MATH2267 Week 2

Factorisation, Sequences and Series

June 15, 2018

Overview

- **Part 1. Polynomial Factorisation**

  Second order polynomials: by hand

  Higher order polynomials: help from Julia
- **Part 2. Sequences and Series**

  Sequences

  Arithmetic progression

  Arithmetic series

  Geometric progression

  Geometric series
- **Part 3. Applications**

  Present and future values

  Compound interests

Part 1

Factorisation

Part 1. Factorisation

1.1 Introduction

- A factor of a polynomial $P(x)$ is any polynomial $Q(x)$ of order ≥ 1 that divides $P(x)$: there exists polynomial $R(x)$ such that $P(x) = Q(x)R(x)$.
- Polynomial factorisation is the decomposition of polynomials into a product of irreducible factors – factors that do not have factors other than themselves.
- For example, $x + 2$ is a factor of the polynomial $x^2 + 2x$ and its factorisation is $x^2 + 2x = (x + 2)x$.
  - **Example.** Factorisation of $x^2 - 4$ is $(x + 2)(x - 2)$.
  - **Example.** Factorisation of $x^4 - 1$ is $(x + 1)(x - 1)(x^2 + 1)$ as all $(x + 1)$, $(x - 1)$ and $(x^2 + 1)$ are irreducible.

Part 1. Factorisation

1.2 Real Roots of Polynomials

Consider a polynomial $P(x)$.

- A real root of $P(x)$ is a real number, say a, such that $P(a) = 0$.
- $x = a$ is a root of P $\iff$ $(x - a)$ is a factor of P
- In many cases, polynomial factorisation can be done by finding the roots.

Part 1. Factorisation

1.3 Factoring quadratic functions

Quadratic functions are 2nd order polynomials.

- Recall: roots of quadratic function $P(x) = ax^2 + bx + c$ are:

$$
\begin{aligned}
x_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\
x_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

  where $(b^2 - 4ac)$ is known as the discriminant.
- If $(b^2 - 4ac) \geq 0$, x_1 and x_2 are real roots and $P(x)$ has factorisation:

$$P(x) = a(x - x_1)(x - x_2)$$

- If $(b^2 - 4ac) < 0$, there is no real root and $P(x)$ is irreducible.

Example. Factorise $x^2 + 5x + 6$.
Solution.
$x^2 + 5x + 6 = 0$ has roots

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-5 + \sqrt{25 - 24}}{2} = -2,$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-5 - \sqrt{25 - 24}}{2} = -3.$$

Thus, $P(x) = x^2 + 5x + 6 = (x + 2)(x + 3)$.

NOTE: $P_1(x) = -x^2 - 5x - 6 = -(x + 2)(x + 3)$ though
$P_1(x) = 0$ has the same roots

- Quadratic function has no real root $\Longrightarrow$ cannot be factorised

Example. Factorise $x^2 + x + 6$.

Solution. The discriminant of $x^2 + x + 6 = 0$ is

$$b^2 - 4ac = 1^2 - 4 \times 1 \times 6 = -23 < 0.$$

$\Rightarrow x^2 + x + 6 = 0$ has no real solution.
$\Rightarrow x^2 + x + 6$ cannot be factorised.

Part 1. Factorisation

1.4 Factorising higher order polynomials

Julia can handle polynomials of higher orders. The following commands are useful:

```
p1=poly([1,2,3,4])   ##lower case "poly"
                     ##defalt variable is x
         #this input specifies p1 as the
         #polynomial with roots 1,2,3 and 4
ps=poly([1,2,3,4],'s')
                     ##specify variable is s
p2=Poly([1,2,3,4])   ##capitalized "Poly"
         #this input specifies p2 as the
         #polynomial 1+2x+3x^2+4x^3
roots(p2)
         #produces the roots of p2
```

Part 1. Factorisation

1.4 Factorising higher order polynomials

Example. Factorise $P(x) = x^3 - x$.
Solution. Input the following lines in Julia:

```
Pkg.add("Polynomials")
Pkg.build("Polynomials")
Pkg.update()
using Polynomials
roots(Poly([0,-1,0,1]))
```

then press **Shif+Enter** you will get the roots:

```
Out [2]: 3-element Array{Float64,1}:
          -1.0
           1.0
           0.0
```

This indicates that the roots of $P(x) = x^3 - x$ are $-1, \ 1$ and 0.
Thus, $P(x) = x(x-1)(x+1)$.

Example. Factorise $P(x) = x^3 + x^2 - x - 1$.
Solution. Using Julia obtain roots as follows:

```
using Polynomials
roots(Poly([-1,-1,1,1]))
```

then press **Shif+Enter** you will get the roots:

```
Out [2]: 3-element Array{Float64,1}:
           1.0
          -1.0
          -1.0
```

This indicates that the roots of $P(x) = x^3 + x^2 - x - 1$ are $1, \ -1$ and -1. Thus, $P(x) = (x+1)^2(x-1)$

Part 1. Factorisation

1.4 Factorising higher order polynomials

Example. Factorise $P(x) = x^3 - x^2 + 3x - 10$.
Solution. Roots: $-0.5 + 2. * im, \ -0.5 - 2. * im, \ 2.0 + 0.0im$.
One real root 2 $\Rightarrow P(x) = (x-2)R(x)$. To obtain $R(x)$,

$$
\begin{array}{r|rrrrrrr}
\multicolumn{1}{r}{} & x^2 & + & x & + & 5 & & \\
\hline
x-2 & x^3 & - & x^2 & + & 3x & - & 10 \\
 & x^3 & - & 2x^2 & & & & \\
\hline
 & & & x^2 & + & 3x & - & 10 \\
 & & & x^2 & - & 2x & & \\
\hline
 & & & & & 5x & - & 10 \\
 & & & & & 5x & - & 10 \\
\hline
 & & & & & & & 0
\end{array}
$$

Thus, $R(x) = x^2 + x + 5$ and $P(x) = (x-2)(x^2 + x + 5)$.

Part 2

Sequence and Series

Part 2. Sequence and Series

Topics

- Arithmetic Progression (AP)
- Arithmetic Series
- Geometric Progression (GP)
- Geometric Series

Part 2. Sequence and Series

2.1 Arithmetic Progression

A sequence is an ordered set of numbers
For example, $2, 4, 6, \ldots$ is an infinite sequence of even numbers.

Arithmetic sequence or arithmetic progression (AP) is a sequence with common difference between adjacent terms

Sequence $3, 8, 13, 18, 23$ has a common difference 5, and is an AP.

In general, if the first term of an AP is a and the common difference is d then the AP is:

$$a, \quad a+d, \quad a+2d, \quad \ldots, \quad a+(n-1)d$$

The *sum* of terms in a sequence is a series. For $t_1, t_2, \cdots, t_n$, $t_1 + t_2 + \cdots + t_n$ is a series.

Example

* Sarah receives $5000 on her 21st birthday from her grandfather.
* Each following year she gets $1000 more than the previous year.
* How much has Sara received in total when she has turned 30?

Solution: Working in thousands (for 10 years)

$$5 + (5 + 1) + (5 + 2) + \cdots + (5 + 9) = 95$$

Calculation is simple but tedious when many terms are involved.

Consider an AP with n terms:

$$a, \;\; a+d, \;\; \cdots, \;\; a+(n-2)d, \;\; a+(n-1)d$$

The sum is

$$S = [a] + [a+d] + \cdots + [a+(n-2)d] + [a+(n-1)d]$$

Sum in reversed order

$$S = [a+(n-1)d] + [a+(n-2)d] + \cdots + [a+d] + [a]$$

Adding up: $\quad 2S = n \times [2a + (n-1)d]$

Dividing by 2: $\quad S = \dfrac{n}{2}[2a + (n-1)d]$

This is just $n \times \dfrac{\text{1st term} + \text{last term}}{2}$

Part 2. Sequence and Series

2.3 Geometric Progression

- A geometric sequence or geometric progression (GP) is a sequence with common ratio between adjacent terms:

$$\frac{\text{term } i}{\text{term } i-1} = \textit{constant}$$

- Sequence $3, 9, 27, 81, 243$ is a GP. Its common ratio is 3.

- In general, if the first term of an GP is a and the common ratio is r then the GP is:

$$a,\ ar,\ ar^2,\ \ldots,\ ar^{n-1}$$

Part 2. Sequence and Series

2.4 Geometric series

Sum of a GP is a geometric Series

Let S_n denote the sum of the first n terms of a GP. Then:

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Now multiply each term by r

$$rS_n = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

Subtracting

$$S_n - rS_n = a - ar^n$$

We can now solve for S_n to get

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

NOTE: $r \neq 1$; (no need for a formula if $r = 1$).

Part 3

Applications

Example

Dube recently inherited \$15k. He invests this sum in an account that pays 10% interest each year. This amount is added to the sum already invested until the account is closed and all the money paid out. Dube would like to know how much money he would receive if he closed the account after five years.

Solution. Dube will have

After 1 year: $\$(15k + 15 \times 0.1)k = 15(1 + 0.1)^1 k$.

After 2 years: $15(1 + 0.1)^1 \times (1 + 0.1)k = 15(1 + 0.1)^2 k$.

After 5 years: $15(1 + 0.1)^5 k$.

Amounts in Dube's account from current year (in \$1000's):

$$15,\ 15(1 + 0.1),\ 15(1 + 0.1)^2,\ \cdots,\ 15(1 + 0.1)^5$$

continued ...

... continued

The last term $15(1+0.1)^5$ is the amount Dube will collect after 5yrs. This is known as the **Future Value** and the \$15k is called the **Present Value.**

In general, relation between (PV) and (FV) after n years (interest rate i) is:

$$FV = PV(1+i)^n$$

and

$$PV = FV(1+i)^{-n}$$

Example

Suppose Dube deposits an extra \$15 (thousand) each year then $a = 15$ and $r = 1.1$. Thus after five years Dube would have:

$$S_6 = \frac{15(1-1.1^6)}{1-1.1} \textit{ (thousand dolars)} = \$115,734.15$$

In fact, Dube's

- Initial investment earns $15(1+0.1)^5$ thousand dollars.
- Each year's extra investment of \$15 thousand earns: $15(1+0.1)^4$, $15(1+0.1)^3$, $15(1+0.1)^2$, $15(1+0.1)^1$, and $15(1+0.1)^0$ thousand dollars.

All these add up to the above amount of $\$115,734.15$.

Example

How much will an investment of \$500 at4.5% annual interest rate pay out after 15 years?

Solution.

Amount pays out after 15 years:

$$500 \times (1 + 0.045)^{15} = \$967.64$$

Example

A bank is offering an annual interest rate of 6% compounded monthly. What will it pay out for an investment of \$10,000 after 5 years?

Solution.

Monthly interest rate $0.06/12 = 0.005$.

Amount it pays out after 5 years (or 60 months)

$$10000 \times (1 + 0.005)^{60} = \$13488.50$$

Example

Ruby will receive a gift of \$2000 in four years time. What is the PV of the gift given that the current cost of living is increasing at 4% per annum?

Solution.

$$PV \times (1+0.04)^4 = 2000$$

$$\Rightarrow \qquad PV = \frac{2000}{(1.04)^4} = \$1709.61$$

Part 3. Applications

3.2 Compound Interest

Example

Jim is saving up to buy a car. He wants to puts a fixed amount per month into a savings account that pays 6% annual interest compounded monthly. How much is the fixed amount if he is to reach his target of \$15000 in 2 years?

Solution.

Monthly interest rate $0.06/12 = 0.005$.

It takes 24 months to reach \$15000.

Let the monthly fixed amount be $\$x$. Then,

$$x(1 + 1.005 + 1.005^2 + \cdots + 1.005^{24}) = 15000$$

$$x(1 - 1.005^{25})/(1 - 1.005) = 15000$$

$$x = 15000(1 - 1.005)/(1 - 1.005^{25}) = \$589.81$$

Jim needs to put \$589.81 per month into his account.

Next Week:

- System of Linear Equations
- Simultaneous Non-linear Equations of 2 Variables