Week 7 Exercise Solutions

Exercise 7.1.2

1. Substitute

   (a) $\lim_{x\to 5} 2x = 10$

   (b) $\lim_{a\to 3} \frac{9}{a} = \frac{9}{3} = 3$

2. Simplify

   (a) $\lim_{x\to 0} \frac{5x}{x} = 5$

   (b) $\lim_{t\to 1} \frac{t^2-1}{t-1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{t-1} = \lim_{t\to 1}(t+1) = 2$

   (c) $\lim_{x\to -4} \frac{x^2+6x+8}{x+4} = \lim_{x\to -4} \frac{(x+4)(x+2)}{x+4} = \lim_{x\to -4}(x+2) = -2$

   (d) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{(1-\sqrt{x})(1+\sqrt{x})}{(1-x)(1+\sqrt{x})} = \lim_{x\to 1} \frac{1-x}{(1-x)(1+\sqrt{x})} \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$

3. Compute (using Julia)

   (a) The solution is given in the file '**Week 7 Exercise Sol.ipynb**'

   (b) Proceed as above to get a result =1.

Exercise 7.1.4

1. $\lim_{x\to\infty} \frac{x^3+1}{x^2+20x} = \infty$ since the degree of the polynomial in the numerator is greater than that of the denominator

2. $$\lim_{x\to\infty} \frac{x^2+20x}{x^3+1} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{20}{x^2}}{1+\frac{1}{x^3}} = 0$$

3. $\lim_{x\to\infty} \frac{-x^3+4}{2x^2+x} = \lim_{x\to\infty} \frac{-x+\frac{4}{x^2}}{2+\frac{1}{x}} = -\infty$

4. $\lim_{x\to\infty} \frac{x^3+4}{-2x^2+x} = \lim_{x\to\infty} \frac{x+\frac{4}{x^2}}{-2+\frac{1}{x}} = -\infty$

5. $\lim_{x\to\infty}(1+\frac{1}{x})^x$. See the file '**Week 7 Exercise Sol.ipynb**'

Exercise 7.2.2

1. $5x$

$$\lim_{h\to 0}\frac{5(x+h)-5x}{h}=\lim_{h\to 0}\frac{5h}{h}=5$$

2. x^3

$$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}=3x^2$$

3. x^2+5x

$$\lim_{h\to 0}\frac{(x+h)^2+5(x+h)-x^2-5x}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2+5x+5h-x^2-5x}{h}$$

$$=\lim_{h\to 0}(2x+h+5)=2x+5$$

Exercise 7.2.4

Use WalframAlpha (`http://www.wolframalpha.com`) to find the derivatives of the following functions.

1. Check all your answers from the Exercises in the previous section.
2. $ln(x)$
3. $ln(x^2)$
4. $3x^2+4x+5$
5. x^3+x
6. $(3x^2+4x+5)(x^3+x)$. Use the results from the above and the Product Rule. Compare the result with that obtained by direct insertion into Wolfram Alpha.
7. $\frac{3x^2+4x+5}{x^3+x}$

Exercise 7.3.1

1. Rates of change

   (a) A population is given by the following function of time $p(t)=t^3-2t+100$. What was the growth rate at $t=10$?
   The growth rate is the rate of change of the population with time. Hence the growth rate $=p'(t)=3t^2-2$ *and* $p'(10)=298$.

(b) Water gushes out of a tap into a bucket. The volume of water in the bucket at any time $t < 10$ is given by $v(t) = 5t$ where v is in litres and t in minutes. What is the rate at which water gushes from the tap?

The rate of flow from the tap is $v'(t) = 5$, a constant for all $t < 10$.

2. Find the maximum of the following functions

   (a) $-x^3 + 10.5x^2$

   Derivative must be zero ie. $-3x^2 + 21x = 0 \implies x = 0 or x = 7$. Plot or look at the seond derivative to determine maximum at x=0

   (b) $-2x^2 + 6x + 12$

   $-4x + 6 = 0 \implies x = \frac{3}{2}$

Exercise 7.3.3

1. $f(x) = 5x^3 + 2x^2 - 3x$

   $f'(x) = 15x^2 + 4x - 3 = 0 \implies x = \frac{1}{3}$ *and* $x = -\frac{3}{5}$ *. Also $f''(x) = 30x + 4$. This is positive when $x = \frac{1}{3}$, indicating a local maximum while a minimum occurs when $x = -\frac{3}{5}$ since f'' is negative at this value.*

2. $f(x) = x^3 - 6x^2 + 12x - 5$. See the file '**Week 7 Exercise Sol.ipynb**'