Essential Mathematics

Week 3 Exercises

1 Systems of Linear Equations

Exercise 3.2.2

1. Solve the following linear systems.

   (a)

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 1 \\ 2x & + & 2y & - & z & = & 3 \\ x & - & 4y & + & 3z & = & 0 \end{array}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & -1 & 3 \\ 1 & -4 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & \\ 0 & 4 & -3 & 1 & R_2 - 2R_1 \\ 0 & -3 & 2 & -1 & R_3 - R_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & \\ 0 & 1 & -1 & 0 & R_2 + R_3 \\ 0 & -3 & 2 & -1 & \end{array}\right]$$

$$\left[\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & \\ 0 & 1 & -1 & 0 & \\ 0 & 0 & -1 & -1 & R_3 + 3R_2 \end{array}\right]$$

Back substitution:

R_3: $-z = -1 \implies z = 1$

R_2: $y - z = 0 \implies y = z = 1$

R_1: $x - y + z = 1 \implies x = 1 - 1 + 1 = 1$

Solution: $x = 1,\ y = 1,\ z = 1.$

(b)

$$\begin{array}{rcrcrcr} x & - & 5y & + & 2z & = & -5 \\ 3x & - & 14y & + & 3z & = & -8 \\ 4x & - & 18y & + & 4z & = & -10 \end{array}$$

Solution:

$$\begin{aligned} x-5y+2z &= -5 \\ 3x-14y+3z &= -8 \\ 4x-18y+4z &= -10 \end{aligned}$$

$$\begin{array}{lrcrcrcr} & x & - & 5y & + & 2z & = & -5 \\ Eq.2-3\times Eq.1 & & & y & - & 3z & = & 7 \\ Eq.3-4\times Eq.1 & & & 2y & - & 4z & = & 10 \end{array}$$

$$\begin{array}{lrcrcrcr} & x & - & 5y & + & 2z & = & -5 \\ & & & y & - & 3z & = & 7 \\ Eq.3-2\times Eq.2 & & & & & 2z & = & -4 \end{array}$$

Thus $z=-2$ from the last equation. Then,
Back substitution: $y+6=-7 \Rightarrow y=1$.
Then, $x=4$.
$x=4,\ y=1,\ z=-2.$

(c)

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 9 \\ 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 0 \\ x_1 & + & x_2 & & & + & 2x_4 & = & 5 \\ 3x_1 & - & 4x_2 & + & 3x_3 & + & x_4 & = & -1 \end{array}$$

Solution:

$$\begin{bmatrix} 1 & 2 & -1 & 3 & | & 9 \\ 2 & -1 & 2 & 1 & | & 0 \\ 1 & 1 & 0 & 2 & | & 5 \\ 3 & -4 & 3 & 1 & | & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & 9 & \\ 0 & -5 & 4 & -5 & -18 & -2R_1 + R_2 \\ 0 & -1 & 1 & -1 & -4 & -R_1 + R_3 \\ 0 & -10 & 6 & -8 & -28 & -3R_1 + R_4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & 9 & \\ 0 & 1 & -2 & 1 & 6 & R_2 - 6R_3 \\ 0 & -1 & 1 & -1 & -4 & \\ 0 & 0 & -2 & 2 & 8 & R_4 - 2R_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & 9 & \\ 0 & 1 & -2 & 1 & 6 & \\ 0 & 0 & -1 & 0 & 2 & R_3 + R_2 \\ 0 & 0 & -2 & 2 & 8 & \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & 9 & \\ 0 & 1 & -2 & 1 & 6 & \\ 0 & 0 & 1 & 0 & -2 & -1 \times R_3 \\ 0 & 0 & 0 & 2 & 4 & R_4 - 2R_3 \end{bmatrix}$$

Back substitution:

R_4: $2x_4 = 4 \implies x_4 = 2$

R_3: $x_3 = -2$

R_2: $x_2 - 2x_3 + x_4 = 6 \implies x_2 + 4 + 2 = 6 \implies x_2 = 0$

R_1: $x_1 + 2x_3 - x_3 + 3x_4 = 9 \implies x_1 = 9 - 0 - 2 - 6 = 1$

Solution: $x_1 = 1$, $x_2 = 0$, $x_3 = -2$, and $x_4 = 2$.

2 Systems of nonlinear equations

Exercise 3.3.3

1. Solve the following simultaneous equations by substitution.

   (a)

$$\begin{aligned}2x + y &= 9\\ xy &= 10\end{aligned}$$

Solution: From Eq.1, $y = 9 - 2x$. Substitute into Eq.2:

$$x(9-2x) = 10 \quad\Rightarrow\quad -2x^2 + 9x - 10 = 0 \quad\Rightarrow\quad x_1 = 2,\ x_2 = 5/2$$

Substitute x values into $y = 9 - 2x$:

when $x = 2$, $y = 9 - 2(2) = 5$

when $x = 5/2$, $y = 9 - 2(5/2) = 4$

Solutions to system: $(2,\ 5)$, $(5/2,\ 4)$

   (b)

$$\begin{aligned}\frac{1}{x^2} - \frac{1}{y^2} &= -16\\ \frac{1}{x} + \frac{1}{y} &= 8\end{aligned}$$

Solution: Let $u = \frac{1}{x}$, $v = \frac{1}{y}$. The system becomes:

$$\begin{aligned}u^2 - v^2 &= -16\\ u + v &= 8\end{aligned}$$

From Eq.2, $v = 8 - u$. Substitute into Eq.1:

$$u^2 - (8-u)^2 = -16 \quad\Rightarrow\quad 16u - 64 = -16 \quad\Rightarrow\quad u = 3$$

Substitute $u = 3$ into $u + v = 8$: $v = 8 - u = 5$

when $u = 3$, $x = 1/3$; when $v = 5$, $y = 1/5$.

Solution: $(x,\ y) = (1/3,\ 1/5)$

2. Solve the following simultaneous equations using graphical method.

(a)

$$x^2 - x - y = 1$$
$$x - y - 1 = 0$$

Solution: The two equations correspond to two functions:

$$y = x^2 - x - 1, \quad \text{and} \quad y = x - 1$$

From their plot (see Figure 1 (left)) we find intersection points $(0, -1)$ and $(2, 1)$.
Solutions: $(0, -1)$, $(2, 1)$

(b)

$$x^2 - y = 3$$
$$y - x = 4$$

Solution: The two equations correspond to two functions:

$$y = x^2 - 3, \quad \text{and} \quad y = x + 4$$

From their plot (see Figure 1 (right)) we find there two solutions. They are approximately $(-2.2,\ 1.8), (3.2,\ 7.2)$. (One can zoom in to find better estimates.) Solutions: $(0, -1)$, $(2, 1)$

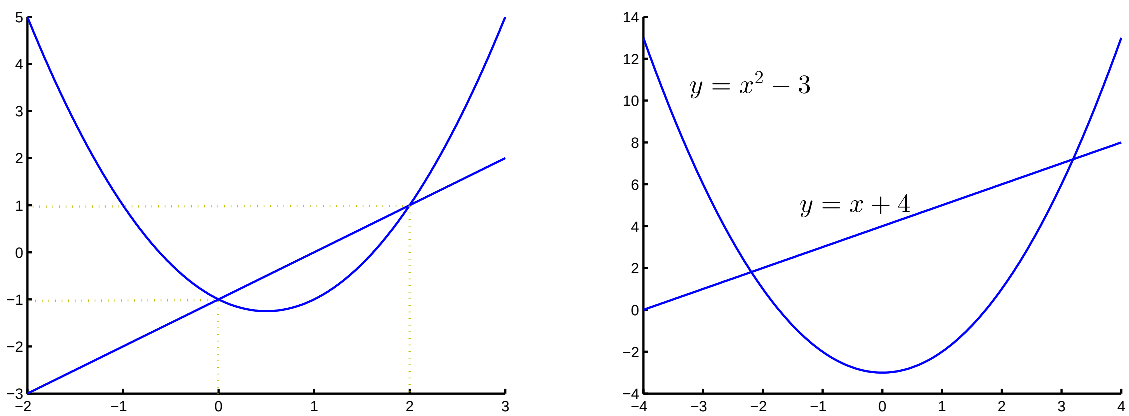

Figure 1: