Essential Mathematics

Week 4 Exercises

Matrix properties and algebra

Exercise 4.2.11

1. Let $\mathbf{A} = \begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & -3 \end{bmatrix}$,

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}.$$

Find:

(a) $5\mathbf{A}$

Solution:

$$5\mathbf{A} = 5\begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 10 & -15 & -5 \\ -5 & 10 & -15 \end{bmatrix}$$

(b) $\mathbf{B}+\mathbf{C}$

Solution:

$$\begin{aligned} \mathbf{B}+\mathbf{C} &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 & 2 \\ 1 & 4 & -6 \\ 3 & 2 & -2 \end{bmatrix} \end{aligned}$$

(c) $(\mathbf{B}+\mathbf{C})^T$

Solution:

$$\begin{aligned} (\mathbf{B}+\mathbf{C})' &= \begin{bmatrix} 3 & -2 & 2 \\ 1 & 4 & -6 \\ 3 & 2 & -2 \end{bmatrix}' \\ &= \begin{bmatrix} 3 & 1 & 3 \\ -2 & 4 & 2 \\ 2 & -6 & -2 \end{bmatrix} \end{aligned}$$

(d) $\mathbf{B}^T + \mathbf{C}^T$

Solution:

$$\begin{aligned} \mathbf{B}^T + \mathbf{C}^T &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}' + \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}' \\ &= \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ -1 & -3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & 1 \\ 3 & -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 & 3 \\ -2 & 4 & 2 \\ 2 & -6 & -2 \end{bmatrix} \end{aligned}$$

(e) Is $(BC)^T = B^T C^T$ true?

Solution:

$$\begin{aligned} BC &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -4 & 7 \\ -1 & 2 & -6 \\ 5 & -1 & 2 \end{bmatrix} \\ (BC)' &= \begin{bmatrix} -1 & -1 & 5 \\ -4 & 2 & -1 \\ 7 & -6 & 2 \end{bmatrix} \\ B'C' &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}' \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}' \\ &= \begin{bmatrix} 9 & -6 & -2 \\ -1 & -1 & 0 \\ -2 & -5 & -3 \end{bmatrix} \\ &\neq (BC)^T \end{aligned}$$

(f) Is $(BC)^T = C^T B^T$ true?

Solution:

$$\begin{aligned} C'B' &= \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}' \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}' \\ &= \begin{bmatrix} 9 & -6 & -2 \\ -1 & -1 & 0 \\ -2 & -5 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & 5 \\ -4 & 2 & -1 \\ 7 & -6 & 2 \end{bmatrix} \end{aligned}$$

Thus, $(BC)' = C'B'$.

2. For the matrix B given in Question 1, complete the following EROs successively:

   (a) Row 1 is added to Row 2

   (b) (-2) time of Row 1 is added to Row 3

   (c) (-3) times of Row 2 is added to Row 3

   (d) Row 3 is divided by 13

   (e) Check if all elements under the diagonal are 0's and all diagonal elements are 1's.

Solution:

Performing the required EROs (a)-(d), we obtain

$$
\begin{array}{lll}
B: & \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix} & \\
\Rightarrow & \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -4 \\ 0 & 3 & 1 \end{bmatrix} & \begin{array}{ll} R_2 + R_1 \to R_2 & (a) \\ R_3 - 2R_1 \to R_3 & (b) \end{array} \\
\Rightarrow & \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 13 \end{bmatrix} & R_3 - 3R_2 \to R_3 \quad (c) \\
\Rightarrow & \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} & R_3/13 \to R_3 \quad (d)
\end{array}
$$

Finding Inverse of a Matrix

Exercise 4.3.5

1. Find the inverse for each of the following matrices.

$$
\begin{aligned}
A &= \begin{bmatrix} 3 & 5 & -12 \\ 0 & 1 & -2 \\ -1 & -2 & 5 \end{bmatrix} \\
B &= \begin{bmatrix} 2 & 1 & -4 \\ 1 & 0 & -2 \\ 2 & 1 & -5 \end{bmatrix} \\
C &= \begin{bmatrix} -5 & -4 & \frac{7}{2} \\ 6 & 5 & -4 \\ 4 & 3 & -\frac{5}{2} \end{bmatrix}
\end{aligned}
$$

Solution:

$$
\begin{aligned}
[A \mid I] &= \left[\begin{array}{rrr|rrr} 3 & 5 & -12 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -1 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{rrr|rrr} 1 & 2 & -5 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 3 & 5 & -12 & 1 & 0 & 0 \end{array}\right] \qquad R_1 \leftrightarrow (-R_3) \\
&= \left[\begin{array}{rrr|rrr} 1 & 2 & -5 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 3 \end{array}\right] \qquad R_3 - 3R_1 \to R_3 \\
&= \left[\begin{array}{rrr|rrr} 1 & 2 & -5 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \qquad R_3 + R_2 \to R_3 \\
&= \left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 5 & 5 & 14 \\ 0 & 1 & 0 & 2 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \qquad \begin{array}{l} R_1 + 5R_3 \to R_1 \\ R_2 + 2R_3 \to R_2 \end{array} \\
&= \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \qquad R_1 - 2R_2 \to R_1 \\
\Rightarrow \qquad A^{-1} &= \left[\begin{array}{rrr} 1 & -1 & 2 \\ 2 & 3 & 6 \\ 1 & 1 & 3 \end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
[B \mid I] &= \left[\begin{array}{rrr|rrr} 2 & 1 & -4 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 2 & 1 & -5 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{rrr|rrr} 1 & 1 & -2 & 1 & -1 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 1 \end{array}\right] \qquad \begin{array}{l} R_1 - R_2 \to R_1 \\ \\ R_3 - 2R_2 \to R_3 \end{array} \\
&= \left[\begin{array}{rrr|rrr} 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 & -2 & 1 \end{array}\right] \qquad R_2 - R_1 \to R_2 \\
&= \left[\begin{array}{rrr|rrr} 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \qquad \begin{array}{r} -R_2 \to R_2 \\ -(R_3 + R_2) \to R_3 \end{array} \\
&= \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 1 & -2 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \qquad R_1 - R_2 + 2R_3 \to R_1 \\
\Rightarrow \qquad B^{-1} &= \left[\begin{array}{rrr} 2 & 1 & -2 \\ 1 & -2 & 0 \\ 1 & 0 & -1 \end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
[C \mid I] &= \left[\begin{array}{ccc|ccc} -5 & -4 & \frac{7}{2} & 1 & 0 & 0 \\ 6 & 5 & -4 & 0 & 1 & 0 \\ 4 & 3 & -\frac{5}{2} & 0 & 0 & 1 \end{array}\right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & -\frac{1}{2} & 1 & 1 & 0 \\ 6 & 5 & -4 & 0 & 1 & 0 \\ 4 & 3 & -\frac{5}{2} & 0 & 0 & 1 \end{array}\right] \quad R_1 + R_2 \to R_1 \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & -\frac{1}{2} & 1 & 1 & 0 \\ 0 & -1 & -1 & -6 & -5 & 0 \\ 0 & -1 & -\frac{1}{2} & -4 & -4 & 1 \end{array}\right] \quad \begin{array}{r} R_2 - 6R_1 \to R_2 \\ R_3 - 4R_1 \to R_3 \end{array} \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & -\frac{1}{2} & 1 & 1 & 0 \\ 0 & 1 & 1 & 6 & 5 & 0 \\ 0 & 0 & \frac{1}{2} & 2 & 1 & 1 \end{array}\right] \quad \begin{array}{r} -R_2 \to R_2 \\ R_3 - R_2 \to R_3 \end{array} \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & 4 & 2 & 2 \end{array}\right] \quad \begin{array}{r} R_1 + R_3 \to R_1 \\ R_2 - 2R_3 \to R_2 \\ 2R_3 \to R_3 \end{array} \\
& \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & 4 & 2 & 2 \end{array}\right] \quad R_1 - R_2 \to R_1 \\
\Rightarrow \qquad C^{-1} &= \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & -2 \\ 4 & 2 & 2 \end{bmatrix}
\end{aligned}
$$

Determinants

Exercise 4.4.3

1. Consider

$$
\begin{aligned}
A &= \begin{bmatrix} 3 & 5 & -12 \\ 0 & 1 & -2 \\ -1 & -2 & 5 \end{bmatrix} \\
B &= \begin{bmatrix} 2 & 1 & -4 \\ 1 & 0 & -2 \\ 2 & 1 & -5 \end{bmatrix} \\
C &= \begin{bmatrix} -5 & -4 & \frac{7}{2} \\ 6 & 5 & -4 \\ 4 & 3 & -\frac{5}{2} \end{bmatrix}
\end{aligned}
$$

Find

(a) The determinants of A, B and C

Solution:

$$
\begin{aligned}
\det(A) &= \begin{vmatrix} 3 & 5 & -12 \\ 0 & 1 & -2 \\ -1 & -2 & 5 \end{vmatrix} \\
& \downarrow \text{ expending in 1st column} \\
&= 3 \begin{vmatrix} 1 & -2 \\ -2 & 5 \end{vmatrix} - 0 \begin{vmatrix} 5 & -12 \\ -2 & 5 \end{vmatrix} - \begin{vmatrix} 5 & -12 \\ 1 & -2 \end{vmatrix} \\
&= 3 - 0 - 2 = 1
\end{aligned}
$$

$$
\begin{aligned}
\det(B) &= \begin{vmatrix} 2 & 1 & -4 \\ 1 & 0 & -2 \\ 2 & 1 & -5 \end{vmatrix} \\
& \downarrow \text{ expending in 2nd row} \\
&= -1 \times \begin{vmatrix} 1 & -4 \\ 1 & -5 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & -4 \\ 2 & -5 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \\
&= 1 + 0 + 0 = 1
\end{aligned}
$$

$$
\begin{aligned}
\det(C) &= \begin{vmatrix} -5 & -4 & \frac{7}{2} \\ 6 & 5 & -4 \\ 4 & 3 & -\frac{5}{2} \end{vmatrix} \\
& \downarrow \text{ expending in 3rd column} \\
&= \frac{7}{2} \times \begin{vmatrix} 6 & 5 \\ 4 & 3 \end{vmatrix} - (-4) \times \begin{vmatrix} -5 & -4 \\ 4 & 3 \end{vmatrix} + (-\frac{5}{2}) \times \begin{vmatrix} -5 & -4 \\ 6 & 5 \end{vmatrix} \\
&= \frac{7}{2} \times (-2) + 4 \times 1 - \frac{5}{2} \times (-1) = -\frac{1}{2}
\end{aligned}
$$

(b) The determinants of A^T, B^T and C^T

Solution:

$$
\begin{aligned}
\det(A^T) &= \begin{vmatrix} 3 & 0 & -1 \\ 5 & 1 & -2 \\ -12 & -2 & 5 \end{vmatrix} \\
& \downarrow \text{ expending in 1st row} \\
&= 3 \times \begin{vmatrix} 1 & -2 \\ -2 & 5 \end{vmatrix} - 0 \times \begin{vmatrix} 5 & -2 \\ -12 & 5 \end{vmatrix} + (-1) \times \begin{vmatrix} 5 & 1 \\ -12 & -2 \end{vmatrix} \\
&= 3 - 0 - 2 = 1
\end{aligned}
$$

$$
\begin{aligned}
\det(B^T) &= \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -4 & -2 & -5 \end{vmatrix} \\
& \downarrow \text{ expending in 2nd column} \\
&= -1 \times \begin{vmatrix} 1 & 1 \\ -4 & -5 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 2 \\ -4 & -5 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} \\
&= 1 + 0 + 0 = 1
\end{aligned}
$$

$$\begin{aligned}
\det(C^T) &= \begin{vmatrix} -5 & 6 & 4 \\ -4 & 5 & 3 \\ \frac{7}{2} & -4 & -\frac{5}{2} \end{vmatrix} \\
&\quad \downarrow \text{ expending in 3rd row} \\
&= \frac{7}{2} \times \begin{vmatrix} 6 & 4 \\ 5 & 3 \end{vmatrix} - (-4) \times \begin{vmatrix} -5 & 4 \\ -4 & 3 \end{vmatrix} + (-\frac{5}{2}) \times \begin{vmatrix} -5 & 6 \\ -4 & 5 \end{vmatrix} \\
&= \frac{7}{2} \times (-2) + 4 \times 1 - \frac{5}{2} \times (-1) = -\frac{1}{2}
\end{aligned}$$

(c) The determinants of A^{-1}, B^{-1} and C^{-1}.

NOTE: Use the results from Exercise 4.3.5 for this question.

Solution: According to solutions to Exercise 4.3.5,

$$\begin{aligned}
\det(A^{-1}) &= \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 6 \\ 1 & 1 & 3 \end{vmatrix} \\
&\quad \downarrow \text{ expending in 1st row} \\
&= 1 \times \begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} - (-1) \times \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\
&= 3 + 0 - 2 = 1
\end{aligned}$$

$$\begin{aligned}
\det(B^{-1}) &= \begin{vmatrix} 2 & 1 & -2 \\ 1 & -2 & 0 \\ 1 & 0 & -1 \end{vmatrix} \\
&\quad \downarrow \text{ expending in 2nd row} \\
&= -1 \times \begin{vmatrix} -1 & -2 \\ 0 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \\
&= 1 + 0 + 0 = 1
\end{aligned}$$

$$\begin{aligned}
\det(C^{-1}) &= \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & -2 \\ 4 & 2 & 2 \end{vmatrix} \\
&\quad \downarrow \text{ expending in 1st row} \\
&= 1 \times \begin{vmatrix} 3 & -2 \\ 2 & 2 \end{vmatrix} - (-1) \times \begin{vmatrix} 2 & -2 \\ 4 & 2 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} \\
&= 1 \times 10 + 1 \times 12 + 3 \times (-8) = -2
\end{aligned}$$