Week 8 Exercise Solutions

1 Exercise 8.1.3

1. Domain
   (a) $x^2 + y^2 > 4$ The term in a square root must be non-negative. Moreover, a zero in the denominator is not allowed hence the strictly greater than sign.
   (b) $4x^2 + y^2 \geq 4$ Zero is allowed in this case hence the greater than or equal sign
   (c) $x^2 + 3y - 2z > 0$ Log function requires a poistive argument
2. Domain and range
   (a) $|x + y|$ Domain: $-\infty < x < \infty,\ -\infty < y < \infty$; Range: $[0, \infty)$ since the mod signs ensure non-negativity.
   (b) $(x + y)^2$ Same as above. The squaring ensures non-negative values for the range.
   (c) $(x + y)^3$ Doamin same as prevbious two problems but range will now include negative values. Range $(-\infty, \infty)$

2 Exercise 8.1.5

First order partial derivatives of the following functions:

1. $z = x^3 + 3x^2y + 3xy^2 + y^3$

$$\begin{aligned} z_x &= 3x^2 + 6xy + 3y^2 \\ z_y &= 3x^2 + 6xy + 3y^2 \end{aligned}$$

2. $f(x, y, z) = x^4y^2 + xz + y^3z^2 + 6z$

$$\begin{aligned} f_x &= 4x^3y^2 + z \\ f_y &= 2x^4y + 3y^2z^2 \\ f_z &= x + 2y^3z + 6 \end{aligned}$$

3. $z = (x^2 + 3y^2 + xy)^4$

$$\begin{aligned} z_x &= 4(x^2 + 3y^2 + xy)^3(2x + y) \\ z_y &= 4(x^2 + 3y^2 + xy)^3(6y + x) \end{aligned}$$

4. $z = e^{x^2+2y}$

$$\begin{aligned} z_x &= 2xe^{x^2+2y} \\ z_y &= 2e^{x^2+2y} \end{aligned}$$

3 Exercise 8.1.7

Second order partial derivatives

1. $z = x^3 + 3x^2y + 3xy^2 + y^3$

$$\begin{aligned} z_x &= 3x^2 + 6xy + 3y^2 \\ z_y &= 3x^2 + 6xy + 3y^2 \\ z_{xx} &= 6x + 6y \\ z_{yy} &= 6x + 6y \\ z_{xy} &= z_{yx} = 6x + 6y \end{aligned}$$

2. $f(x, y, z) = x^4y^2 + xz + y^3z^2 + 6z$

$$\begin{aligned} f_x &= 4x^3y^2 + z \\ f_y &= 2x^4y + 3y^2z^2 \\ f_z &= x + 2y^3z + 6 \\ f_{xx} &= 12x^2y^2 \\ f_{yy} &= 2x^4 + 6yz^2 \\ f_{zz} &= 2y^3 \\ f_{xy} &= f_{yx} = 8x^3y \\ f_{xz} &= f_{zx} = 1 \\ f_{yz} &= f_{zy} = 6y^2z \end{aligned}$$

3. $z = (x^2 + 3y^2 + xy)^4$

$$\begin{aligned} z_x &= 4(x^2 + 3y^2 + xy)^3(2x + y) \\ z_y &= 4(x^2 + 3y^2 + xy)^3(6y + x) \\ z_{xx} &= 12(x^2 + 3y^2 + xy)^2(2x + y)^2 + 8(x^2 + 3y^2 + xy)^3 \\ z_{yy} &= 12(x^2 + 3y^2 + xy)^2(6y + x)^2 + 24(x^2 + 3y^2 + xy)^3 \\ z_{xy} &= z_{yx} = 12((x^2 + 3y^2 + xy)^2(6y + x)(2x + y) + 4((x^2 + 3y^2 + xy)^3 \end{aligned}$$

4. $z = e^{x^2+2y}$

$$\begin{aligned} z_x &= 2xe^{x^2+2y} \\ z_y &= 2e^{x^2+2y} \\ z_{xx} &= 2e^{x^2+2y} + 4x^2e^{x^2+2y,} \\ z_{yy} &= 4e^{x^2+2y} \\ z_{xy} &= 4xe^{x^2+2y} \end{aligned}$$

4 Exercise 8.2.4

Find all the critical points of the following functions and classify them.

1. $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

   Critical points

$$\begin{aligned} f_x &= 6xy - 6x = 0 \\ f_y &= 3x^2 + 3y^2 - 6y = 0 \end{aligned}$$

   Simplifying

$$\begin{aligned} 6x(y-1) = 0 \quad &\Rightarrow \quad x = 0 \text{ or } y = 1 \\ & \quad\ \ x = 0 \Rightarrow 3y^2 - 6y = 0 \Rightarrow 3y(y-2) = 0 \Rightarrow y = 0 \text{ or } y = 2 \\ & \quad\ \ y = 1 \Rightarrow 3x^2 + 3 - 6 = 0 \Rightarrow x^2 = 1 \Rightarrow x = +1 \text{ or } x = -1 \end{aligned}$$

   Thus there are four critical points $(0, 0), (0, 2), (-1, 1), (1, 1)$

   Classification

   To classify the critical points the second order partial derivatives are needed.

$$f_{xx} = 6y - 6, \quad f_{yy} = 6y - 6, \quad f_{xy} = 6x$$

   Now at each critical point calculate $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (6y - 6)^2 - 36x^2$.

$$\begin{aligned} &D(0,0) = 36 > 0 \implies \text{extremum} \\ &D(0,2) = 36 > 0 \implies \text{extremum} \\ &D(-1,1) = -36 < 0 \implies \text{saddle point} \\ &D(1,1) = -36 < 0 \implies \text{saddle point} \end{aligned}$$

Determine the nature of the two extrema

$$f_{xx}(0,0) = -6 < 0, \quad f_{yy}(0,0) = -6 < 0$$

$$f_{xx}(0,2) = 6 > 0, \quad f_{yy}(0,2) = 6 > 0$$

We conclude that there is a maximum at $(0,0)$ and a minimum at $(0,2)$

2. $f(x,y) = e^{-(x^2+y^2)}$

Critical points

$$\begin{aligned} f_x &= -2xe^{-(x^2+y^2)} = 0 \;\Rightarrow\; x = 0 \\ f_y &= -2ye^{-(x^2+y^2)} = 0 \;\Rightarrow\; y = 0 \end{aligned}$$

$(0,0)$ is the only critical point.

Classification

$$\begin{aligned} f_{xx} &= 4x^2e^{-(x^2+y^2)} - 2e^{-(x^2+y^2)} = e^{-(x^2+y^2)}(4x^2-2) \\ f_{yy} &= 4y^2e^{-(x^2+y^2)} - 2e^{-(x^2+y^2)} = e^{-(x^2+y^2)}(4y^2-2) \\ f_{xy} &= 4xye^{-(x^2+y^2)} \\ f_{xx}(0,0) &= -2 < 0 \\ f_{yy}(0,0) &= -2 < 0 \\ D(0,0) &= f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = (-2)(-2) - 0^2 = 4 > 0 \end{aligned}$$

Therefore, function has a maximum at $(0,0)$ with maximum value 1.

3. $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$

Critical points

$$\begin{aligned} f_x &= 6x^2 + 6y^2 - 150 = 0 \\ f_y &= 12xy - 9y^2 = 0 \end{aligned}$$

Simplifying the above equations

$$\begin{aligned} x^2 + y^2 &= 25 \\ 3y(4x-3y) &= 0 \end{aligned}$$

The second equation tells us that either $y = 0$ or $4x - 3y = 0$

$$y = 0 \implies x^2 = 25 \implies x = \pm 5$$

$$4x - 3y = 0 \implies y = \tfrac{4}{3}x$$

Substituting into the first equation gives

$$x^2 + (\tfrac{4}{3}x)^2 = 25 \implies \tfrac{25}{9}x^2 = 25 \implies x^2 = 9 \implies x = \pm 3 \text{ and hence } y = \pm 4$$

Thus there are four critical points (-5,0), (5, 0), (3,4), (-3, -4).

Classification

To classify the critical points the second order partial derivatives are needed.

$$f_{xx} = 12x, \quad f_{yy} = 12x - 18y, \quad f_{xy} = 12y, \quad D = 12x(12x - 18y) - 144y^2$$

At $(-5, 0)$:

$$D(-5,0) = 144(-5)^2 > 0 \implies (-5,0) \text{ extremum}$$
$$f_{xx}(-5,0) < 0,\ f_{yy} < 0 \implies \text{ maximum}$$

At $(5, 0)$:

$$D(5,0) = 144(5^2) > 0 \implies \text{ extremum}$$
$$f_{xx}(5,0) > 0,\ f_{yy}(5,0) > 0 \implies \text{ minimum}$$

At $(3, 4)$:

$$D(3,4) < 0 \implies \text{ saddle point}$$

At $(-3, -4)$:

$$D(-3,-4) < 0 \implies \text{ saddle point}$$