MATH2267 Week 9

Numerical Methods

Semester 2, 2018

Overview

Part 1: Newton's Method for Roots
- Introduction
- Algorithm

Part 2: Monte-Carlo Method for Area
- Introduction
- Random Numbers (in $\mathbf{R}$) and Rand Points (in $\mathbf{R}_2$)
- Area of Plane Region

Part 1

Newton's Method for Roots

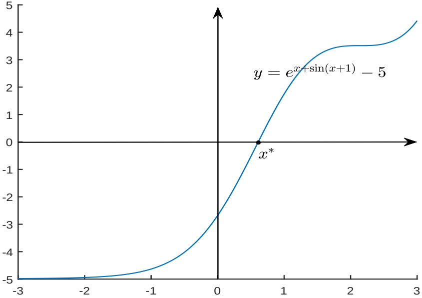
Part 1: Newton Method

1.1. Introduction

A root of a real-valued function $f(x)$ is value of x, say x^*, such that $f(x^*) = 0$. Roots of $f(x)$ are also called zeros.

Graphically, roots of $f(x)$ are points on the real axis where the graph of $y = f(x)$ crosses the real axis.

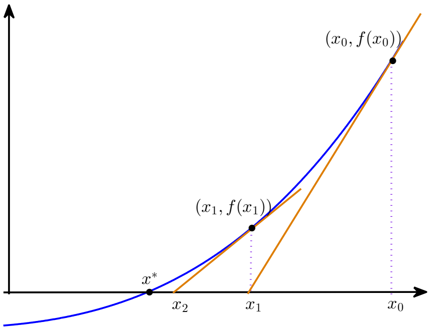
In the following figure, x^* is a root of $e^{x+\sin(x+1)-2}$.



Part 1: Newton Method

1.1. Introduction

Newton's method is an iterative method for finding roots (or zeroes) of a real-valued function $f(x)$. Procedure demonstrated by figure shows it produces $\{x_i\}$ such that $x_i \to x^*$ (true root).



Part 1: Newton Method

1.1. Introduction

Suppose $f(x)$ has a root in $[a, b]$. $f'(x)$ exists.

- ✓ Let $x_0 \in [a,\ b]$ be an approximate root.
- ✓ Find x-intercept, x_1, of tangent line at $(x_0, f(x_0))$

$$y = f'(x_0)\,(x - x_0) + f(x_0) \quad (\ \text{tangent}) :$$

$$f'(x_0)\,(x_1 - x_0) + f(x_0) = 0 \implies x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- ✓ Find x-intercept, x_2, of tangent line at $(x_1, f(x_1))$:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

- ✓ Repeat this until requirement met.

Part 1: Newton Method

1.2. Algorithm

Newton's method for functions $f(x)$ of one variable is as follows:

1. Find initial guess x_0 (guess and check) for a root.
2. Calculate a sequence of approximate roots $x_1, \ x_2, \ \cdots$ according to

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0, 1, 2, \cdots .$$

3. Terminates when required accuracy/number of iteration is reached

- An exact root x^* may never be reached.
- $x_n \to x^*$ if initial guess is sufficiently close to root x^*.

Example

Perform 3 iterations of Newton's method to find an approximate root x_3 for the function $f(x) = x^3 - 2$, starting from $x_0 = 1.5$.

Solution: $f'(x) = 3x^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{1.5^3 - 2}{3(1.5^2)} = 1.2963$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.260932$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.259922$$

This is a very good approximation to the true root

$$x^* = \sqrt[3]{2} \approx 1.259921049894873.$$

Part 2

Monte-Carlo Method

1. Introduction
2. Random Numbers in Julia
3. Area of Plane Region

Part 2: Monte-Carlo Method

2.1. Introduction

Monte Carlo simulation is a computerized mathematical technique used in a wide range fields such as finance, management, energy, manufacturing, engineering, research and development, insurance, oil & gas, transportation, and environment.

Monte Carlo methods follow the following pattern:

1. Define a domain of possible inputs
2. Generate inputs randomly
3. Perform a deterministic computation on the inputs
4. Aggregate the results

Part 2: Monte-Carlo Method

2.2. Random Numbers in Julia

How to generate random points box $[a, b] \times [c, d] \subset \mathbf{R}_2$.

- In Julia,

  ```
  x = rand(3,7)
  ```

  returns an 3×7 matrix containing pseudorandom values drawn from the standard uniform distribution on $(0, 1)$.

  ```
  x = rand(5,1)
  ```

  returns a vector in $\mathbf{R}^5$, elements randomly drown in $(0, 1)$

  ```
  x = rand(9)
  ```

  returns a 9-element (1-D) array randomly drown in $(0, 1)$.

- To generate a random value in interval $[a, b]$:

  ```
  r=a+(b-a)*rand(1)
  ```

  To generate a random point p in box $[a, b] \times [c, d] \subset \mathbf{R}_2$:

  ```
  p=[a c]+[b-a d-c].*rand(1,2)
  ```

Part 2: Monte-Carlo Method

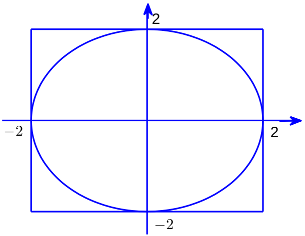
2.3. Area of Plane Region

A particular example is to find the area of a finite region on the plane given by $f(x,y) \leq 0$.

Example

Find the area of the circular disk $x^2 + y^2 \leq 4$.

Solution: Consider the square of size 4 as shown in figure.



Solution: continued....

Create N random points in the square. Count the number of those that fall in the circle, denoted by n. Then,

$$n/N \approx \frac{\text{area cricle}}{\text{area square}} \quad \text{(when } N \text{ large)}$$

$$\text{area circle} \approx \frac{n}{N} \times (\text{area square}) = \frac{16n}{N}$$

```
N=100000, n=0
for i in 1:N
    x=[-2 -2]+4*rand(1,2)
    if x[1,1]^2+x[1,2]^2<=4
        n=n+1
    end
end
AreaOfCircle=(n/N)*16
```

Next Week:

- Modeling with Ordinary Differential Equation
  - Introduction to ODE
  - Solving ODE by Julia
  - Application