Essential Mathematics
Week 5 Notes

RMIT

Semester 2, 2018

Contents

Chapter 5

Vectors

5.1 Introduction

A vector is a quantity with a magnitude (or length) and a direction. Examples of vector are force, velocity, acceleration, and electric field intensities.

Quantities having only magnitude are called scalars. Examples of scalar are temperature, time, and distance.

Geometrically, a vector is represented by a directed line segment. The direction of the vector is indicated by an arrow pointing from one end (the tail) to the other (the nose or head). The magnitude of the vector is indicated by its length.

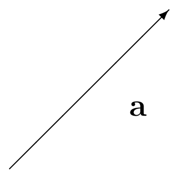


Figure 5.1: Geometric representation of vector

We use bold face letters such as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$ to represent vectors.

In 3D (xyz) space, we start with three basic vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, all with magnitude 1, as shown in Figure 5.2.

A general 3D vector $\mathbf{v}$ has the following *component form* $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where a, b and c are the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of $\mathbf{v}$ respectively. For example, -7 is the $\mathbf{j}$ component and 15 is the $\mathbf{k}$ component of $2\mathbf{i} - 7\mathbf{j} + 15\mathbf{k}$.

In 2D (xy) space, we have only the two basic vectors $\mathbf{i}$ and $\mathbf{j}$, and a general 2D vector $\mathbf{v}$ has the component form $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

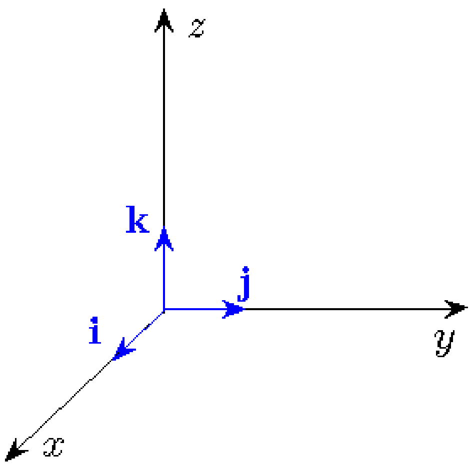


Figure 5.2: The vectors **i**, **j**, **k**

5.2 Vector Operations

5.2.1 Multiplication by Scalars

For a scalar λ (a real number in this course) and a vector $\mathbf{a}$, $\lambda\mathbf{a}$ is a vector of length $|\lambda|$ times the length of $\mathbf{a}$ and points in the same direction if $\lambda > 0$ and the opposite direction if $\lambda < 0$.

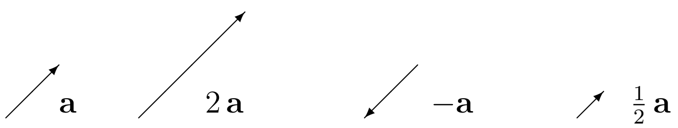


Figure 5.3: Multiplication by scalar

By formula, if $\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then for any scalar λ,

$$\boxed{\lambda\mathbf{a} = (\lambda a)\mathbf{i} + (\lambda b)\mathbf{j} + (\lambda c)\mathbf{k}.}$$

Example 5.2.1

1. $\lambda = 3,\ \mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$

$$\begin{aligned}\lambda\mathbf{a} &= 3(2\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}) \\ &= (3\times 2)\mathbf{i} + (3\times(-1))\,\mathbf{j} + (3\times 3)\,\mathbf{k} = 6\mathbf{i} - 3\,\mathbf{j} + 9\,\mathbf{k}\end{aligned}$$

2. $\lambda = -5,\ \mathbf{a} = 3\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}$

$$\begin{aligned} \lambda\mathbf{a} &= (-5)(3\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}) \\ &= (-5\times 3)\mathbf{i} + (-5\times 2)\,\mathbf{j} + (-5\times(-4))\,\mathbf{k} = -15\mathbf{i} - 10\,\mathbf{j} + 20\,\mathbf{k} \end{aligned}$$

5.2.2 Vector Addition and Subtraction

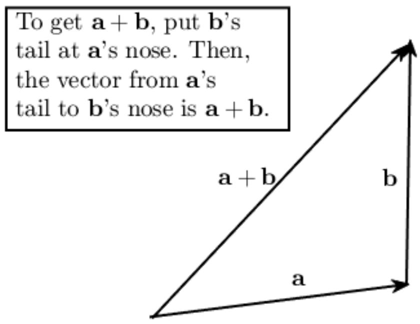


Figure 5.4: Vector addition

Geometrically, as shown by Figure 5.4, two vectors are added by bringing the starting point of one to the end point of the other – the sum is the third side of the triangle. Equation $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ is used to subtract vectors.

By formula, if $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, then

$$\boxed{\mathbf{a} + \mathbf{b} = (a_1 + b_1)\,\mathbf{i} + (a_2 + b_2)\,\mathbf{j} + (a_3 + b_3)\,\mathbf{k}.}$$

Similarly,

$$\boxed{\mathbf{a} - \mathbf{b} = (a_1 - b_1)\,\mathbf{i} + (a_2 - b_2)\,\mathbf{j} + (a_3 - b_3)\,\mathbf{k}.}$$

Example 5.2.2 *For* $\mathbf{a} = \mathbf{i} - 2\,\mathbf{j} - 5\,\mathbf{k}$ *and* $\mathbf{b} = -\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$*, find* $2\,\mathbf{a},\ \mathbf{a} + \mathbf{b},\ 3$ *and* $\mathbf{a} - \mathbf{b}$.

Solution.

$$\begin{aligned} 2\,\mathbf{a} &= 2(\mathbf{i} - 2\,\mathbf{j} - 5\,\mathbf{k}) = 2\mathbf{i} - 4\,\mathbf{j} - 10\,\mathbf{k} \\ \mathbf{a} + \mathbf{b} &= (\mathbf{i} - 2\,\mathbf{j} - 5\,\mathbf{k}) + (-\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}) = \mathbf{j} - 3\,\mathbf{k} \\ 3\,\mathbf{a} - \mathbf{b} &= 4\mathbf{i} - 9\,\mathbf{j} - 17\,\mathbf{k} \end{aligned}$$

5.2.3 Position Vector

If $P(a, b, c)$ is some point then the vector from the origin O to P, denoted by $\overrightarrow{OP}$, is the *position vector* of P (Figure 5.5).

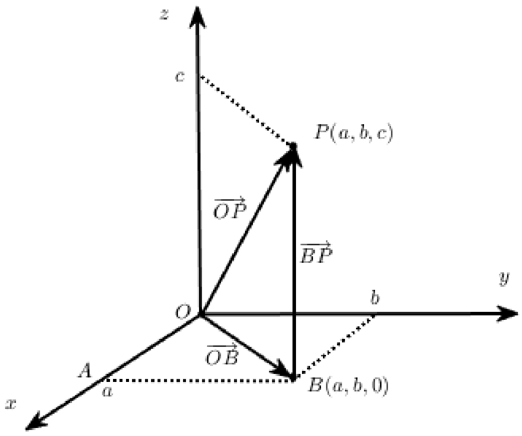


Figure 5.5: Components of a position vector

As shown in Figure 5.5, $\overrightarrow{OB} = a\mathbf{i} + b\mathbf{j}$ and $\overrightarrow{BP} = c\mathbf{k}$. Thus,

$$\begin{aligned} \overrightarrow{OP} &= \overrightarrow{OB} + \overrightarrow{BP} \\ &= a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}. \end{aligned}$$

In a vector $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, a, b and c are called the components of $\mathbf{v}$.

5.2.4 Magnitude

The *magnitude (length)* of $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, denoted $|\mathbf{v}|$, is

$$\boxed{|\mathbf{v}| = |a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}| = \sqrt{a^2 + b^2 + c^2}.}$$

This can be shown as follows. In Figure 5.5, $\mathbf{v} = \overrightarrow{OP}$. We can see from the figure that $|\overrightarrow{OA}| = |a|$, $|\overrightarrow{AB}| = |b|$ and $|\overrightarrow{BP}| = |c|$, so using the Pythagoras Theorem on triangle OAB gives

$$|\overrightarrow{OB}| = \sqrt{|a|^2 + |b|^2} = \sqrt{a^2 + b^2}$$

and from triangle OBP

$$|\mathbf{v}| = |\overrightarrow{OP}| = \sqrt{\left(\sqrt{a^2 + b^2}\right)^2 + |c|^2} = \sqrt{a^2 + ab^2 + c^2}$$

Magnitude has the following property. For vector $\mathbf{v}$ and scalar λ,

$$\boxed{|\lambda\mathbf{v}| = \sqrt{(\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2}}$$

In fact,

$$\begin{aligned} |\lambda\mathbf{v}| &= |(\lambda a)\mathbf{i} + (\lambda b)\mathbf{j} + (\lambda c)\mathbf{j}| = \sqrt{(\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2)} \\ &= \sqrt{\lambda^2}\sqrt{a^2+b^2+c^3} = |\lambda||\mathbf{v}| \quad \text{as stated.} \end{aligned}$$

We note that, in general,

$$\boxed{|\mathbf{a}-\mathbf{b}| \neq |\mathbf{a}| - |\mathbf{b}|.}$$

Example 5.2.3

1. $|3\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}| = \sqrt{3^2+2^2+(-4)^2} = \sqrt{9+4+16} = \sqrt{29}$
2. $|\frac{1}{\sqrt{2}}(2\mathbf{i} - \,\mathbf{j} + 3\,\mathbf{k})| = \frac{1}{\sqrt{2}}\sqrt{2^2+(-1)^2+3^2} = \sqrt{\frac{14}{2}} = \sqrt{7}$

5.2.5 Vector between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are two points, what is $\overrightarrow{AB}$, the vector from A to B?

From the tail nose rule, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, we obtain

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \quad \text{('last' minus 'first')} \\ &= (b_1-a_1)\,\mathbf{i} + (b_2-a_2)\,\mathbf{j} + (b_3-a_3)\,\mathbf{k} \end{aligned}$$

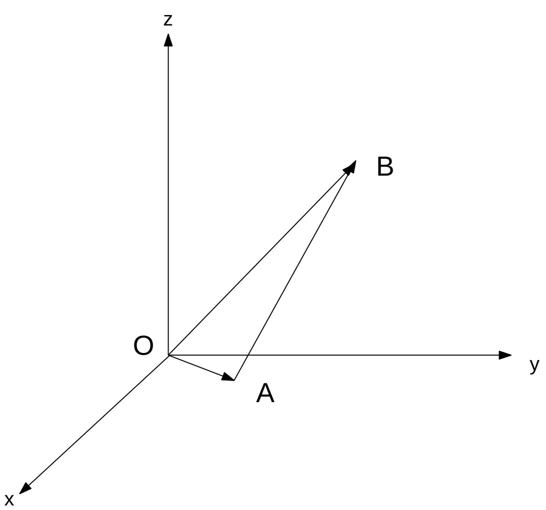


Figure 5.6: Vector Projection

Example 5.2.4 *Find the vector from* $A(1,2,-3)$ *to* $B(3,5,-6)$.

Solution. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\mathbf{i}+5\mathbf{j}-6\mathbf{k}) - (\mathbf{i}+2\mathbf{j}-3\mathbf{k}) = 2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$

5.2.6 Unit Vectors

A *unit vector* is a vector of magnitude 1.

The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors.

In general, if $\mathbf{a}$ is any vector other than $\mathbf{0}$, then

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \quad \text{is a unit vector in the direction of } \mathbf{a}.$$

Example 5.2.5 *Find the unit vector in the direction of* $\overrightarrow{AB} = 2\,\mathbf{i} + 3\,\mathbf{j} - 3\,\mathbf{k}$.

Solution. $\widehat{\overrightarrow{AB}} = \overrightarrow{AB}/\sqrt{4+9+9} = (2/\sqrt{22})\mathbf{i} + (3/\sqrt{22})\mathbf{j} - (3/\sqrt{22})\mathbf{k}$

5.2.7 Parallel Vectors

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if there is a scalar α such that

$$\mathbf{a} = \alpha\mathbf{b}.$$

5.2.8 Exercise

1. For $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$ and $\mathbf{b} = -5\mathbf{i} - 7\mathbf{j} + \mathbf{k}$, find
   (a) $4\mathbf{a} + 2\mathbf{b}$
   (b) $|\mathbf{a}|$
   (c) $\hat{\mathbf{a}}$ (i.e. the unit vector of $\mathbf{a}$)
2. Given points $P(1, -2, -5)$ and $Q(3, 5, -2)$, find $\overrightarrow{OP}$ $\quad \overrightarrow{OQ}$ and $|\overrightarrow{PQ}|$.
3. Check if the give pairs of vectors are parallel.
   (a) $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -10\mathbf{i} + 15\mathbf{j} - 5\mathbf{k}$
   (b) $\mathbf{a} = 1\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 6\mathbf{j} + 21\mathbf{k}$

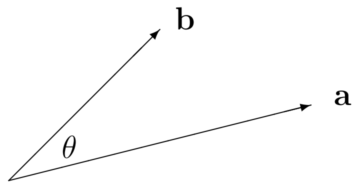
5.3 Scalar or Dot Product

5.3.1 Scalar or Dot Product

The *scalar product*, also called *dot product*, of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.}$$

where θ is the angle between the vectors.



If $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, it can be shown that

$$\boxed{\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.}$$

5.3.2 Exercise

Find $\mathbf{a}\cdot\mathbf{b}$ if $\mathbf{a} = 2\,\mathbf{i} - 3\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{b} = 6\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$.

5.3.3 Orthogonal Vectors

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal* (perpendicular) if and only if

$$\mathbf{a}\cdot\mathbf{b} = 0.$$

This follows from

$$\mathbf{a}\cdot\mathbf{b} = 0 \iff |\mathbf{a}||\mathbf{b}|\cos\theta = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$

5.3.4 Exercises

1. Are the vectors $\mathbf{a} = 3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{b} = 3\,\mathbf{i} + 5\,\mathbf{j} - 2\,\mathbf{k}$ orthogonal?
2. Find the value of λ such that $\mathbf{a} = -\mathbf{i} + 6\,\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \lambda\,\mathbf{i} + 2\,\mathbf{j} - 7\,\mathbf{k}$ are orthogonal.

5.3.5 Properties of Dot Product

1. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
2. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
3. $\mathbf{a}\cdot\mathbf{0} = 0$
4. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
5. $(\lambda\mathbf{a})\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(\lambda\mathbf{b})$ where λ is a scalar.

As an exercise, we shall prove Property No. 4.

Proof

$$
\begin{aligned}
\text{Let} \quad \mathbf{a} &= a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} \\
\mathbf{b} &= b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k} \\
\text{and} \quad \mathbf{c} &= c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}. \\
\mathbf{b} + \mathbf{c} &= (b_1 + c_1)\,\mathbf{i} + (b_2 + c_2)\,\mathbf{j} + (b_3 + c_3)\,\mathbf{k}. \\
\text{Therefore} \quad \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\
&= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
&= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}
\end{aligned}
$$

5.3.6 Angle between Vectors

From $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ one obtains $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ and hence

$$\boxed{\theta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.}$$

5.3.7 Exercise

Find the angle between the vectors $\mathbf{a} = 2\,\mathbf{i} + 3\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$.

5.3.8 Scalar Projection

The *scalar projection* of $\mathbf{a}$ in the direction of $\mathbf{b}$ is simply the (scalar) component of $\mathbf{a}$ in the direction of $\mathbf{b}$.

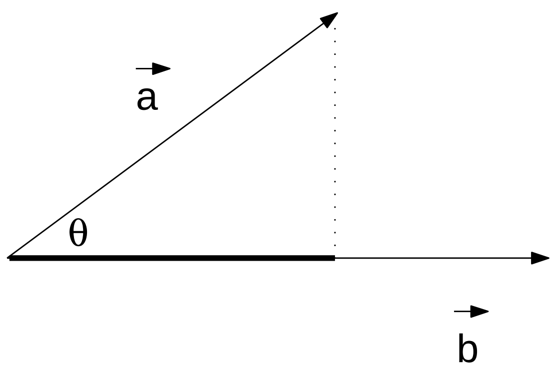


Figure 5.7: Scalar Projection

$$\text{Scalar projection of } \mathbf{a} = |\mathbf{a}| \cos\theta \qquad = |\mathbf{a}| \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\boxed{\text{Scalar projection of } \mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}}$$

Note: If $\theta > 90\circ$, the scalar projection will be negative.

5.3.9 Vector Projection

The *vector projection* of $\mathbf{a}$ in the direction of $\mathbf{b}$ is simply $\hat{\mathbf{b}}$ multiplied by the scalar projection. i.e., $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$. Thus,

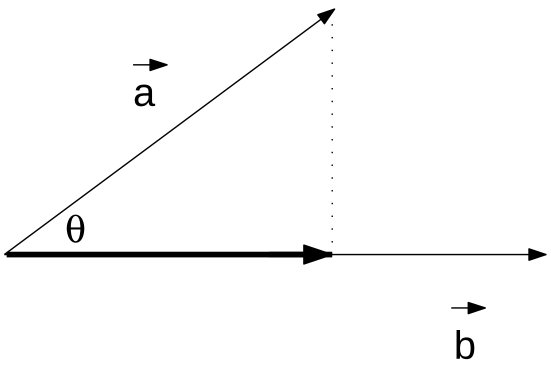


Figure 5.8: Vector Projection

$$\boxed{\text{“vector projection of } \mathbf{a} \text{ in the direction of } \mathbf{b}\text{”} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}.}$$

5.3.10 Exercise

Find the scalar and vector projection of $5\,\mathbf{i} - 7\,\mathbf{j} - 6\,\mathbf{k}$ in the direction of $2\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k}$.

5.3.11 *Resolution into Orthogonal Vectors (Optional topic)

The process of representing a given vector into the sum of two vectors in two given directions is called *vector resolution*.

The *orthogonal resolution* of vector $\mathbf{a}$ into $\mathbf{b}$ and $\mathbf{b}$'s orthogonal direction is to find $\mathbf{x}$ and $\mathbf{y}$ such that

$$\mathbf{a} = \mathbf{x} + \mathbf{y}$$

where $\mathbf{x}$ is parallel to $\mathbf{b}$ and $\mathbf{y}$ is perpendicular (orthogonal) to $\mathbf{b}$.

Now $\mathbf{x}$ is the vector projection of $\mathbf{a}$ in the direction of $\mathbf{b}$, that is,

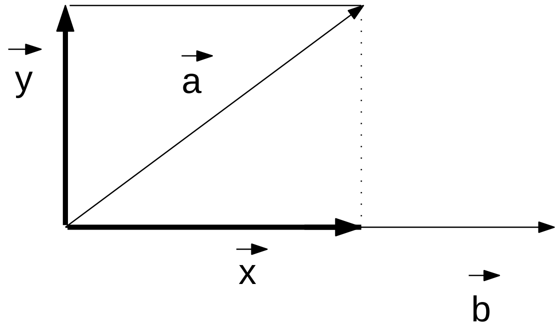


Figure 5.9: Vector Projection

$$\boxed{\begin{aligned} \mathbf{x} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} \\ \mathbf{y} &= \mathbf{a}-\mathbf{x}. \end{aligned}}$$

5.3.12 *Exercise (Optional)

Resolve the vector $7\,\mathbf{i} - 5\,\mathbf{j} + \mathbf{k}$ into orthogonal vectors, one of which is parallel to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

5.4 *Vector Product or Cross Product (Optional)

The *vector product* or *cross product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\boxed{\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}.}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a unit vector perpendicular (orthogonal) to both $\mathbf{a}$ and $\mathbf{b}$ that follows the Right-Hand Rule: stretch one's right-hand fingers in $\mathbf{a}$'s direction and curl them toward $\mathbf{b}$ through angle θ, then $\hat{\mathbf{n}}$ in the thumb direction.

Note: $\mathbf{a}\times\mathbf{b}$ is perpendicular to both the vector $\mathbf{a}$ and the vector $\mathbf{b}$.

It can be shown that if $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, then

$$\begin{aligned} \mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \\ &= (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k} \end{aligned}$$

Note that the determinant multiplying $\mathbf{i}$ is obtained by crossing out the row and the column through $\mathbf{i}$, etc. These 2×2 determinants are evaluated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

5.4.1 *Exercise (Optional)

1. If $\mathbf{a} = 2\,\mathbf{i} - 5\,\mathbf{j} + 7\,\mathbf{k}$ and $\mathbf{b} = 3\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.
2. If $\mathbf{a} = -\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\,\mathbf{i} + 2\mathbf{j} - 6\,\mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$ and verify that it is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

5.4.2 *Properties of Cross Product (Optional)

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $\mathbf{a} \times \mathbf{a} = \mathbf{0} \quad (\sin 0 = 0)$
3. $\mathbf{a} \times \mathbf{0} = \mathbf{0}$
4. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
   $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$ (order is important)
5. $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b})$ where λ is a scalar