MATH2267 Week 8

Integration and Multivariate Calculus

Semester 2, 2018

Overview

Part 1: Integration (1 variable)

- Antiderivative/Indefinite Integral
- Definite Integral

Part 2: Functions of Two Variables

- Domain, Range
- Partial Derivatives

Antiderivatives

Given a function $f(x)$. If there is $F(x)$ such that $F'(x) = f(x)$, then $F(x)$ is called an antiderivative of $f(x)$.

Note that

$$\begin{array}{ll} \text{if} & \dfrac{d}{dx}F(x) = f(x) \\ \text{then} & \dfrac{d}{dx}(F(x) + c) = f(x), \quad \text{for any constant } c \end{array}$$

Thus, if $f(x)$ has antiderivative, it has infinitely many antiderivatives.

Indefinite Integrals

The set of all antiderivatives of $f(x)$, if exist, is called the indefinite integral of $f(x)$, denoted by

$$\int f(x)\,dx$$

If $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + c$$

where c is an arbitrary constant.

Rules of Integration

1. $\int kf(x)\,dx = k\int f(x)\,dx$;
2. $\int \{f(x)+g(x)\}\,dx = \int f(x)\,dx + \int g(x)\,dx$;
3. $\int \{f(x)-g(x)\}\,dx = \int f(x)\,dx - \int g(x)\,dx$.

Example. Since $(x^3)' = 3x^2$, or $(\frac{1}{3}x^3)' = x^2$,

$$\begin{aligned}\int x^2\,dx &= \frac{1}{2+1}x^{2+1} + c\\ &= \frac{1}{3}x^3 + c\end{aligned}$$

More generally

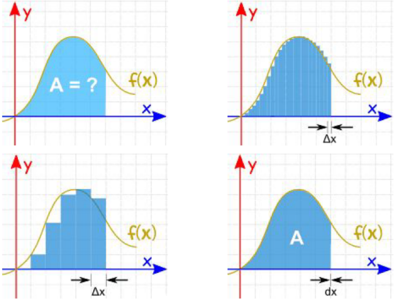
$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$$

Definite Integrals

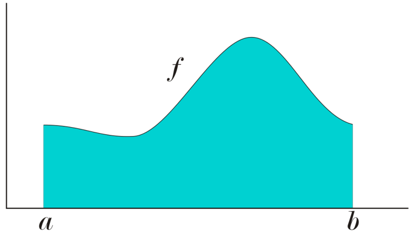
Definition. Divide $[a, b]$: $a = x_0 < x_1 < \cdots < x_n = b$. The definite integral of $f(x)$ on $[a, b]$ is defined as

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(c_k)\Delta_i$$

if limit exists, wqhere $\Delta_i = x_i - x_{i-1}$ and $\Delta = \max\{\Delta_i\}$.



$\int_a^b f(x)\,dx$ is the **area under the curve** $y = f(x),\ a \leq x \leq b$:



Fundamental Theorem of Calculus If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Example. Find $\int sin(x)\,dx$

Solution: An antiderivative of $\sin(x)$ is $-\cos(x)$. Thus,

$$\int sin(x)\,dx = -\cos(x) + C$$

One can use WolframAlpha to check this out:

Inputing

`integral sin(x)`

the result will appear.

Example. Evaluate $\int_0^3 e^{2x}\,dx$.

Solution: An antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$. Thus,

$$
\begin{aligned}
& \int_0^3 e^{2x}\,dx \\
= \quad & \left.\frac{1}{2}e^{2x}\right|_0^3 \\
= \quad & \frac{1}{2}(e^6 - e^0) \\
= \quad & \frac{e^6 - 1}{2}
\end{aligned}
$$

Checking by WolframAlpha:

```
integral [0,3] e^x*sin(x)
```

Week 8 Part 1

Integration

Example. Find the area under the curve

$$y = \sin(x), \quad 0 \leq x \leq \pi/2.$$

Solution: The area is Area$= \displaystyle\int_0^{\pi/2} \sin(x)\, dx$.

An antiderivative of $\sin(x)$ is $-\cos(x)$. Thus,

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/2} \sin(x)\, dx \\
&= -\cos(x)\big|_0^{\pi/2} \\
&= -\cos(\pi/2) - [-\cos(0)] \\
&= 0 - (-1) \\
&= 1 \quad (\text{check with WolframAlpha.})
\end{aligned}$$

Numerical Integration

One of the numerical integration methods, adaptive Gauss-Kronrod method, is available in Julia (v.0.6).

Example. Find the area under the curve

$$y = \sin^4(x + \sqrt{2}), \quad 0 \le x \le \pi/2.$$

Solution: Area$= \int_0^{\pi/2} \sin^4(x + \sqrt{2})\, dx$. Code & output is:

```
In [1]   import QuadGK.quadgk
         function f(x)
         sin^4(x+sqrt{2})
         end
         quadgk(f, 0, pi/2)
Out[1]   (0.74308449..., 1.32286881...e-9)
```

NOTE: 0.74308449... is the area found, 1.32286881...e-9 is the accuracy

Part 2

Function of Two Variables

- Notations, Domain, Range
- Partial Derivatives
- Higher Partial Derivatives (limited to 2_{nd} Order)

Week 8 Part 2

Function of Two Variables

Notations: $f(x, y)$, $g(x, y)$, etc

The region $D \subseteq \mathbb{R}^2$ on which function $f(x, y)$ is defined is the domain. The set of all values of $f(x, y)$ is the range.

For example, domain of $f(x, y) = \sqrt{x} + \sqrt{y}$ is

$$\{(x, y) \mid x \geq 0, y \geq 0\} \quad \text{(upper right quadrant)}$$

Another example: Domain of $\sqrt{1 - x^2 - y^2}$ is

$$\{(x, y) \mid x^2 + y^2 \leq 1\} \quad \text{(the unit disk)}$$

$\ln(x + y + 3z - 3)$ is a function of 3 variables. Its domain is

$$\{(x, y, z) \mid x + y + 3z > 3\} \quad \text{(half space)}$$

Week 8 Part 2

Function of Two Variables

Consider $z = f(x, y)$.

- The partial derivative of f w.r.t. x, denoted

$$\frac{\partial z}{\partial x} \equiv \frac{\partial f(x, y)}{\partial x} \equiv f_x(x, y)$$

  is the derivative of $f(x, y)$ w.r.t. x when y is fixed.

- Similarly, we have partial derivative of f w.r.t. y

$$\frac{\partial z}{\partial y} \equiv \frac{\partial f(x, y)}{\partial y} \equiv f_y(x, y)$$

- $f_x(x, y)$ and $f_y(x, y)$ are called first order partial derivatives.
- The partial derivatives of $f_x(x, y)$ and $f_y(x, y)$ are second order partial derivatives.

Week 8 Part 2

Function of Two Variables

Example 1. Find all 1st and 2nd order partial derivatives for $z = x^3y^2$.

Solution. $z_x = 3x^2y^2$ and $z_y = 2x^3y$

$$z_{xx} = (z_x)_x = (3x^2y^2)_x = 6xy^2$$
$$z_{yy} = (z_y)_y = (2x^3y)_y = 2x^3$$
$$z_{xy} = (z_x)_y = (3x^2y^2)_y = 6x^2y$$

Note: $z_{xy} = z_{yx}$ for well behaved functions.

Examples using WolframAlpha (try each of the following!):

```
partial derivatives x^2-4xy-x+y^3
d^2/dx^2(x^2-4xy-x+y^3)
d/dx(d/dy(x^2-4xy-x+y^3))
d^2/dy^2(x^2-4xy-x+y^3)
```

Example 2. Find all 1st and 2nd order partial derivatives for $w = 6x^3 + xyz + sin(2y) + y^2z^3$.

Solution.
$$w_x = 18x^2 + yz$$
$$w_y = xz + 2cos(2y) + 2yz^3$$
$$w_z = xy + 3y^2z^2$$
$$w_{xx} = 36x$$
$$w_{yy} = -4sin(2y) + 2z^3$$
$$w_{zz} = 6y^2z$$
$$w_{xy} = w_{yx} = z$$
$$w_{xz} = w_{zx} = y$$
$$w_{yz} = w_{zy} = x + 6yz^2$$

Next Week

- Numerical Methods
  - Newton Method
  - Monte-Carlo Method