Essential Mathematics
Week 3 Notes

RMIT

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Contents

Chapter 3

Simultaneous Equations

3.1 Introduction

This chapter will be focused on solving two types of simultaneous equations.

The first type of simultaneous equations is the so-called systems of linear equations. We can be efficiently solved by Gaussian elimination method.

3.2 Systems of Linear Equations

Note: This topic was covered earlier in the course. Notes are given here to see the topic in the context of matrices and for revision.

3.2.1 Introduction

Systems of linear equations arise in many areas, far more than we have time and space to list. Here is one example which has some applicability to computer graphics.

Curve Fitting

Suppose we have three distinct points $P(x_0, y_0)$, $Q(x_1, y_1)$ and $R(x_2, y_2)$ in two-dimensional space and we wish to find the **unique parabola** which passes through these points. (See the sketch below.)

The parabola will have equation

$$y = ax^2 + bx + c$$

and our task is to find the unknowns a, b and c. Since we want point P to lie on the parabola, we must have

$$x_0^2 a + x_0 b + c = y_0$$

(Remember, a, b and c are the unknowns here.) Similarly for Q and R we obtain

$$x_1^2 a + x_1 b + c = y_1$$

and

$$x_2^2 a + x_2 b + c = y_2.$$

Together we have a system of three equations in our three unknowns a, b and c.

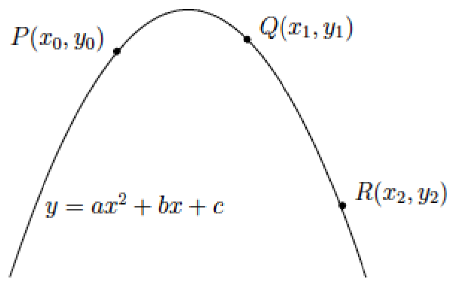

Figure 3.1: Curve fitting example

A Simple Example

Consider the following pair of simultaneous equations in two variables

$$\begin{array}{rcrcr} 2x & - & 3y & = & 5 \\ 5x & + & 4y & = & 1 \end{array}$$

We may solve this system many ways. One possibility is to multiply equation 1 by 5 and equation 2 by 2:

$$\begin{array}{rcrcr} 10x & - & 15y & = & 25 \\ 10x & + & 8y & = & 2 \end{array}$$

Now we subtract the first equation from the second to obtain

$$23y = -23$$

Hence $y = -1$. We now substitute back into the original first equation:

$$2x + 3 = 5$$

so that $x = 1$.

We can simplify the above calculation and make it clearer using matrices.

Our system of equations is equivalent to the **matrix system**

$$\begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Check this yourself by multiplying out the left hand side and equating the left and right hand sides.

We can write this in an even more compact form as

$$\left[\begin{array}{rr|r} 2 & -3 & 5 \\ 5 & 4 & 1 \end{array}\right].$$

where the first column is associated with x and the second column is associated with y. This is known as an **augmented matrix** or **tableau**.

We repeat the solution using the augmented matrix. We subtract 5 times row 1 (equation 1) from 2 times row 2 (equation 2) and place the result of this in row 2 to obtain

$$\left[\begin{array}{rr|r} 2 & -3 & 5 \\ 0 & 23 & -23 \end{array}\right] \qquad 2R_2 - 5R_1 \to \text{new } R_2$$

The new R_2 can be interpreted as $23y = -23$ so that $y = -1$. Substituting back into R_1 we have $x = 1$.

A Harder Example

Suppose we now have 3 equations in 3 variables:

$$\begin{array}{rcrcrcr} x & - & 5y & + & 2z & = & -5 \\ 3x & - & 14y & + & 3z & = & -8 \\ 4x & - & 5y & + & 3z & = & 5 \end{array}$$

This system represents the intersection of three planes in 3D space.

How do we solve this system without going round in circles? Here is a strategy that works. Take equations 1 and 2 together. Subtract 3 times equation 1 from equation 2 to obtain equation $2'$:

$$y \;-\; 3z \;=\; 7$$

Take equations 1 and 3 together. Subtract 4 times equation 1 from equation 3 to obtain equation $3'$:

$$15y \;-\; 5z \;=\; 25$$

Multiply through this equation by $1/5$ to obtain equation $3''$:

$$3y \;-\; z \;=\; 5$$

Take the new equations $2'$ and $3''$ together.

$$\begin{array}{rcrcr} y & - & 3z & = & 7 \\ 3y & - & z & = & 5 \end{array}$$

Subtract 3 times equation $2'$ from equation $3''$ to obtain

$$8z \;=\; -16$$

Clearly $z = -2$. Substitute back into equation $2'$ to find y :

$$y + 6 = 7$$

so that $y = 1$. Finally, substituting back into equation 1 gives

$$x - 5 - 4 = -5$$

so that $x = 4$. The three planes meet at the point $(4, 1, -2)$.

The augmented matrix approach makes the solution strategy clear.

$$\left[\begin{array}{rrr|r} 1 & -5 & 2 & -5 \\ 3 & -14 & 3 & -8 \\ 4 & -5 & 3 & 5 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 2 & -5 \\ 0 & 1 & -3 & 7 \\ 0 & 15 & -5 & 25 \end{array}\right] \qquad \begin{array}{l} R_2 - 3R_1 \to R_2 \\ R_3 - 4R_1 \to R_3 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 2 & -5 \\ 0 & 1 & -3 & 7 \\ 0 & 3 & -1 & 5 \end{array}\right] \qquad \tfrac{1}{5}R_3 \to R_3$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 2 & -5 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 8 & -16 \end{array}\right] \qquad R_3 - 3R_2 \to R_3$$

At each step you should be able to recognise the same equations as discussed above in the algebraic method.

The strategy is clear. If we can use **elementary row operations** (like $R_3 - 3R_2 \to R_3$) to transform the augmented matrix into **echelon form** (with zeros in the bottom left corner), then we can read off the value of the last variable and substitute back to find the rest one by one.

Elementary Row Operations

There are 3 types of legal elementary row operations:

- swap two rows ($R_i \leftrightarrow R_2$)
- multiply through a whole row by a scalar ($kR_i \to R_i$)
- add or subtract a multiple of one row from another ($R_i \pm kR_j \to R_i$)

Worked Example 1

Use elementary row operations to solve the following system:

$$\begin{array}{rcrcrcr} 2x & + & 6y & + & z & = & 10 \\ x & - & y & + & 2z & = & 4 \\ 4x & + & 4y & + & 5z & = & 18 \end{array}$$

The augmented matrix is

$$\left[\begin{array}{rrr|r} 2 & 6 & 1 & 10 \\ 1 & -1 & 2 & 4 \\ 4 & 4 & 5 & 18 \end{array}\right]$$

Swap rows 1 and 2:

$$\left[\begin{array}{rrr|r} \boxed{1} & -1 & 2 & 4 \\ 2 & 6 & 1 & 10 \\ 4 & 4 & 5 & 18 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

We call the boxed element a **pivot**. We now try to get a column of zeros beneath the pivot, using the row that contains the pivot.

$$\left[\begin{array}{rrr|r} \boxed{1} & -1 & 2 & 4 \\ 0 & 8 & -3 & 2 \\ 0 & 8 & -3 & 2 \end{array}\right] \qquad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 4R_1 \to R_3 \end{array}$$

Select the next pivot and zero beneath it

$$\left[\begin{array}{rrr|r} \boxed{1} & -1 & 2 & 4 \\ 0 & \boxed{8} & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 - R_2 \to R_3$$

We've got a bonus here with a whole row of zeros! We have reduced our system of three equations to an effective system of two equations in three unknowns. That is, the process has shown that one of the original equations was **redundant**.

We have not enough equations (constraints) to determine all of the variables uniquely. One of the three variables will remain unconstrained, i.e. can take any value. The best choice for the unconstrained variable is always a variable (column) not associated with a pivot. The obvious choice here is z, so we let $z = t$ where $t \in R$. t is called a **parameter**.

If $z = t$ we substitute back into R_2 to find y:

$$8y - 3t = 2$$

so that $y = \frac{1}{4} + \frac{3}{8}t$. Now substitute back into R_1 to find x:

$$x - (\frac{1}{4} + \frac{3}{8}t) + 2t = 4$$

so that $x = \frac{17}{4} - \frac{13}{8}t$. The whole solution is thus

$$x = \frac{17}{4} - \frac{13}{8}t, \quad y = \frac{1}{4} + \frac{3}{8}t, \quad z = t \quad \text{for } t \in R.$$

which is the equation of a line in three dimensions.

Discussion

If the last row in the original augmented matrix was

$$4 \quad 4 \quad 5 \,\big|\, 17$$

what would happen? In fact, everything would proceed as before, except that the last row would become

$$0 \quad 0 \quad 0 \,\big|\, -1$$

This equation has **no solution**. We could interpret this as three planes having no common point of intersection.

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Worked Example 2

Use elementary row operations to solve the following system:

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & x_3 & - & x_4 & = & 5 \\ 3x_1 & - & 3x_2 & + & 4x_3 & - & 2x_4 & = & 21 \\ -2x_1 & + & 2x_2 & - & 5x_3 & - & x_4 & = & -28 \end{array}$$

Note that we have less equations than unknowns so we will definitely need at least one parameter.

The augmented matrix (with first pivot) is

$$\left[\begin{array}{rrrr|r} \boxed{1} & -1 & 1 & -1 & 5 \\ 3 & -3 & 4 & -2 & 21 \\ -2 & 2 & -5 & -1 & -28 \end{array}\right]$$

Zero in first column and select the next pivot:

$$\left[\begin{array}{rrrr|r} \boxed{1} & -1 & 1 & -1 & 5 \\ 0 & 0 & \boxed{1} & 1 & 6 \\ 0 & 0 & -3 & -3 & -18 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{array}$$

Zero in third column (because there is nothing to be done in second column):

$$\left[\begin{array}{rrrr|r} \boxed{1} & -1 & 1 & -1 & 5 \\ 0 & 0 & \boxed{1} & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 - 3R_2 \to R_3$$

We are left with 2 equations in 4 unknowns so we need $4 - 2 = 2$ parameters. Call the parameters s and t. The best choices for the parameters are $x_2 = s$ and $x_4 = t$.

We now substitute back to find x_1 and x_3 in terms of s and t.

$$x_3 + t = 6$$

so that $x_3 = 6 - t$, and

$$x_1 - s + (6 - t) - t = 5$$

so that $x_1 = -1 + s + 2t$.

Our solution may be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

3.2.2 Exercise

1. Solve the following linear systems.

   (a)

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 1 \\ 2x & + & 2y & - & z & = & 3 \\ x & - & 4y & + & 3z & = & 0 \end{array}$$

   Answer: $x_1 = 1,\ x_2 = 1,\ x_3 = 1.$

   (b)

$$\begin{array}{rcrcrcr} x & - & 5y & + & 2z & = & -5 \\ 3x & - & 14y & + & 3z & = & -8 \\ 4x & - & 18y & + & 4z & = & -10 \end{array}$$

   Answer: $x_1 = 4,\ x_2 = 1,\ x_3 = -2.$

   (c)

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 9 \\ 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 0 \\ x_1 & + & x_2 & & & + & 2x_4 & = & 5 \\ 3x_1 & - & 4x_2 & + & 3x_3 & + & x_4 & = & -1 \end{array}$$

   Answer: $x_1 = 0,\ x_2 = -1,\ x_3 = -2,\ x_4 = 3.$

3.3 Nonlinear Simultaneous Equations (2 variables)

A system of equations is nonlinear if at least one of the equations in the system is a nonlinear equation. In this section, we are going to learn two methods for solving nonlinear simultaneous equations involving 2 variables. They are the substitution method and the graphical method.

3.3.1 Substitution

Consider the nonlinear system given by

$$\begin{aligned} x^2 + y^2 &= 100 \\ x \;+\; y &= 14 \end{aligned}$$

To solve this system of equations, we use the substitution method as shown step by step below.

1. From one of the two equations work out one variable in terms of the the other variable.

   For the current system, we find $y = 14 - x$ from the 2nd equation.

2. Substitute the variable into the other equation equation to obtain an equation involving only one variable.

   For the current system, we substitute $y = 14 - x$ into $x^2 + y^2 = 100$ to obtain

   $$x^2 + (14 - x)^2 = 100$$

   or

   $$2x^2 - 28x + 96 = 0.$$

3. Solve the resulted equation.

   For the current problem, we solve $2x^2 - 28x + 96 = 0$ and obtain solutions $x_1 = 6$, $x_2 = 8$.

4. Substitute the solutions into the other equation, and find the corresponding values of the other variable. The original system will be solved.

   For the current problem, we substitute $x_1 = 6$ and $x_2 = 8$ into $y = 14 - x$ and obtain $y_1 = 8$ and $y_2 = 6$, respectively. Thus, the solutions to the nonlinear system are

   $$(x, y) = (6, 8) \quad \text{and} \quad (x, y) = (8, 6).$$

Example 3.3.1 *Solve the nonlinear system given by*

$$x^2 - 2x - y + 1 = 0 \tag{3.1}$$
$$x + \sqrt{y} - 5 = 0 \tag{3.2}$$

Solution. From (3.2), $y = (5 - x)^2$. Substitute into (3.1) and obtain

$$x^2 - 2x - (5 - x)^2 + 1 = 0$$

or

$$8x - 24 = 0$$

which has one solution $x = 3$.

Substitute $x = 3$ into $y = (5 - x)^2$ we get $y = (5 - 3)^2 = 4$.

Finally, the only solution to the system is $(x, y) = (3, 4)$.

Example 3.3.2 *Solve the system of equations.*

$$3x^2 + 4x - y = 7 \tag{3.3}$$
$$2x - y = -1 \tag{3.4}$$

Solution: Begin by solving for y in (3.4) to obtain $y = 2x + 1$. Next, substitute this expression for y into (3.3) and solve for x.

$$\begin{aligned} & 3x^2 + 4x - (2x + 1) = 7 \\ \Rightarrow \quad & 3x^2 + 2x - 8 = 0 \\ & x_1 = \frac{-2 + \sqrt{2^2 - 4 \times 3 \times (-8)}}{2 \times 3} = \frac{4}{3} \\ & x_2 = \frac{-2 - \sqrt{2^2 - 4 \times 3 \times (-8)}}{2 \times 3} = -2 \end{aligned}$$

Back-substituting these values of x into $y = 2x + 1$ to solve for the corresponding values of y:

$$y_1 = \frac{11}{3}, \quad y_2 = -3.$$

The above leads to solutions: $(\frac{4}{3},\ \frac{11}{3})$ and $(-2,\ -3)$.

3.3.2 Graphical Method

Example 3.3.3 *Solve the system of equations.*

$$y = \ln x \tag{3.5}$$
$$x + y = 1 \tag{3.6}$$

Solution: From the graph, we see that $(1, 0)$ is the solution of the system. Check this solution in (3.5-3.6):

$$0 = \ln 1 \quad \text{correct}$$
$$1 + 0 = 1 \quad \text{correct}$$

We will find applications of the material in finding optimal values for functions of two variables.

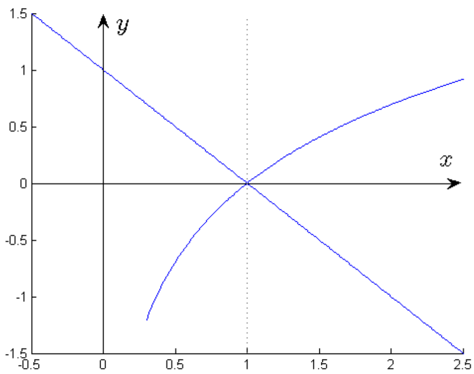

Figure 3.2: Graphical solution of nonlinear system (3.5-3.6)

3.3.3 Exercise

1. Solve the following simultaneous equations by substitution.

   (a)

$$\begin{aligned} 2x + y &= 9 \\ xy &= 10 \end{aligned}$$

   (Solution: $(2, 5)$)

   (b)

$$\begin{aligned} \frac{1}{x^2} - \frac{1}{y^2} &= -16 \\ \frac{1}{x} + \frac{1}{y} &= 8 \end{aligned}$$

   (Hint: Put $\frac{1}{x} = u$ and $\frac{1}{y} = v$ and solve for u and v.)

   (Solution: $(1/3, 1/5)$)

2. Solve the following simultaneous equations using graphical method.

   (a)

$$\begin{aligned} x^2 - x - y &= 1 \\ x - y - 1 &= 0 \end{aligned}$$

   (Solutions: $(0, -1)$, $(2, 1)$)

   (b)

$$\begin{aligned} x^2 - y &= 3 \\ y - x &= 4 \end{aligned}$$

   (There is no solution.)