MATH2267 Week 6

Part 1

More Vectors and Matrices

Overview

- Eigenvalues and eigenvectors

  Requirements:
  - Hand-computing eigenvals/eigenvecs of 2×2 matrices
  - Julia computing eigenvals/eigenvecs of square matrices
- Leslie matrix

  Requirements:
  - Modelling age structured population, Leslie Matrix
  - Determine population behavior

Consider $n \times n$ square matrix A.

If a non-zero vector (column n-vector) X and scalar λ satisfy

$$AX = \lambda X,$$

then, X is called an eigenvector of A, and

λ is called an eigenvalue associated with X.

For an eigenvector X,

- $-2X$, $-X$, $3X$, (aX, $a \neq 0$) are eigenvectors.

Finding eigenvalues and eigenvectors of A

It can be shown that

λ is an eigenvalue and X is associated eigenvector if

1. λ satisfy the characteristic equation $\det(A - \lambda I) = 0$
2. X is a non-zero solution of linear system $(A - \lambda I)X = \mathbf{0}$

Example Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the corresponding eigenvalue?

Solution: $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus, the given vector is an eigenvector and the corresponding eigenvalue is 2.

Let A be the matrix above. Try, in Julia,

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eigvals(A)
eigvecs(A)
eig(A)
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Example Find the eigenvalues and eigenvectors for the matrix:

$$A = \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix}$$

Solution: Step 1. Finding eigenvalues:

- Finding $\det(A - \lambda I)$:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 4 \\ 4 & -5-\lambda \end{vmatrix} \\ &= (1-\lambda)(-5-\lambda) - 16 \\ &= \lambda^2 + 4\lambda - 21 \end{aligned}$$

- Solving $\det(A - \lambda I) = 0$ (use formula):

  The solutions of $(\lambda - 3)(\lambda + 7) = 0$ are $\lambda = 3$ and $\lambda = -7$

  eigenvalues: $\lambda = 3$ and$\lambda = -7$

Step 2. Finding eigenvectors: $\begin{bmatrix} 1-\lambda & 4 \\ 4 & -5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

1. For $\lambda = 3$, system (1) becomes

$$\begin{bmatrix} 1-3 & 4 \\ 4 & -5-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ 4 & -8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rcl} R1/(-2) & \to & R1 \\ R2+2R1 & \to & R2 \end{array}$$

$x - 2y = 0$. For any $y = t$, $x = 2t$. Thus,

$$X = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Choose any $t \neq 0$, especially $t = 1$, we find that

$X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 3$.

$$\begin{bmatrix} 1-\lambda & 4 \\ 4 & -5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. For $\lambda = -7$ system (1) becomes

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 4 & 0 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{rcl} R1/(4) & \to & R1 \\ R2 - R1/2 & \to & R2 \end{array}$$

$2x + y = 0$. For any $y = -2t$, $x = t$. Thus,

$$X = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Choose $t = 1$, we obtain

$X = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \leftarrow$ eigenvector corresponding to $\lambda = -7$

Week 6: More Vectors and Matrices

Part 1: Eigenvalues and Eigenvectors

Summary

Finding eigenvalues and eigenvectors of A

(e.g. $A = \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix}$)

1. Solving the characteristic equation $\det(A - \lambda I) = 0$.

   Solving $\det(A - \lambda I) = 0$
   gives eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -7$.

2. For each λ, determining the corresponding eigenvector.

   * For $\lambda_1 = 3$, solve $(A - 3I)X = \mathbf{0}$ for non-zero sol.
     $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ – an eigenvector a.w $\lambda_1 = 3$
   * For $\lambda_1 = -7$, solve $(A - (-7)I)X = \mathbf{0}$ for non-zero sol.
     $X = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ – an eigenvector a.w. $\lambda_1 = -7$

Week 6: More Vectors and Matrices

Part 2: Leslie Matrix

Age-structured modelling A type of structured population model is the Lesie model in which a population breaks up into a number of different groups by age, or stage of development. The size of each group changes over time depending linearly on the sizes of other groups.

There are two requirements for this topic.

- Modelling: obtain math eqns for each group's population
- Analysis: determine behavior of population over time

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Part 2.2: Leslie Matrix

Example: Consider the female population of a species that only lives for four years. The animal matures and becomes fertile after exactly one year. The specific fecundity rates (f_i) and specific survival rates (s_i) for each age group i are given in the table below.

group index i	1	2	3	4
age (years)	0-1	1-2	2-3	3-4
f_i (fraction per year)	0.0	0.2	0.9	0.9
s_i (fraction per year)	0.6	1.0	1.0	0.0

Let $x_i(t)$ = "the number of animals of group i in year t.
The initial populations $x_i(0)$ are known for all groups.

$x_1(t+1)$ is determined by the fecundity rate of each group
$x_2(t+1)$ be qual to the number that survive from group 1, ...

Thus,

$$
\begin{aligned}
x_1(t+1) &= 0.0x_1(t) \quad +0.2x_2(t) \quad +0.9x_3(t) \quad +0.9x_4(t) \\
x_2(t+1) &= 0.6x_1(t) \\
x_3(t+1) &= \qquad\qquad\quad 1.0x_2(t) \\
x_4(t+1) &= \qquad\qquad\qquad\qquad\quad 1.0x_3(t)
\end{aligned}
$$

Week 6: More Vectors and Matrices

Part 2: Leslie Matrix

Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.0 & 0.2 & 0.9 & 0.9 \\ 0.6 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

then the equations in the previous slide can be written concisely as:

$$X(t+1) = AX(t)$$

The matrix A is known as a Leslie matrix.

$$X(1) = AX(0), \quad X(2) = AX(1) = A^2X(0), \quad \cdots \quad X(n) = A^nX(0)$$

It is beyond the scope of this course but it can be shown that whether the populations survives or not depends on the largest eigenvector of A, λ:
The population is

increasing if $\lambda > 1$

declining if $\lambda < 1$

stable if $\lambda = 1$

Question: The above population model is

$$X(t+1) = AX(t)$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 0.0 & 0.2 & 0.9 & 0.9 \\ 0.6 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

For an initial population, how would you expect the population to behave over time? Does the population survive?
Solution: Largest eigenvalue of A (by Julia): $1.05613 > 1$
Thus, the population will survive.