 5 -4 ; 4 3 -5]
b=[0 ; 2 ; -1]

x=inv(A)*b

x
```

Output (solution) for x:

$$
\begin{array}{r} -4.2000 \\ 5.6000 \\ 0.2000 \end{array}
$$

4. Consider

$$
\begin{aligned}
A &= \begin{bmatrix} 3 & 5 & -12 \\ 0 & 1 & -2 \\ -1 & -2 & 5 \end{bmatrix} \\
B &= \begin{bmatrix} 2 & 1 & -4 \\ 1 & 0 & -2 \\ 2 & 1 & -5 \end{bmatrix} \\
C &= \begin{bmatrix} -5 & -4 & \frac{7}{2} \\ 6 & 5 & -4 \\ 4 & 3 & -\frac{5}{2} \end{bmatrix}
\end{aligned}
$$

.

Using Julia, find:

(a) A^{-1}, B^{-1} and C^{-1}

(b) The determinants of A, B and C

(c) The determinants of A^T, B^T and C^T

(d) The determinants of A^{-1}, B^{-1} and C^{-1}.

(e) What can you conclude from (b), (c) and (d) above?

Solution:

```
A = [3 5 -12 ; 0 1 -2 ; -1 -2 5]
B = [2 1 -4 ; 1 0 -2 ; 2 1 -5]
C = [-5 -4 7/2 ; 6 5 -4 ; 4 3 -5/2]

inv(A)
B^-1
inv(C)

det(A)
det(B)
det(C)

det(A')
det(B')
det(C')

det(A^-1)
det(inv(B))
det(inv(C))
```