Essential Mathematics

Week 7 Notes

RMIT

Semester 2, 2018

Contents

Chapter 7

Limits, Derivatives and Applications

7.1 Limits

7.1.1 Introduction

What happens to the expression $\frac{x}{x+1}$ as x approaches 4? If we look at a graph we can see that the answer is 0.8.

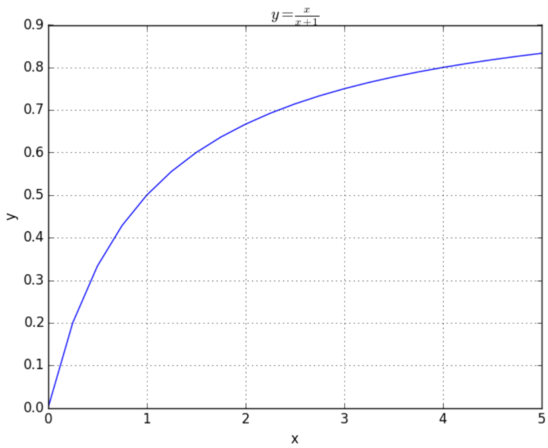

Figure 7.1: Graph of $\frac{x}{x+1}$

Did we need a graph to find the answer? Why not just substitute $x = 4$ into the expression? This gives $\frac{4}{4+1} = \frac{4}{5} = 0.8$ as before!

Let us try this approach with the each of the following two expressions

1. $\frac{3(x-4)}{x-4}$
2. $\frac{2(x-4)}{x-4}$

In both cases we get the result $\frac{0}{0}$ which is undefined. These two problems might be solved by simply first cancelling the $(x-4)$ terms. But consider what happens to the following function as x approaches 2:

$$f(x) = \frac{x^2 - 4}{x - 2}$$

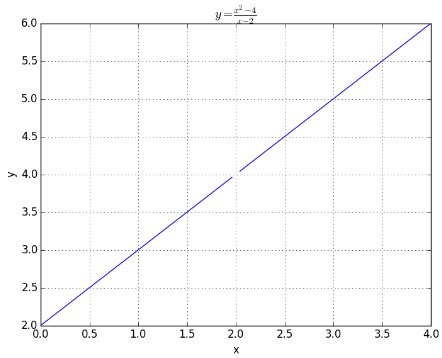

Figure 7.2: Graph of $\frac{x^2-4}{x-2}$

From the figure we see that although the graph is undefined at $x = 2$ the function approaches 4 as x approaches 2. Note also that we conclude this whether x approaches 2 from either larger or smaller values. We say that the *limit of* $f(x)$ *as* x *approaches* 2 *is equal to* 4. Mathematically we write this as follows:

$$\lim_{x \to 2} f(x) = 4$$

7.1.2 Exercises

Methods for calculating a limit depend on the problem.

1. Substitute
   (a) $\lim_{x \to 5} 2x$
   (b) $\lim_{a \to 3} \frac{9}{a}$
2. Simplify
   (a) $\lim_{x \to 0} \frac{5x}{x}$
   (b) $\lim_{t \to 1} \frac{t^2-1}{t-1}$
   (c) $\lim_{x \to -4} \frac{x^2+6x+8}{x+4}$
   (d) $\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}$

   Hint: Multiply top and bottom by $(1 + \sqrt{x})$.
3. Compute (using Julia)

   Use the following two methods to find the limits below:

i Plot a graph as shown in figure 7.2

ii Print out values

(a) $\lim_{x\to 0} \frac{x}{\sin(x)}$

(b) $\lim_{x\to 0} \frac{\sin^2(x)}{x^2}$

7.1.3 Limits at Infinity

Consider

$$\lim_{x\to\infty} \frac{1}{x}$$

Trying the substitution method we get $\frac{1}{\infty}$ which is not defined. Infinity is not a number so the substitution method cannot be used to calculate the limit. But by evaluating the function for larger and larger values of x it becomes apparent that the limit is equal to zero. This is expressed below with a few more examples of limits to infinity.

- $\lim_{x\to\infty} \frac{1}{x} = 0$
- $\lim_{x\to\infty} 3x = \infty$
- $\lim_{x\to\infty} -2x = -\infty$
- $\lim_{x\to\infty} \frac{2}{3x} = 0$ The numerator is constant while the denominator tends to infinity.
- $\lim_{x\to\infty} \frac{x^4+3x+2}{x^2+20x} = \infty$. We conclude this because the degree of the polynomial in the numerator (=4) is greater than that of the denominator(=2).
- $\lim_{x\to\infty} \frac{3x^2+2}{2x^2+10} = \lim_{x\to\infty} \frac{3+\frac{1}{x^2}}{2+\frac{10}{x^2}} = \frac{3}{2}$ since the last term in both numerator and denominator tend to zero as $x \to \infty$.

7.1.4 Exercises

1. $\lim_{x\to\infty} \frac{x^3+1}{x^2+20x}$
2. $\lim_{x\to\infty} \frac{x^2+20x}{x^3+1}$
3. $\lim_{x\to\infty} \frac{-x^3+4}{2x^2+x}$
4. $\lim_{x\to\infty} \frac{x^3+4}{-2x^2+x}$
5. $\lim_{x\to\infty}(1+\frac{1}{x})^x$ (This needs to be done by computation)

7.2 Derivatives

7.2.1 Introduction

A car travels past a certain distance marker at 1pm. The distance (in km) from the marker is recorded at all times over the next hour and is shown by the blue line in figure 7.3 below. It can be shown that this blue line can be represented by the equation $f(x) = -2x^4 + 64x$ where x represents time.

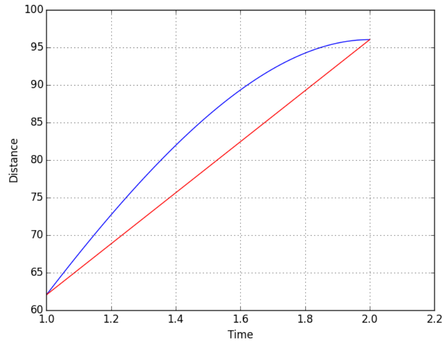

Figure 7.3: Distance travelled against time

What is the average speed of the vehicle over the first observation hour? Average speed is distance travelled divided by time elapsed and is equal to the slope of the red line shown. From the figure or by calculation { $(f(2.0) - f(1.0))/(2.0 - 1.0)$ } the average speed over the hour is 34 $km.hr^{-1}$. Following a similar procedure (see figure 7.4) we find the average speed over the interval [1.0, 1.5] to be 47.75 $km.hr^{-1}$.

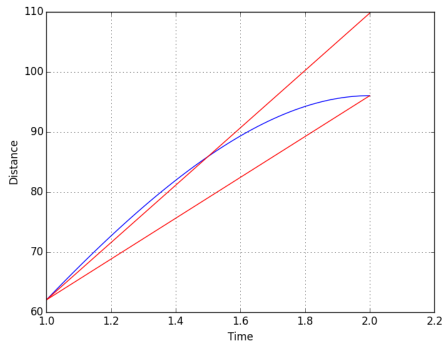

Figure 7.4: Distance travelled against time

A smaller interval [1.0, 1.1] (see figure 7.5) gives the average speed over the first 0.1 hr after 1pm as 54.72 $km.hr^{-1}$.

Traffic police might be interested in the instantaneous speed at 1pm. How do we determine that? Consider the interval [1.0, 1.0+h] where h is small. Using the same approach as previously the average velocity over this interval is given by:

$$\frac{f(1.0+h) - f(1.0)}{1.0+h-1.0} = \frac{f(1.0+h) - f(1.0)}{h}$$

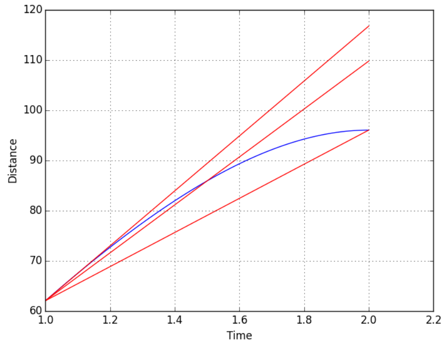

Figure 7.5: Distance travelled against time

To get the instantaneous velocity at 1pm we need to consider this quantity as h approaches zero. In other words we need to calculate the following limit.

$$\lim_{h\to 0} \frac{f(1.0+h)-f(1.0)}{h}$$

We call this the **derivative** of f at $x = 1$. Graphically this is the slope of the tangent to the function at 1pm (ie at $x = 1.0$). Evaluating:

$$\lim_{h\to 0} \frac{f(1.0+h)-f(1.0)}{h} = \lim_{h\to 0} \frac{[-2(1+h)^4+64]-[-2+64]}{h}$$

$$= \lim_{h\to 0} \frac{-2(1+4h+6h^2+4h^3+h^4)+64+64h-(-2+64)}{h}$$

$$= \lim_{h\to 0} \frac{56h-12h^2-8h^3-2h^4}{h} = 56$$

So the speed at 1pm is 56 $km.hr^{-1}$.

More generally, we define **the derivative of a function f at x by:**

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Either of the following is used to denote the derivative of $f(x)$:

$f'(x)$

$\frac{df(x)}{dx}$

Example 7.2.1 *Calculate the derivative of the function $f(x) = x^2$.*

Solution.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} 2x+h = 2x$$

7.2.2 Exercises

Use the definition, as demonstrated in the example above, to find the derivative of the following:

1. $5x$
2. x^3
3. $x^2 + 5x$.

7.2.3 Rules for Finding Derivatives

The example given above involved a simple function (x^2). Finding the derivative of log and trigonometric functions can be harder. Then there are more complicated functions such as those comprising the product or quotient of two functions. In this course we will not learn how to handle all these cases but we will need such derivatives at times. Fortunately we can access WolframAlpha
http://www.wolframalpha.com

from any browser. Simply type in "derivative x^2", for example, and the result will appear. Try the following:

derivative x^n

derivative $f(x)g(x)$

derivative $f(x)/g(x)$

derivative $f(g(x))$

derivative $exp(f(x))$

derivative $ln(f(x))$

1. $f(x) = c \quad f'(x) = 0$ where c is a constant

   Example $f = 8 \quad f' = 0$

2. $f(x) = cx^n \quad f'(x) = ncx^{n-1}$ where c and n are constants.

   Example $f(t) = 6t^5 \quad f'(t) = 30t^4$

3. $f(x) = g(x) + h(x) \quad f'(x) = g'(x) + h'(x)$

   Example $f(x) = x^3 + 6x^2 \quad f'(x) = 3x^2 + 12x$

   Example $f(u) = u^4 + 3u^3 - 5u^2 + 2u + 6 \quad f'(u) = 4u^3 + 9u^2 - 10u + 2$

4. $f(x) = g(x)h(x) \quad f'(x) = g'(x)h(x) + g(x)h'(x)$

   Example $f(x) = (x^2+3x)(x^3+1) \quad f'(x) = (2x+3)(x^3+1)+(x^2+3x)(3x^2)$

5. $f(x) = \frac{g(x)}{h(x)} \quad f'(x) = \frac{g'(x)h(x)-g(x)h'(x)}{h^2(x)}$

   Example $f(x) = \frac{x^3}{x^2+5} \quad f'(x) = \frac{3x^2(x^2+5)-x^3(2x)}{(x^2+5)^2}$

6. $f(x) = h(g(x)) \quad f'(x) = h'(g(x))g'(x)$

   Example $f(x) = (x^3 + 3x)^4 \quad f'(x) = 4(x^3 + 3x)^3(3x^2 + 3)$

7. $f(x) = e^{g(x)} \quad f'(x) = g'(x)e^{g(x)}$

   Example $f(x) = e^{x^3} \quad f'(x) = 3x^2e^{x^3}$

8. $f(x) = \ln(g(x)) \quad f'(x) = \frac{g'(x)}{g(x)}$

   Example $f(x) = \ln(x^3) \quad f'(x) = \frac{3x^2}{x^3} = \frac{3}{x}$

There are also a number of derivatives of special functions such as the trig functions but these can usually be looked up quite easily.

7.2.4 Exercises

Use WolframAlpha

http://www.wolframalpha.com

to find the derivatives of the following functions.

1. Check all your answers from the Exercises in the previous section.
2. $ln(x)$
3. $ln(x^2)$
4. $3x^2 + 4x + 5$
5. $x^3 + x$
6. $(3x^2 + 4x + 5)(x^3 + x)$. Use the results from the above and the Product Rule. Compare the result with that obtained by direct insertion into Wolfram Alpha.
7. $\frac{3x^2+4x+5}{x^3+x}$

7.3 Applications of the Derivative

Rate of Change

We have already seen that the change in distance over a change in time gives velocity. So if we have a function $d(t)$ that gives distance as a function, the derivative will give us a new function, $v(t)$, that represents velocity over time. Differentiating this will give acceleration, $a(t)$. Thus:

$$d(t) = -t^3 + 6t^2 - 0.2t$$

$$d'(t) = v(t) = -3t^2 + 12t - 0.2$$

$$v'(t) = a(t) = -6t + 12$$

These functions are shown graphically in figure 7.6 below.

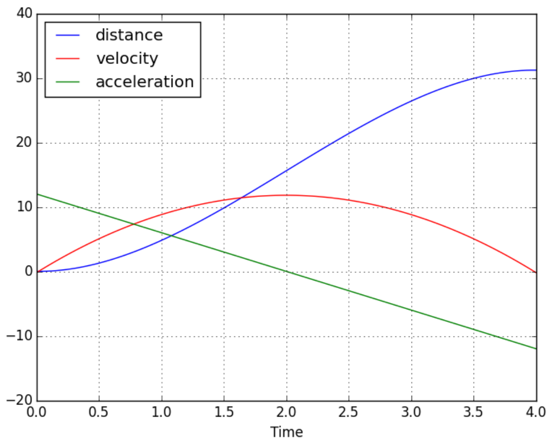

Figure 7.6: Application of the derivative

Optimisation

Observe figure 7.6 again. The acceleration is zero at $t = 2.0$. We also note that the velocity reaches a maximum at this time. What is the connection? The slope of the tangent to a curve must be zero at a local maximum. But the slope of the tangent is just the derivative.

So, in general, a local maximum of a function $f(x)$, the derivative must be zero: $f'(x) = 0$.

This is also true for minimum: if $f(x)$ has a local minimum at x_0 then $f'(x_0) = 0$.

A value of x at which $f'(x) = 0$ is called a critical point of x. Thus, maxima and minima of a function $f(x)$ can only occur ar critical points.

NOTE: The above discussion is for differentiable functions defined on $(-\infty, \infty)$. A function defined over a finite interval may have a maximum at a boundary where the derivative is non-zero (eg the velocity over the interval $[0, 1.0]$ reaches a maximum at $t = 1.0$ where the derivative is still positive). In the later case, critical points will include end points of the domain of the function.

7.3.1 Exercises

1. Rates of change
   (a) A population depends on time is given by $p(t) = t^3 - 2t + 100$. What was the growth rate at $t = 10$?
   (b) Water gushes out of a tap into a bucket. The volume of water in the bucket at any time $t < 10$ is given by $v(t) = 5t$ where v is in litres and t in minutes. What is the rate at which water gushes from the tap?
2. Find the maximum of the following functions
   (a) $-x^3 + 10.5x^2$
   (b) $-2x^2 + 6x + 12$

7.3.2 The Second Derivative

We have already implicitly introduced the concept of the second derivative. We had a function for distance travelled against time. We differentiated this and obtained velocity. Then we differentiated the velocity function to obtain acceleration. Thus velocity is the first derivative of the distance function and acceleration is the second derivative of the distance function.

Given a function $f(x)$ we denote the derivative by $f'(x)$ and the second derivative by $f''(x)$. An alternative notation given below is also used.

$$f'(x) \equiv \frac{d\,f(x)}{dx} \text{ and } f''(x) \equiv \frac{d^2 f(x)}{dx^2}$$

Application of the Second Derivative

Consider the following function shown in figure **??**:

$$f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x + 4$$

We have already noted that at a local maximum the derivative of a function is zero. But this is also true of a minimum. In figure **??**, for example, we see that the derivative (green line) is zero at both $x = 1$ and $x = 4$. In this case we can easily see that the function has a local maximum at 1 and a local minimum at 4 but it will not always be possible or easy to plot a function. So how can we know whether we have a maximum or minimum?

Optimality – Second Derivative Test

When a function has zero derivative at some point x then

- if the second derivative is negative at x, there is a local maximum at x
- if the second derivative is positive at x, there is a local minimum at x
- the test fails if the second derivative is zero at x

In summary, we have the following method for **finding local minima and maxima** of a function $f(x)$ defined over $(-\infty, \infty)$:

1. Finding all critical points (i.e. find solutions of $f'(x) = 0$)
2. Second Derivative Test:

   For each critical point x_0,

   if $f''(x_0) < 0$, there is a local maximum at x_0

   if $f''(x_0) > 0$, there is a local minimum at x_0

   if $f''(x_0) = 0$, the test fails

   Note: Local minima and maxima only occur at critical points

7.3.3 Exercises

Find the maxima and minima of the following functions:

1. $f(x) = 5x^3 + 2x^2 - 3x$
2. $f(x) = x^3 - 6x^2 + 12x - 5$ In cases where the second derivative test fails plot the function to try to understand why.