MATH2267 Week 3

Simultaneous Equations

June 15, 2018

Overview

- **Part 1. System of Linear Equations**
  - Introduction
  - Gaussian Elimination
- **Part 2. Nonlinear Simultaneous Equations**
  - Substitution Method
  - Graphical Method

Part 1a

Linear Systems

A system of linear equations (linear system in brief) involves a number linear equations of some fixed unknowns. For example

$$\begin{array}{rrrcr} x & -3y & +2z & = & 1 \\ -2x & +3y & -4z & = & -8 \\ 3x & -y & +2z & = & 7 \end{array}$$

Here we have 3 linear equations in 3 unknowns in the system.

A set of values of the unknowns, if satisfy all the equations simultaneously, is called a solution (or a set of solutions). Solving a linear system is to find all sets of solutions.

Gaussian elimination method:

very efficient for solving linear systems

can be carried out manually or by computer codes

based on 3 types of elementary operations

Elementary operations:

- Two equations are interchanged
- One equation is multiplied by a nonzero number
- A multiple of one equation is added to another

These operations do not change the solutions of the system.

Main Idea Consider:

$$\begin{array}{rcrcrcr} x & - & y & & & = & -1 \\ & & y & - & z & = & -1 \\ & & & & z & = & 3 \end{array}$$

The system is solved easily:

We can find $z = 3$ from last eqn.

Then, we can find $y = z - 1 = 3 - 1 = 2$ from 2nd eqn.

Then, we can find $x = y - 1 = 2 - 1 = 1$ from 1st eqn.

Idea of Gaussian elimination:

- reduce system to simple form (similar to above)
- solve the system (back substitution)

Example. Solve:

$$\begin{array}{rcrcrcrr} x & - & 5y & + & 2z & = & -5 & (5)\\ x & - & 14y & + & 3z & = & -8 & (6)\\ 4x & - & 18y & + & 4z & = & -10 & (7) \end{array}$$

Solution:

$$\begin{array}{rrcrcrcrr} & x & - & 5y & + & 2z & = & -5 & (8)\\ (6)-(5) & & - & 9y & + & z & = & -3 & (9)\\ (7)-4(5) & & & 2y & - & 4z & = & 10 & (10) \end{array}$$

$$\begin{array}{rrcrcrcrr} & x & - & 5y & + & 2z & = & -5 & (11)\\ & & & 2y & - & 4z & = & 10 & (12)\\ (9)\leftrightarrow(10) & & - & 9y & + & z & = & -3 & (13) \end{array}$$

... continued

$$
\begin{array}{rrrrrrrrrl}
& x & - & 5y & + & 2z & = & -5 & (11) \\
& & & 2y & - & 4z & = & 10 & (12) \\
& & - & 9y & + & z & = & -3 & (13) \\
\\
& x & - & 5y & + & 2z & = & -5 & (14) \\
\frac{1}{2}(12) & & & y & - & 2z & = & 5 & (15) \\
& & - & 9y & + & z & = & -3 & (16) \\
\\
& x & - & 5y & + & 2z & = & -5 & (17) \\
& & & y & - & 2z & = & 5 & (18) \\
(16)+9(15) & & & & - & 17z & = & 42 & (19)
\end{array}
$$

$$
\begin{array}{lll}
(19) \Rightarrow & & z = -42/17 \\
(18) \Rightarrow & -9y - 42/17 = -3 & \Rightarrow y = 1/17 \\
(17) \Rightarrow & x - 5(1/17) - 84/17 = -5 & \Rightarrow x = 4/17
\end{array}
$$

Part 1b

Linear Systems (continued)

In previous example, our operations affected only the constants rather than x, y, z. To be more efficient, we solve without writing unknowns.

To facilitate this, we need operations on rows of numbers:

Consider tow row:

R_1: 2 1 2

R_2: 1 0 1

operation $2R_2$ results in new row: 2 0 2

operation $R_1 - 2R_2$ results in new row: 0 1 0

Rows of n numbers can be operated as above.

Example. Solve

$$\begin{array}{rcrcrcr} x & - & 3y & + & 2z & = & 1 \\ -2x & + & 3y & - & 4z & = & -8 \\ 3x & - & y & + & 2z & = & 7 \end{array}$$

Solution: Consider only the coefficients and the RHS:

x	y	z	rhs
1	−3	2	1
−2	3	−4	−8
3	−1	2	7

... continued

x	y	z	rhs	
1	-3	2	1	
-2	3	-4	-8	
3	-1	2	7	
1	-3	2	1	
0	-3	0	-6	$R_2 + 2R_1$
0	8	-4	4	$R_3 - 3R_1$
1	-3	2	1	
0	1	0	2	$R_2/(-3)$
0	8	-4	4	
1	-3	2	1	
0	1	0	2	
0	0	-4	-12	$R_3 - 8R_2$
1	-3	2	1	
0	1	0	2	
0	0	1	3	$R_3/(-4)$

... continued

x	y	z	rhs
1	-3	2	1
0	1	0	2
0	0	1	3

Last row: $\Longrightarrow z = 3$.

Second row: $\Longrightarrow y = 2$.

First row: $x = 1 + 3y - 2z = 1 + 6 - 6 = 1$.

Solution to linear system: $x = 1, \ y = 2, \ z = 3$.

Example. Solve

$$\begin{array}{rrrrcr} x_1 & +2x_2 & -x_3 & +3x_4 & = & 9 \\ 2x_1 & -x_2 & +2x_3 & +x_4 & = & 0 \\ x_1 & +x_2 & & +2x_4 & = & 5 \\ 3x_1 & -4x_2 & +3x_3 & +x_4 & = & -1 \end{array}$$

x_1	x_2	x_3	x_4	rhs	
1	2	-1	3	9	
2	-1	2	1	0	
1	1	0	2	5	
3	-4	3	1	-1	
1	2	-1	3	9	
0	-5	4	-5	-18	$-2R_1 + R_2$
0	-1	1	-1	-4	$-R_1 + R_3$
0	-10	6	-8	-28	$-3R_1 + R_4$
1	2	-1	3	9	
0	-5	4	-5	-18	
0	0	$\frac{1}{5}$	0	$-\frac{2}{5}$	$-\frac{1}{5}R_2 + R_3$
0	0	-2	2	8	$-2R_2 + R_4$
1	2	-1	3	9	
0	-5	4	-5	-18	
0	0	$\frac{1}{5}$	0	$-\frac{2}{5}$	
0	0	0	2	4	$10R_3 + R_4$

... continued

x_1	x_2	x_3	x_4	rhs
1	2	-1	3	9
0	-5	4	-5	-18
0	0	$\frac{1}{5}$	0	$-\frac{2}{5}$
0	0	0	2	4

Back substitution

R_4: $2x_4 = 4 \quad \implies \; x_4 = 2$

R_3: $\frac{1}{5}x_3 = -\frac{2}{5} \quad \implies \; x_3 = -2$

R_2: $-5x_2 + 4x_3 - 5x_4 = -18$

$$\implies \; x_2 = \frac{-18-4x_3+5x_4}{-5} = \frac{-18+8+10}{5} = 0$$

R_1: $x_1 + 2x_3 - x_3 + 3x_4 = 9$

$$\implies \; x_1 = 9 - 2x_2 + x_3 - 3x_4 = 9 - 0 - 2 - 6 = 1$$

Solution: $x_1 = 1,\; x_2 = 0,\; x_3 = -2$, and $x_4 = 2$.

Standard notations – Matrix

Example. Solve

$$\begin{array}{rcrcrcr} x & - & 2y & - & 3z & = & -7 \\ 3x & - & 4y & - & 2z & = & -8 \\ 3x & - & 2y & - & z & = & -1 \end{array}$$

Only coefficients/rhs change with ERO. So consider augmented coefficient matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -7 \\ 3 & -4 & -2 & -8 \\ 3 & -2 & -1 & -1 \end{array}\right]$$

... continued

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & -7 \\ 3 & -4 & -2 & -8 \\ 3 & -2 & -1 & -1 \end{array}\right] \quad \leftarrow \quad \begin{array}{l}\text{the}\\ \text{augmented}\\ \text{matrix}\end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & -7 \\ 0 & 2 & 7 & 13 \\ 0 & 4 & 8 & 20 \end{array}\right] \quad \begin{array}{r} \\ R2-3\,R1 \\ R3\text{-}3\,R1 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & -7 \\ 0 & 2 & 7 & 13 \\ 0 & 0 & -6 & -6 \end{array}\right] \quad \begin{array}{l} \\ \\ R3\text{-}2\,R2 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & -7 \\ 0 & 2 & 7 & 13 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ R3/(-6) \end{array}$$

... continued

Apply back-substitution to the last system

Last matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -7 \\ 0 & 2 & 7 & 13 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Last row: $z = 1$

Second row: $2y = 13 - 7z = 13 - 7 = 6. \implies y = 3.$

First row: $x - 2y - 3z = -1.$

Sub. $y = 3$, $z = 1$ into eqn:

$$x - 2 \times 3 - 3 \times 1 = -7 \implies x = 2$$

Solution to linear system: $x = 2, \; y = 3, \; z = 1.$

Part 2

Nonlinear Systems (2 variables)

Part 2. Nonlinear Systems

2.1. Substitution

Example. Solve the nonlinear system:

$$\begin{aligned} x^2 + y^2 &= 100 \\ x + y &= 14 \end{aligned}$$

Solution:

1. Find $y = 14 - x$ from the 2nd equation.
2. Substitute $y = 14 - x$ into $x^2 + y^2 = 100$:

$$\begin{aligned} & x^2 + (14 - x)^2 = 100 \\ \Rightarrow \quad & 2x^2 - 28x + 96 = 0 \\ \Rightarrow \quad & x^2 - 14x + 48 = 0 \end{aligned}$$

3. Solve $2x^2 - 28x + 96 = 0$: $x_1 = 6$, $x_2 = 8$.
4. Substitute $x_1 = 6$ and $x_2 = 8$ into $y = 14 - x$ and obtain $y_1 = 8$ and $y_2 = 6$, respectively.
   Solutions are $(x, y) = (6, 8)$ and $(x, y) = (8, 6)$.

Example. Solve the nonlinear system:

$$x^2 - 2x - y + 1 = 0 \quad (1)$$
$$x + \sqrt{y} - 5 = 0 \quad (2)$$

Solution:

From (2), $y = (5 - x)^2$.

Substitute into (1):

$$\begin{aligned} & x^2 - 2x - (5 - x)^2 + 1 = 0 \\ \Rightarrow \quad & 8x - 24 = 0 \\ \Rightarrow \quad & x = 3 \end{aligned}$$

Substitute $x = 3$ into $y = (5 - x)^2$: $y = (5 - 3)^2 = 4$.

4. Finally, solution is $(x, y) = (3, 4)$.

Example. Solve the nonlinear system:

$$3x^2 + 4x - y = 7 \tag{3}$$
$$2x - y = -1 \tag{4}$$

Solution: From (4), $y = 2x + 1$.

Substitute $y = 2x + 1$ into (3):

$$3x^2 + 4x - (2x + 1) = 7 \ \Rightarrow 3x^2 + 2x - 8 = 0$$

$$x_1 = \frac{-2 + \sqrt{2^2 - 4 \times 3 \times (-8)}}{2 \times 3} = \frac{4}{3}$$

$$x_2 = \frac{-2 - \sqrt{2^2 - 4 \times 3 \times (-8)}}{2 \times 3} = -2$$

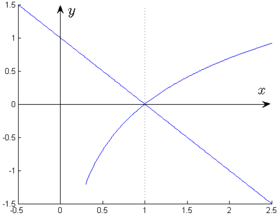
Substituting into $y = 2x + 1$:

$y_1 = \frac{11}{3}, \quad y_2 = -3.$

Solutions are $(\frac{4}{3}, \ \frac{11}{3})$ and $(-2, \ -3)$

Example. Solve the system using graphical method.

$$y = \ln x \tag{5}$$
$$x + y = 1 \tag{6}$$



Solution: Graph shows there one solution $(1, 0)$.
Check: $0 = \ln 1$ correct $1 + 0 = 1$ correct
We will find applications in optimization.

Next Week

- Matrices
  Matrix Algebra
  Inverse Matrix
  Determinant