Essential Mathematics

Week 4 Notes

RMIT

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Contents

Chapter 4

Matrices

4.1 Introduction

A matrix is a **rectangular array** of numbers (or, in some cases, symbols) arranged in rows and columns. Typically, square brackets are used to denote a matrix, e.g.

$$A = \begin{bmatrix} 1 & 13 & 5 & 7 \\ -4 & 8 & 2 & 6 \\ 9 & 0 & -7 & 25 \end{bmatrix}.$$

The numbers (or symbols) in the matrix are referred to as **elements** or entries. Their position in the matrix has significance.

Matrices are often described in terms of their **order** (or **size**, or **shape**).

The matrix given above has 3 **rows** and 4 **columns**. Its order is consequently 3×4, and we refer to it as a 3×4 matrix. A matrix which has an equal number n of rows and columns (i.e. is a $n \times n$ matrix) is called a **square** matrix, e.g.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 8 & 4 \\ 7 & 6 & 9 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}.$$

A matrix with a single column is called a *column matrix*. Column matrices are often used to represent vectors. For example, the vector $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is often written in "component" form as (x, y, z) but in matrix form as the 3×1 matrix

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Matrices are used in numerous applications. In fact, just about every problem involving many variables can be formulated in terms of matrices. A system of linear equations can be represented and solved compactly using matrices. Another application example is given below.

4.1.1 Example

The matrix

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

can be used to perform a 45° anticlockwise rotation of points in the xy plane. To find out how this is done, you need to understand matrix multiplication (and linear mappings) – topics which are to follow shortly.

Similar transformations are possible in three dimensions. The 3×3 matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

can be used to perform a 90° rotation of points in three dimensions about the y axis.

4.2 Matrix Properties and Algebra

4.2.1 Notation

A general $m \times n$ matrix A may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where the element in the ith row and jth column is denoted a_{ij}. Two matrices A and B are equal if they are the same order and $a_{ij} = b_{ij}$ for all i and j.

4.2.2 Addition and Subtraction

Two matrices may be added or subtracted if they are of the same order (i.e. have the same size or shape). Addition or subtraction is done element-wise. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 9 & 0 \\ 3 & 1 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 5 \\ 6 & 9 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 8 & 11 & 3 \\ 7 & 6 & 10 \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} -6 & -7 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$

but $A + C$ and $A - C$ do not exist.

4.2.3 Multiplication by Scalar

Given a matrix A, we calculate kA by multiplying every element of A by the scalar k. For example, using A defined immediately above, we have

$$7A = \begin{bmatrix} 7 & 14 & 21 \\ 28 & 35 & 42 \end{bmatrix}.$$

4.2.4 Multiplication of Two Matrices

Not all matrices may be multiplied! Two matrices may be multiplied together only if they are **compatible**, that is, the number of columns of the first matrix must equal the number of rows of the second matrix. If A is a $m \times n$ matrix and B is a $p \times q$ matrix then we can calculate AB only if $n = p$.

To understand why this is so we need to understand the process of matrix multiplication. Consider the following example. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 \\ 3 & 1 \\ 0 & 4 \end{bmatrix}.$$

Suppose we multiply A and B to form C. We form the element in the first row and first column of C by performing a **dot product** between the first row of A and the first column of B.

$$AB = \begin{bmatrix} \color{red}{1} & \color{red}{2} & \color{red}{3} \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} \color{red}{7} & 9 \\ \color{red}{3} & 1 \\ \color{red}{0} & 4 \end{bmatrix} = \begin{bmatrix} \color{red}{13} & \\ & \end{bmatrix},$$

since $(1, 2, 3) \cdot (7, 3, 0) = 7 + 6 + 0 = 13$. To get the element in the first row and second column of C we perform a dot product between the first row of A and the second column of B:

$$AB = \begin{bmatrix} \color{red}{1} & \color{red}{2} & \color{red}{3} \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & \color{red}{9} \\ 3 & \color{red}{1} \\ 0 & \color{red}{4} \end{bmatrix} = \begin{bmatrix} 13 & \color{red}{23} \\ & \end{bmatrix}.$$

Similarly, we get c_{21}:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 \\ 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 23 \\ 43 & \end{bmatrix}.$$

and finally c_{22}:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 \\ 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 23 \\ 43 & 65 \end{bmatrix} = C.$$

Clearly, the product AB can be performed only if the dot products can be performed, i.e. if the length of each row in A equals the length of each column in B, or in other words, the number of columns of A matches the number of rows of B.

Notice that the order of C is 2×2, that is, it has the same number of rows as A and the same number of columns as B. This is true in general. If A is $m \times n$ and B is $n \times p$ then AB will be a $m \times p$ matrix.

4.2.5 Examples

- **Problem**

  Given

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 3 & 8 \\ 9 & 2 \end{bmatrix}$$

  calculate AB and BA. What do you notice?

  Solution

  A is a 2×3 matrix and B is a 3×2 matrix, so AB will be a 2×2 matrix and BA will be a 3×3 matrix. Immediately it is clear that $AB \neq BA$ in this case since they are different sizes.

$$AB = \begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 8 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 45 & 33 \\ 51 & 32 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 0 & 1 \\ 3 & 8 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 5 \\ 51 & 25 & 52 \\ 21 & 31 & 46 \end{bmatrix}.$$

∎

- **Problem**

  Given

  $$A = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

  calculate AB and BA.

  Solution

  A is a 1×3 matrix and B is a 3×1 matrix, so AB will be a 1×1 matrix and BA will be a 3×3 matrix. Again $AB \neq BA$.

  $$AB = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = [34].$$

  $$BA = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 10 \\ 6 & 12 & 30 \\ 4 & 8 & 20 \end{bmatrix}.$$

  ∎

- **Problem**

  Given

  $$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 3 \\ -3 & 1 & 6 \end{bmatrix} \quad \textbf{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

  calculate AX.

  Solution

  A is a 3×3 matrix and X is a 3×1 matrix, so AX will be a 3×1 matrix.

  $$AX = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 3 \\ -3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 5z \\ 2x + 4y + 3z \\ -3x + y + 6z \end{bmatrix}$$

  ∎

4.2.6 Powers of a Matrix

Integer powers of a matrix, e.g. A^2, A^3 etc, are obtained by multiplying A by itself the required number of times. For example, let

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}.$$

Then
$$A^2 = AA = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 13 & 24 \\ 32 & 61 \end{bmatrix}.$$

Clearly $A^3 = AAA = A^2A$. Equally well we could have $A^3 = AAA = AA^2$.

Thus
$$A^3 = \begin{bmatrix} 13 & 24 \\ 32 & 61 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 13 & 24 \\ 32 & 61 \end{bmatrix} = \begin{bmatrix} 109 & 207 \\ 276 & 523 \end{bmatrix}.$$

Note that this is one of the rare situations in which matrix multiplication is **commutative**.

4.2.7 Transpose of a Matrix

The **transpose** of a matrix A, denoted A^T is the matrix obtained when the rows and columns of A are interchanged. For example, if
$$A = \begin{bmatrix} 6 & 3 & 5 \\ 4 & 7 & 2 \end{bmatrix}$$
then
$$A^T = \begin{bmatrix} 6 & 4 \\ 3 & 7 \\ 5 & 2 \end{bmatrix}.$$

Clearly $(A^T)^T = A$. It is also a property of transpose that the transpose of a product AB is the product of the transposes **in the reverse order**, i.e $(AB)^T = B^TA^T$.

Example

Given
$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 3 \end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$
verify that $(AB)^T = B^TA^T$.

Solution
$$AB = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 16 \end{bmatrix}.$$

Thus,
$$(AB)^T = \begin{bmatrix} 15 & 16 \end{bmatrix}.$$

Now,

$$B^T A^T = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 16 \end{bmatrix}.$$

as required. ∎

4.2.8 Special Matrices

Zero Matrix

A **zero matrix** is a matrix of any size in which every element is zero, e.g.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

For any matrix A, there is a zero matrix O of the same size with the property that

$$A + O = O + A = A.$$

Identity Matrix

An **identity matrix** is a square matrix with zeros everywhere except along the **leading diagonal** where every element is a 1. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For any matrix A there is an identity matrix I such that $AI = A$. There is also an identity matrix I such that $IA = A$. The identity matrix functions like the number 1 in real arithmetic.

Diagonal Matrix

A **diagonal matrix** is a square matrix with zeros everywhere except along the **leading diagonal** where there are some non-zero elements, e.g.

$$\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Upper Triangular Matrix

An **upper triangular matrix** is a square matrix with zeros everywhere below the **leading diagonal** , e.g.

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

Lower Triangular Matrix

A **lower triangular matrix** is a square matrix with zeros everywhere above the **leading diagonal** , e.g.

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 6 & 4 & 3 \end{bmatrix}.$$

Symmetric Matrix

A **symmetric matrix** A is a square matrix with the property that $A^T = A$.

$$\begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 & 0 \\ 4 & 2 & 7 \\ 0 & 7 & 5 \end{bmatrix}.$$

Skew-Symmetric Matrix

A **skew-symmetric matrix** A is a square matrix with the property that $A^T = -A$.

$$\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 4 & -6 \\ -4 & 0 & -7 \\ 6 & 7 & 0 \end{bmatrix}.$$

Note the zeros down the leading diagonal.

4.2.9 Inverse of a Square Matrix

The **inverse** of a square matrix A, if it exists, is denoted by A^{-1}. The inverse A^{-1} has the property that

$$AA^{-1} = A^{-1}A = I.$$

The inverse of a matrix functions like the **reciprocal** of a real number. Just as some numbers, e.g. 0, do not have a reciprocal, many matrices do not have an inverse. We will discuss how to find an inverse in a later topic.

The inverse of the product AB, i.e. $(AB)^{-1}$ is related to the individual inverses A^{-1} and B^{-1} by the following:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Note the similarity with the formula for the transpose of a product.

4.2.10 Matrix Algebra

We may solve equations and manipulate expressions involving variables which represent matrices just as we solve equations and manipulate expressions involving real variables. However, the fact that matrix multiplication is not commutative in general does lead to some differences. For example, if a and b are real-valued variables, it is well known that

$$(a+b)^2 = a^2 + b^2 + 2ab \quad \text{and} \quad (a+b)(a-b) = a^2 - b^2.$$

Suppose A and B represent matrices. Then

$$\begin{aligned} (A+B)^2 &= (A+B)(A+B) \\ &= A^2 + AB + BA + B^2 \end{aligned}$$

which cannot be simplified further since $AB \neq BA$. Similarly,

$$(A+B)(A-B) = A^2 - AB + BA - B^2$$

cannot be simplified further, unlike for real variables.

4.2.11 Exercise

1. Let $\mathbf{A} = \begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & -3 \end{bmatrix}$,

   $$\mathbf{B} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}.$$

   Find:

   (a) $5\mathbf{A}$

   (b) $\mathbf{B} + \mathbf{C}$

   (c) $(\mathbf{B} + \mathbf{C})^T$

   (d) $\mathbf{B}^T + \mathbf{C}^T$

   (e) Is $(BC)^T = B^T C^T$ true?

   (f) Is $(BC)^T = C^T B^T$ true?

2. For the matrix B given in Question 1, complete the following EROs successively:

   (a) Row 1 is added to Row 2

   (b) (-2) time of Row 1 is added to Row 3

   (c) (-3) times of Row 2 is added to Row 3

   (d) Row 3 is divided by 13

   (e) Check if all elements under the diagonal are 0's and all diagonal elements are 1's.

4.3 Finding the Inverse of a Matrix

Given a general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

it is easy to find the inverse A^{-1} using the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note: the inverse of A will only exist if the **determinant** of A, i.e. $ad - bc$ is not zero.

4.3.1 Example

Use the formula to determine the inverse of

$$A = \begin{bmatrix} 4 & 9 \\ 3 & 8 \end{bmatrix}$$

and verify that your answer is correct.

Using the formula,

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 8 & -9 \\ -3 & 4 \end{bmatrix}.$$

Check:

$$AA^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 9 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 8 & -9 \\ -3 & 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = I.$$

Similarly $A^{-1}A = I$ as required.

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4.3.2 Elementary Matrices

While there is no formula for finding the inverse of general $n \times n$ matrix, we may determine the inverse relatively easily using elementary row operations. Before doing this we first need to define an **elementary matrix**.

Suppose we have a system of equations represented by the following augmented matrix:

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 0 \\ 3 & 6 & 4 & 2 & 0 \\ 2 & 0 & 3 & 1 & 0 \end{array}\right]$$

and we wish to perform the elementary row operation $R_2 - 3R_1 \to R_2$. Pretty clearly the result is

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 0 \\ 0 & 0 & -5 & -10 & 0 \\ 2 & 0 & 3 & 1 & 0 \end{array}\right].$$

The same result can be obtained by **pre-multiplying** the original augmented matrix by the **elementary matrix** which represents the operation $R_2 - 3R_1 \to R_2$.

Now, the elementary matrix which does this job is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let's check that it works:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 3 & 6 & 4 & 2 & 0 \\ 2 & 0 & 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & -5 & -10 & 0 \\ 2 & 0 & 3 & 1 & 0 \end{array}\right]$$

as required!

In fact, the elementary matrix E which represents the row operation $R_2 - 3R_1 \to R_2$ is very easily obtained. All we need to do is apply $R_2 - 3R_1 \to R_2$ to the 3×3 identity matrix. The same applies to any elementary row operation.

4.3.3 Example

Determine a 3×3 elementary matrix E which swaps rows 2 and 3, i.e. performs the operation $R_2 \leftrightarrow R_3$.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

4.3.4 Inverse of a General Matrix

We are now in a position to determine the inverse of a general matrix A using elementary row operations. The method works as follows:

Create an augmented matrix by appending A to the identity of the same size.

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

Apply the first elementary row operation E_1 to both parts of the augmented matrix:

$$\left[\begin{array}{c|c} E_1 A & E_1 I \end{array}\right]$$

Continue applying elementary row operations:

$$\left[\begin{array}{c|c} E_nE_{n-1}\ldots E_3E_2E_1A & E_nE_{n-1}\ldots E_3E_2E_1I \end{array}\right]$$

If A has been reduced to the identity, i.e.

$$E_nE_{n-1}\ldots E_3E_2E_1A = I$$

then it is clear that

$$A^{-1} = E_nE_{n-1}\ldots E_3E_2E_1$$

i.e. the product of the elementary matrices must be the inverse of A. In fact, at the last step we must effectively have an augmented matrix which looks like this:

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

In other words if we reduce A to the identity, then the identity I will be transformed into A^{-1}. The process of using elementary row operations to reduce A to the identity is called **Gauss-Jordan** elimination.

Example

Use elementary row operations to determine the inverse of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

We begin by creating the augmented matrix:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right].$$

Zero beneath the pivot in row 1:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array}$$

Zero beneath the pivot in row 2:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 2 & 1 & 0 & 0 \\ 0 & \boxed{-3} & -2 & -2 & 1 & 0 \\ 0 & 0 & -5 & -2 & -2 & 3 \end{array}\right] \quad 3R_3 - 2R_2 \to R_3$$

Divide through the third row by -5:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 2 & 1 & 0 & 0 \\ 0 & \boxed{-3} & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right] \quad R_3/(-5) \to R_3$$

Zero **above** the pivot in the third row:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 1/5 & -4/5 & 6/5 \\ 0 & \boxed{-3} & 0 & -6/5 & 9/5 & -6/5 \\ 0 & 0 & \boxed{1} & 2/5 & 2/5 & -3/5 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_3 \to R_1 \\ R_2 + 2R_3 \to R_2 \end{array}$$

Divide through the second row by -3:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 1/5 & -4/5 & 6/5 \\ 0 & \boxed{1} & 0 & 2/5 & -3/5 & 2/5 \\ 0 & 0 & \boxed{1} & 2/5 & 2/5 & -3/5 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_3 \to R_1 \\ R_2 + 2R_3 \to R_2 \end{array}$$

Zero **above** the pivot in the second row:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & -3/5 & 2/5 & 2/5 \\ 0 & \boxed{1} & 0 & 2/5 & -3/5 & 2/5 \\ 0 & 0 & \boxed{1} & 2/5 & 2/5 & -3/5 \end{array}\right] \quad R_1 - 2R_2 \to R_1$$

and we are finished!

The inverse of A is

$$\begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix} \quad \text{or} \quad \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

∎

Example

Use elementary row operations to determine the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We begin by creating the augmented matrix:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right].$$

Zero beneath the pivot in row 1:

$$\left[\begin{array}{rrr|rrr} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 - 4R_1 \to R_2 \\ R_3 - 7R_1 \to R_3 \end{array}$$

Zero beneath the pivot in row 2 (disaster is looming):

$$\left[\begin{array}{rrr|rrr} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{-3} & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right] \quad R_3 - 2R_2 \to R_3$$

Clearly this A can never be reduced to the identity as we have an entire row of zeros in the third row, and no chance of a pivot. Thus the given A does not have an inverse.

4.3.5 Exercise

1. Find the inverse for each of the following matrices.

$$\begin{aligned} A &= \begin{bmatrix} 3 & 5 & -12 \\ 0 & 1 & -2 \\ -1 & -2 & 5 \end{bmatrix} \\ B &= \begin{bmatrix} 2 & 1 & -4 \\ 1 & 0 & -2 \\ 2 & 1 & -5 \end{bmatrix} \\ C &= \begin{bmatrix} -5 & -4 & \frac{7}{2} \\ 6 & 5 & -4 \\ 4 & 3 & -\frac{5}{2} \end{bmatrix} \end{aligned}$$

4.4 Determinants

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a general 2×2 matrix. The **determinant** of A is

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For example

$$\begin{vmatrix} 2 & 5 \\ 9 & 4 \end{vmatrix} = 8 - 45 = -37.$$

and

$$\begin{vmatrix} 8 & -4 \\ 2 & 1 \end{vmatrix} = 8 - (-8) = 16.$$

4.4.1 General 3 by 3 Determinants

Let

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

be a general 3×3 matrix. The determinant of A can be expressed in terms of smaller 2×2 determinants in the following way:

$$detA = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Here we have expanded using the **first row**. Each 2×2 determinant is obtained above by ignoring the row and column containing the corresponding a_i. So for a_1 which is in row 1 and column 1 we ignore row 1 and column 1:

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}$$

Note also the minus sign attached to the second, i.e. a_2, term. To work out the sign attached to each a_i we calculate $(-1)^{row+column}$ where *row* is the row containing a_i and *column* is the column containing a_i. The signs are given by the following sign diagram:

$$\begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} \Longrightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The following example may make this clearer.

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 6 & 2 \end{vmatrix} &= 2\begin{vmatrix} 4 & 7 \\ 6 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 7 \\ 0 & 2 \end{vmatrix} + 5\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix} \\
&= 2(8-42) - 3(2-0) + 5(6-0) \\
&= -68 - 6 + 30 \\
&= -44
\end{aligned}$$

In fact, we may expand using any row or column we choose. For example, using the **first column** instead of the first row we have

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 6 & 2 \end{vmatrix} &= 2\begin{vmatrix} 4 & 7 \\ 6 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 6 & 2 \end{vmatrix} + 0\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} \\
&= 2(8-42) - 1(6-30) + 0(21-20) \\
&= -68 + 24 + 0 \\
&= -44
\end{aligned}$$

Using the **second column** we have

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 6 & 2 \end{vmatrix} &= -3\begin{vmatrix} 1 & 7 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} - 6\begin{vmatrix} 2 & 5 \\ 1 & 7 \end{vmatrix} \\
&= -3(2-0) + 4(4-0) - 6(14-5) \\
&= -6 + 16 - 54 \\
&= -44
\end{aligned}$$

Given that we may expand by any row or column it makes sense and saves effort if we expand using the row or column with the most zeros.

Upper Triangular Matrix

Let

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

be a general **upper-triangular** matrix.

Expanding repeatedly using the first column we have

$$det A = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = a\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} = adf$$

which is just the production the entries on the leading diagonal.

4.4.2 Properties of Determinants

Let A and B be two $n \times n$ matrices. It can be shown that

$$\det AB = \det A \det B$$

In particular, if A is invertible,

$$\det AA^{-1} = \det A \det A^{-1} = \det I = 1$$

Thus

$$\det A^{-1} = \frac{1}{\det A}.$$

Clearly then, having a determinant of zero is incompatible with having an inverse.

In fact, there is a theorem which states that

$$\det A = 0 \iff A \text{ has no inverse.}$$

4.4.3 Exercise

1. Consider

$$\begin{aligned} A &= \begin{bmatrix} 3 & 5 & -12 \\ 0 & 1 & -2 \\ -1 & -2 & 5 \end{bmatrix} \\ B &= \begin{bmatrix} 2 & 1 & -4 \\ 1 & 0 & -2 \\ 2 & 1 & -5 \end{bmatrix} \\ C &= \begin{bmatrix} -5 & -4 & \frac{7}{2} \\ 6 & 5 & -4 \\ 4 & 3 & -\frac{5}{2} \end{bmatrix} \end{aligned}$$

Find

(a) The determinants of A, B and C

(b) The determinants of A^T, B^T and C^T

(c) The determinants of A^{-1}, B^{-1} and C^{-1}.

NOTE: Use the results from Exercise 4.3.5 for this question.