RealAnalysis

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Chapter 1

Cardinality

Definition (Cardinality) *Let A, B be two (non-empty) sets, we say $|A| \leq |B|$ if there exists a injective map $f:\ A \to B$*
Similarly we have $|A| \leq |B| \Leftrightarrow$ there exists a surjective map $g : B \to A$
We say $|A| = |B|$

Lemma (Set Decomposition Under Mapping) *Let $f : X \to Y$, $g : Y \to X$ there exists decomposition such that,*

$$X = A \cup A^{\sim}, Y = B \cup B^{\sim}$$

where $f(A) = B, g(B^{\sim}) = A^{\sim}, A \cap A^{\sim} = \emptyset$ and $B \cap B^{\sim} = \emptyset$

Proof *For a subset E of X, W.O.L.G. $Y \backslash f(E) \neq \emptyset$, if E satisfy,*

$$E \cap g(Y \backslash f(E)) = \emptyset$$

we call E a seperate set of X, now denote the set of all seperate set as Γ, and make union,

$$A = \bigcup_{E \in \Gamma} E$$

We have $A \in \Gamma$, Actually for any $E \in \Gamma$, as $A \supset E$, so from,

$$E \cap g(Y \backslash f(E)) = \emptyset$$

we know that $E \cap g(Y \backslash f(A)) = \emptyset$, since A is larger than E, so actually $g(Y \backslash f(A)) \subseteq g(Y \backslash f(E))$, thus $A \cap g(Y \backslash f(A)) = \emptyset$. This shows that A is also a seperate set in X and the largest element in Γ.
Now let $f(A) = B, Y \backslash B = B^{\sim}$ and $g(B^{\sim}) = A^{\sim}$. First we know that.

$$Y = B \cup B^{\sim}$$

Secondly, as $A \cap A^{\sim} = \emptyset \Leftrightarrow A \cap g(Y \backslash f(A)) = \emptyset$, so we know $A \cup A^{\sim} = X$.
We can assume $A \cup A^{\sim} \neq X$, then there exists $x_0 \in X$ such that $x_0 \notin A \cup A^{\sim}$.
Let $A_0 = A \cup x_0$ we have,

$$B = f(A) \subset f(A_0), B^{\sim} \supset Y \backslash f(A_0)$$

so that $A^{\sim} \supset g(Y \backslash f(A_0))$, which means A and $g(Y \backslash f(A_0))$ do not intersect. So

$$A_0 \cap g(Y \backslash f(A_0)) = \emptyset$$

(A_0 多了一个元素 x_0，但该元素在 $A^{\sim}$ 中不存在并且 $A^{\sim} \supset A_0^{\sim} = g(B_0^{\sim}) = g(Y \backslash f(A_0))$) which is contradict to A is the largest element in Γ

Theorem (Schröder–Bernstein theorem) *If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$*

Proof *We need to show if there exists an injective map $f : X \to Y$ and an injective map $g : Y \to X$ then there exists a bijective map $h : X \to Y$.*

Define $X = A \cup A^{\sim}, Y = B \cup B^{\sim}, f(A) = B(Surjective\ to\ B), g(B^{\sim}) = A^{\sim}(Surjective\ to\ A^{\sim})$(Using decomposition lemma)
For any $a \in X$, define a map h.

$$h(x) = \begin{cases} f(x) & x \in A \\ g^{-1}(x) & x \in A^{\sim} \end{cases}$$

which shows $X \sim Y$

Definition (Arithmetic Operation of Cardinal Number) *Suppose* a, b *are two cardinal numbers, where* $a = |A|$, $b = |B|$.

1. $a + b \triangleq |A \cup B|$ *where* A, B *are disjoint sets.*
2. $a \cdot b \triangleq |A \times B|$
3. $a^b = |A^B| = \prod_B A$ *where* $A^B = \{$*all maps* $\phi : B \to A\}$

Proposal $c = m^{\aleph_0}, \forall m \in \mathbb{N}, m \geq 2$

Proof *We view R.H.S. as* $|\{0,1,2,...,m-1\}^{\mathbb{N}}| = |\{f : \mathbb{N} \to \{0,1,...,m-1\}\}|$. *Actually, R.H.S. can be viewed as a map from* i_{th} *digit index to the* i_{th} *digit itself and L.H.S. as* $|(0,1]|$.
Recall that $\forall r \in (0,1]$ *we have a sequence* $\{r_n\}$, *each* $r_n \in \{0,1,...,m-1\}$ *such that*

$$r = \sum_{n=1}^{\infty} \frac{r_n}{m_n}$$

(闭区间套: the principle of nested intervals)
The sequence $\{r_n\}$ *is unique if we require it has infinite many non-zero numbers.*
This means that we have an injective map.

$$\Phi : (0,1] \to \{0,1,...,m-1\}^{\mathbb{N}}$$

$$Im\Phi = \{f\text{: there are infinitely many } n \text{ with } f(n) \neq 0\}$$

Let $A_N = \{f : \exists\ N \text{ s.t. } f(n) = 0\ \forall n > N\}$, *so* $|A_N| = m^N < \infty$

$$|(Im\Phi)^{\complement}| = |\bigcup_{N=0}^{\infty} A_N| = \aleph_0$$

So, now we have shown,

$$L.H.S. = c = c + \aleph_0 = m_0^{\aleph} = R.H.S.$$

Definition (Power Set) $P(A) \triangleq \{$*subsets of* $A\} \triangleq \{0,1\}^A = \{f : A \to \{0,1\}\}$

Theorem ($\boldsymbol{P(A) > |A|}$) *First, $P(A) \geq |A|$, because there exists a injective map $f : a \to \{a\}$. we will show $|P(A)| \neq |A|$, hence $|P(A)| > |A|$.*
Otherwise, $|P(A)| = |A|$ and hence there exists a bijective map $\phi : A \to P(A)$.
Consider the subset $B = \{a \in A | a \notin \phi(a)\}$, thus $B \in P(A)$.
Let b be the pre-image of B under ϕ.
If $b \in B$ then by the construction of B, then $b \notin \phi(b) = B$.
If $b \notin B$ then by the construction of B, then $b \in \phi(b) = B$.
Therefore, such bijective map ϕ does not exist.

Chapter 2

The Second Chapter