Essential Mathematics

Week 8 Notes

RMIT

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Contents

Chapter 8

Integration, Multivariate Calculus

8.1 Functions of Several Variables

8.1.1 Functions of Several Variables

In week 7 we described a function as a mapping from a set X (the domain) to another set Y (the range). In some cases an element of the domain may best be described by two or more variables. For example, consider the surface of the earth. We identify a point on this surface by latitude and longitude. A function giving the altitude at any point is a function of both latitude and longitude ie of two variables. Similarly, the growth rate of a plant might be a function of factors such as soil moisture, temperature, nitrogen and phosphorus concentrations, etc ie a function of several variables.

8.1.2 Domain

Earlier we obtained the interval for the domain of a function of a single variable. The domain of a function of two variables will be a region. For example, consider the function $f(x, y) = \sqrt{x} + \sqrt{y}$. Since a number under the square root sign must be non-negative the domain of this function is the region given by $x \geq 0, y \geq 0$ ie the upper right quadrant of the plane.

Another example: Find the domain of $\sqrt{1 - x^2 - y^2}$. The domain consists of all points that satisfy $x^2 + y^2 \leq 1$ ie. all points on and within the unit circle.

A similar approach can be used for functions of more than two variables. For example, the domain of the function $\ln(x+y+3z-3)$ is given by $x+y+3z > 3$.

8.1.3 Exercises

1. Find the domain of the following functions:

(a) $\frac{1}{\sqrt{x^2+y^2-4}}$

(b) $\sqrt{4x^2+y^2-4}$

(c) $\ln(x^2+3y-2z)$

2. Find the domain and range of the following functions:

(a) $|x+y|$

(b) $(x+y)^2$

(c) $(x+y)^3$

8.1.4 Partial Derivatives

Recall that the derivative of a function $f(x)$ is the rate of change of f with respect to x. For a function of two or more variables we might want to know the rate of change of one of the variables while keeping the others fixed. For example, we might have a function for the growth rate of a greenhouse crop that depends on soil moisture, temperature, and nitrate concentration. How will the growth rate be affected if we increase the temperature while keeping all other variables constant? To answer questions like this we use a **partial derivative**.

Consider a function of two variables $z = f(x,y)$. The *partial derivative of f with respect to x* is defined below:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

Similarly, the *partial derivative of f with respect to y* is defined as:

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Below are alternative notations for the partial derivative:

$$\frac{\partial z}{\partial x} \equiv \frac{\partial f(x,y)}{\partial x} \equiv f_x(x,y), \quad \frac{\partial z}{\partial y} \equiv \frac{\partial f(x,y)}{\partial y} \equiv f_y(x,y)$$

The above are sometimes referred to as *first order partial derivatives.* Higher order partial derivatives are discussed later.

Example 8.1.1 *Find all the first order partial derivatives of the following functions:*

1. $z = x^3y^2$

Solution $z_x = 3x^2y^2$ *and* $z_y = 2x^3y$

2. $w = 6x^3 + xyz + sin(2y) + y^2z^3$

Solution $w_x = 18x + yz$ *and* $w_y = xz + 2cos(2y) + 2yz^3$ *and* $w_z = xy + 3y^2z^2$

3. $z = (x+2y)^3y^5$

Solution $z_x = 3(x+2y)^2y^5$ *and* $z_y = 3(x+2y)^2 2y^5 + (x+2y)^3 5y^4$

8.1.5 Exercises

Find all the first order partial derivatives of the following functions:

1. $z = x^3 + 3x^2y + 3xy^2 + y^3$
2. $f(x, y, z) = x^4y^2 + xz + y^3z^2 + 6z$
3. $z = (x^2 + 3y^2 + xy)^4$
4. e^{x^2+2y}

8.1.6 Higher Order Partial Derivatives

Higher order partial derivatives have various applications such as in optimisation discussed in the next section. Calculating them is no more difficult than calculating the second derivative of a univariate function. The *second order partial derivatives* are calculated in the examples below.

Example Referring to Examples 8.1.1:

1. $z_{xx} = 6xy^2, \quad z_{yy} = 2x^3, \quad z_{xy} = z_{yx} = 6x^2y$
2. $w_{xx} = 18, \quad w_{yy} = -4sin(2y)+2z^3, \quad w_{zz} = 6y^2z, \quad w_{xy} = w_{yx} = z, \quad w_{xz} = w_{zx} = y, \quad w_{yz} = w_{zy} = x + 6yz^2$
3. $z_{xx} = 6(x + 2y)y^5, \quad z_{yy} = 24(x + 2y)y^5 + 30(x + 2y)^2y^4 + 30(x + 2y)^2y^4 + 20(x + 2y)^3y^3, \quad z_{xy} = z_{yx} = 12(x + 2y)y^5 + 15(x + 2y)^2y^4$

8.1.7 Exercises

Calculate all the second order partial derivatives for the problems given in Exercises 8.1.5.

8.2 *Optimisation (Optional Topic)

8.2.1 Introduction

Recall: A function of one variable, $f(x)$, has a **critical value** where $f'(x) = 0$. At a point a where $f'(a) = 0$ there is a local maximum if $f''(a) < 0$ and a local minimum if $f''(a) > 0$. We have not discussed the case where $f'(a) = f''(a) = 0$. (in one of the tutorial problems this yielded a 'point of inflection' but this result is not general).

For functions of two variables a critical point can be a local maximum, a local minimum, or a third possibility a **saddle point**. The term **extremum** is used to refer to either a maximum and or a minimum. Analogous to finding critical

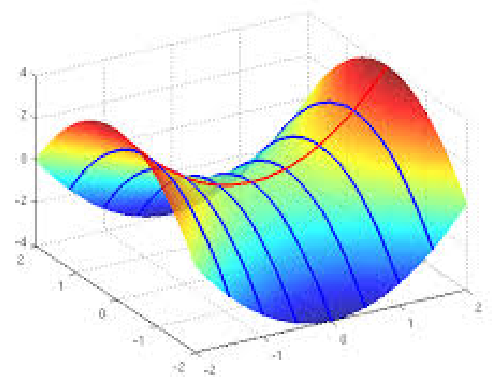

Figure 8.1: Example of a saddle point

points of a univariate function, criteria are needed to find critical points and then to classify them.

8.2.2 Critical Points and Their Optimality

Finding critical points Set both $f_x = 0$ and $f_y = 0$. This will give two equations in two unknowns x and y. Solve these equations for x and y. There will usually be more than one solution.

Classifying Minimum and Maximum Let (a, b) be a critical point obtained as described above. then (a, b):

- is a **saddle point** if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b)
- is an **extremum** if $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b). In this case (a, b) is:
  - a **maximum** if $f_{xx} < 0$ and $f_{yy} < 0$ at (a, b)
  - a **minimum** if $f_{xx} > 0$ and $f_{yy} > 0$ at (a, b)
- could be determined if $f_{xx}f_{yy} - f_{xy}^2 = 0$. **The test fails** .

8.2.3 Examples

Find all the critical points of the following functions and classify them.

- $f(x, y) = x^2 + y^2$ It is clear that this function is bowl-shaped as the terms increase as they move away from the origin. So we expect a minimum at the origin. Let us see whether the mathematical criteria show this.

  Solution $f_x = 2x = 0$ and $f_y = 2y = 0$ so $(0, 0)$ is a critical point. $f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 0^2 = 4 > 0$ so the point is an extremum. Further $f_{xx} = 2 > 0, \quad f_{yy} = 2 > 0$ therefore we have a minimum as expected.

- $f(x, y) = 4 + x^3 + y^3 - 3xy$

Solution $f_x = 3x^2 - 3y, \quad f_y = 3y^2 - 3x, \quad f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3$

Thus critical points will be the solutions of the following system of equations:

$$f_x = 3x^2 - 3y = 0$$
$$f_y = 3y^2 - 3x = 0$$

There are two solutions to this system of equations. The critical points are $(0,0)$ and $(1,1)$.

Let $D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. At $(0,0)$ $D = -9 < 0$ so this must be a saddle point. At $(1,1)$ $D = 27 > 0$ so this must be an extremum. Since $f_{xx}(1,1) = 6 > 0$ and $f_{yy}(1,1) = 6 > 0$ this point is a minimum.

8.2.4 Exercises

Find all the critical points of the following functions and classify them.

1. $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$
2. $f(x,y) = e^{-(x^2+y^2)}$
3. $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$

8.3 Integration

8.3.1 Antiderivatives

Let f be defined on $[a,b]$. Any function F such that

$$F'(x) = f(x) \text{ for all } x \in (a,b)$$

is called the antiderivative of f.

Since $\frac{d}{dx}(F(x)+c) = F'(x) + 0 = f(x)$ for any constant c, $F(x)+c$ is also an antiderivative of f.

8.3.2 Indefinite Integrals

The set of all antiderivatives of $f(x)$ is denoted by the indefinite integral

$$\int f(x)\,dx$$

so that

$$\int f(x)\,dx = F(x) + c \quad \text{(where } c \text{ is an arbitrary constant).}$$

8.3.3 Rules of Integration

(These follow from the rules of differentiation.)

1. $\int kf(x)\,dx = k\int f(x)\,dx$;
2. $\int \{f(x)+g(x)\}\,dx = \int f(x)\,dx + \int g(x)\,dx$;
3. $\int \{f(x)-g(x)\}\,dx = \int f(x)\,dx - \int g(x)\,dx$.

Example

$$\begin{aligned}\int x^2\,dx &= \frac{1}{2+1}x^{2+1}+c\\ &= \frac{1}{3}x^3+c\end{aligned}$$

More generally

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$$

8.4 Definite Integrals

8.4.1 Definition

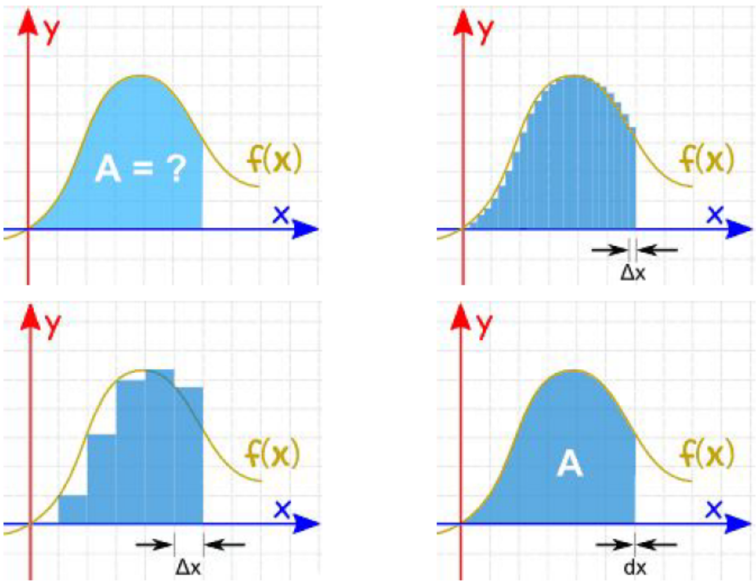

Figure 8.2: Area under curve (order: top left, bottom left, top right, bottom right)

Suppose f is defined on $[a,b]$ and that $[a,b]$ is subdivided into n intervals of length Δx by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

where $\Delta x = x_k - x_{k-1} \quad \forall k$.

If f is continuous on $[a, b]$, then it can be shown that

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(c_k)\Delta x$$

exists and has the same value for any choice of $c_k \in [x_{k-1}, x_k]$ and we define

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(c_k)\Delta x.$$

Note that as $\Delta x \to 0$, the number of strips $n \to \infty$.

Remarks

- If $f(x) > 0$ for all $x \in [a, b]$, then the integral equals the area beneath the curve.
- $\int_a^b f(x)\,dx$ is a number called the definite integral of f over $[a, b]$.
- a and b are the limits or terminals of integration.
- x is a dummy variable

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(\alpha)\,d\alpha, \text{ etc .}$$

8.4.2 Properties of Definite Integrals

- $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ where k is a constant
- $\int_a^b \Big[f(x) \pm g(x)\Big]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0$
- $f(x) \leq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$
- $(b-a)m \leq \int_a^b f(x)\,dx \leq (b-a)M$ where $m = \min_{a \leq x \leq b} f(x)$ and $M = \max_{a \leq x \leq b} f(x)$.

8.4.3 Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$