Essential Mathematics

Week 6 Exercises

The following are solutions to Exercises in Week 6 Notes.

6.2.3 Exercise

Given the data in figure 1, determine whether the population will increase.

Birth and Survival Rates for Female New Zealand Sheep
[from G. Caughley, "Parameters for Seasonally Breeding Populations," *Ecology* 48(1967)834-839]

Age (years)	Birth Rate	Survival Rate
0-1	0.000	0.845
1-2	0.045	0.975
2-3	0.391	0.965
3-4	0.472	0.950
4-5	0.484	0.926
5-6	0.546	0.895
6-7	0.543	0.850
7-8	0.502	0.786
8-9	0.468	0.691
9-10	0.459	0.561
10-11	0.433	0.370
11-12	0.421	0.000

Figure 1: Sheep data for a Leslie matrix

Solution The Leslie matrix is

$$A = \begin{bmatrix} 0.000 & 0.045 & 0.391 & 0.472 & 0.484 & 0.546 & 0.543 & 0.502 & 0.468 & 0.459 & 0.433 & 0.421 \\ 0.845 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.975 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.965 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.950 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.926 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.895 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.850 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.786 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.691 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.561 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.370 & 0.000 \end{bmatrix}$$

Using Julia we find that the (real) eigenvalues are $\lambda_1 = 1.17557, \qquad \lambda_2 = -0.65855$ (There are 10 other complex eigenvalues.)

Larest eigenvalue λ_1 is larger than 1, the population increase.

6.3 Exercise

1. In each case, determine whether or not $\mathbf{X}$ is an eigenvector of $\mathbf{A}$. If it is, state the corresponding eigenvalue.

   (a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 6 & 2 & -4 \\ -1 & 4 & 1 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (b) $\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 2 & -1 \\ -2 & 1 & 2 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

   (c) $\mathbf{A} = \begin{bmatrix} 3 & 7 & -2 \\ -2 & -3 & 1 \\ 6 & 4 & -4 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

   Solution.

   (a)

$$\begin{aligned} \mathbf{AX} &= \begin{bmatrix} 2 & -1 & 3 \\ 6 & 2 & -4 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \\ &= 4\mathbf{X} \end{aligned}$$

   So, $\mathbf{X}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue $\lambda = 4$

(b)

$$\begin{aligned}
\mathbf{AX} &= \begin{bmatrix} 1 & -1 & 3 \\ 3 & 2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ -2 \\ -2 \end{bmatrix} \\
&\neq \lambda\mathbf{X} \quad \text{for any } \lambda
\end{aligned}$$

Thus, $\mathbf{X}$ is not an eigenvector of $\mathbf{A}$.

(c)

$$\begin{aligned}
\mathbf{AX} &= \begin{bmatrix} 3 & 7 & -2 \\ -2 & -3 & 1 \\ 6 & 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \\
&= -\mathbf{X} \qquad (\lambda = -1)
\end{aligned}$$

Thus, $\mathbf{X}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue $\lambda = -1$

2. Find the eigenvalues and eigenvectors of the following matrices ((a) (b) by hand, (c)-(e) by Julia):

(a) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} -5 & 2 & -7 \\ -4 & 4 & -4 \\ 1 & -2 & 3 \end{bmatrix}$

(c) $\mathbf{B} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 1 & -3 \\ -1 & 1 & 5 \end{bmatrix}$

(d) $\mathbf{C} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\mathbf{D} = \begin{bmatrix} 3 & 2 & 0 \\ -2 & -4 & -2 \\ 2 & 7 & 5 \end{bmatrix}$

Solution.

(a)

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= \lambda^2 - 4\lambda + 3\end{aligned}$$

Solving $\det(A-\lambda I)=0$ (i.e. $\lambda^2-4\lambda+3=0$):

$$\lambda = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{4\pm\sqrt{16-4\times1\times3}}{2} = \frac{4\pm2}{2} = 2\pm1,$$

we obtain the eigenvalues $\lambda_1=1,\ \lambda_2=3$.

For $\lambda_1=1$, consider $(A-\lambda_1 I)X=\mathbf{0}$, which is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The non-zero solution $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1=1$.

For $\lambda_2=3$, consider $(A-\lambda_2 I)X=\mathbf{0}$, which is

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The non-zero solution $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2=3$.

(b)

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} -5-\lambda & 2 & -7 \\ -4 & 4-\lambda & -4 \\ 1 & -2 & 3-\lambda \end{vmatrix} \\ &= -(\lambda+5)(\lambda-4)(\lambda-3)+8(\lambda+5)-2(4\lambda-8)-7(\lambda+4) \\ &= (\lambda-2)(\lambda-4)(\lambda+4)\end{aligned}$$

Solving $(\lambda-2)(\lambda-4)(\lambda+4)=0$ obtains eigenvalues $\lambda_1=2,\ \lambda_2=4,\ \lambda_3=-4$.

* For $\lambda_1=2$, $(A-\lambda_1 I)X=\mathbf{0} \implies \begin{bmatrix} -7 & 2 & -7 \\ -4 & 2 & -4 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We solve the system in the following without using the RHS because it is all zeros.

$$\begin{bmatrix} -7 & 2 & -7 \\ -4 & 2 & -4 \\ 1 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ -4 & 2 & -4 \\ -7 & 2 & -7 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & -6 & 0 \\ 0 & -12 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solutions: $\begin{array}{l} x = -t \\ y = 0 \\ z = t \end{array}$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

eigenvector: $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

* For $\lambda_2 = 4$, $(A - \lambda_2 I)X = \mathbf{0} \implies \begin{bmatrix} -9 & 2 & -7 \\ -4 & 0 & -4 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -9 & 2 & -7 \\ -4 & 0 & -4 \\ 1 & -2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ -4 & 0 & -4 \\ -9 & 2 & -7 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & -8 & -8 \\ 0 & -16 & -16 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solutions: $\begin{array}{l} x = -t \\ y = -t \\ z = t \end{array}$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

eigenvector: $X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

* For $\lambda_3 = -4$, $(A - \lambda_3 I)X = \mathbf{0} \implies \begin{bmatrix} -1 & 2 & -7 \\ -4 & 8 & -4 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & 2 & -7 \\ -4 & 8 & -4 \\ 1 & -2 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solutions: $\begin{array}{l} x = 2t \\ y = t \\ z = 0 \end{array}$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

eigenvector: $X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

The remaining problems are solved using Julia.

(c) $\lambda_1 = 1$, $\mathbf{X_1} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$;

$\lambda_2 = 2$, $\mathbf{X_2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$;

$\lambda_3 = 3$, $\mathbf{X_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(d) $\lambda_1 = -1$, $\mathbf{X_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$;

$\lambda_2 = 1$, $\mathbf{X_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$;

$\lambda_3 = 1$, $\mathbf{X_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(e) $\lambda_1 = -1$, $\mathbf{X_1} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$;

$\lambda_2 = 2$, $\mathbf{X_2} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$;

$$\lambda_3 = 3,\ \mathbf{X_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$