Essential Mathematics
Week 2 Notes

RMIT

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Contents

2 Factorization, Sequences and Series **1**

2.1 Polynomial Factorization 1

2.1.1 Exercise 4

2.2 Sequences and Series 5

2.2.1 Arithmetic Sequence 5

2.2.2 Series 5

2.2.3 Sum of an AP 5

2.2.4 Exercises 6

2.2.5 Geometric Progression 6

2.2.6 Exercises 7

2.3 Sum of Geometric Progression 7

2.3.1 Exercises 8

Chapter 2

Factorization, Sequences and Series

2.1 Polynomial Factorization

In mathematics and computer algebra, factorization of polynomials is the process of expressing a polynomial as the product of irreducible factors. For example, $x^2 - 1 = (x-1)(x+1)$ so the polynomial $x^2 - 1$ is factorized as the product of $x - 1$ and $x + 1$, both of which are irreducible.

Example 2.1.1 *Consider the polynomial $P(x) = x^3 + x^2 - x - 1$. It can be expressed as*

$$\begin{aligned} P(x) &= (x^3 - x) + (x^2 - 1) = x(x^2 - 1) + (x^2 - 1) \\ &= (x+1)(x^2 - 1) = (x+1)^2(x-1) \end{aligned} \tag{2.1}$$

We say that $P(x)$ has factors $x + 1$, $x - 1$, $x^2 - 1$, $x^2 + 2x + 1$ $(= (x+1)^2)$ and $P(x)$ it self. The factors $x + 1$ and $x - 1$ are irreducible as they cannot be further factorized. On the other hand, the factor $x^2 + 2x + 1$ is reducible as it can be further factorized as $x^2 + 2x + 1 = (x + 1)(x + 1)$.

In mathematics and computer algebra, the factorization of polynomials is the expression of a polynomial as the product of irreducible factors. Thus, the starting example of this section gives an example of factorization. There are several methods for factorizing a polynomial. There is no one method for doing this in general. One of the factorization methods is to find the zeros of the polynomial (a zero of a polynomial $P(x)$ is a root of the equation $P(x) = 0$), as we will see below through examples.

A linear term $x - a$ is a factor of polynomial $P(x)$ if and only if $x = a$ is a root of $P(x) = 0$. Linear factors are always irreducible.

It is easy to factorize a quadratic function.

Example 2.1.2 *Factorize $x^2 + 5x + 6$.*

Solution. In order to factorize $x^2 + 5x + 6$, we consider that $x^2 + 5x + 6 = 0$ has roots

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-5 + \sqrt{25 - 24}}{2} = -2,$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-5 - \sqrt{25 - 24}}{2} = -3.$$

Thus, both $x + 2$ and $x + 3$ are factors the polynomial and $P(x) = x^2 + 5x + 6 = (x + 2)(x + 3)$.

If a quadratic function has no root, then it cannot be factorized.

Example 2.1.3 *Factorize* $x^2 + x + 6$.

Solution. The discriminant of $x^2 + x + 6 = 0$ is

$$b^2 - 4ac = 1^2 - 4 \times 1 \times 6 = -23 < 0.$$

Thus, $x^2 + x + 6 = 0$ doesn't have a real solution. Hence, $x^2 + x + 6$ cannot be further factorized.

For polynomials of order 3, the situation becomes more complicated. One can first find the the roots of a given polynomial. Sine $(x - a)$ is factor of a polynomial $P(x)$ if and only if $x = a$ is a root of $P(x)$, we can use the roots to factorize in some cases. Polynomial roots can be found two ways using Julia: by the 'roots' command using the **Polynomials** package, and by plotting the polynomial. We show these methods by by examples.

Example 2.1.4 *Factorize* $P(x) = x^3 - x$.

Solution. Input the following lines in Julia:

```
using Polynomials
roots(Poly([0,-1,0,1]))
```

then press **Shif+Return** you will get the roots:

```
Out [2]: 3-element Array{Float64,1}:
          -1.0
           1.0
           0.0
```

This indicates that the roots of $P(x) = x^3 - x$ are -1, 1 and 0. Thus, $P(x) = x(x - 1)(x + 1)$.

Example 2.1.5 *Factorise* $P(x) = x^3 + x^2 - x - 1$.

Solution.

```
using Polynomials
roots(Poly([-1,-1,1,1]))
```

then press **Shif+Return** you will get the roots:

```
Out [2]: 3-element Array{Float64,1}:
           1.0
          -1.0
          -1.0
```

This indicates that the roots of $P(x) = x^3 - x$ are 1, -1 and -1. Thus, $P(x) = (x+1)^2(x-1)$

In case the number of zeros found is less than the order of the polynomial, one can factorize the polynomial as in the following examples.

Example 2.1.6 *Factorize* $P(x) = x^3 - 3x^2 + 3x - 1$.

Solution. In this example, there is only one zero $x = 1$ found in the graph (Figure 2.1). This is a case where the third order polynomial has ONE solution $x = 1$. However, the x-axis is TANGENT to the curve at the zero $x = 1$. In this case we conclude that $x = 1$ is a zero that is repeated 3 times, and

$$P(x) = (x-1)^3$$

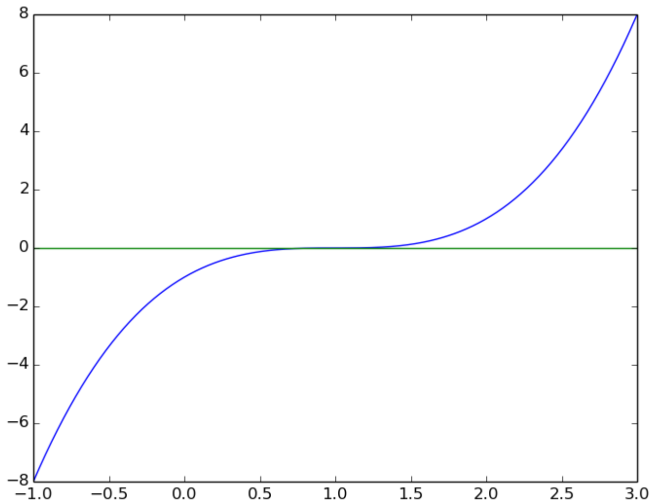

Figure 2.1: Finding zeros by Julia

Example 2.1.7 *Factorize* $P(x) = x^3 - x^2 + x - 1$.

Solution. Again, there is only one zero $x = 1$ found by the graphic method (Figure 2.2). This is a case where a third order polynomial has only ONE solution $x = 1$. However, the x-axis is NOT TANGENT to the curve at the zero $x = 1$. In this case we conclude that $x = 1$ is a zero that is not repeated. The other factor of the polynomial can be found using long division:

$$\begin{array}{r|rrrrr}
 & x^2 & +1 & & & \\
\hline
x-1 & x^3 & -x^2 & +x & -1 & \\
 & x^3 & -x^2 & & & - \\
\hline
 & & & +x & -1 & \\
 & & & x & -1 & - \\
\hline
 & & & 0 & 0 &
\end{array}$$

$$P(x) = (x-1)(x^2+1)$$

in which it is clear that x^2+1 is irreducible.

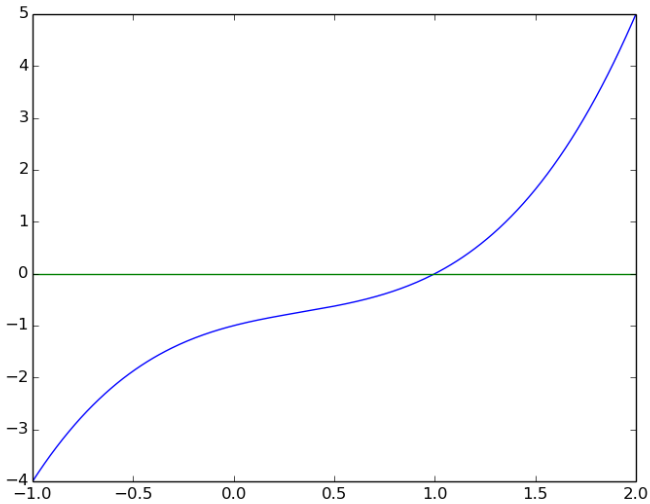

Figure 2.2: Finding zeros by Julia

2.1.1 Exercise

Factorize each of the given polynomials using given method.

1. $x^2 - 18x + 80$
2. $x^3 + 8$
3. $3x^4 - 3x^3 - 36x^2$
4. $x^4 - 25$
5. $x^4 + x^2 - 20$

2.2 Sequences and Series

2.2.1 Arithmetic Sequence

A sequence is a set of numbers in some order. For example,

$$2, 4, 6, ...$$

is an infinite sequence of even numbers. A sequence that has a common difference between any two adjacent terms is known as an arithmetic sequence or *arithmetic progression (AP)* . In the example:

$$3, 8, 13, 18, 23$$

the number 5 is the common difference between any two adjacent numbers in the sequence. In general, if the first term of an AP is a and the common difference is d then the AP is:

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+(n-1)d$$

2.2.2 Series

The *sum* of terms in a sequence is known as a *series*. Suppose $t_i, i = 1, 2, \ldots, n$ are terms of a sequence then

$$t_1 + t_2 + \cdots + t_n$$

is a series.

Example 2.2.1 *Sara has a rich grandfather who gives her $5000 on her 21st birthday. He also promises to give her more money every year on her birthday until she is 30 years old. He plans to increase the present by an extra $1000 each year compared with the amount of the previous year. How much money has Sara received in total by the time she has turned 30?*

Solution Let us work in thousands of dollars. We need to find the sum: $5 + (5+1)+(5+2)+(5+3)+(5+4)+(5+5)+(5+6)+(5+7)+(5+8)+(5+9)$ This is simple enough to calculate but very tedious. In the next section we find a more efficient way to sum an AP.

2.2.3 Sum of an AP

Suppose we have an AP with n terms, a first term a and a common difference d. Denoting the sum by S we get:

$$S = a + [a+d] + [a+2d] + \cdots + [a+(n-2)d] + [a+(n-1)d]$$

We can also write the terms in reverse order.

$$S = [a + [n-1)d] + [a + (n-2)d] + \cdots + [a + 2d] + [a + d] + a$$

If we now add these two sums together we get $2S$ on the left hand side. On the right hand side we add the first term of the top series to the first term of the second series, then the second terms, and so on. Thus:

$$2S = [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d] + [2a + (n-1)d]$$

We note that we have n identical terms on the right hand side. Dividing both sides by 2 we get;

$$S = \frac{n}{2}[2a + (n-1)d]$$

This can also be written in the following way:

$$S = \frac{n}{2}[a + a + (n-1)d]$$

It is clear that this is just $\frac{n}{2}$ multiplied by the sum of the first term and last term of the series ie the sum of a and $[a + (n-1)d]$.

Example 2.2.2 *Let us now return to the example of Sara in the previous section. In this case $a = 5$ and the common difference between successive terms is $d = 1$. We have 10 payments so $n = 10$. The total amount Sara receives is thus:*

$$S = \frac{10}{2}(5 + 14) = 95 \textit{ (thousand dollars)}$$

2.2.4 Exercises

Find the sum of the first 30 terms of the sequence 2,4,6,... Ans: 930

2.2.5 Geometric Progression

Example 2.2.3 *Dube recently inherited \$15k. He invests this sum in an account that pays 10% interest each year. This amount is added to the sum already invested until the account is closed and all the money paid out. Dube would like to know how much money he would receive if he closed the account after five years.*

Solution At the end of the first year Dube will have \$15k plus the interest earned for one year ie 15k(1 +0.1) = 16.5k. Similarly at the end of the second year he will have $16.5k(1 + 0.1)$ or written another way $1k(1 + 0.1)^2$. Starting with the current year, let us write down the sequence of amounts in the account at the end of each year (in thousands of dollars).

$$15, 15(1 + 0.1), 15(1 + 0.1)^2, \cdots , 15(1 + 0.1)^5$$

The last term is the amount Dube will collect after five years. This is known as the **Future Value** and the \$15k is called the **Present Value**. More generally with an annual interest rate i, the relationship between the present value (PV) and the future value (FV) after n years is given by:

$$FV = PV(1+i)^n$$

and

$$PV = FV(1+i)^{-n}$$

The sequence above is what is known as a geometric progression (GP). You might notice there is a common ratio between any two consecutive elements of the sequence. In general, if a is the first term and r is the common ratio then the geometric progression can be expressed as follows:

$$a, ar, ar^2, ar^3, \cdots$$

where the nth term is given by ar^{n-1}.

2.2.6 Exercises

1. Write down the first few terms of a GP which starts with 5 and has a common ratio of 2.
2. Is the sequence $2, -6, 18, -54, \cdots$ a GP and, if so, what is the common ratio?
3. A GP has first term 4 and common ration 2. What is the 20th term?

Ans: **1.** 5, 20, 40, $\cdots$, **2.** yes, $r = -3$, **3.** $4 * 2^{19} = 2097152$

2.3 Sum of Geometric Progression

Let S_n denote the sum of the first n terms of a GP. Then:

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Now multiply each term by r

$$rS_n = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

Subtracting

$$S_n - rS_n = a - ar^n$$

We can now solve for S_n to get

$$S_n = \frac{a(1-r^n)}{1-r}$$

Of course, this will not hold for $r = 1$ but in this case the formula would not be needed anyway.

Example 2.3.1 *Suppose Dube deposits an extra \$15 (thousand) each year then* $a = 15$ *and* $r = 1.1$*. Thus after five years Dube would have:*

$$S_6 = \frac{15(1 - 1.1^6)}{1 - 1.1} \text{ (thousand dolars)} = \$115,734.15 \text{ (dolars)}$$

Regular payments into an interest-earning account is known as an **annuity** . The financial formulae all work for periods of other durations such as monthly. *But the interest must be expressed in terms of the period.* For example, an annual interest rate of 6% compounded monthly for the purposes of the formula needs to be converted to 0.06/12 =0.005 and n would now represent the number of months.

2.3.1 Exercises

1. How much will an investment of \$1000 at 5% annual interest rate pay out after 10 years?
2. A bank is offering an annual interest rate of 5% compounded monthly. What will it pay out for an investment of \$1000 after 10 years?
3. Ruby will receive a gift of \$1000 in five years time. What is the PV of the gift given that the current cost of living is increasing at 3% per annum?
4. Jim is saving up to buy a car. He puts \$300 per month into a savings account that pays 6% annual interest compounded monthly. How long will it take to reach his target of \$15000?