

Differentiating $f(x) = x$

Part 2

A simple, Intuitive, and Beautiful Solution

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Contents

1	Purpose of this Document	1
2	Breaking Down the Question	2
3	Solving the Question	3
3.1	Solving the Inner Half-Derivative	3
3.2	Solving the Outer Half-Derivative	4
4	Conclusion	6

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1 Purpose of this Document

I was bored.

2 Breaking Down the Question

Question: Solve $\frac{dx}{dx}$

While this may appear quite difficult at first glance, there is a hidden—yet simple—solution to solve it. After spending years on this question, I was amazed by its beauty once I had the answer.

Instead of tackling the problem as a standard derivative (these tend to be quite difficult), you can instead break it into fractional derivatives. While there are an infinite number of possibilities, splitting it into two half-derivatives produces the nicest solution.

Braking this into two half-derivatives can be solved with two different formulas. The first being as follows:

$$\frac{d^\alpha x^n}{dx^\alpha} = \frac{\Pi(n)}{\Pi(n-\alpha)} x^{n-\alpha}, \quad \Pi(n) = \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt$$

This allows us to take the α -order derivative of a function $f(x) = x^n$.

The second formula is a simple differintegral, and is given by:

$${}_a D_t^\alpha f(t) = \begin{cases} \frac{d^\beta}{dt^\beta} \left[\beta - \alpha \int_a^t f(x) dx^{\beta-\alpha} \right], & \alpha > 0 \\ |\alpha| \int_a^t f(x) dx^{|\alpha|}, & \alpha < 0 \end{cases}$$

Where $\beta = \lceil \alpha \rceil$. Applying the Riemann-Liouville fractional integral, we get the following for positive α :

$$\begin{aligned} {}_a D_t^\alpha f(t) &= \frac{d^\beta}{dt^\beta} \left[\frac{1}{\Gamma(\beta-\alpha)} \int_a^t (t-s)^{\beta-\alpha-1} f(s) ds \right] \\ {}_a D_t^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \end{aligned}$$

This means that we can write out the question as follows:

$$\begin{aligned} \frac{dx}{dx} &= \frac{d^{1/2}}{dx^{1/2}} \left[\frac{d^{1/2} x}{dx^{1/2}} \right] \\ &= \frac{d^{1/2}}{dx^{1/2}} \left[\frac{\Pi(1)}{\Pi(1-\frac{1}{2})} x^{1-\frac{1}{2}} \right] \\ &= \frac{d}{dx} \left[\frac{1}{\Gamma(1-\frac{1}{2})} \int_a^t (t-s)^{1-\frac{1}{2}-1} \left(\frac{\Pi(1)}{\Pi(1-\frac{1}{2})} s^{1-\frac{1}{2}} \right) ds \right] \\ &= \frac{d}{dx} \left[\left(\int_0^\infty t^{-\frac{1}{2}} e^{-t} dt \right)^{-1} \int_a^t (t-s)^{-\frac{1}{2}} \left(\frac{\int_0^\infty t e^{-t} dt}{\int_0^\infty t^{\frac{1}{2}} e^{-t} dt} \sqrt{s} \right) ds \right] \end{aligned}$$

3 Solving the Question

3.1 Solving the Inner Half-Derivative

Now that we have the necessary formulas, we can get started on the question.

$$\begin{aligned}\frac{dx}{dx} &= \frac{d^{1/2}}{dx^{1/2}} \left[\frac{d^{1/2}}{dx^{1/2}} \right] \\ &= \frac{d^{1/2}}{dx^{1/2}} \left[\frac{\Pi(1)}{\Pi\left(1 - \frac{1}{2}\right)} x^{1-\frac{1}{2}} \right] \\ &= \frac{d^{1/2}}{dx^{1/2}} \left[\sqrt{x} \left(\int_0^\infty x^{\frac{1}{2}} e^{-x} dx \right)^{-1} \right]\end{aligned}$$

So now we will need to solve the integral, which is very simple. First, we need to use Integration by Parts.

$$\begin{aligned}\int_0^\infty x^{\frac{1}{2}} e^{-x} dx &= \left(-e^{-x} \sqrt{x} \Big|_0^\infty + \int_0^\infty \frac{1}{2e^x \sqrt{x}} dx \right) \\ \therefore \int_0^\infty x^{\frac{1}{2}} e^{-x} dx &= \int_0^\infty \frac{1}{2} x^{-\frac{1}{2}} e^{-x} dx\end{aligned}$$

Now we can let $u = x^{\frac{1}{2}}$. Note that this is an even function.

$$\begin{aligned}\int_0^\infty \frac{1}{2} x^{-\frac{1}{2}} e^{-x} dx &= \int_0^\infty e^{-u^2} du \\ &= \frac{1}{2} \int_{\mathbb{R}} e^{-u^2} du\end{aligned}$$

Now we'll define this to be I , and do some simple manipulation.

$$\begin{aligned}2I &= \int_{\mathbb{R}} e^{-x^2} dx \\ 4I^2 &= \int_{\mathbb{R}} e^{-x^2} dx \cdot \int_{\mathbb{R}} e^{-y^2} dy \\ &= \iint_{\mathbb{R} \times \mathbb{R}} e^{-x^2 - y^2} dx dy \\ &= \int_{\mathbb{R}^2} e^{-|z|^2} dz \\ \iint_D f(x, y) dA &= \iint_S f(g(u, v), h(u, v)) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dB \\ &= \int_0^{2\pi} \int_{\mathbb{R}_0^+} e^{-r^2} \begin{vmatrix} \frac{\partial r \cos \phi}{\partial r} & \frac{\partial r \cos \phi}{\partial \phi} \\ \frac{\partial r \sin \phi}{\partial r} & \frac{\partial r \sin \phi}{\partial \phi} \end{vmatrix} dr d\phi \\ &= \int_0^{2\pi} \int_{\mathbb{R}_0^+} e^{-r^2} \begin{vmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{vmatrix} dr d\phi \\ &= \int_{\mathbb{R}_0^+} \int_0^{2\pi} r e^{-r^2} d\phi dr \\ &= \pi \int_{\mathbb{R}_0^+} 2r e^{-r^2} dr \\ 4I^2 &= \pi \\ \therefore I &= \frac{\sqrt{\pi}}{2}\end{aligned}$$

Substituting this back in gives:

$$\begin{aligned}\frac{dx}{dx} &= \frac{d^{1/2}}{dx^{1/2}} \left[\sqrt{x} \left(\int_0^\infty x^{\frac{1}{2}} e^{-x} dx \right)^{-1} \right] \\ &= \frac{d^{1/2}}{dx^{1/2}} \left[\frac{2\sqrt{x}}{\sqrt{\pi}} \right]\end{aligned}$$

3.2 Solving the Outer Half-Derivative

Now we will take our result from section 3.1 and substitute it into our Riemann-Liouville integral.

$$\begin{aligned}\frac{d^{1/2}}{dx^{1/2}} \left[\frac{2\sqrt{x}}{\sqrt{\pi}} \right] &= \frac{d}{dx} \left[\frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x-t)^{-\frac{1}{2}} \frac{2\sqrt{t}}{\sqrt{\pi}} dt \right] \\ &= \frac{2}{\sqrt{\pi}\Gamma(\frac{1}{2})} \cdot \frac{d}{dx} \left[\int_0^x \frac{\sqrt{t}}{\sqrt{x-t}} dt \right]\end{aligned}$$

Now we'll solve the integral. We'll perform two substitutions. The first being $u = \sqrt{t}$, $dt = 2u du$. This gives the following:

$$\int \frac{\sqrt{t}}{\sqrt{x-t}} dt = 2 \int \frac{u^2}{\sqrt{x-u^2}} du$$

Now we'll let $u = \sqrt{x} \sin v$, $du = \sqrt{x} \cos v dv$.

$$\begin{aligned}2 \int \frac{u^2}{\sqrt{x-u^2}} du &= 2 \int \frac{x\sqrt{x} \cos v \sin^2 v}{\sqrt{x-x\sin^2 v}} dv \\ &= 2x \int \sin^2 v dv\end{aligned}$$

This is where knowing your reduction formulas comes in handy for you. The reduction formula for $\int \sin^n x dx$ is as follows:

$$\int \sin^n x dx = \frac{n-1}{n} \int \sin^{n-2} x dx - \frac{\sin^{n-1} x \cos x}{n}$$

Now using that and undoing our substitutions, we get:

$$\begin{aligned}
2 \int \frac{u^2}{\sqrt{x-u^2}} du &= 2x \int \sin^2 v dv \\
&= xv - x \sin v \cos v \\
&= x \sin^{-1} \left(\frac{u}{\sqrt{x}} \right) - x \sin \left(\sin^{-1} \left(\frac{u}{\sqrt{x}} \right) \right) \cos \left(\sin^{-1} \left(\frac{u}{\sqrt{x}} \right) \right) \\
&= x \sin^{-1} \left(\frac{u}{\sqrt{x}} \right) - \frac{xu}{\sqrt{x}} \sqrt{1 - \frac{u^2}{x}} \\
&= x \sin^{-1} \left(\sqrt{\frac{t}{x}} \right) - x \sqrt{\frac{t}{x}} \sqrt{1 - \frac{t}{x}} \\
\Rightarrow \int_0^x \sqrt{\frac{t}{x-t}} dt &= \left(x \sin^{-1} \left(\sqrt{\frac{t}{x}} \right) - x \sqrt{\frac{t}{x}} \sqrt{1 - \frac{t}{x}} \right) \Big|_0^x \\
&= x \sin^{-1} 1 \\
&= \frac{x\pi}{2} \\
\therefore \frac{dx}{dx} &= \frac{2}{\sqrt{\pi}\Gamma(\frac{1}{2})} \cdot \frac{d}{dx} \left(\frac{x\pi}{2} \right) \\
&= 1
\end{aligned}$$

4 Conclusion

You have just wasted your time reading this.