Essential Mathematics
Week 10 Notes

RMIT

Semester 2, 2018

Contents

Chapter 10

Modelling with Differential Equations

10.1 Introduction

In this chapter some basic ordinary differential equation (ODE) models will be introduced. The use of Julia to solve these models will be described. All the models will be solved in this way.

Recall that the derivative represents a *rate of change*. So $\frac{dx}{dt}$ is the rate of change of x with respect to time t. We are interested in solving **ordinary differential equation** (ODE) in the following form

$$\dot{x}(t) = f(x(t), t), \quad 0 < t < t_{\text{end}} \tag{10.1}$$

$$x(0) = x_0 \tag{10.2}$$

where $\dot{x}(t)$ is a popular notation for $\frac{dx}{dt}$ (especially when t is time), (10.1) and (10.2) are respectively the ODE and the initial condition, and (10.1) and (10.2) together is called an initial value problem. We refer to x as the state variable. Equation (10.1) is called a first order ODE as its highest order derivative involved in the equation is 1.

We call the derivatives of functions of one variable ordinary derivatives. An ordinary differential equation does not involve partial derivatives.

10.2 Modeling with 1st Order ODE

10.2.1 An Example

Consider water flowing into a bathtub at a fixed rate of r litres per minute. Let $v(t)$ represent the volume of water in the tub at any time t. Then the rate of change of the volume of water in the tub is given by the differential equation:

$$\frac{dv}{dt} = r$$

To solve this differential equation and find the volume v at any time t, the initial volume $v(0)$ and the inflow rate r need to be specified. Suppose the tub is empty initially, ie $v(0) = 0.0$ and the inflow rate is $r = 3$ litres per minute.

10.2.2 Solving the System with Julia

The problem is solved using the Julia file **ODEtub_18S2.ipynb** . Details with discussion are give below.

For this course the package **DifferentialEquations** will be used to solve all differential equation problems. We will use **Plots** to plot most of our results (Note: Currently, **PyPlot** and **Plots** do not work together!). They may not be available for installation on your laptop. So, use Juliabox. In JuliaBox, we commence our code with the following:

```
using(‘‘DifferentialEquations")
using Plots
```

The way we solve ODEs using the package DifferentialEquations is demonstrated by solving examples as shown in the following tutorial files for week 10:

```
ODEtub_18S2.ipynb
ODEexp_18S2.ipynb   (Optional)
ODEpredprey_18S2.ipynb
ODEpest_18S2.ipynb   (Optional)
Week 10 Exercise Sol.ipynb
```

10.2.3 Exercise

Suppose the bathtub plug is not closed and water runs out at a rate of 1 litre per minute. Add an outflow to the system. Do not simply reduce the inflow. Now solve again and plot the result to check that the output makes sense.

10.2.4 *Exponential Growth (Optional)

There will be no births or deaths if a population is zero, and the annual number of births will be greater in a large city than a small town. The growth rate of a population is usually some function of the population. Thus:

$$\frac{dp}{dt} = f(p) \tag{10.3}$$

The simplest case of 10.3 is where the growth rate of a population is proportional to the population.

$$\frac{dp}{dt} = r\,p \tag{10.4}$$

Some numerical examples of solution to this equation are given in the Julia file **ODEexp_18S2.ipynb**.

Is the World Population growing at an exponential rate?

Population[2]		
Years Passed	**Year**	**Billion**
-	1800	1
127	1927	2
33	1960	3
14	1974	4
13	1987	5
12	1999	6
12	2011	7
14	*2025**	8
18	*2043**	9
40	*2083**	10
* UNFPA United Nations Population Fund estimate 31.10.2011		

Figure 10.1: World Population data

It can be shown that the specific growth rate, r in equation 10.4 can be calculated from the population values at two different times, say t_0 and t_1 as follows:

$$r = \frac{1}{t_1 - t_0} \ln(\frac{p(t_1)}{p(t_0)})$$

Using this equation let us look at the data.

World Population growth data

$$\begin{bmatrix} Year & Population & Growth\ rate \\ 1200 & 0.4 & - \\ 1800 & 1 & 0.00153 \\ 1927 & 2 & 0.00546 \\ 1960 & 3 & 0.01229 \\ 1974 & 4 & 0.02055 \\ 1987 & 5 & 0.01716 \\ 1999 & 6 & 0.01519 \\ 2011 & 7 & 0.01285 \end{bmatrix}$$

Clearly, the world population is growing faster than exponential growth.

A number of environmental factors can influence the population growth rate. Potato blight in Ireland caused widespread famine and death during the 19th century. Today the population is 4.7 million still far short of the peak attained before the famine. Other influences of environmental factors affecting the growth rates include increased births nine months after the blackouts in the NE parts of the USA during the 1960's, and during the curfews in Chile during the 1970's.

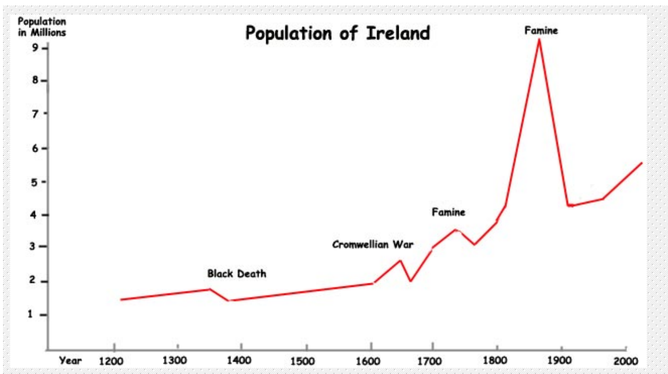

Figure 10.2: Population of Ireland

10.2.5 *Exercise (Optional)

Use the Julia file **ODEexp_18S2.ipynb** for this exercise. Use the 1800 world population as the initial value. For each case below plot the world population to 2011 and note the final population compared with the recorded 2011 population:

- the growth rate remains constant at the value recorded in 1800
- the growth from 1800 onwards is equal to that recorded in 2011

10.2.6 * Density-dependent Growth (Optional)

The simplest situation of environmental influence on growth rates is the case of density-dependent growth. Consider an island where there is only sufficient grazing produced to sustain k animals. We refer to k as the *carrying capacity*. As the population approaches k more effort will be needed by each individual to compete for their share of the food available. There would possibly be increased fighting resulting in mortality. The stressed situation might also affect fecundity rates. On the other hand when population numbers are low there will be an abundance of grazing available to an individual and growth rates per individual are likely to be at a maximum. Let r be the maximal specific growth then the differential equation for this population can be given by:

$$\frac{dp}{dt} = rpf(p) \tag{10.5}$$

The function f should have the following properties:

- $f = 1$ when $p << k$ ie no density effect at low densities
- $f \leq 1$ as r is the maximum specific growth rate
- f must decrease as density $(= p/k)$ increases
- $f = 0$ when $p = k$ ie when $p/k = 1$

The simplest function that meets these properties is the linear function $f(p) = 1 - p/k$. Substituting this into the differential equation yields:

$$\frac{dp}{dt} = rp(1 - \frac{p}{k}) \tag{10.6}$$

This is the well-known **logistic equation**. It can be used for systems where the rate of change of a system slows as the state variable approaches a limit or goal. Two examples are given in the exercises below.

10.2.7 *Exercise (Optional)

Assume the logistic equation holds for each of the following situations. Modify the Julia file **ODEexp_18S2.ipynb** to represent logistic growth. Then use your modified file to plot the time trajectory of the state variable for the following cases.

1. Suppose a toilet cistern takes 20 litres of water. Experiment with different values of r to determine the value required for a cistern to refill in 5 minutes after flushing.

2. Look at the data for ownership of *tablet computers* in the US at

   http://www.pewinternet.org/data-trend/mobile/device-ownership/

   By trial and error try to find an r and k that gives similar results to the data. What is the maximum percentage of the population that will eventually own tablet computers?

10.3 Systems of Differential Equations

10.3.1 Introduction

Where two or more state variables interact dynamically we have a system of differential equations. For a system of n state variables we write this as follows:

$$\dot{x_1} = f_1(x_1, x_2, \ldots, x_n) \tag{10.7}$$
$$\dot{x_2} = f_2(x_1, x_2, \ldots, x_n) \tag{10.8}$$
$$\ldots \tag{10.9}$$
$$\dot{x_n} = f_1(x_n, x_2, \ldots, x_n) \tag{10.10}$$

This can be written more concisely in vector form. Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

then the system equations are:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

Note: An ODE involving higher order derivatives can also be transformed to a system of equations of the form given above.

10.3.2 Using Julia to Solve Systems of Equations

The DifferentialEquations package also includes solvers for system of ODE's.

10.3.3 Predator-prey System

As an example of a system of differential equations consider a predator-prey system. Let y_1 and y_2 represent the populations of prey and predators, respectively. This system might be represented by the following differential equations:

$$\dot{y_1} = r_1 y_1 - b y_2 y_1 \tag{10.11}$$
$$\dot{y_2} = c y_2 y_1 - d y_2 \tag{10.12}$$

The first equation above states that the prey population will grow exponentially in the absence of the predator. The second term in the equation is the rate of loss due to predation. The growth rate of the predator (second equation above) depends on predation. In the absence of prey the predator dies off (last term). This system with certain parameter values has been solved in the Julia file **ODEpred-prey_18S2.ipynb** . Note that the populations oscillate and that a peak in the predator population follows a peak in the prey population.

10.3.4 Phase Plane

It is often useful to plot one state variable against another to get a phase-plane diagram. A phase-plane diagram is shown in figure 10.3. It is clear that the system cycles with both populations going through periodic peaks and troughs. By looking at the time plots in the Julia file we can deduce that the phase plane plot with time will move in an anticlockwise direction.

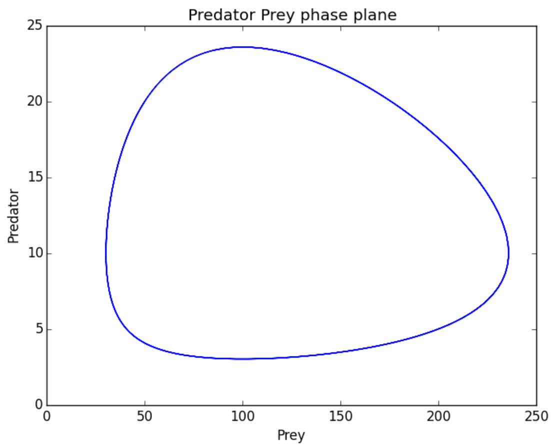

Figure 10.3: Predator-prey phase plane

10.3.5 Exercises

1. Use the predator-prey model given in the Julia file ODEpredprey_18S2.ipnb for the following:
   (a) Investigate what happens, particularly to the phase plane, with different sets of initial values
   (b) In the absence of predators the prey population grows exponentially. Modify the first term on the right hand side of the prey equation to reflect density dependent growth (as in the logistic equation). For this modified system note the change to the phase plane.
2. If two species are in *competition* for the same resource the system can be described by the following differential equations:

$$\dot{x_1} = r_1 x_1 (1 - \frac{x_1 + x_2}{k})$$
$$\dot{x_2} = r_2 x_2 (1 - \frac{x_1 + x_2}{k})$$

   Explore this system with $k = 100, r_1 = 0.9, r_2 = 1.1$ and various initial values. Do these two species enjoy a peaceful co-existence?

10.3.6 *Larger Systems - An Application (Optional)

Problem

A crop of sugarcane will be ready for harvesting in 12 months time. A stalk borer consumes the cane at a certain rate depending on the density of cane. The damage caused by the borer is unacceptable and two means of controlling this pest have been proposed: (1) biological control through the introduction of a parasitoid and (2) chemical control. The first method is much cheaper and also more desirable from environmental considerations but there is more confidence in the efficacy of chemical control. To facilitate making a decision, a model of the system with the parasitoid has been formulated. The aim of the model is to answer the following question:
Will the introduction of the parasitoid population ensure that the pest is sufficiently controlled to ensure that an acceptable biomass level of cane is achieved at harvest time?

Solution

Without parasitoids

Let us first consider the cane and borer populations which we measure in terms of biomass, x_1 and x_2, respectively. This type of system is often referred to as a 'host-parasite' problem. It can be modelled in a similar way as the predator-prey system. However, in this problem there is a short time horizon of 12 months. It will not be possible for the borer to destroy all of the cane during this period so we do not need to consider a decline in the borer population due to lack of food (cane). Thus:

$$\dot{x_1} = r_1 x_1 (1 - \frac{x_1}{k}) - a_1 x_2 \frac{x_1}{x_1 + a_2} \tag{10.13}$$
$$\dot{x_2} = r_2 x_2 \frac{x_1}{x_1 + a_2} \tag{10.14}$$

This system is implemented in the file **ODEpest_18S2.ipynb** where you can also find the parameter and initial values. This is also apparent from the phase-plane diagram 10.5.

From figure 10.4 we observe that the cane biomass after 12 months is actually lower than at the start.

With parasitoids

The borer and parasitoid form a predator prey system. The complete cane-borer-parasitoid system can be modelled by the following equations:

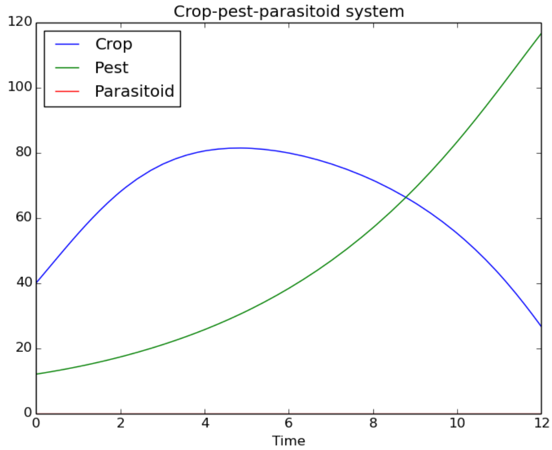

Figure 10.4: Cane borer system

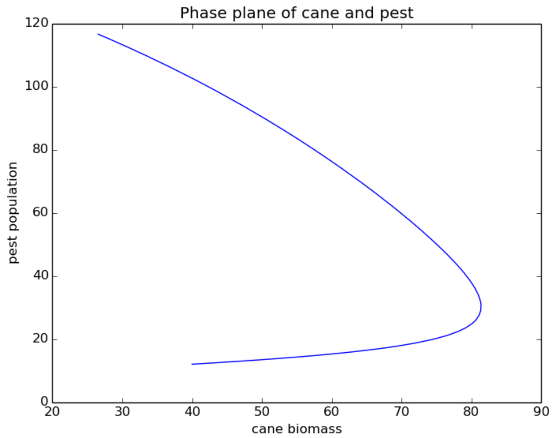

Figure 10.5: Cane borer phase plane

$$\dot{x_1} = r_1 x_1 (1 - \frac{x_1}{k}) - a_1 x_2 \frac{x_1}{x_1 + a_2} \tag{10.15}$$

$$\dot{x_2} = r_2 x_2 \frac{x_1}{x_1 + a_2} - b_1 x_3 \frac{x_2}{x_2 + b_2} \tag{10.16}$$

$$\dot{x_3} = r_3 x_3 \frac{x_2}{x_2 + b_2} - g_1 x_3 e^{g_2} \tag{10.17}$$

The file **ODEpest-18S2.ipynb** contains parameter values for this system. The effect of the parasitoid on the borer and hence the cane is shown in the figures 10.6 and 10.7 below.

10.3.7 *Exercises (Optional)

Use the file ODEpest_18S2.ipynb to solve the following problems. Read Section 8.3.6 before you start.

1. Using the procedure in earlier exercises

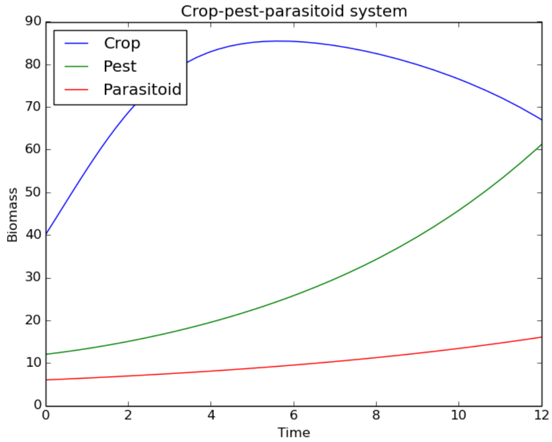

Figure 10.6: Cane, borer, parasitoid system

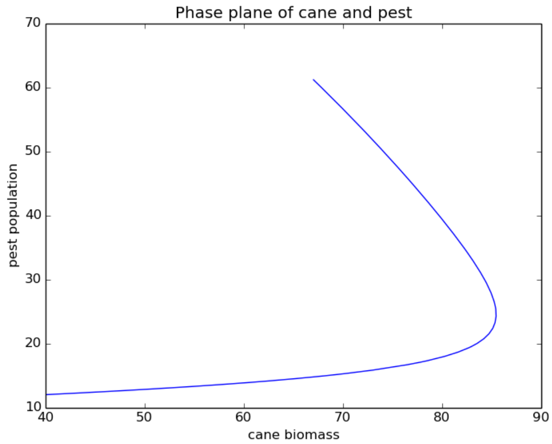

Figure 10.7: Cane borer phase plane

(a) plot the phase-plane for the borer and parasitoid;

(b) plot the phase-pane for the cane and parasitoid.

2. The sugar mill pays \$10 per unit of cane biomass. Parasitoids must be reared in a laboratory and cost \$30 per biomass unit. Is it worth releasing 6 biomass units of parasitoids? {Hint: $x[end, 1]$ will yield the final value of cane.}

3. With the prices above is it worth releasing more parasitoids? What would be a good number to release?