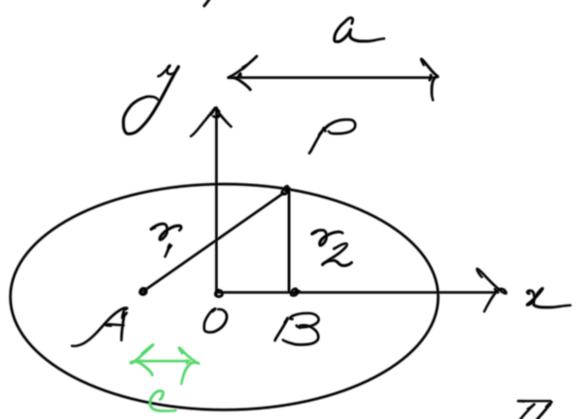
Elliptic Functions and Elliptic Integrals



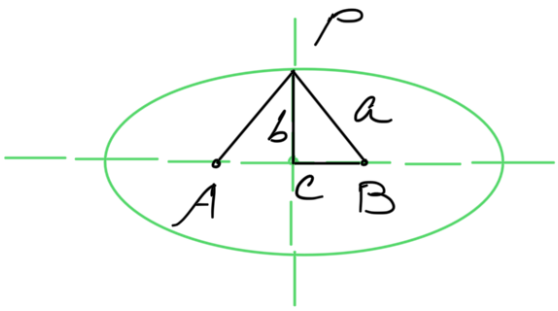
The two fixed points A and B are called the focii of the ellipse. $\forall P$ on the boundary of the ellipse $(r_1 + r_2)$ is constant. Let $P = (a, 0)$.

$r_1 = a + c$

$r_2 = a - c$

$\Rightarrow \boxed{r_1 + r_2 = 2a} \; \forall P$

And when $P = (0, b)$, $b^2 + c^2 = a^2$



$\Rightarrow \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2 = 1$

$\Rightarrow \left(\frac{b}{a}\right)^2 + \varepsilon^2 = 1$

$\boxed{\varepsilon = \frac{c}{a}}$ is called the eccentricity of the ellipse.

We will do a coordinate transformation such that b gets normalized to 1. So

$$c = \sqrt{a^2 - 1}$$

$$\Rightarrow \varepsilon = \frac{\sqrt{a^2-1}}{a}.$$

In the context of elliptic functions, ε is also called the modulus (k).

$$\Rightarrow \boxed{k = \frac{\sqrt{a^2-1}}{a}}$$

Jacobi Elliptic functions are defined as

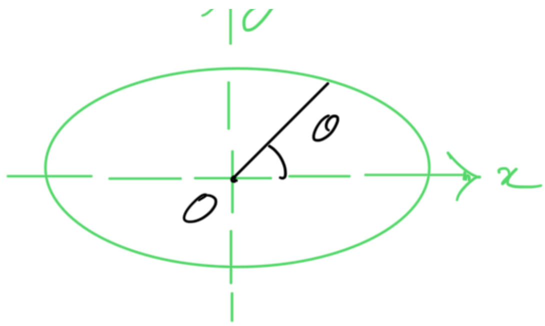
$$\boxed{sn(u, K) = y} \quad \text{and} \quad \boxed{cn(u, K) = \frac{x}{a}}$$

$$\Rightarrow \boxed{sn^2(u, K) + cn^2(u, K) = 1} \text{———} (1)$$

We will define $\boxed{dn(u, K) = \frac{r}{a}}$. It has the maximum value of 1 when $u = 0$ $\left(\text{or } P = (a, 0)\right)$.

$$\frac{1}{a} \leq dn(u, K) \leq 1$$

Here $\boxed{du = r \cdot d\theta}$.



From $x^2 + y^2 = r^2$

$$\Rightarrow \{a \cdot cn(u, K)\}^2 + \{sn(u, K)\}^2 = \{a \cdot dn(u, K)\}^2$$

$$\Rightarrow cn^2(u, K) + \{sn(u, K) \cdot (1-K)\}^2 = dn^2(u, K)$$

$$\Rightarrow 1 - \{K \cdot sn(u, K)\}^2 = dn^2(u, K)$$

$$\Rightarrow \boxed{dn^2(u, K) + K^2 \cdot sn^2(u, K) = 1.} \quad (2)$$

where $\boxed{a = \dfrac{1}{\sqrt{1 - K^2}}}$

Differentials

$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow d\theta \cdot \frac{d}{d\theta}(\tan\theta) = \frac{x \cdot dy - y \cdot dx}{x^2}$$

$$\Rightarrow d\theta = \frac{x \cdot dy - y \cdot dx}{x^2} \cdot \cos^2\theta$$

$$= \frac{x \cdot dy - y \cdot dx}{x^2} \cdot \left(\frac{x}{r}\right)^2$$

$$\Rightarrow d\theta = \frac{x \cdot dy - y \cdot dx}{r^2}$$

$$\left(\frac{x}{a}\right)^2 + y^2 = 1 \quad \Rightarrow \frac{2x \cdot dx}{a^2} + 2y \cdot dy = 0$$

$$\Rightarrow dx = -\frac{ya^2}{x} \cdot dy.$$

$$du = r \cdot d\theta = \frac{x \cdot dy - y \cdot dx}{r}$$

$$\Rightarrow du = \left(\frac{x}{r}\right) \cdot dy - \left(\frac{y}{r}\right) \cdot \left(-\frac{ya^2}{x} dy\right)$$

$$\Rightarrow du = \frac{x^2 + (ay)^2}{xr} \cdot dy$$

$$\Rightarrow du = \frac{a^2}{xr} dy.$$

$$\Rightarrow dy = x \quad r$$

$$\frac{\quad}{du} \;-\; \frac{\quad}{a} \cdot \frac{\quad}{a}$$

$$\Rightarrow \boxed{\frac{d}{du} sn(u, K) = cn(u, K) \cdot dn(u, K)}$$

And using $sn^2(u, K) + cn^2(u, K) = 1$

$$\Rightarrow 2\, sn(u, K) \cdot d\{sn(u, K)\}$$
$$+ 2\, cn(u, K) \cdot d\{cn(u, K)\} = 0$$

$$\Rightarrow \boxed{\frac{d}{du}\{cn(u, K)\} = -sn(u, K) \cdot dn(u, K)}$$

From (2), we get,

$$\boxed{\frac{d}{du}\{dn(u, K)\} = -K^2 \cdot sn(u, K)\, cn(u, K)}$$

Consider $\frac{d}{du}\{sn(u, K)\} = cn(u, K) \cdot dn(u, K)$

$$\Rightarrow \left\{\frac{d}{du} sn(u, K)\right\}^2 = cn^2(u, K) \cdot dn^2(u, K)$$

$$= \left\{1 - sn^2(u, K)\right\} \cdot \left\{1 - K^2 \cdot sn^2(u, K)\right\}$$

$$= 1 + K^2 sn^4(u, K) - \left(1 + K^2\right) sn^2(u, K).$$

It turns out that any elliptic function zn satisfies a differential equation of this format

$$\boxed{\left\{\frac{d}{du} zn(u, K)\right\}^2 = \alpha \cdot zn^4(u, K) + \beta \cdot zn^2(u, K) + \gamma}$$

$$\Rightarrow \frac{d}{du}\left(\frac{d}{du} zn\right)^2 = 2 \frac{d\, zn}{du} \cdot \frac{d^2 zn}{du^2}$$

$$\Rightarrow \left(4\alpha \cdot zn + 2\beta \cdot zn\right) \frac{d}{du} zn = 2 \frac{d}{du} zn \cdot \frac{d^2}{du^2} zn$$

$$\Rightarrow \boxed{\frac{d^2}{du^2} zn(u, K) = 2\alpha \cdot zn(u, K) + \beta \cdot zn(u, K)}$$

Solving differential equations

$F_s = -\rho x + s \cdot x^3$ is the restoration force between two molecules. We will try to find the potential energy

$$U = U_0 - \int F_s \, dx.$$

Taking the reference point to be $x = 0$,

$$\Rightarrow U = -p\frac{x^2}{2} + s\frac{x^4}{4}.$$

$$\Rightarrow E_{total} = \frac{1}{2} m\left(\frac{dx}{dt}\right)^2 + \left(-p\frac{x^2}{2} + s\frac{x^4}{4}\right)$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2E_{total}}{m} + \left(\frac{sx^4}{2m} - \frac{px^2}{m}\right)$$

We know that elliptic functions satisfy differential equations of the format

$\left\{\frac{d}{dt} x(t)\right\}^2 = Ax^4 + Bx^2 + C$. So, let

$$x(t) = a \cdot zn\,(bt, K).$$

$$\Rightarrow \left(\frac{dx}{\cdot}\right)^2 = a^2\left\{b \cdot zn'(bt, K)\right\}^2.$$

$$\Big(\quad dt \Big) \qquad \Big(\qquad \Big)$$

$$= (ab)^2 \cdot \left(\alpha\, zn^4 + \beta \cdot zn^2 + \gamma\right)$$

$$\Rightarrow Ax^4 + Bx^2 + C = \left(\frac{b^2\alpha}{a^2}\right)x^4 + \left(b^2\beta\right)\cdot x^2 + a^2 b^2 \gamma$$

$$\Rightarrow A = \alpha \cdot \left(\frac{b}{a}\right)^2, \quad B = b^2\beta \quad \text{and} \quad C = (ab)^2\gamma.$$

$\Rightarrow \dfrac{AC}{B^2} = \dfrac{\alpha\gamma}{\beta^2}$ and we can also determine which elliptic function zn is.

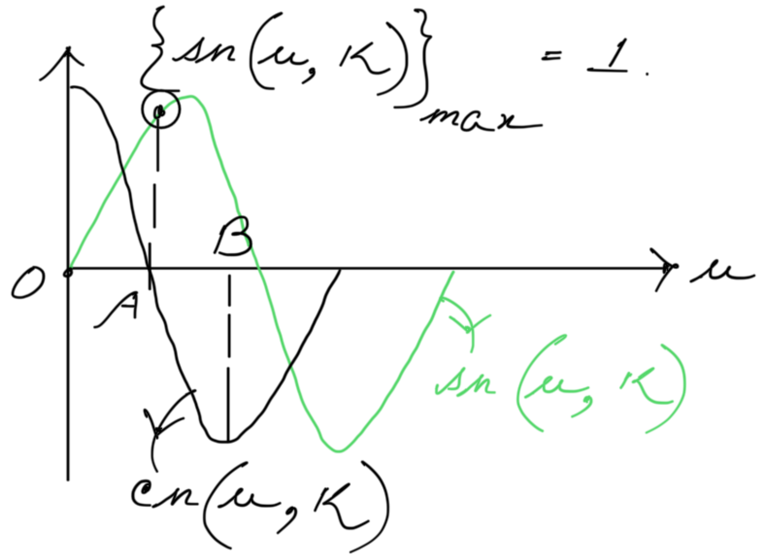
$$b = \sqrt{\frac{B}{b}} \quad \text{and} \quad a = \sqrt{\frac{C\beta}{B\gamma}}$$

$$\Rightarrow x(t) = \sqrt{\frac{C\beta}{B\gamma}} \cdot zn\left(\sqrt{\frac{B}{b}}\, t, K\right)$$

$$\Rightarrow x(t) = \sqrt{\frac{C\beta}{B\gamma}} \cdot zn\left(\sqrt{\frac{B}{b}}\,(t - t_0), K\right)$$

considering boundary conditions.

Shape of Elliptic functions



We know, $\frac{d}{du} sn(u,K) = cn(u,K) \cdot dn(u,K)$

$$\Rightarrow \frac{d}{du} sn(u,K) = \sqrt{\{1 - sn^2(u,K)\}\{1 - K^2 sn^2(u,K)\}}$$

$$\Rightarrow du = \frac{d(sn)}{\sqrt{(1-sn^2)(1-K^2 sn^2)}}$$

$$\Rightarrow u = \int_0^u du = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-K^2y^2)}} = K(K) = OA$$

Replacing y by $-y$ in $K(K)$, we can show that $OA = AB$. Thus periodicity of

$sn(u,k)$ is $4K(k)$.

The integral $\boxed{\displaystyle\int_0^y \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}}$ is called

a Legendre elliptic integral.