MATH2267 Sample Examination, Semester 2, 2018

Q1. [$10 \times 3 = 30$ **marks**]

Complete ANY TEN of the following questions. If more than ten questions are answered, only the first ten that appear in your answer book will be marked. (No working is required.)

(1) List all elements of the set $\{x \mid |x-2|^3 < 8,\ x \text{ is integer}\}$.

Solution: $\{1, 2, 3\}$

(2) List all elements in $\{x \mid |x-1| \le 1\} \cap \{x \mid |x+1| \ge 2\}$.

Solution: $1 \le x \le 2$ (or $x \in [1, 2]$)

(3) Solve the inequality $7x - 5 < 3x + 7$.

Solution: $7x - 3x < 7 + 5 \quad \Rightarrow \quad 4x < 12 \quad \Rightarrow \quad x < 3$

(4) Solve to the inequality $\frac{x-7}{x+4} < 0$.

Solution: Inequality holds if and only if $x-7$ and $x+4$ have different signs, which means either

$$x - 7 > 0 \text{ and } x + 4 < 0 \quad \rightarrow \quad \text{no solution}$$

or

$$x - 7 < 0 \text{ and } x + 4 > 0 \quad \Rightarrow \quad -4 < x < 7$$

thus, solution: $x \in (-4, 7)$

(5) Find the sum of the first 20 terms in the arithmetic progression $5, 7, 9, \cdots$.

Solution: sum$= 20 \times (5 + 5 + 19 \times 2)/2 = 480$

(6) Find the sum of the first 13 terms in the geometric progression $3, 6, 12, \cdots$.

Solution: sum$= \frac{3\times(2^{13}-1)}{2-1} = 24573$

(7) Find the inverse of $\begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{rr|rr} 0 & -2 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rr|rr} 5 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{rr|rr} 1 & 0 & 0 & 1/5 \\ 0 & 1 & -1/2 & 0 \end{array}\right]$$

$$\Rightarrow \quad \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1/5 \\ -1/2 & 0 \end{bmatrix}$$

(7) Is $\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ the inverse of $\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$?

Solution: YES

(9) Let $f(x)$ and $g(x)$ have domains $[0, 3)$ and $[2, 5]$, respectively. If $g(x) \neq 0$ for all $x \in [2, 5]$, what is the domain of $f(x)/g(x)$.

Solution: $[2, 3)$, which is the intersection of the both domains

(10) Find the domain of $f(x) = \dfrac{x^6 - 1}{x^2 - 7}$.

Solution: Function is well defined everywhere except for the points where $x^2 - 7 = 0$. Thus, the domain is $\mathbf{R} \setminus \{-\sqrt{7}, \sqrt{7}\}$.

(11) Find the domain of $f(x, y) = \frac{x-3}{y-3}$.

Solution: Function not defined at points $(x, 3)$. Thus, the domain is $\{(x, y) \mid x \in \mathbf{R}, y \in \mathbf{R} \setminus \{3\}\}$

(12) Find the range of $f(x) = \dfrac{x^2 - 6x + 9}{|x-3|}$.

Solution: $f(x) = \dfrac{x^2 - 6x + 9}{|x-3|} = \dfrac{|x-3|^2}{|x-3|} = |x-3|$ when $x - 3 \neq 0$. We see that the range is $(0, \infty)$.

(13) Find the roots of $x^2 + 7x + 12$.

Solution: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-7 \pm \sqrt{49-48}}{2} = \dfrac{-7 \pm 1}{2}$. Thus, two solutions: $-3,\ -4$

(14) For a 3×3 matrix, is the operation "replacing Row 3 by Row 2" an elementary row operation?

Solution: NO

(15) For a 3×3 matrix, is the operation "multiplying Row 3 by 0" an elementary row operation?

Solution: NO

(16) Is matrix $\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$ in reduced row echelon form?

Solution: NO (leading non-zero element must be 1)

(17) Which of the following definite integrals represents the area (in the 1_{st} quadrant) under the curve $y = \log(x+1),\ 0 \le x \le 1$?

(i) $\int_0^1 \log(x)\, dx$, (ii) $\int_0^1 \log(y+1)\, dy$

✓(ii). $\int_0^1 \log(y+1)\, dy$ is the area.

(18) What are the solutions to the nonlinear system $x - y^3 = 0, \quad x - 9y = 0$?

Solution: Sub $x = y^3$ into $x - 9y = 0$:

$$y^3 - 9y = y(y^2 - 9) = y(y-3)(y+3) = 0 \ \Rightarrow\ y = 0,\ y = 3,\ y = -3$$

solutions: $(0,0)$, $(27,3)$, $(-27,-3)$

(19) Find $\frac{\partial f}{\partial x}$ for $f(x,y) = \sqrt{1 + x^2 + \sin(y)}$.

Solution: $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{1+x^2+\sin(y)}}$

(20) Find $\frac{\partial f}{\partial y}$ for $f(x,y) = \sqrt{1 + x^2 + \sin(y)}$.

Solution: $\frac{\partial f}{\partial y} = \frac{\cos(y)}{2\sqrt{1+x^2+\sin(y)}}$

(21) The unit vector in the direction of $\boldsymbol{u} = (-2, 5, -4)$.

Answer: $\frac{1}{3\sqrt{5}}(-2,\ 5,\ -4)$

(22) The angle between $\boldsymbol{u} = (-2, 5, -4)$ and $\boldsymbol{v} = (3, -2, 6)$.

Answer: 2.5902 (radians) (or 148.41°)

For the remaining questions (Q2 ∼ Q6), full marks may not be awarded unless sufficient working is shown.

Q2. [$6 + (2 + 7 + 3) = 18$ **marks**] Solve the following questions.

(a) Solve the nonlinear system $x - y^3 = 0, \quad x - 9y = 0$.

Solution: Sub $x = y^3$ into $x - 9y = 0$:

$$y^3 - 9y = y(y^2 - 9) = y(y-3)(y+3) = 0 \ \Rightarrow\ y = 0,\ y = 3,\ y = -3$$

solutions: $(0,0)$, $(27,3)$, $(-27,-3)$

(b) Consider the following linear system:

$$\begin{array}{rcrcrcr} x & - & 3y & - & z & = & 2 \\ x & + & 2y & + & z & = & -5 \\ 3x & - & y & + & z & = & 6 \end{array}$$

i. Write down the augmented matrix of the linear system.

ii. Reduce the augmented matrix to echelon form.

iii. Solve the system by 'back substitution'.

Solution:

i.

$$\begin{bmatrix} 1 & -3 & -1 & 2 \\ 1 & 2 & 1 & -5 \\ 3 & -1 & 1 & 6 \end{bmatrix}$$

ii.

$$\begin{bmatrix} 1 & -3 & -1 & 2 \\ 1 & 2 & 1 & -5 \\ 3 & -1 & 1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 5 & 2 & -7 \\ 0 & -7 & -2 & 21 \end{bmatrix} \qquad \begin{matrix} \\ -R_1 + R_2 \\ -3R_2 + R_3 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 5 & 2 & -7 \\ 0 & -2 & 0 & 14 \end{bmatrix} \qquad \begin{matrix} \\ \\ R_2 + R_3 \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & 21 \\ 0 & -2 & 0 & 14 \end{bmatrix} \qquad \begin{matrix} \\ 2R_3 + R_2 \\ \\ \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -3 & -1 & 2 \\ 0 & 1 & 2 & 21 \\ 0 & 0 & 4 & 56 \end{bmatrix} \qquad \begin{matrix} \\ \\ 2R_2 + R_3 \end{matrix}$$

iii.

$$\begin{aligned} &4z = 56, \\ &y + 2z = 21, \\ &x - 3y - z = 2. \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} &z = 14, \\ &y + 28 = 21, \\ &x - 3y - 14 = 2. \end{aligned}$$

The solution is

$$\begin{aligned} x &= -5 \\ y &= -7 \\ z &= 14 \end{aligned}$$

Q3. $[4 \times 3 = 12$ **marks**]

Find each of the following limits.

(a) $\displaystyle\lim_{t\to\infty} \frac{t^2+4t-2}{t^3-4t^2}$

Solution: The limit

$$= \lim_{t\to\infty} \frac{1+4/t-2/t^2}{t-4} = \frac{1}{\infty} = 0$$

(b) $\displaystyle\lim_{x\to 2} \frac{x^3-4x^2+4x}{x-2}$

Solution: The limit

$$= \lim_{x\to 2} \frac{x(x-2)^2}{x-2} = \lim_{x\to 2} \frac{x(x-2)}{1} = 0$$

(c) $\displaystyle\lim_{x\to 1} \frac{x^5+5x^4+3x}{2x^4-3x^3}$

Solution: The limit

$$= \lim_{x\to 1} \frac{x^5+5x^4+3x}{2x^4-3x^3} = \frac{1+5+3}{2-3} = -9$$

(d) $\displaystyle\lim_{t\to\infty} \frac{t^7+3t^5-8}{t+100t^6}$

Solution: The limit

$$= \lim_{t\to\infty} \frac{t+3/t-8/t^6}{1/t^5+100} = \frac{\infty}{100} = \infty$$

Q4. $[2+2+3+3 = 10$ **marks**]

Find the indicated derivative for each of the following functions.

(a) For $f(x) = \sin(2x^2+1) - 6x^{3/2} - 3x$, find $f'(x)$.

Solution: $f'(x) = \cos(2x^2+1)(2x^2+1)' - 6 \times (3/2)x^{1/2} - 3 = 4x\cos(2x^2+1) - 9x^{1/2} - 3$

(b) For $f(x) = e^x \cos(x+1) - 1$, find $f'(x)$.

Solution: $f'(x) = (e^x)'\cos(x+1) + e^x(\cos(x+1))' - (1)' = e^x\cos(x+1) - e^x\sin(x+1)$

(c) Find $\dfrac{\partial f}{\partial x}$ for $f(x,y) = \sqrt{1+x^2+\sin(y)}$.

Solution: $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{1+x^2+\sin(y)}}$

(d) Find $\dfrac{\partial f}{\partial y}$ for $f(x,y) = \sqrt{1+x^2+\sin(y)}$.

Solution: $\dfrac{\partial f}{\partial y} = \dfrac{\cos(y)}{2\sqrt{1+x^2+\sin(y)}}$

Q5. $[4+4 = 8$ **marks**]

Let $f(x) = 3x^4 - 24x^2 - 5$.

(i) Find the critical points of $f(x)$;

(ii) Classify each critical point found in (i).

Solution:

(i) $f'(x) = 12x^3 - 48x = 12x(x^2-4) = 0 \quad \Rightarrow \quad$ critical points: $-2,\ 0,\ 2$

(ii) $f''(x) = 36x^2 - 48$

$f''(0) = -48 < 0 \quad \Rightarrow \quad x = 0$ is a maximum point (max value: -5);

$f''(-2) = 144 - 48 = 96 > 0 \quad \Rightarrow \quad x = -2$ is a minimum point (min value: -53);

$f''(2) = 144 - 48 = 96 > 0 \quad \Rightarrow \quad x = 2$ is a minimum point (min value: -53).

Q6. $[4 + 4 + 6 = 14$ **marks**]

Consider

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

(a) Find $\det(A)$.

(b) Is $\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$ an eigenvector of A? If yes, what is the corresponding eigenvalue?

(c) Find the eigenvalues of B.

Solution:

(a) $\det(A) = 4 \begin{vmatrix} -2 & 0 \\ -1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = 4(2) - 2(-2) + 2(0) = 12.$

(b) $\begin{bmatrix} 4 & 2 & 2 \\ 2 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ -9 \end{bmatrix} = -3 \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}.$

Thus, $\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$ is an eigenvector of A and the corresponding eigenvalue is -3.

(c) $\det(\lambda I - B) = \begin{vmatrix} \lambda - 5 & 1 \\ 1 & \lambda - 5 \end{vmatrix} = (\lambda - 5)^2 - 1 = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6).$

Solving $\lambda^2 - 10\lambda + 24 = 0$ we obtain eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 6$.

Q7. [**8 marks**]

Perform 3 iterations of Newton's method to find an approximate solution to the equation $x^2 - \log(x) - 15 = 0$, starting from $x_0 = 5$.

Solution:

$f(x) = x^2 - \log(x) - 15,\ f'(x) = 2x - 1/x,\ x_0 = 5$

$$x_1 = x_0 - \frac{f(x)}{f'(x)} = 5 - \frac{5^2 - \log(5) - 15}{2(5) - 1/5} \approx 4.143820195$$

$$x_2 = x_1 - \frac{x_1^2 - \log(x_1) - 15}{2x_1 - 1/x_1} \approx 4.0506561204$$

$$x_3 = x_2 - \frac{x_2^2 - \log(x_2) - 15}{2x_2 - 1/x_2} \approx 4.04951840285$$