# The Riemann Hypothesis

We will mostly be looking at the Riemann Zeta function

$$\boxed{\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}} \quad \text{where } s \in \mathbb{C}.$$

Let $s=1$. $\zeta(1) = \sum_{n \geq 1} \frac{1}{n}$ diverges to infinity.

Notice that for any $s > 1$, $\zeta(s)$ converges.

## Eulers Theorem

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

$$= \sum_{n \geq 1} \left( \prod_{i=1}^{k_n} \frac{1}{p_i^{\alpha_i s}} \right)$$

We have used the fundamental theorem of arithmetic that any number $n$ can be written as the product of prime numbers

So $n = \prod_{i=1}^{K_n} \frac{1}{p_i^{a_i}}$.

$$\Rightarrow \zeta(s) = \prod_{\text{prime } p} \left( \sum_{K \geq 0} \frac{1}{p^{Ks}} \right)$$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

$$\Rightarrow \frac{1}{2^s} \zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots$$

$$\Rightarrow \left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots$$

Notice that all the $\frac{1}{x^s}$ where $x$ is a multiple of 2, has gone away. Similarly we can eliminate the $\frac{1}{x^s}$ terms where $x$ is a multiple of 3.

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \cdots$$

$$\Rightarrow \prod \left(1 - \frac{1}{\ldots}\right) \zeta(s) = 1$$

$$\prod_{\text{prime } p}\left(1-\frac{1}{p^s}\right)\zeta(s) = 1$$

$$\Rightarrow \boxed{\zeta(s) = \prod_{\text{prime } p}\left(1-\frac{1}{p^s}\right)^{-1}}$$

Since $\zeta(1) = \prod_{\text{prime } p}\left(1-\frac{1}{p}\right)$ diverges, we can infer that there are infinitely many primes.