Quotient Rings

Let R be a ring. Since $(R,+)$ is an Abelian Group, all of its subgroups are normal. I be an ideal of R. So $(I,+)$ is a subgroup of $(R,+)$.

$$R/I = \{a+I \mid a \in R\}.$$

We already know that $(R/I, +)$ is a group.

Lets define a multiplication operation for R/I as $\boxed{(a+I)(b+I) = ab\,I.}$

First, we need to show that if $a+I = a'+I$ and $b+I = b'+I$, then $a'b'+I = ab+I$.

Proof

$$a+I = a'+I$$

$$\Rightarrow \{a+x \mid x \in I\} = \{a+y \mid y \in I\}$$

$\Rightarrow$ For any x on the LHS there exists a y on the RHS such that

$$a + x = a' + y$$

$$\Rightarrow a - a' = y - x \in I.$$

Similarly $b - b' \in I$.

Now since I is an ideal and $(a - a') \in I$ and $b \in R$ $\Rightarrow (a - a')b \in I$

$$\Rightarrow (ab - a'b) \in I$$

Similarly $(a'b - a'b') \in I$

$$\Rightarrow (ab - a'b) + (a'b - a'b') \in I$$

$$\Rightarrow ab - a'b' \in I$$

$$\Rightarrow ab + I = a'b' + I.$$

We claim that $(R/I, +, \cdot)$ is a ring and there is a natural ring homomorphism $\varphi : R \to R/I$ given by $\varphi(a) = a + I$ where

I [illegible]

$I = \mathrm{Ker}(\varphi)$.

Proof

$$\varphi(a+b) = (a+b) + I$$
$$= (a+I) + (b+I)$$
$$= \varphi(a) + \varphi(b)$$

$\varphi(1) = (1+I)$ which is the multiplicative identity of the ring $(R/I, +, \cdot)$.

$$\varphi(ab) = ab + I$$
$$= (a+I)(b+I)$$
$$= \varphi(a)\cdot\varphi(b)$$

Thus φ is a ring homomorphism.

Now $\mathrm{Ker}(\varphi) = \{a \in R \mid \varphi(a) = 0 \text{ in } R/I\}$

$$= \{a \in R \mid \varphi(a) = 0 + I\}$$
$$= \{a \in R \mid a + I = 0 + I\}$$
$$= \{a \in R \mid a \in I\}$$

$\Rightarrow \mathrm{Ker}(\varphi) = I$.

Let $R = \mathbb{R}[x]$. We will prove that $\mathbb{R}[x]/I$ is isomorphic to $\mathbb{C}$.

Proof/ We already showed that there is a ring homomorphism φ from $\mathbb{R}[x]$ to $\mathbb{R}[x]/I$.

Let $\psi : \mathbb{R}[x]/I \xrightarrow{\sim} \mathbb{C}$ such that

$\psi(f(x) + I) = f(i)$. We will first show that ψ is well defined.

Let $f(x) + I = g(x) + I$

$$\Rightarrow f(x) - g(x) \in I.$$

I be all the multiples of (x^2+1), which is represented by (x^2+1)

$$\Rightarrow f(x) - g(x) = (x^2+1) \cdot h(x) \quad , h(x) \in \mathbb{R}[x]$$

When $x = i$, $f(x) - g(x) = (i^2+1) \cdot h(x) = 0$

$$\Rightarrow f(x) = g(x).$$

So ψ is a well defined map.

$\psi(1+I) = 1$ which is the multiplicative identity of $\mathbb{C}$.

$$\psi((f+I)+(g+I)) = \psi((f+g)+I)$$
$$= (f+g)\, i$$
$$= f(i) + g(i)$$
$$= \psi(f+I) + \psi(g+I).$$

$$\psi((f+I)(g+I)) = \psi(fg+I)$$
$$= (fg)\, i = f(i)\cdot g(i)$$
$$= \psi(f+I)\cdot \psi(g+I).$$

Thus ψ is a ring homomorphism.

$$\mathrm{Ker}(\psi) = \left\{ (f+I) \in \mathbb{R}[x]/I \mid \psi(f+I) = 0 \right\}$$

$$= \left\{ (f+I) \in \mathbb{R}[x]/I \mid f(i) = 0 \right\}$$

Now $f(i) = 0$

$$\Rightarrow a_n i^n + \cdots + a_1 i + a_0 = 0$$

$$\Rightarrow (a_n i^n + \cdots + a_1 i + a_0)^* = 0$$

$\Rightarrow a_n(-i^n) + \cdots + a_1(-i) + a_0 = 0$

$\Rightarrow f(-i) = 0$ or $f(i^*) = 0$

If $f(i) = 0$, that means $f(x)$ is divisible by $(x-i)$. Similarly if $f(-i) = 0$, that means $f(x)$ is divisible by $(x+i)$. So

$$Ker(\psi) = \{ f + I \mid f(x) \text{ is divisible by } (x^2+1) \}$$

$$= \{ f(x) + I \mid f(x) \in I \}$$

$$= \{ 0 + I \}$$

$\Rightarrow \psi$ is injective (one to one).

Let $z = a + ib \in \mathbb{C}$, $a, b \in \mathbb{R}$. Now,

$(a+bx) + I \in \frac{\mathbb{R}[x]}{I}$. Its image is $(a+bi)$.

Thus ψ is onto.

$\Rightarrow \psi$ is thus a ring isomorphism.