# Differentiating $f(x) = x$

## ***Testing PDF***

A simple, Intuitive, and Beautiful Solution

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# Contents

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# 1 Purpose of this Document

I was bored.

## 2 Breaking Down the Question

**Question:** Solve $\dfrac{\mathrm{d}x}{\mathrm{d}x}$

While this may appear quite difficult at first glance, there is a hidden—yet simple—solution to solve it. After spending years on this question, I was amazed by its beauty once I had the answer.

Instead of tackling the problem as a standard derivative (these tend to be quite difficult), you can instead break it into fractional derivatives. While there are an infinite number of possibilities, splitting it into two half-derivatives produces the nicest solution.

Braking this into two half-derivatives can be solved with two different formulas. The first being as follows:

$$\frac{\mathrm{d}^\alpha x^n}{\mathrm{d}x^\alpha} = \frac{\Pi(n)}{\Pi(n-\alpha)} x^{n-\alpha}, \qquad \Pi(n) = \Gamma(n+1) = \int_0^\infty t^n e^{-t}\,\mathrm{d}t$$

This allows us to take the $\alpha$-order derivative of a function $f(x) = x^n$.

The second formula is a simple differintegral, and is given by:

$$_a\mathrm{D}_t^\alpha f(t) = \begin{cases} \dfrac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\left[ {}_{\beta-\alpha}\displaystyle\int_a^t f(x)\,\mathrm{d}x^{\beta-\alpha}\right], & \alpha > 0 \\ {}_{|\alpha|}\displaystyle\int_a^t f(x)\,\mathrm{d}x^{|\alpha|}, & \alpha < 0 \end{cases}$$

Where $\beta = \lceil \alpha \rceil$. Applying the Riemann-Liouville fractional integral, we get the following for positive $\alpha$:

$$_a\mathrm{D}_t^\alpha f(t) = \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\left[\frac{1}{\Gamma(\beta-\alpha)}\int_a^t (t-s)^{\beta-\alpha-1} f(s)\,\mathrm{d}s\right]$$
$$_a\mathrm{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s$$

This means that we can write out the question as follows:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}x} &= \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\frac{\mathrm{d}^{1/2}x}{\mathrm{d}x^{1/2}}\right] \\
&= \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\frac{\Pi(1)}{\Pi(1-\frac{1}{2})}x^{1-\frac{1}{2}}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\Gamma(1-\frac{1}{2})}\int_a^t (t-s)^{1-\frac{1}{2}-1}\left(\frac{\Pi(1)}{\Pi(1-\frac{1}{2})}s^{1-\frac{1}{2}}\right)\mathrm{d}s\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\int_0^\infty t^{-\frac{1}{2}}e^{-t}\,\mathrm{d}t\right)^{-1}\int_a^t (t-s)^{-\frac{1}{2}}\left(\frac{\int_0^\infty te^{-t}\,\mathrm{d}t}{\int_0^\infty t^{\frac{1}{2}}e^{-t}\,\mathrm{d}t}\sqrt{s}\right)\mathrm{d}s\right]
\end{aligned}$$

# 3 Solving the Question

## 3.1 Solving the Inner Half-Derivative

Now that we have the necessary formulas, we can get started on the question.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}x} &= \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\right] \\ &= \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\frac{\Pi(1)}{\Pi\left(1-\frac{1}{2}\right)}x^{1-\frac{1}{2}}\right] \\ &= \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\sqrt{x}\left(\int_0^\infty x^{\frac{1}{2}}e^{-x}\,\mathrm{d}x\right)^{-1}\right]\end{aligned}$$

So now we will need to solve the integral, which is very simple. First, we need to use Integration by Parts.

$$\begin{aligned}\int_0^\infty x^{\frac{1}{2}}e^{-x}\,\mathrm{d}x &= \left(-e^{-x}\sqrt{x}\Big|_0^\infty + \int_0^\infty \frac{1}{2e^x\sqrt{x}}\,\mathrm{d}x \\ \therefore \int_0^\infty x^{\frac{1}{2}}e^{-x}\,\mathrm{d}x &= \int_0^\infty \frac{1}{2}x^{-\frac{1}{2}}e^{-x}\,\mathrm{d}x\end{aligned}$$

Now we can let $u = x^{\frac{1}{2}}$. Note that this is an even function.

$$\begin{aligned}\int_0^\infty \frac{1}{2}x^{-\frac{1}{2}}e^{-x}\,\mathrm{d}x &= \int_0^\infty e^{-u^2}\,\mathrm{d}u \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-u^2}\,\mathrm{d}u\end{aligned}$$

Now we'll define this to be $I$, and do some simple manipulation.

$$\begin{aligned}2I &= \int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x \\ 4I^2 &= \int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x \cdot \int_{\mathbb{R}} e^{-y^2}\,\mathrm{d}y \\ &= \iint\limits_{\mathbb{R}\times\mathbb{R}} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} e^{-|z|^2}\,\mathrm{d}z \\ \iint\limits_D f(x,y)\,\mathrm{d}A &= \iint\limits_S f\Big(g(u,v),h(u,v)\Big)\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\,\mathrm{d}B \\ &= \int_0^{2\pi}\int_{\mathbb{R}_0^+} e^{-r^2}\begin{vmatrix}\frac{\partial r\cos\phi}{\partial r} & \frac{\partial r\cos\phi}{\partial\phi} \\ \frac{\partial r\sin\phi}{\partial r} & \frac{\partial r\sin\phi}{\partial\phi}\end{vmatrix}\,\mathrm{d}r\,\mathrm{d}\phi \\ &= \int_0^{2\pi}\int_{\mathbb{R}_0^+} e^{-r^2}\begin{vmatrix}\cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi\end{vmatrix}\,\mathrm{d}r\,\mathrm{d}\phi \\ &= \int_{\mathbb{R}_0^+}\int_0^{2\pi} re^{-r^2}\,\mathrm{d}\phi\,\mathrm{d}r \\ &= \pi\int_{\mathbb{R}_0^+} 2re^{-r^2}\,\mathrm{d}r \\ 4I^2 &= \pi \\ \therefore\ I &= \frac{\sqrt{\pi}}{2}\end{aligned}$$

Substituting this back in gives:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}x} &= \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\sqrt{x}\left(\int_0^{\infty} x^{\frac{1}{2}} e^{-x}\,\mathrm{d}x\right)^{-1}\right] \\ &= \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\frac{2\sqrt{x}}{\sqrt{\pi}}\right]\end{aligned}$$

## 3.2 Solving the Outer Half-Derivative

Now we will take our result from section 3.1 and substitute it into our Riemann-Liouville integral.

$$\begin{aligned}\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}\left[\frac{2\sqrt{x}}{\sqrt{\pi}}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-t)^{-\frac{1}{2}}\frac{2\sqrt{t}}{\sqrt{\pi}}\,\mathrm{d}t\right] \\ &= \frac{2}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^x \frac{\sqrt{t}}{\sqrt{x-t}}\,\mathrm{d}t\right]\end{aligned}$$

Now we'll solve the integral. We'll perform two substitutions. The first being $u=\sqrt{t},\ \mathrm{d}t = 2u\ \mathrm{d}u$. This gives the following:

$$\int \frac{\sqrt{t}}{\sqrt{x-t}}\,\mathrm{d}t = 2\int \frac{u^2}{\sqrt{x-u^2}}\,\mathrm{d}u$$

Now we'll let $u=\sqrt{x}\sin v,\ \mathrm{d}u = \sqrt{x}\cos v\ \mathrm{d}v$.

$$\begin{aligned}2\int \frac{u^2}{\sqrt{x-u^2}}\,\mathrm{d}u &= 2\int \frac{x\sqrt{x}\cos v\sin^2 v}{\sqrt{x-x\sin^2 v}}\,\mathrm{d}v \\ &= 2x\int \sin^2 v\,\mathrm{d}v\end{aligned}$$

This is where knowing your reduction formulas comes in handy for you. The reduction formula for $\int \sin^n x\,\mathrm{d}x$ is as follows:

$$\int \sin^n x\,\mathrm{d}x = \frac{n-1}{n}\int \sin^{n-2} x\,\mathrm{d}x - \frac{\sin^{n-1} x\cos x}{n}$$

Now using that and undoing our substitutions, we get:

$$\begin{aligned}
2\int \frac{u^2}{\sqrt{x-u^2}}\,\mathrm{d}u &= 2x\int \sin^2 v\,\mathrm{d}v \\
&= xv - x\sin v\cos v \\
&= x\sin^{-1}\left(\frac{u}{\sqrt{x}}\right) - x\sin\left(\sin^{-1}\left(\frac{u}{\sqrt{x}}\right)\right)\cos\left(\sin^{-1}\left(\frac{u}{\sqrt{x}}\right)\right) \\
&= x\sin^{-1}\left(\frac{u}{\sqrt{x}}\right) - \frac{xu}{\sqrt{x}}\sqrt{1-\frac{u^2}{x}} \\
&= x\sin^{-1}\left(\sqrt{\frac{t}{x}}\right) - x\sqrt{\frac{t}{x}}\sqrt{1-\frac{t}{x}} \\
\implies \int_0^x \sqrt{\frac{t}{x-t}}\,\mathrm{d}t &= \left(x\sin^{-1}\left(\sqrt{\frac{t}{x}}\right) - x\sqrt{\frac{t}{x}}\sqrt{1-\frac{t}{x}}\right|_0^x \\
&= x\sin^{-1} 1 \\
&= \frac{x\pi}{2} \\
\therefore\ \frac{\mathrm{d}x}{\mathrm{d}x} &= \frac{2}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\pi}{2}\right) \\
&= 1
\end{aligned}$$

## 4 Conclusion

You have just wasted your time reading this.