# Solution to Week 5 Exercises

The following are solutions to Exercises in Week 5 Notes.

### 5.2.8 Exercise

1. For $\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$ and $\boldsymbol{b} = -5\mathbf{i} - 7\boldsymbol{j} + \boldsymbol{k}$, find

   (a) $4\boldsymbol{a} + 2\boldsymbol{b}$

   **Solution** $4\boldsymbol{a} + 2\boldsymbol{b} = (12\boldsymbol{i} - 8\boldsymbol{j} + 32\boldsymbol{k}) + (-10\boldsymbol{i} - 14\boldsymbol{j} + 2\boldsymbol{k}) = 2\boldsymbol{i} - 22\boldsymbol{j} + 34\boldsymbol{k}$

   (b) $|\boldsymbol{a}|$

   **Solution** $|\boldsymbol{a}| = \sqrt{3^2 + (-2)^2 + 8^2} = \sqrt{9 + 4 + 64} = \sqrt{77}$

   (c) $\hat{\boldsymbol{a}}$ (i.e. the unit vector of $\boldsymbol{a}$)

   **Solution** $\hat{\boldsymbol{a}} = \boldsymbol{a}/|\boldsymbol{a}| = \dfrac{3}{\sqrt{77}}\boldsymbol{i} - \dfrac{2}{\sqrt{77}}\boldsymbol{j} + \dfrac{8}{\sqrt{77}}\boldsymbol{k}$

2. Given points $P(1, -2, -5)$ and $Q(3, 5, -2)$, find $\overrightarrow{OP}$ $\overrightarrow{OQ}$ and $|\overrightarrow{PQ}|$.

   **Solution**

   $\overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} - 5\boldsymbol{k}$

   $\overrightarrow{OQ} = 3\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$

   $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 2\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$ $\quad |\overrightarrow{PQ}| = \sqrt{2^2 + 7^2 + 3^2} = \sqrt{62}$

3. Check if the give pairs of vectors are parallel.

   (a) $\boldsymbol{a} = 2\mathbf{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = -10\mathbf{i} + 15\boldsymbol{j} - 5\boldsymbol{k}$

   **Solution** Looking at $\boldsymbol{i}$ components: $-10 = -5(2)$.

   $\boldsymbol{b} = -5\boldsymbol{a}$? $-5\boldsymbol{a} = -10\mathbf{i} + 15\boldsymbol{j} - 5\boldsymbol{k} = \boldsymbol{b}$

   Thus, $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

   (b) $\boldsymbol{a} = 1\mathbf{i} + 2\boldsymbol{j} - 7\boldsymbol{k}$, $\boldsymbol{b} = 3\mathbf{i} + 6\boldsymbol{j} + 21\boldsymbol{k}$

   **Solution** Looking at $\boldsymbol{i}$ components: $3 = 3(1)$.

   $3\boldsymbol{a} = -3\mathbf{i} + 6\boldsymbol{j} - 21\boldsymbol{k} \neq \boldsymbol{b}$

   Thus, $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

### 5.3.2 Exercise

Find $\boldsymbol{a} \cdot \boldsymbol{b}$ if $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = 6\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$.

**Solution**

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 \times 6 + (-3) \times 1 + 2 \times (-2) = 12 - 3 - 4 = 5$$

### 5.3.4 Exercise

1. Are the vectors $\boldsymbol{a} = 3\,\mathbf{i} - \boldsymbol{j} + 2\,\boldsymbol{k}$ and $\boldsymbol{b} = 3\,\mathbf{i} + 5\,\boldsymbol{j} - 2\,\boldsymbol{k}$ orthogonal?

   Key to solution: $\boldsymbol{a}$ orthogonal to $\boldsymbol{b} \;\Leftrightarrow\; \boldsymbol{a} \cdot \boldsymbol{b} = 0$

   **Solution** $\boldsymbol{a} \cdot \boldsymbol{b} = 9 - 5 - 4 = 0$ Thus, $\boldsymbol{a} \perp \boldsymbol{b}$

2. Find the value of $\lambda$ such that $\boldsymbol{a} = -\boldsymbol{i} + 6\,\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \lambda\,\boldsymbol{i} + 2\,\boldsymbol{j} - 7\,\boldsymbol{k}$ are orthogonal.

   **Solution**

   $\boldsymbol{a} \cdot \boldsymbol{b} = -\lambda + 12 + 7 = 19 - \lambda$

   $\boldsymbol{a} \perp \boldsymbol{b}$ if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = -\lambda + 12 + 7 = 19 - \lambda = 0$ or $\lambda = 19$

   Thus, when $\lambda = 19$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal.

### 5.3.7 Exercise

Find the angle between the vectors $\boldsymbol{a} = 2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 5\,\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\,\boldsymbol{j} + \boldsymbol{k}$.

**Solution**

$\boldsymbol{a} \cdot \boldsymbol{b} = 2 - 6 + 5 = 1$

$|\boldsymbol{a}| = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$ $|\boldsymbol{b}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

Angle$= cos^{-1}(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}) = cos^{-1}(\frac{1}{\sqrt{38\times 6}}) \approx 86.20^\circ$

### 5.3.10 Exercise

Find the scalar and vector projections of $\boldsymbol{a} = 5\,\boldsymbol{i} - 7\,\boldsymbol{j} - 6\,\boldsymbol{k}$ in the direction of $\boldsymbol{b} = 2\,\boldsymbol{i} + 6\boldsymbol{j} - 3\,\boldsymbol{k}$.

**Solution**

$\boldsymbol{a} \cdot \boldsymbol{b} = 10 - 42 + 18 = -14, \;\; |\boldsymbol{b}| = \sqrt{4 + 36 + 9} = 7$

scalar projection$= \boldsymbol{a} \cdot \boldsymbol{b}/|\boldsymbol{b}| = -14/7 = -2$

vector projection$= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}\,\frac{\boldsymbol{b}}{|\boldsymbol{b}|} = -2 \times \frac{1}{7}(2\,\boldsymbol{i} + 6\,\boldsymbol{j} - 3\,\boldsymbol{k}) = -\frac{4}{7}\boldsymbol{i} - \frac{12}{7}\boldsymbol{j} + \frac{6}{7}\boldsymbol{k}$

### 5.3.12 Exercise

Resolve the vector $\boldsymbol{a} = 7\,\boldsymbol{i} - 5\,\boldsymbol{j} + \boldsymbol{k}$ into orthogonal vectors, one of which is parallel to $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

**Solution**

$\boldsymbol{a} \cdot \boldsymbol{b} = 7 - 5 + 1 = 3, \;\; |\boldsymbol{b}|^2 = 1^2 + 1^2 + 1^2 = 3$

"vector projection of $\boldsymbol{a}$ in $\boldsymbol{b}$"

$$= \boldsymbol{x} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}\,\boldsymbol{b} = \frac{3}{3}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

Let $\boldsymbol{y} = \boldsymbol{a} - \boldsymbol{x} = 6\boldsymbol{i} - 6\boldsymbol{j}$. Then,

$$\boldsymbol{a} = \boldsymbol{x} + \boldsymbol{y} = (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + (6\boldsymbol{i} - 6\boldsymbol{j})$$

### 5.4.1 Exercise

1. If $\boldsymbol{a} = 2\,\boldsymbol{i} - 5\,\boldsymbol{j} + 7\,\boldsymbol{k}$ and $\boldsymbol{b} = 3\,\boldsymbol{i} - \boldsymbol{j} - 2\,\boldsymbol{k}$, find $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$.

   **Solution**

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -5 & 7 \\ 3 & -1 & -2 \end{vmatrix} \\
&= \begin{vmatrix} -5 & 7 \\ -1 & -2 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 2 & 7 \\ 3 & -2 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 2 & -5 \\ 3 & -1 \end{vmatrix} \boldsymbol{k} \\
&= 17\boldsymbol{i} + 25\boldsymbol{j} + 13\boldsymbol{k} \\
\boldsymbol{b} \times \boldsymbol{a} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -1 & -2 \\ 2 & -5 & 7 \end{vmatrix} \\
&= \begin{vmatrix} -1 & -2 \\ -5 & 7 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 3 & -2 \\ 2 & 7 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 3 & -1 \\ 2 & -5 \end{vmatrix} \boldsymbol{k} \\
&= -17\boldsymbol{i} - 25\boldsymbol{j} - 13\boldsymbol{k}
\end{aligned}$$

2. If $\boldsymbol{a} = -\boldsymbol{i} - 2\,\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 3\,\boldsymbol{i} + 2\boldsymbol{j} - 6\,\boldsymbol{k}$, find $\boldsymbol{a} \times \boldsymbol{b}$ and verify that it is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$.

   **Solution**

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -2 & 1 \\ 3 & 2 & -6 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 1 \\ 2 & -6 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} -1 & 1 \\ 3 & -6 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} \boldsymbol{k} \\
&= 10\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}
\end{aligned}$$

   Now, $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = (10\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}) \cdot (-\boldsymbol{i} - 2\,\boldsymbol{j} + \boldsymbol{k}) = 0$

   $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{b} = (10\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}) \cdot (3\,\boldsymbol{i} + 2\boldsymbol{j} - 6\,\boldsymbol{k}) = 0$

   Thus, $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.