MATH 2267
Week 1

Sets, Functions and Inequalities

Overview

- Part 1: Sets
  - What are sets?
  - How can we use them?
- Part 2: Functions
  - What are functions?
  - Types of Functions
  - Standard Functions and their properties
    - Exponential
    - Logarithm
    - Trigonometric Functions
- Part 3: Inequalities
  - An introduction
  - Solving inequalities

Part 1: Sets

What are sets?

- A set is <u>*a collection of distinct objects called elements*</u>

- Examples:
  - $\{1,2,3,4\}$ is a set of numbers
  - {apples, bananas, figs, oranges} is a set of fruits

- These are **NOT** sets:
  - {all young workers} ⟵ objects not clearly specified
  - $\{1,2,3,4,\sqrt{9}\}$ ⟵ objects not unique ($\sqrt{9} = 3$)

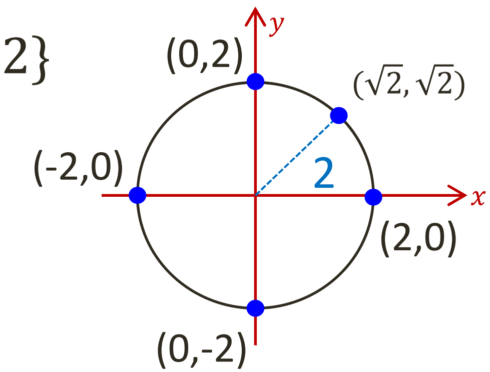
Part 1: Sets

How do we write them?

- List all of the objects:
  - $\{1,2,3,4\}$
  - $\{2,4,6,8, \dots\}$

- Use Set-Builder Notation:
  - {*variables* | *conditions that elements of set must satisfy*}

  - $\{n \mid n \text{ is a positive integer, and n } \leq 4\}$ = {1,2,3,4}
  - $\{x \mid -2 \leq x < 4\}$
  - $\{(x, y) \mid \sqrt{x^2 + y^2} = 2\}$



Part 1: Sets
Standard Sets

- ϕ is the empty set
  - Contains no elements
  - Every set contains the empty set
- $\mathbb{N}$ is the set of natural numbers $\{1,2,3,\ldots\}$
  - Some textbooks include 0, so be careful
- $\mathbb{Z}$ is the set of integers $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$
- $\mathbb{Q}$ is the set of rational numbers
  - Can be written as $\frac{a}{b}$ where a and b are integers
- $\mathbb{R}$ is the set of real numbers
  - Any “standard” number (that you’ll come across in this course)

Part 1: Sets

Standard Sets

- Can specify points in 2-, 3- or higher-dimensional space:
  - $\mathbb{R}^2$ is the set of all real points on a 2-dimensional plane (x,y)
  - $\mathbb{R}^3$ is the set of all real points in a 3-dimensional space (x,y,z)

- Can do the same with other sets:
  - $\mathbb{N}^2$ would be the set of pairs of natural numbers (e.g. (1,1), (1,2))
  - $\mathbb{R} \times \mathbb{Q}$ would be the set of pairs of a real number with a rational number (e.g. (1,1), $(\pi, 0.5)$, etc)

- Use in Set-Builder Notation:
  - Declare what sort of values we want:
    - $\{n \in \mathbb{N} \mid \mathrm{n} \ \leq 4\}$ = $\{1,2,3,4\}$
    - $\{\ (x, y) \in \mathbb{R}^2 | \sqrt{x^2 + y^2} = 2\}$
  - Don't have to use this style:
    - $\{x \in \mathbb{Z} \mid x < 3\} = \{x \mid x \in \mathbb{Z} \text{ and } x < 3\}$

∈ means "element of"

Part 1: Sets

Sets of Real Numbers

- Can write intervals in shorthand
  - $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
  - $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

  - $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
  - $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

  - $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$
  - $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$

  - $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$
  - $[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$

(means "don't include end point"

[means "include end point"

Part 1: Sets

Comparing Sets

- $\in$ means "element of"
- $\notin$ means "not an element of"

- $\subseteq$ means subset, or "is contained by"
  - $\mathbb{N} \subseteq \mathbb{Z}$
  - $\{1,2,3\} \subseteq \{1,2,3\}$
  - $A \subseteq A$
- $\subsetneq$ means proper subset, or "is contained by, but not equal to"
  - $\{1,2,3\} \subsetneq \{1,2,3,4\}$
  - $\mathbb{N} \subsetneq \mathbb{Z}$
- $\subset$ means one of these, depending on how things are written
  - Different people use it slightly differently, so be careful
  - We'll be using it as being meaning proper subset (same definition as $\subsetneq$)

- Supersets, meaning "contains"
  - $\supseteq$ (Superset) : $\{1,2,3,4\} \supseteq \{1,2,3,4\}$
  - $\supsetneq$ (Proper Superset) : $\{1,2,3,4\} \supsetneq \{1,2,3\}$
  - $\supset$ is equally ambiguous as $\subset$: For this class, use it as meaning $\supsetneq$

Part 1: Sets

Operations on Sets

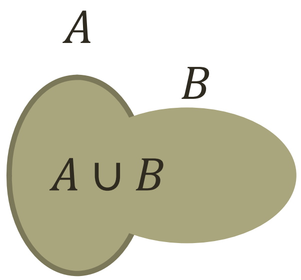
- $A \cup B$, the **union** of A and B, equals:
  - $\{x \mid x \in A \text{ or } x \in B\}$
- Union of infinitely many sets:

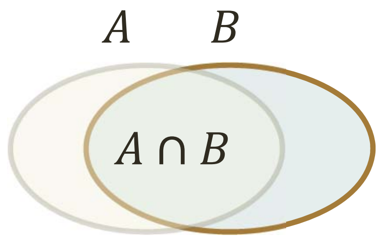
$$A \cup B \cup C \cup D \cup \ \ldots$$

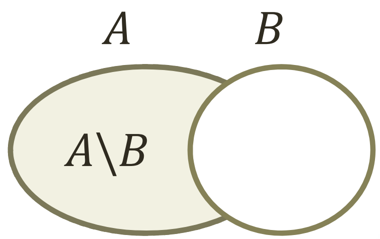
- $A \cap B$, **intersection** of A and B, equals
  - $\{x \mid x \in A \text{ and } x \in B\}$
- Intersection of infinitely many sets:

$$A \cap B \cap C \cap D \cap \ \ldots$$

- $A \backslash \mathrm{B}$, difference of A and B, equals
  - $\{x \mid x \in A \text{ and } x \notin B\}$







Part 1: Sets

Examples

- Write the elements of $\{x \in \mathbb{N} \mid x < 5\}$
  - $\{1,2,3,4\}$

- Are the following statements true?
  - $\pi \in \mathbb{Q}$
  - $(1, \sqrt{2}) \in \mathbb{R}^2$
  - $0.5 \in \{x \in \mathbb{Z} \mid 2x = 1\}$
  - $\frac{1}{3} \in \{x \in \mathbb{R} \mid 9x^2 = 1\}$

Part 1: Sets
Examples

- Write the elements of $\{x \in \mathbb{N} \mid x < 5\}$
  - $\{1,2,3,4\}$

- Are the following statements true?
  - $\pi \in \mathbb{Q}$ FALSE
  - $(1, \sqrt{2}) \in \mathbb{R}^2$ TRUE
  - $0.5 \in \{x \in \mathbb{Z} \mid 2x = 1\}$ FALSE ($0.5 \notin \mathbb{Z}$, RHS is empty)
  - $\frac{1}{3} \in \{x \in \mathbb{R} \mid 9x^2 = 1\}$ TRUE

Part 1: Sets

Examples

- Describe all the subsets of each of {dog, cat, budgie}
  - ϕ, {dog}, {cat}, {budgie}, {dog, cat}, {dog, budgie}, {cat, budgie}, {dog, cat, budgie}

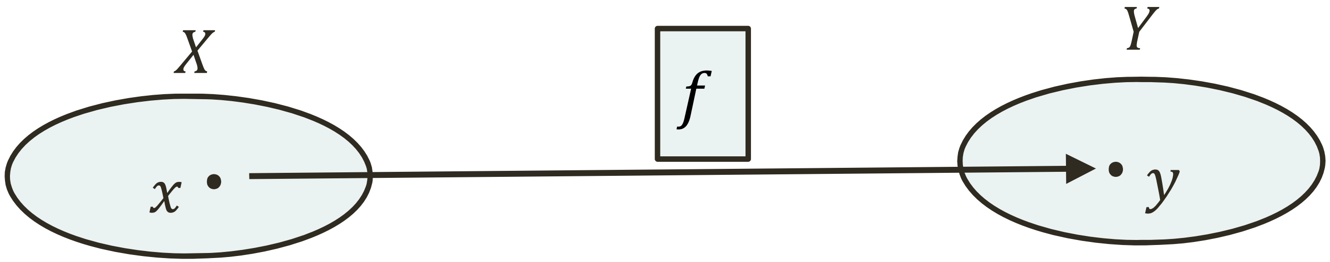
- Using interval notation, write:
  - $\mathbb{R} \setminus \{3\}$
    - $(-\infty, 3) \cup (3, \infty)$
  - $\{x \in \mathbb{R} \mid x^2 < 4\}$
    - $(-2,2)$
  - $\{x \in \mathbb{R} \mid x(x-1) \geq 0\}$
    - $(-\infty, 0] \cup [1, \infty)$

MATH 2267
Week 1: Part 2

Functions

Part 2: Functions

Notation and Terminology



- X, Y are two sets
- A function $f: X \to Y$ is a relation such that for each $x \in X$, f assigns a unique $y \in Y$ to it, denoted by $y = f(x)$.
- x is called the independent variables
- y is called the dependent variable
- X is the domain of f
- $R = \{y \in Y | y = f(x) \text{ for some } x \in X\}$ is called the range of f
- A function $f: X \to Y$ is called a real function or real-valued function.

Part 2: Functions

Examples of functions

- Power function (p is some real constant)

$$f(x): D \to \mathbb{R}, \qquad f(x) = x^p$$

  the domain $D \subset \mathbb{R}$ depends on the value of p

- Polynomial

  For $a_0, a_1, \dots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$, function $p(x): \mathbb{R} \to \mathbb{R}$ given by

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ \dots + a_1 x + a_0$$

  Is call a polynomial, its order is n if $a_n \neq 0$.

- Rational function ($p(x)$ and $q(x)$ are polynomials)

$$r(x): \mathbb{R} \to \mathbb{R}, \qquad r(x) = \frac{p(x)}{q(x)}$$

Part 2: Functions

Some notes on finding domain/range

- If you can't calculate a function's value at a point, then that point isn't in the domain. e.g.
  - $f(x) = \frac{\sin(x)}{x}$ has domain $\mathbb{R}\backslash\{0\}$, because we can't divide by zero
  - $g(x) = \sqrt{x}$ has domain $[0, \infty)$, because we can't take the square root of a negative number
    - This holds for real numbers. Complex numbers let you do this, but that's outside of this course

- The range can often be determined by plotting the function
  - We'll explore this in the tutorial

Part 2: Functions

Exponential Function

- $f(x) = b^x$ (where b is some constant in $(0, \infty)$)

- b: base
- x: exponent (power)

- The most common choice for b is $e = 2.71828\ldots$
  - e is called Euler's number, we'll look at it next week in Limits

- $b = 10$ is also common, and is referred to as scientific notation. Julia uses this:
  - -1.5e6 means -1.5×10^6 (-1,500,000)
  - 9e-15 means 9×10^{-15} (0.000,000,000,000,009)

Part 2: Functions

Exponential Function

- Properties
  - $b^x > 0$
  - $b^x \times b^y = b^{x+y}$
  - $\frac{b^x}{b^y} = b^{x-y}$
  - $b^0 = 1$ $\quad\quad \frac{b^x}{b^x} = b^{x-x} \Rightarrow 1 = b^0$
  - $b^{-x} = \frac{1}{b^x}$ $\quad\quad \frac{1}{b^x} = \frac{b^0}{b^x} = b^{0-x} = b^{-x}$
  - $(ab)^x = a^x b^x$
  - $(b^x)^y = b^{xy}$
  - $b^{\frac{1}{n}} = \sqrt[n]{b}$ $\quad\quad \sqrt{b^2} = (b^2)^{\frac{1}{2}} = b^{2\times\frac{1}{2}} = b^1 = b$
  - If $b > 1$, then b^x grows as x increases
  - If $b < 1$, then b^x decreases as x increases

Part 2: Functions

Exponential Function

- Example
  - Turn $\frac{\sqrt[3]{2^x}\,4^x}{4^3 4^{x/3}}$ into a power of 2
    - $\frac{\sqrt[3]{2^x}\,4^x}{4^3 4^{x/3}} = \frac{(2^x)^{\frac{1}{3}}\,(2^2)^x}{(2^2)^3 (2^2)^{\frac{x}{3}}}$
    - $= \frac{2^{\frac{x}{3}} 2^{2x}}{2^6 2^{\frac{2x}{3}}}$
    - $= 2^{(\frac{x}{3}+2x-\frac{2x}{3}-6)}$
    - $= 2^{(\frac{6x}{3}-\frac{x}{3}-6)}$
    - $= 2^{(\frac{5x}{3}-6)}$

Part 2: Functions

Logarithmic Functions

- The log function with base b is defined by
  - $\log_b x = y \Leftrightarrow x = b^y$
  - That is, the log function reverses the exponential function, and is it's inverse. This means that when combined, they cancel each other out:
    - $\log_b(b^y) = y$ and $b^{\log_b x} = x$
  - Any base b (> 0) is allowed
    - $b = e = 2.718 \ldots$ is the most important and popular
    - $b = 10$ and $b = 2$ are also sometimes used

- The natural log function $\log_e x$ is written as $\log(x)$ or $\ln(x)$
- Julia uses log(x) for $\log_e x$ and log(b,x) for $\log_b x$

Part 2: Functions

Logarithmic Functions

- Properties
  - $\log_b(xy) = \log_b x + \log_b y$

    $b^x b^y = b^{x+y}$
  - $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

    $\frac{b^x}{b^y} = b^{x-y}$
  - $\log_b(x^y) = y \log_b x$

Part 2: Functions

Logarithmic Functions

- Example
  - Express $\ln\left(2\sqrt[3]{9}\right)$ in terms of $\ln 2$ and $\ln 3$

    - $\ln\left(2\sqrt[3]{9}\right) = \ln\left(2\left(3^2\right)^{\frac{1}{3}}\right)$
    - $= \ln\left(2 \times 3^{\frac{2}{3}}\right)$
    - $= \ln 2 + \ln\left(3^{\frac{2}{3}}\right)$
    - $= \ln 2 + \frac{2}{3}\ln 3$

Part 2: Functions

Logarithmic Functions

- Example
  - Simplify $e^{-2\ln x}$

$$\begin{aligned} e^{-2\ln x} &= \frac{1}{e^{2\ln x}} \\ &= \frac{1}{e^{\ln x^2}} \\ &= \frac{1}{x^2} \end{aligned}$$

  - Simplify $\log_{10}(0.01 \times 10^x)$

$$\begin{aligned} \log_{10}(0.01 \times 10^x) &= \log_{10}(10^{-2}10^x) \\ &= \log_{10} 10^{x-2} \\ &= x - 2 \end{aligned}$$

Part 2: Functions

Logarithmic Functions

- Example
  - Solve the equation $\ln(y-1) = 2$ for y

$$\ln(y-1) = 2 \;\Rightarrow e^{\log(y-1)} = e^2$$
$$y - 1 = e^2$$
$$y = e^2 + 1$$

Part 2: Functions

Logarithmic Functions

- Example
  - Find y if $\ln(x^2-1)-2\ln(x+1)=\ln y$
    - $\ln y=\ln(x^2-1)-2\ln(x+1)$
    - $\ln y=\ln(x^2-1)\ -\ln\left((x+1)^2\right)$
    - $\ln y=\ln\left(\frac{x^2-1}{(x+1)^2}\right)$
  - Since $\frac{x^2-1}{(x+1)^2}=\frac{(x+1)(x-1)}{(x+1)^2}=\frac{x-1}{x+1}$
  - We have $\ln y=\ln\frac{x-1}{x+1}\ \Rightarrow y=\frac{x-1}{x+1}$

Part 2: Functions

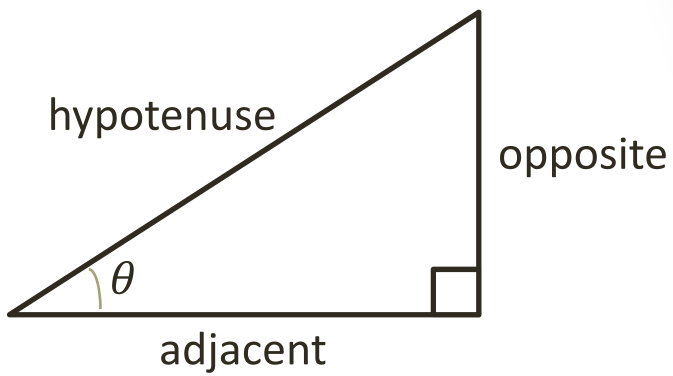
Logarithmic Functions

- Conversion (Change of Base)
  - $y = \log_b x$ can be expressed in terms of $\log_a x$ according to the log conversion formula:
    - $\log_b x = \dfrac{\log_a x}{\log_a b}$
- Example
  - Evaluate $\log_7 23$ using the natural logarithm function on your calculator.
    - $\log_7 23 = \dfrac{\log_e 23}{\log_e 7} = \dfrac{\ln 23}{\ln 7} \approx 1.61$

Part 2: Functions

Trigonometric Functions

- Sine, cosine and tangent Functions
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
  - NOTE
    - $(\sin\theta)^n$ is written as $\sin^n\theta$
    - $\sin\theta^n$ usually represents $\sin(\theta^n)$
  - Converting between degrees and radians
    - $\text{radians} = \dfrac{\pi}{180} \times \text{degrees}$
    - $\text{degrees} = \dfrac{180}{\pi} \times \text{radians}$



θ is restricted to $(0^{\text{o}}, 90^{\text{o}})$

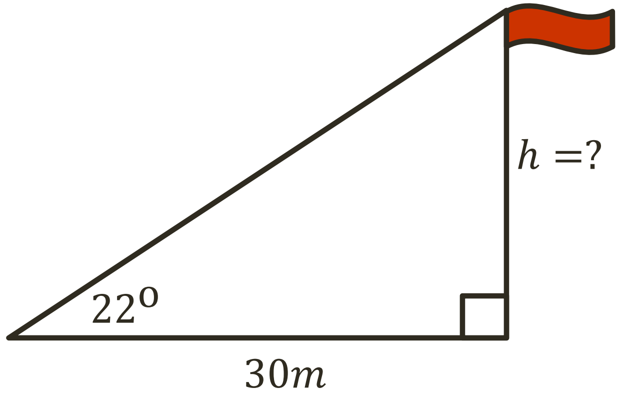
θ is restricted to $\left(0, \frac{\pi}{2}\right)$

Part 2: Functions

Examples

- The angle of inclination of the top of a flagpole from a distance of 30m is 22 degrees. How high is the flagpole?

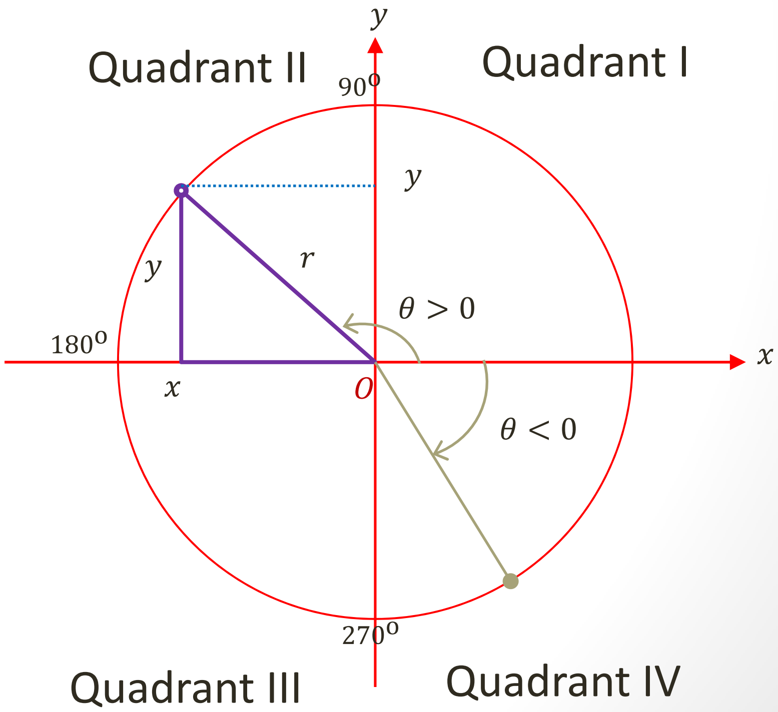
  - $\tan 22^{\mathrm{o}} = \frac{h}{30}$
  - $h = 30\tan 22^{\mathrm{o}} \approx 12.12m$



Part 2: Functions

Trigonometric Functions

- Extension to arbitrary angle
  - $\sin\theta = \frac{y}{r}, \cos\theta = \frac{x}{r}, \tan\theta = \frac{y}{x}$
  - Signs for $\sin\theta$, $\cos\theta$ and $\tan\theta$ change according to the signs of x and y
- Important identities:
  - $\sin^2\theta + \cos^2\theta = 1$
  - $\tan\theta = \frac{\sin\theta}{\cos\theta}$



Part 2: Functions

Inverse functions asin and acos

- What are the values of $\theta \in [0^{o}, 360^{o})$ such that $\cos\theta = 1/2$?
  - $\operatorname{acos}\frac{1}{2} = 60^{o}$
  - $\cos\theta > 0$, so quadrants 1 and 4
  - $\theta = 60^{o}, (360 - 60)^{o} = 300^{o}$

- Solve $\sin\theta = -\frac{1}{2}$ for $\theta \in [0,2\pi)$
  - $\operatorname{asin} -\frac{1}{2} = -\frac{\pi}{6} \left(= 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}\right)$
  - $\sin\theta < 0$ so quadrants 3 and 4
  - $\theta = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$
  - $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Calculators can usually help here!

$\operatorname{asin} x = \theta$
(i.e. $\sin^{-1} x = \theta$)

$-90^{o} \leq \theta \leq 90^{o}$
$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
(Same applies to tan)

$\operatorname{acos}\ x = \theta$
(i.e. $\cos^{-1} x = \theta$)

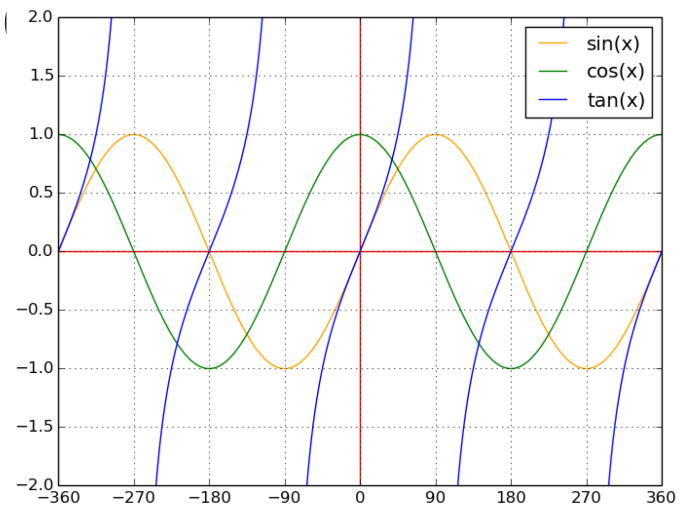
$0 \leq \theta \leq 180^{o}$
$0 \leq \theta \leq \pi$

Part 2: Functions

Periodic Functions

- If a function has a repeating pattern, it's said to be periodic
  - The size of a single instance of the pattern is called its period. More formally,
    - If for all values of x,
      $$f(x) = f(x+T)$$
      $f(x)$ is a periodic function with period T

  - $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all periodic functions
    - $\sin\theta$ and $\cos\theta$ both have period 2π (
    - $\tan\theta$ has period π (180 degrees)

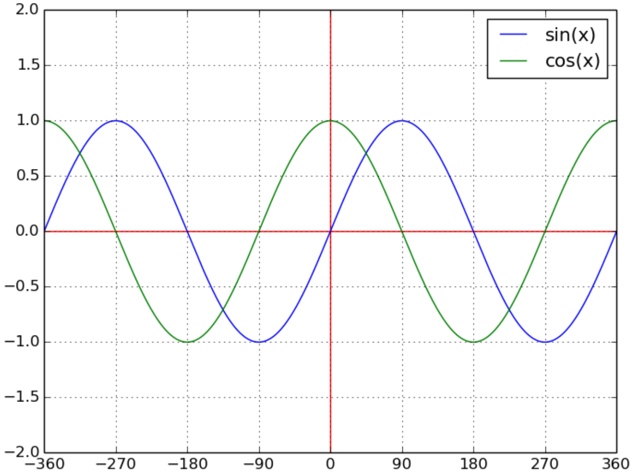


Part 2: Functions

Odd and Even Functions

- Odd and even functions are "reflected" in the line $x = 0$
  - $f(x)$ is **even** if $f(x) = f(-x)$ ----- symmetric about y axis
  - $f(x)$ is **odd** if $f(x) = -f(-x)$ ----- symmetric about origin

- $\cos(x)$ is even as $\cos\theta = \cos(-\theta)$
- $\sin(x)$ is odd as $\sin\theta = -\sin(-\theta)$



Part 2: Functions
Examples

- Classify the following functions as odd, even or neither
  - $f(x) = x$
  - $f(x) = x^2$
  - $f(x) = x^3$
  - $f(x) = e^x$
  - $f(x) = \tan(x)$ hint: $\tan(x) = \frac{\sin(x)}{\cos(x)}$

Part 2: Functions

Examples

- Classify the following functions as odd, even or neither
  - $f(x) = x$ ODD $f(-x) = -x = -f(x)$
  - $f(x) = x^2$ EVEN $f(-x) = (-x)^2 = x^2 = f(x)$
  - $f(x) = x^3$ ODD $f(-x) = (-x)^3 = -x^3 = -f(x)$
  - $f(x) = e^x$ NEITHER $f(-x) = e^{-x} = \frac{1}{e^x} \neq e^x = f(x)$
  - $f(x) = \tan(x)$ hint: $\tan(x) = \frac{\sin(x)}{\cos(x)}$ ODD
    - $f(-x) = \tan(-x\ ) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan(x) = -f(x)$

Part 2: Functions

Increasing and Decreasing Functions

- A function is said to be an increasing function if for any values $x_2 \geq x_1$,
  - $f(x_2) \geq f(x_1)$
- Likewise, a function is decreasing if:
  - $f(x_2) \leq f(x_1)$
- Example:
  - $f(x) = e^x$ is an increasing function
  - $f(x) = e^{-x}$ is a decreasing function
- We'll touch on this more next week

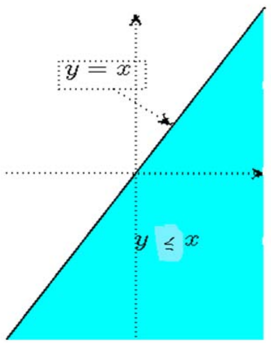
MATH 2267
Week 1: Part 3

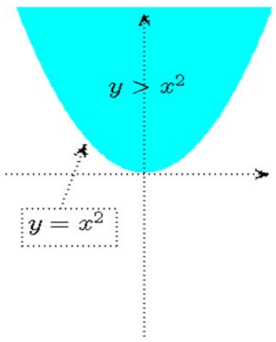
Inequalities

Part 3: Inequalities

Introduction

- Inequalities are relations between unequal quantities
- Each inequality has one of the 4 possible inequality signs
  - $3 > 2$ "greater than" inequality
  - $|x| \geq x$ "greater or equal to" inequality
  - $|x| < 3$ "less than" inequality
  - $|y - 1| \leq x$ "less or equal to" inequality
- Geometry:
  - $y \leq x$: all points $(x, y) \in R^2$ on or under line $y = x$
  - $y > x^2$: all points $(x, y) \in R^2$ above the curve $y = x^2$





Part 3: Inequalities

Introduction

- Properties of Inequalities after operations
  - An inequality will remain unchanged by adding or subtracting equal quantities to or from both sides

$$3 > 2 \quad \Longrightarrow \quad \begin{array}{r} 3 + a > 2 + a \\ \text{and } 3 - a > 2 - a \end{array} \quad \text{for any real } a$$

  - Multiplying or dividing both sides of an inequality by a positive number the inequality will not change

$$3 > 2 \quad \Longrightarrow \quad \begin{array}{r} 3a > 2a \\ \text{and } \frac{3}{a} > \frac{2}{a} \end{array} \quad \text{for any } a > 0$$

  - Multiplying or dividing both sides of an inequality by a negative number the inequality will change direction

$$3 > 2 \quad \Longrightarrow \quad \begin{array}{r} -2 \times 3 < -2 \times 2 \\ \text{and } -\frac{3}{2} < -\frac{2}{2} \end{array}$$

  - Swap the left and right hand sides of the inequality the inequality sign changes direction

$$3 > 2 \quad \Longrightarrow \quad 2 < 3$$

Part 3: Inequalities

Solving Inequalities

- Solving inequalities involves some manipulations such as the operations mentioned in the last slide.

- Example: Solve $-9 > -3x$.

  Multiply by −1 (change ">" to "<"):

  $$9 < 3x$$

  Swap sides (change "<" to ">"):

  $$3x > 9$$

  Divide by 3 (">" remains):

  $$x > 3$$ ⟵ the solution

Part 3: Inequalities

Solving Inequalities

Example. Solve inequality for x: $\frac{2x-3}{9} > 1$

Solution.

Multiply by 9 (“>” remain unchanged):

$$2x - 3 > 9$$

Add 3 (“>” remain unchanged):

$$2x > 12$$

Divide by 2 (“>” remain unchanged):

$x > 4$ $\longleftarrow$ the solution

Part 3: Inequalities

Solving Inequalities

Example. Solve inequality for x in $\frac{x-3}{x+5} > 0$

Solution. Both numerator and denominator must have the same sign $\Longrightarrow$ (1) or (2) below must hold.

(1) $x - 3 > 0$ and $x + 5 > 0$

(2) $x - 3 < 0$ and $x + 5 < 0$

(1) $\Longrightarrow x > 3$ and $x > -5 \Rightarrow x > 3$

(2) $\Longrightarrow x < 3$ and $x < -5 \Rightarrow x < -5$

The solution is: $x < -5$ or $x > 3.$

or $x \in (-\infty, -5) \cup (3, \infty)$

Next Week

- Polynomial factorization
- Sequences and series
- Arithmetic and Geometric Progressions
- Compound interests