Essential Mathematics
Week 6 Notes

RMIT

Semester 2, 2018

Contents

Chapter 6

6.1 Eigenvalues and Eigenvectors

6.1.1 Introduction

Matrices have special directions which satisfy the matrix equation

$$AX = \lambda X \tag{6.1}$$

where λ is a scalar constant associated with each particular X. The special direction vectors X are called **eigenvectors** and the associated special λ values are called **eigenvalues**. Since any multiple of X clearly satisfies (6.1) we can think of the eigenvectors as describing **invariant lines** under the given mapping, i.e lines which do not move under the mapping.

It is clear that for rotations in 3D there is an eigenvector associated the axis of rotation, and for reflections in 2D an eigenvector is associated with the mirror line.

Consider the following example:

Example

Determine the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix}$$

and interpret the mapping defined by A physically.

Firstly, $\det A = -21$ so that some **reflection and stretching** is involved. (Remember a pure reflection has a determinant of -1.)

The equation defining the eigenvalues and eigenvectors is

$$AX = \lambda X$$

so that we have

$$\begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

Writing this a system of equations we have

$$\begin{aligned} x + 4y &= \lambda x \\ 4x - 5y &= \lambda y \end{aligned}$$

and collecting like terms on one side we have

$$\begin{aligned} (1-\lambda)x + 4y &= 0 \\ 4x + (-5-\lambda)y &= 0 \end{aligned}$$

Note that this system of equations is equivalent to the matrix system

$$\begin{bmatrix} 1-\lambda & 4 \\ 4 & -5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or in other words

$$(A - \lambda I)X = O.$$

Here we strike our first hurdle. We have 2 equations and 3 unknowns! There seems to be no way to find a **unique** solution for x, y and λ, however the task is not quite as difficult as it seems. In fact we don't really want a unique solution at all here!

For almost every choice of λ, the matrix $A - \lambda I$ will have an inverse. This means that we may solve for X using matrix algebra:

$$X = (A - \lambda I)^{-1} O = O$$

so that there *is* a unique solution $X = O$. Now, the solution $X = O$ is obvious by inspection and is called the **trivial solution**. If we are looking for special directions we definitely do not want the trivial solution so these λ values are no good to us.

Non-trivial solutions will only occur for the values of λ such that $A - \lambda I$ has **no inverse**. From the determinants topic we know that the matrix $A - \lambda I$ will have no inverse if

$$\det(A - \lambda I) = 0.$$

This is called the **characteristic equation** of A.

Now, for the current example,

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 4 \\ 4 & -5-\lambda \end{vmatrix} \\ &= (1-\lambda)(-5-\lambda) - 16 \\ &= \lambda^2 + 4\lambda - 21 \\ &= (\lambda - 3)(\lambda + 7) \end{aligned}$$

Thus $\det(A - \lambda I) = 0$ if $\lambda = 3$ or $\lambda = -7$.

For each of these special λ values, the eigenvalues, we may now solve the system of equations to determine the associated interesting, non-trivial solution:

We return to the system

$$(A - \lambda I)X = O$$

and solve this system for each λ using an augmented matrix.

For $\lambda = 3$ we have

$$\left[\begin{array}{rr|r} \boxed{\text{-2}} & 4 & 0 \\ 4 & -8 & 0 \end{array}\right]$$

and, following the first row operation:

$$\left[\begin{array}{rr|r} \boxed{\text{-2}} & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad R_2 + 2R_1 \to R_2$$

If we now let $y = t$ we have $x = 2t$, so that the solution is

$$X = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

for any t.

For $\lambda = -7$ we have

$$\left[\begin{array}{rr|r} \boxed{8} & 4 & 0 \\ 4 & 2 & 0 \end{array}\right]$$

and, following the first row operation:

$$\left[\begin{array}{rr|r} \boxed{8} & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad 2R_2 - R_1 \to R_2$$

If we now let $y = t$ we have $x = -(1/2)t$, so that the solution is

$$X = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = t^* \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

for any t^*.

Geometrical Interpretation

For $\lambda = 3$ the corresponding solution is $x = 2t$, $y = t$. If we eliminate t we have $y = (1/2)x$.

For $\lambda = -7$ the corresponding solution is $x = -t$, $y = 2t$. If we eliminate t we have $y = -2x$.

Points on these lines will always remain on these lines under the mapping defined by A. For example consider point P on $y = -2x$. Since $\lambda = -7$, points on this line will satisfy $AX = -7X$, so that point P will move to P' on the other side of the origin and the distance from P' to the origin will be 7 times that of P.

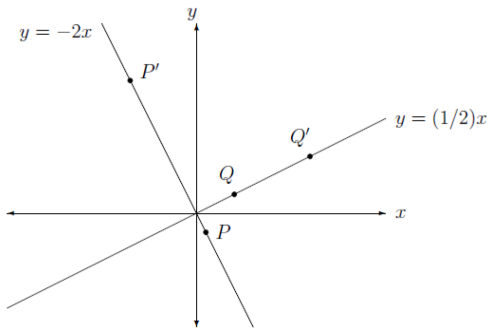

Figure 6.1: Eigenvectors

Similarly, for point Q on $y = (1/2)x$, we have $\lambda = 3$. All points on this line will satisfy $AX = 3X$, and point Q will move to Q' on the same side of the origin but at a distance from the origin three times greater.

From the sketch below it is clear that $y = (1/2)x$ behaves something like a mirror while $y = -2x$ is a line perpendicular to the "mirror".

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Summary

Given a matrix A we determine the eigenvalues and eigenvectors by

- solving the characteristic equation $\det(A - \lambda I) = 0$. This is a polynomial equation in terms of λ. For 2×2 matrices the characteristic polynomial $\det(A - \lambda I)$ is a quadratic and for 3×3 matrices it is a cubic.

- for each λ, determining the corresponding eigenvector.

Example

Determine the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 7 & 0 \\ 2 & 0 & 5 \end{bmatrix}.$$

We begin by solving the characteristic equation $\det(A-\lambda I)=0$:

$$\begin{aligned}
\det(A-\lambda) &= \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 7-\lambda & 0 \\ 2 & 0 & 5-\lambda \end{vmatrix} \\
&= (7-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} \\
&\qquad \text{expanding by second row} \\
&= (7-\lambda)\left[(2-\lambda)(5-\lambda)-4\right] \\
&= (7-\lambda)\left[\lambda^2-7\lambda+6\right] \\
&= (7-\lambda)(\lambda-1)(\lambda-6)
\end{aligned}$$

Thus, $\det(A-\lambda I)=0$ if $\lambda=1$, 6 or 7.

For $\lambda=1$ we solve $(A-\lambda I)X=O$:

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & 6 & 0 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right].$$

After the first set of row operations we have

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{r} \\ R_2/6 \to R_2 \\ R_3-2R_1 \to R_3 \end{array}$$

We let $z=t$. The second equation tells us that $y=0$, while the first says $x=-2t$. Hence the solution is

$$X=t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda=6$ we solve $(A-\lambda I)X=O$:

$$\left[\begin{array}{ccc|c} \boxed{-4} & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right].$$

After the first set of row operations we have

$$\left[\begin{array}{ccc|c} \boxed{-4} & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad 2R_3+R_1 \to R_3$$

We let $z=t$. The second equation tells us that $y=0$, while the first says $x=(1/2)t$. Hence the solution is

$$X=t\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \tau\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad (\tau=t/2).$$

For $\lambda = 7$ we solve $(A - \lambda I)X = O$:

$$\left[\begin{array}{ccc|c} \boxed{-5} & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right].$$

Swap rows 2 and 3:

$$\left[\begin{array}{ccc|c} \boxed{-5} & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

Zero in first column:

$$\left[\begin{array}{ccc|c} \boxed{-5} & 0 & 2 & 0 \\ 0 & 0 & \boxed{14} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad 5R_2 + 2R_1 \to R_2$$

We let $y = t$. The first and second equations tell us that $x = 0$ and $z = 0$. Hence the solution is

$$X = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The three eigenvalues are 1, 6 and 7. The corresponding eigenvectors are

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

These three eigenvectors are described as **linearly independent** since none of them can be obtained as a linear combination of the others.

There is a theorem which states that if A is a symmetric matrix with distinct real eigenvalues then the eigenvectors are not only linearly independent, but mutually orthogonal.

Example

Determine with reasons which, if any, of the following vectors

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \textbf{and} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

are eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Firstly, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is never an eigenvector. The zero vector is the trivial solution.

Secondly,

$$A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Thus $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda = 0$.

Lastly,

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Thus $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda = 3$.

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Diagonalisation

Suppose we have a matrix A which has eigenvalues λ_1, λ_2 and λ_3, and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. If we construct a matrix P with the eigenvectors of A as columns, i.e.

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$$

then the product $P^{-1}AP$ is a **diagonal** matrix, i.e.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

(note the order of the eigenvalues along the diagonal).

Note that only those $n \times n$ matrices with n eigenvectors may be diagonalised.

Often the set of eigenvectors of a matrix A provides a natural (and better) set of coordinate axes for the mapping represented by A. The process of diagonalisation is closely related to the process of rotating the axes to line up with the eigenvectors of A (e.g. lining up the coordinate axes with the axis of rotation, or the mirror line).

Example

Find a matrix P which diagonalises the matrix $A = \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix}$.

From worked example 1, A has eigenvectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

We construct P using the eigenvectors as columns:

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Now, the inverse of P is easy to write down using the formula:

$$P^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

Check that P diagonalises A by calculating $P^{-1}AP$:

$$\begin{aligned} P^{-1}AP &= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 6 & 3 \\ 7 & -14 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 15 & 0 \\ 0 & -35 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}. \end{aligned}$$

6.2 Leslie Matrix

6.2.1 Introduction

Consider the female population of a species that only lives for four years. Suppose that the animal matures and becomes fertile after exactly one year. The specific fecundity rates (f_i) and specific survival rates (s_i) for each age group i are given in the table below.

group index i	1	2	3	4
age (years)	0-1	1-2	2-3	3-4
f_i (fraction per year)	0.0	0.2	0.9	0.9
s_i (fraction per year)	0.6	1.0	1.0	0.0

Let $x_i(t)$ be the number of animals in the ith age group in year t. Then the number of animals in group 1 next year will be determined by the fecundity rate of each group, given by the equation:

$$x_1(t+1) = 0.0x_1(t) + 0.2x_2(t) + 0.9x_3(t) + 0.9x_4(t)$$

The number of animals in group 2 next year will be qual to the number that survive from group 1. Thus:

$$x_2(t+1) = 0.6x_1(t).$$

Similarly,

$$x_3(t+1) = 1.0x_2(t) \text{ and } x_4(t+1) = 1.0x_3(t).$$

The population can be represented by a vector containing the numbers in each age class:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

With this notation the equations above can be written concisely in vector-matrix form as follows:

$$X(t+1) = AX(t)$$

where the matrix A is given by:

$$A = \begin{bmatrix} 0.0 & 0.2 & 0.9 & 0.9 \\ 0.6 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

The matrix A is known as a Leslie matrix. The matrix A projects or maps the population $X(t)$ into the next year's population, $X(t+1)$. Continuing this process the population after n years is given by:

$$X(t+n) = A^n X(t)$$

It is beyond the scope of this course but it can be shown that this process converges to the eigenvector of A corresponding to its largest eigenvalue. If X is an eigenvector of matrix A then $AX = \lambda X$ where λ is the corresponding eigenvalue. As λ is a scalar its value tells us about the future prospects of the population.

The population is

increasing if $\lambda > 1$

declining if $\lambda < 1$

stable if $\lambda = 1$

Examples

Problem 1. What is the long term fate of the population above?

Solution See the Julia file Matrices.ipynb. The largest eigenvalue is approximately 1.05 which is greater than 1, hence the population will increase.

Problem 2. The Leslie Matrix for a population is given below. (a) Check that the population will survive and (b) will it be able to withstand 10% hunting of individuals in group 3?

$$\begin{bmatrix} 0.5 & 2.4 & 1.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.8 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \end{bmatrix}$$

Solution (a) The largest eigenvalue is > 1. (b) Changing to survival rate of group 3 to 0.4 (from 0.5) we find that the largest eigenvalue of the modified matrix is still > 1 so the population can be harvested in this way.

6.2.2 General Leslie Matrix

In general for a population of n age groups, the associated Leslie matrix A has the form

$$A = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}$$

and the population dynamics is given by:

$$X(t+1) = AX(t)$$

where $X = [x_1,\ x_2,\ \cdots,\ x_n]^T$.

6.2.3 Exercise

Given the data in figure 6.2, determine whether the population will increase.

6.2.4 Harvesting

From the Leslie matrix model we note that:

$$i = 2, 3, \ldots, n \quad x_i(t+1) = s_{i-1}x_{i-1}(t)$$

If a proportion, h_i, of each group i is removed (due to translocation, harvesting, or hunting) then we get:

$$i = 2, 3, \ldots, n \quad x_i(t+1) = s_{i-1}x_{i-1}(t) - h_{i-1}x_{i-1}(t) = S_{i-1}x_{i-1}(t)$$

where $S_i = s_i - h_i$ is the net survival rate that includes removal. To ensure a sustainable population the h_i values should be chosen so that the largest eigenvalue of the system 6.2.2 with all s_i replaced by S_i still has its largest eigenvalue greater than one.

Birth and Survival Rates for Female New Zealand Sheep
[from G. Caughley, "Parameters for Seasonally Breeding Populations," *Ecology* 48(1967)834-839]

Age (years)	Birth Rate	Survival Rate
0-1	0.000	0.845
1-2	0.045	0.975
2-3	0.391	0.965
3-4	0.472	0.950
4-5	0.484	0.926
5-6	0.546	0.895
6-7	0.543	0.850
7-8	0.502	0.786
8-9	0.468	0.691
9-10	0.459	0.561
10-11	0.433	0.370
11-12	0.421	0.000

Figure 6.2: Sheep data for a Leslie matrix

6.3 Exercises

1. In each case, determine whether or not $\mathbf{X}$ is an eigenvector of $\mathbf{A}$. If it is, state the corresponding eigenvalue.

   (a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 6 & 2 & -4 \\ -1 & 4 & 1 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (b) $\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 2 & -1 \\ -2 & 1 & 2 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

   (c) $\mathbf{A} = \begin{bmatrix} 3 & 7 & -2 \\ -2 & -3 & 1 \\ 6 & 4 & -4 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

2. Find the eigenvalues and eigenvectors of the following matrices:

   (a) $\mathbf{A} = \begin{bmatrix} -5 & 2 & -7 \\ -4 & 4 & -4 \\ 1 & -2 & 3 \end{bmatrix}$

   (b) $\mathbf{B} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 1 & -3 \\ -1 & 1 & 5 \end{bmatrix}$

   (c) $\mathbf{C} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\mathbf{D} = \begin{bmatrix} 3 & 2 & 0 \\ -2 & -4 & -2 \\ 2 & 7 & 5 \end{bmatrix}$

ANSWERS

1. (a) $\mathbf{AX} = 4\mathbf{X}$ so $\mathbf{X}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda = 4$
   (b) $\mathbf{X}$ is not an eigenvector of $\mathbf{A}$ since $\mathbf{AX} \neq \lambda\mathbf{X}$ for any λ
   (c) $\mathbf{AX} = -\mathbf{X}$ so $\mathbf{X}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda = -1$

2. (a) $\lambda_1 = -4$, $\mathbf{X_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$;

   $\lambda_2 = 2$, $\mathbf{X_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$;

   $\lambda_3 = 4$, $\mathbf{X_3} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

   (b) $\lambda_1 = 1$, $\mathbf{X_1} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$;

   $\lambda_2 = 2$, $\mathbf{X_2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$;

   $\lambda_3 = 3$, $\mathbf{X_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

   (c) $\lambda_1 = -1$, $\mathbf{X_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$;

   $\lambda_2 = 1$, $\mathbf{X_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$;

   $\lambda_3 = 1$, $\mathbf{X_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

   (d) $\lambda_1 = -1$, $\mathbf{X_1} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$;

   $\lambda_2 = 2$, $\mathbf{X_2} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$;

$$\lambda_3 = 3,\ \mathbf{X_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$