MATH2267 Week 10

Ordinary Differential Equations - Modelling and Solving

Semester 2, 2018

Overview

Part 1:
Ordinary Differential Equations (ODE)
- Introduction
- Julia Codes
- Examples and Applications

Part 2:
Systems of Ordinary Differential Equations
- Introduction
- Julia Codes
- Examples and Applications

Examples are solved using Julia

Part 1

Ordinary Differential Equations

Part 1. Ordinary Differential Equations

Introduction

The following is an ODE

$$\frac{dx}{dt} = f(x, t)$$

where $x(t)$ is an unknown function. Solving the ODE is to find $x(t)$ satisfying the ODE.

Terminology and notation

- Dot notation for the time derivative: $\dot{x}(t) \equiv x'(t)$
- Autonomous ODE: $\dot{x} = f(x)$ ($f(x)$ instead of $f(x, t)$)
- Initial Value problem (IVP): Adding to ODE initial value

$$\begin{aligned} \dot{x} &= f(x, t) \\ x(0) &= x_0 \end{aligned}$$

- ODE as opposed to PDE (partial differential equation)

A Simple ODE: $\dot{v}(t) = r$, where r: constant; $v(t)$: unknown function.

Integrating:

$$v(t) = \int r\, dt = rt + C \quad (-\infty < C < \infty) \quad \text{(general solution)}$$

For solution to be unique, add initial condition:

$$\dot{v} = r$$
$$v(0) = 1$$

Apply $v(0) = 1$ to $v(t) = rt + C \Rightarrow$ unique solution to IVP:

$$v(t) = rt + 1$$

Part 1. Ordinary Differential Equations

Solving with Julia

Initial value problem is easy to solve in Julia.

Using Julia - we need the ODE and PyPlot packages.

You are familiar with package “PyPlot”

For Version 6.0 of Julia, just:

```
Pkg.add("ODE")
using ODE
```

We need two ODE solvers in ODE package:

“ODE.ode45” for IVP involving a single ODE

“ODE.ode23” for IVP involving a system of ODE’s

Part 1. Ordinary Differential Equations

Solving with Julia

Example Julia code for solving IVP of type

$$\dot{v} = f(v, t)$$
$$v(0) = v_0$$

Example: $\dot{v}(t) = 3, \quad v(0) = 0.0$

```
function f(t,v)
    3.0
end
tspan=0.0:5.0 # specify the times range on
              # which solution is required
v0=0.0        # specify the initial value
              # of v (ie empty bathtub)
T,v = ODE.ode45(f,v0,tspan)
              # calling the solver "ODE.ode45"
```

Part 1. Ordinary Differential Equations

Examples

Example 1. Exponential Growth

Population growth rate is proportional to the population x

$$\dot{x} = r \cdot x$$

Many applications

- capital investment with a fixed interest rate etc

Weaknesses facing Environmental influences

- Recession
- Famine
- Blackouts

Applications

- World population (Exercise 10.2.7)

Example 2. Density-dependent growth

r: maximum specific growth rate, $x(t)$: population
A general population model:

$$\dot{x} = r \cdot x \cdot f(x, t)$$

Density-dependent growth:

$$\dot{x} = r \cdot x \cdot f(\rho)$$

where population density $\rho = x/k$, carrying capacity k.
$f(\rho) = 1 - \rho/k$ gives logistic equation: $\dot{x} = r \cdot x \cdot (1 - x/k)$.

- Toilet cistern, Tablet computer (Exercise 10.2.7)

Part 2

Systems of ODE's

Part 2. Systems ODE's

Introduction

The following is a system ODE's

$$\begin{aligned}\dot{x}_1(t) &= f_1(x_1, x_2, t)\\ \dot{x}_2(t) &= f_2(x_1, x_2, t)\end{aligned}$$

Writing

$$\mathbf{x}(t) = \left[\begin{array}{c} x_1(t)\\ x_2(t)\end{array}\right], \quad \mathbf{f}(\mathbf{x}, t) = \left[\begin{array}{c} f_1(\mathbf{x}, t)\\ f_2(\mathbf{x}, t)\end{array}\right]$$

where $f_i(\mathbf{x}, t) = f_i(x_1, x_2, t)$, the above system becomes compact form:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t)$$

Adding to system initial values obtain IVP:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, t)\\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned}$$

Part 2. Systems of ODE's

Solving with Julia

Example:

$$\dot{x}_1(t) = x_1 + x_2$$
$$\dot{x}_2(t) = x_1 - x_2$$
$$x_1(0) = 0.0$$
$$x_2(0) = 1$$

```
function f(x,t)
    dotx=similar(x)
    dotx[1]=x[1]+x[2]
    dotx[2]=x[1]-x[2]
end
tspan=0.0:10.0
x0=[0.0; 1.0]
tval,xval = ODE.ode23(f,x0,tspan)
```

Example 1. Predator-prey system

$$\begin{aligned} \textit{Prey} \quad \dot{y}_1 &= r_1 y_1 - b y_2 y_1 \\ \textit{Predator} \quad \dot{y}_2 &= c y_2 y_1 - d y_2 \end{aligned}$$

Exponential growth term

$$\begin{aligned} \dot{y}_1 &= r_1 y_1 - b y_2 y_1 \\ \dot{y}_2 &= c y_2 y_1 - d y_2 \end{aligned}$$

Predation term

$$\begin{aligned} \dot{y}_1 &= r_1 y_1 - b y_2 y_1 \\ \dot{y}_2 &= c y_2 y_1 - d y_2 \end{aligned}$$

Part 2. Systems of ODE's

Examples

After solution, plotting y_2 against y_1 obtain phase portrait in phase plane:

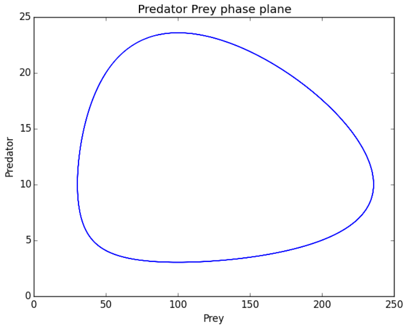

Figure: Predator-prey phase plane