MATH2267 Week 4

Matrices

Semester 2, 2018

MATH2267 Week 4

Overview

Part 1

Matrix Algebra

Week 4: Matrices

Part 1: Matrix Algebra

- Terminology
- Addition, subtraction and scalar multiplication
- Multiplication
- Powers and Transpose
- Elementary row operations

Week 4: Matrices

1.1 Terminology

A matrix is a rectangular array of numbers, arranged in rows and columns.
Matrices are denotes by capital letters, and are represented by square brackets (some textbooks uses round brackets). For example, a matrix A can be give as follows.

$$A = \begin{bmatrix} 1 & 13 & 5 & 7 \\ -4 & 8 & 25 & 6 \\ 9 & 0 & -7 & 2 \end{bmatrix}$$

A has 3 rows and 4 columns.
1×1 matrix treated number, doesn't need brackets.
Common terms:

- elements: numbers in array, $(2,3)$ element is 25
- rows: 3 rows in example, each has 4 elements
- columns: 4 columns in example, each has 3 elements
- order: matrix of order $m \times n$ has m rows and n columns

Matrices in Special Shapes

- zero matrix: all elements are zero
- square matrix: number of rows $=$ number of columns

  Example: 2×2 square matrix $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$
- column matrix: number of columns $= 1$

  Example: 2×1 column matrix $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
- row matrix: number of rows $= 1$

  Example: 2×2 square matrix $\begin{bmatrix} 3 & 5 & 7 \end{bmatrix}$
- two matrices are equal: if they have same size and same elements in same position

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 9 & 0 \\ 3 & 1 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 5 \\ 6 & 9 \end{bmatrix}$$

Addition

$$A + B = \begin{bmatrix} 8 & 11 & 3 \\ 7 & 6 & 10 \end{bmatrix}$$

Subtraction

$$A - B = \begin{bmatrix} -6 & -7 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$

Note: $A + C$ does not exist. Exists only when have same order.

Multiplication by scalar

$$7A = \begin{bmatrix} 7 & 14 & 21 \\ 28 & 35 & 42 \end{bmatrix}.$$

Multiplication

Compatible matrices A and B has product AB:

$$A \text{ is } m \times p,\ B \text{ is } p \times n \Rightarrow AB \text{ is } m \times n$$

AB defined $\iff A_{\text{number of columns}} = B_{\text{number of rows}}$

First, row matrix multiply column matrix:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 4 + 2 \times 5 + 3 \times 6 \end{bmatrix} = 32$$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i$$

Week 4: Matrices

1.3 Matrix Multiplication

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix}$$

The 1st row of A multiply the 3rd column of B:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix} = 1 \times 5 + 2 \times 8 + 3 \times 2 = 27$$

Week 4: Matrices

1.3 Matrix Multiplication

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix}$$

The product AB is a 2×3 matrix.

$(i,\ j)$ element of AB = (ith row of A) multiply (jth column of B)

Thus,

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} & & \\ & & \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix}$$

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Thus,

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix}$$

The product AB is a 2×3 matrix.

$(i,\ j)$ element of AB = (ith row of A) multiply (jth column of B)

Thus,

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 23 & \\ & & \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix}$$

The product AB is a 2×3 matrix.

$(i,\ j)$ element of AB = (ith row of A) multiply (jth column of B)

Thus,

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 23 & 27 \\ & & \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix}$$

The product AB is a 2×3 matrix.

$(i,\ j)$ element of AB = (ith row of A) multiply (jth column of B)

Thus,

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 23 & 27 \\ 43 & & \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix}$$

The product AB is a 2×3 matrix.

$(i,\ j)$ element of AB = (ith row of A) multiply (jth column of B)

Thus,

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 23 & 27 \\ 43 & 65 & \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix}$$

The product AB is a 2×3 matrix.

(i, j) element of AB = (ith row of A) multiply (jth column of B)

Thus,

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 & 5 \\ 3 & 1 & 8 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 23 & 27 \\ 43 & 65 & 72 \end{bmatrix}$$

Noncommutativeness

$$A = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = [34].$$

$$BA = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 10 \\ 6 & 12 & 30 \\ 4 & 8 & 20 \end{bmatrix}.$$

Thus, $AB \neq BA$ in general.

Week 4: Matrices

1.4 Powers and Transpose

- $A^2 = \underbrace{AA}_{2}$ if A is square
- $A^n = \underbrace{AAA\ldots A}_{n}$
- Associativeness $A(BC) = (AB)C$
- Transpose of A: A^T (In Julia, use A' for A^T)

$$A = \begin{bmatrix} 6 & 3 & 5 \\ 4 & 7 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 6 & 4 \\ 3 & 7 \\ 5 & 2 \end{bmatrix}$$

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$

Week 4: Matrices

1.5 Special Square Matrices

- Diagonal of $A = \begin{bmatrix} 7 & 9 & 6 \\ 3 & 1 & 5 \\ 0 & 4 & 2 \end{bmatrix}$ is the red colored elements.
- Diagonal matrix: square matrix whose non-diagonal elements are 0.
- The $n \times n$ identity matrix, denoted by I or I_n, is the $n \times n$ diagonal matrix with diagonal elements equal 1.

  $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the 3×3 identity matrix.

  $IA = A$, $BI = B$ are always true if the products are define.
- symmetric matrix: A is symmetric if $A^T = A$

Week 4: Matrices

1.6 Elementary Row Operations (EROs)

There are 3 EROs.

ERO 1. Interchange of two rows

Example. Interchanging row 2 and row 3 of

$$A = \begin{bmatrix} 1 & -5 & 2 \\ -2 & 1 & -3 \\ 5 & 4 & 8 \end{bmatrix}$$

results in

$$A_1 = \begin{bmatrix} 1 & -5 & 2 \\ 5 & 4 & 8 \\ -2 & 1 & -3 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 & 2 \\ -2 & 1 & -3 \\ 5 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 2 \\ 5 & 4 & 8 \\ -2 & 1 & -3 \end{bmatrix} = A_1$$

ERO 2. Multiplying one row by a NON-ZERO number

Example. Multiplying row 2 of

$$A = \begin{bmatrix} 1 & -5 & 2 \\ -2 & 1 & -3 \\ 5 & 4 & 8 \end{bmatrix}$$

by 4 results in

$$A_2 = \begin{bmatrix} 1 & -5 & 2 \\ -8 & 4 & -12 \\ 5 & 4 & 8 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & 2 \\ -2 & 1 & -3 \\ 5 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 2 \\ -8 & 4 & -12 \\ 5 & 4 & 8 \end{bmatrix} = A_2$$

ERO 3. Adding a multiple of one row to another row

Example. Adding -5 times Row 1 to Row 3 in

$$A = \begin{bmatrix} 1 & -5 & 2 \\ -2 & 1 & -3 \\ 5 & 4 & 8 \end{bmatrix}$$

results in

$$A_3 = \begin{bmatrix} 1 & -5 & 2 \\ -2 & 1 & -3 \\ 0 & 29 & -2 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -5 & 2 \\ -2 & 1 & -3 \\ 5 & 4 & 8 \end{bmatrix}=\begin{bmatrix} 1 & -5 & 2 \\ -2 & 1 & -3 \\ 0 & 29 & -2 \end{bmatrix}=A_3$$

EROs may be indicated as follows

$$\begin{bmatrix} 1 & -5 & 2 \\ 3 & -14 & 3 \\ 4 & -5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & -3 \\ 0 & 15 & -5 \end{bmatrix} \qquad \begin{array}{l} R_2 - 3R_1 \to R_2 \\ R_3 - 4R_1 \to R_3 \end{array}$$

$$\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & -3 \\ 0 & 3 & -1 \end{bmatrix} \qquad \tfrac{1}{5}R_3 \to R_3$$

$$\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 8 \end{bmatrix} \qquad R_3 - 3R_2 \to R_3$$

Part 2

Inverse Matrix and Determinant

Week 4: Matrices

Part 2: Inverse Matrix and Determinant

- Finding inverse matrix
- Determinant and properties
- Matrix in reduced row echelon form (RREF)

Week 4: Matrices

Part 2.1: Finding Inverse Matrix

A: square matrix

$$A \text{ invertible} \Leftrightarrow \exists\, B \text{ such that } AB = BA = I$$

To find A^{-1}:

1. $\boxed{\text{If } [\,A \mid I\,] \xrightarrow{EROs} [\,I \mid B\,] \Rightarrow B = A^{-1}}$

2. $\boxed{A \xrightarrow{EROs} I \text{ impossible} \Rightarrow A \text{ not invertible.}}$

Part 2.1: Finding Inverse Matrix

Example. To find A^{-1} for

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Solution. Start with $[\,A \mid I\,]$:

$$\left[\begin{array}{cc|cc} \boxed{1} & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

Purpose is to reduce A to I by EROs

$$\Longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & \boxed{-3} & -2 & 1 \end{array}\right] \qquad R_2 - 2R_1$$

$$\Longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right] \qquad \begin{array}{l} R_1 + (2/3)R_2 \\ R_2/(-3) \end{array}$$

$$\Longrightarrow A^{-1} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$$

Example. $A = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 1 & 3 \\ 3 & 1 & 5 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

Obtain 1 at (1,1) and 0 under it

$$\begin{array}{r} (-1)R_1 \to R_1 \\ R_2 + 2R_1 \to R_2 \\ R_3 + 3R_1 \to R_3 \end{array} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 \end{array}\right]$$

Example continued...

Obtain 1 at (2,2) and 0 under it

$$R_3 - R_2 \to R_3 \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

Obtain 1 at (3,3) and 0 above it

$$\begin{array}{l} R_1 - R_3 \to R_1 \\ R_2 - R_3 \to R_2 \end{array} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\Longrightarrow$$

$$A^{-1} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Week 4: Matrices

Part 2.2: Determinants

Consider square matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The **determinant** of A, denoted by

$$\det(A), \ \det A, \ |A|, \ \text{or} \ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Notations apply to higher order determinants.

Week 4: Matrices

Part 2.2: Determinants

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

A 3×3 determinant can be expressed in terms of 2×2 determinants:

- Using 1st row:

$$\det A = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

- Or using other rows or columns (see examples).

Example.

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 6 & 2 \end{vmatrix} = 2\begin{vmatrix} 4 & 7 \\ 6 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 7 \\ 0 & 2 \end{vmatrix} + 5\begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix}$$

$$= 2(8-42) - 3(2-0) + 5(6-0) = -44$$

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 6 & 2 \end{vmatrix} = -1\begin{vmatrix} 3 & 5 \\ 6 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix}$$

$$= -1(6-30) + 4(4-0) - 7(12-0) -44$$

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 6 & 2 \end{vmatrix} = 0\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} - 6\begin{vmatrix} 2 & 5 \\ 1 & 7 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= 0 - 6(14-5) + 2(8-3) = -44$$

Example continued... Expand along a column

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 6 & 2 \end{vmatrix} = 2\begin{vmatrix} 4 & 7 \\ 6 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 6 & 2 \end{vmatrix} + 0\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix}$$

$$= 2(8-42) - 1(6-30) + 0 = -44$$

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 6 & 2 \end{vmatrix} = -3\begin{vmatrix} 1 & 7 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} - 6\begin{vmatrix} 2 & 5 \\ 1 & 7 \end{vmatrix}$$

$$= -3(2-0) + 4(4-0) - 6(14-5) = -44$$

Adding sign according to $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ i.e. $(-1)^{\text{row+column}}$

To save time expand by the row or column with the most zeros.

Properties of determinants

- Some EROs on A can effect the value of $|A|$.
- For square matrices A and B of same size,

$$\det(AB) = \det(A)\det(B)$$

Especially, $\det(I) = \det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$. Hence

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

- For square matrix A,

$$A \text{ invertible} \iff \det(A) \neq 0$$

Week 4: Matrices

Part 2.3: Reduced Row Echelon Form

Row Echelon Matrix

A matrix A is a row echelon matrix if

1. The first non-zero element in a row, named leading entry, is 1.
2. Each leading 1 is in a column to the right of the leading 1 in the previous row.
3. Rows of zero elements, if any, are below non-zero rows.

Reduced Row Echelon Matrix

A matrix A is a reduced row echelon matrix if

1. It is a row echelon matrix.
2. Any leading 1 is the only non-zero element in its column.

Example of row echelon matrix

$$\begin{bmatrix} 1 & 5 & -12 \\ 0 & 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -4 & \frac{7}{2} \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of reduced row echelon matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any matrix A is reducible to rref by EROs.
Note: The rank of A, denoted by rank(A), is defined as

"the number of non-zero rows in the RREF of A"

Finding the rank of A by Julia:

```
rank(A)      #(shift+inter)
```

Next Week

- Vectors

  Introduction

  Basic operations

  Dot product, projections

  Cross product
- More Matrices

  Eigenvalues and eigenvectors

  Leslie matrix