Singularities

We are only interested in isolated singularities. We can write the analytic function as Laurent series around an isolated singularity, convergent on a disk punctuated at the singularity.

Let f be analytic on D, with an isolated singularity at z_0. In $|z-z_0| = r$ in D,

$f(z) = \sum_{K=-\infty}^{\infty} a_K (z-z_0)^K$. Based on the behaviour of $f(z)$ we have 3 types of singularities.

$\exists\, c$ such that $|f(z)| < c$.

① Removable singularity

This occurs when $a_K = 0 \; \forall \; K < 0$. $L(z)$ then becomes a Power series. z_0 appears to be a singularity but actually is not. For example

$$f(z) = \frac{\sin z}{z} \text{ at } z = 0.$$

f has a removable singularity at z_0 only if $f(z)$ is bounded around z_0.

Proof/ If f has removable singularity at z_0, then for $|z - z_0| < \varepsilon$, $f(z) \leq \sum_{k=0}^{\infty} a_k \varepsilon^k$ which converges.

If $f(z)$ is bounded around z_0,

$$a_k = \frac{1}{2\pi i} \int_{|w-z_0|=\varepsilon} \frac{f(w)}{(w-z_0)^{k+1}} \, dw$$

$$\Rightarrow |a_k| \leq O\left(\frac{1}{\varepsilon^k}\right)$$

$a_k = 0 \ \forall k < 0$ since $\varepsilon \to 0$.

② Poles

$\forall c, \exists \delta$ such that $|f(z)| > c$.

When $a_k \neq 0$ for finitely many $k < 0$. The order of a pole at z_0 is the largest n for which $a_{-n} \neq 0$.

$f(z)$ has a pole at z_0 only if $\lim_{z \to z_0} |f(z)| = \infty$

Proof If f has a pole of order n at $z = z_0$,

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k.$$

Let $g(z) = (z - z_0)^n f(z)$ which is $\neq 0$ at z_0

$$f(z) = \frac{g(z)}{(z - z_0)^n}$$

$$\Rightarrow \lim_{z \to z_0} |f(z)| = \infty.$$

Conversely, let $\lim_{z \to z_0} |f(z)| = \infty$.

We define $g(z) = \frac{1}{f(z)}$. $g(z)$ is analytic around z_0 since $f(z) \neq 0$ around z_0. Also it is bounded around z_0. $g(z)$ thus may have a removable singularity at z_0.

Since $\lim_{z \to z_0} |f(z)| = \infty$, $g(z_0) = 0$. $g(z)$ thus has an isolated zero of order n at z_0.

$$\Rightarrow g(z) = (z - z_0)^n h(z) \text{ where } h(z) \neq 0$$

$$\Rightarrow f(z) = (z - z_0)^{-n} \cdot \frac{1}{h(z)}$$

$$= (z - z_0)^{-n} \cdot \sum_{K \geq 0} a_K (z - z_0)^K$$

$$= \sum_{K=-n}^{\infty} a_{K+n} (z - z_0)^K$$

$f(z)$ thus has a pole of order n at z_0.

③ <u>Essential Singularity</u>

When $a_K \neq 0$ for infinitely many $K < 0$, then z_0 is called an essential singularity.

An example is $\sum_{K \geq 0} \frac{1}{z^K}$ or $\sqrt{e}$.

$n \geq 0$

$f(z)$ has an essential singularity at z_0, only if $\forall\ \omega \in \mathbb{C}$, there is a sequence $\{u_n\}$ with $\lim\limits_{n \to \infty} u_n = z_0$ such that $\lim\limits_{n \to \infty} f(u_n) = \omega$.

Proof

$\forall \varepsilon, \exists \delta$ and z such that $|z - z_0| \leq \delta$ and $|f(z) - \omega| \leq \varepsilon$.

Let $\omega \in \mathbb{C}$ such that $\exists \delta$ and ε and $|f(z) - \omega| \geq \varepsilon$ for every $|z - z_0| \leq \delta$. We will define $g(z) = \dfrac{1}{f(z) - \omega}$. Consider $g(z)$ in $|z - z_0| \leq \delta$. $g(z) \leq \dfrac{1}{\varepsilon}$

$\Rightarrow g(z)$ is bounded in $|z - z_0| < \delta$.

$\Rightarrow g$ may have a removable singularity at z_0. This means $g(z)$ is completely analytic in $|z - z_0| < \delta$. Let $g(z) = (z - z_0)^n h(z)$ where $h(z) \neq 0$

$h(z) \neq 0$.

$$\Rightarrow f(z) = w + (z - z_0)^{-n} \cdot \frac{1}{h(z)}$$

$\Rightarrow f(z)$ has a pole of order n at z_0.