MATH2267 Week 5

Vectors

Semester 2, 2018

Overview

- Vectors

  Introduction to vectors

  Basic operations

  Scalar or dot product

  Scalar projection

  Vector projection

  Vector product or cross product

Week 5: Vectors

1: Introduction

Physical quantities are either scalars or vectors.

Vectors have both magnitude and direction

Example: forces, velocity, electric field, ...

Scalars have only magnitude

Example: temperature, time, distance, ...

Vectors are represented:

geometrically, by directed line segments;

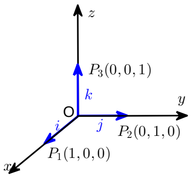
algebraically, by bold face letters ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{u}$, $\boldsymbol{v}$, etc.).

or $\vec{a}$, $\underset{\sim}{a}$ e.t.c (in hand writing)

$\boldsymbol{i}$: from $O(0,0,0)$ to $P_1(1,0,0)$

$\boldsymbol{j}$: from $O(0,0,0)$ to $P_2(0,1,0)$

$\boldsymbol{k}$: from $O(0,0,0)$ to $P_3(0,0,1)$

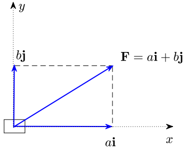


Week 5: Vectors

1: Introduction

A 3-D vector $\boldsymbol{a}$ has three components:

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$$



or

$$\boldsymbol{a} = (a_1,\ a_2,\ a_3) \qquad \text{(component form)}$$

where $a_1,\ a_2,\ a_3$ are real numbers.

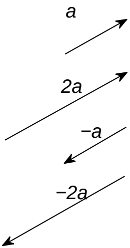
2-D vectors have two components: $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$,
or in component form: $\boldsymbol{a} = (a_1,\ a_2)$.

Multiplication of vector by a scalar:

For $\boldsymbol{a} = (a_1,\ a_2,\ a_3)$, $\lambda \in \mathbb{R}$:

$$\lambda \boldsymbol{a} = (\lambda a_1, \lambda a_2, \lambda a_3) \qquad \text{(parallel to } \boldsymbol{a}\text{)}$$

Geometric interpretation:



Week 5: Vectors

2: Basic Vector Operations

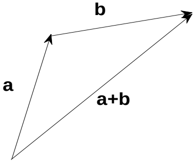
Addition and subtraction:

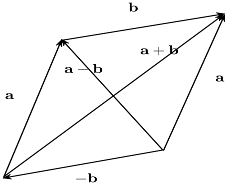
For $\boldsymbol{a} = (a_1,\ a_2,\ a_3)$ and $\boldsymbol{b} = (b_1,\ b_2,\ b_3)$.

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$$

$$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3)$$

- Note: $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$
- Geometrically, vector addition follows Tail-Nose Rule:





Example. For $\boldsymbol{a} = (3,\ 5,\ -7),\ \boldsymbol{b} = (-2,\ 4,\ 11)$,

(1)
$$\begin{aligned} 3\boldsymbol{a} &= 3\,(3,\ 5,\ -7) \\ &= (3\times 3,\ 3\times 5,\ 3\times(-7)) \\ &= (9,\ 15,\ -21) \end{aligned}$$

(2)
$$\begin{aligned} \boldsymbol{a}+\boldsymbol{b} &= (3,\ 5,\ -7)+(-2,\ 4,\ 11) \\ &= (3-2,\ 5+4,\ -7+11) \\ &= (1,\ 9,\ 4) \end{aligned}$$

(3)
$$\begin{aligned} 2\boldsymbol{a}-\boldsymbol{b} &= 2\,(3,\ 5,\ -7)-(-2,\ 4,\ 11) \\ &= (6,\ 10,\ -14)-(-2,\ 4,\ 11) \\ &= (6+2,\ 10-4,\ -14-11) \\ &= (8,\ 6,\ -25) \end{aligned}$$

Week 5: Vectors

2: Basic Vector Operations

The magnitude (length) of a vector:

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad (\text{also } \|\boldsymbol{a}\|)$$

The unit vector in the direction of $\boldsymbol{a}$:

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Example. Let $\boldsymbol{a} = (3,\ 5,\ -7)$. Then,

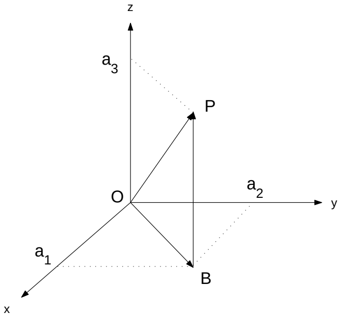
1. $|\boldsymbol{a}| = \sqrt{3^2 + 5^2 + (-7)^2} = \sqrt{9 + 25 + 49} = \sqrt{83}$
2. $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \dfrac{1}{\sqrt{83}}(3,\ 5,\ -7) = (\dfrac{3}{\sqrt{83}},\ \dfrac{5}{\sqrt{83}},\ -\dfrac{7}{\sqrt{83}})$

Position Vector

Given point $P(a_1, a_2, a_3)$, the position vector of P is:

$$\overrightarrow{OP} = a_1\,\boldsymbol{i} + a_2\,\boldsymbol{j} + a_3\,\boldsymbol{k} = (a_1,\ a_2,\ a_3)$$

a_1, a_2, a_3 are components of the vector.



Example. Given point $P(1, -2, -5)$, find $\overrightarrow{OP}$ and $|\overrightarrow{OP}|$.

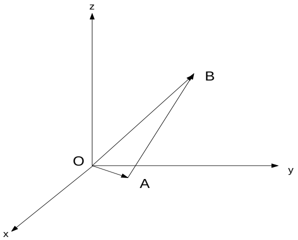
Solution.

$$\begin{aligned} \overrightarrow{OP} &= \boldsymbol{i} - 2\mathbf{j} - 5\boldsymbol{k} \\ |\overrightarrow{OP}| &= \sqrt{1^2 + (-2)^2 + (-5)^2} = \sqrt{30} \end{aligned}$$

Vector between Two Points

Vector from $A(a_1, a_2, a_3)$ to $B(b_1, b_2, b_3)$ is given by the tail-nose rule, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$:

$$\boxed{\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}} \quad \text{('last' minus 'first')}$$



Example. Find the vector from $A(1, 2, -3)$ to $B(3, 5, -6)$.
Solution.

$$
\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
&= (3, 5, 6) - (1, 2, -3) = (3 - 1,\ 5 - 2,\ -6 - (-3)) \\
&= (2,\ 3,\ -3) \\
\overrightarrow{BA} &= \overrightarrow{OA} - \overrightarrow{OB} \\
&= (1, 2, -3) - (3, 5, 6) = (1 - 3,\ 2 - 5,\ -3 - (-6)) \\
&= (-2,\ -3,\ 3) = -\overrightarrow{AB}
\end{aligned}
$$

Week 5: Vectors

3: Dot Product

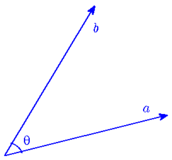
The scalar product or dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

If $\boldsymbol{a} = (a_1,\ a_2,\ a_3)$ and $\boldsymbol{b} = (b_1,\ b_2,\ b_3)$, then

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.}$$



Example. Find $\boldsymbol{a} \cdot \boldsymbol{b}$ for

$$\boldsymbol{a} = (2,\ -3,\ 2) \ \text{ and } \ \boldsymbol{b} = (6,\ 1,\ -2).$$

Solution.

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= 2 \times 6 + (-3) \times 1 + 2 \times (-2) \\ &= 12 - 3 - 4 \\ &= 5 \end{aligned}$$

Angle between Two Vectors

$$\begin{aligned} & \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta \\ \Longrightarrow \quad & \cos\theta = (\boldsymbol{a} \cdot \boldsymbol{b})/(|\boldsymbol{a}||\boldsymbol{b}|) \end{aligned}$$

$$\Longrightarrow \quad \boxed{\theta = \cos^{-1} \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}}$$

Example. Let $\boldsymbol{a} = (2, 3, 5)$ and $\boldsymbol{b} = (1, -2, 1)$. Find the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= 2(1)+3(-2)+5(1)=1,\\
|\boldsymbol{a}||\boldsymbol{b}| &= \sqrt{4+9+25}\sqrt{1+4+1}=2\sqrt{57}\\
\theta &= \cos^{-1}\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\\
&= \cos^{-1}\frac{1}{2\sqrt{57}}\\
&= 1.5045 \text{ (radians)} \quad (= 86.2027^\circ)
\end{aligned}$$

Julia works in degree by adding ‘d’ to the function name:
input degree values: sind(30), cosd(45), tand(60)
output degree values: asind(0.5), acosd(1), atand($\frac{\sqrt{2}}{2}$)

Orthogonal Vectors

Non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal iff

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0$$

as

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

Example. Are $\boldsymbol{a} = (3, -1, 2)$ and $\boldsymbol{b} = (3, 5, -2)$ orthogonal?
Solution.

$$\boldsymbol{a} \cdot \boldsymbol{b} = 3 \times 3 + (-1) \times 5 + 2 \times (-2) = 9 - 5 - 4 = 0,$$

$\Longrightarrow$ $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal.

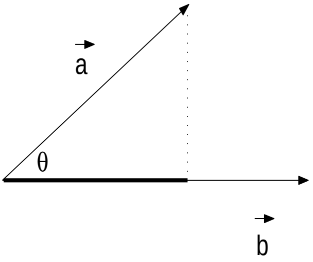
Week 5: Vectors

3: Projections

The scalar projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is simply the (scalar) component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.
"Scalar proj. $\boldsymbol{a}$ on $\boldsymbol{b}$" $= |\boldsymbol{a}|\cos\theta = |\boldsymbol{a}|(\boldsymbol{a}\cdot\boldsymbol{b})/(|\boldsymbol{a}||\boldsymbol{b}|)$

$$\boxed{(\text{Scalar proj. of } \boldsymbol{a} \text{ in } \boldsymbol{b}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}}$$

Note: If $\theta > 90^\circ$, scalar projection is negative.



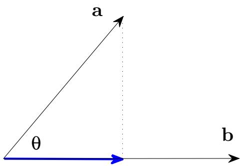
Week 5: Vectors

3: Projections

The vector projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is simply the scalar projection $(\boldsymbol{a}\cdot\boldsymbol{b})/|\boldsymbol{b}|$ multiplied by $\hat{\boldsymbol{b}} = \boldsymbol{b}/|\boldsymbol{b}|$, which is

$$\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}\,\hat{\boldsymbol{b}}\left(=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}\,\frac{\boldsymbol{b}}{|\boldsymbol{b}|}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}\,\boldsymbol{b}\right)$$

Thus, "Vector proj. of $\boldsymbol{a}$ in $\boldsymbol{b}$" $= \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}\,\boldsymbol{b}$



Example. Find the scalar and vector projections of $\boldsymbol{a} = (5, -7, -6)$ in the direction of $\boldsymbol{b} = (2, 6, -3)$.

Solution.

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{b} &= 10 - 42 + 18 = -14 \\
|\boldsymbol{b}| &= \sqrt{2^2 + 6^2 + (-3)^2} = \sqrt{49} = 7 \\
\text{scalar proj.} &= (\boldsymbol{a} \cdot \boldsymbol{b})/|\boldsymbol{b}| \\
&= -14/7 = -2 \\
\text{vector proj.} &= \left[(\boldsymbol{a} \cdot \boldsymbol{b})/|\boldsymbol{b}|^2\right] \boldsymbol{b} \\
&= -[14/49]\, \boldsymbol{b} \\
&= -[2/7]\, (-2, -6, 3) \\
&= (-4/7, -12/7, 6/7)
\end{aligned}$$

NOTE: Cross product is not required.

Next Week 6:

More Vectors and Matrices

- Eigenvalues and eigenvectors

  Requirements:
  - Hand-computing eigenvals/eigenvecs of 2×2 matrices
  - Julia computing eigenvals/eigenvecs of square matrices
- Leslie matrix

  Requirements:
  - Modelling age structured population, Leslie Matrix
  - Determine population behavior