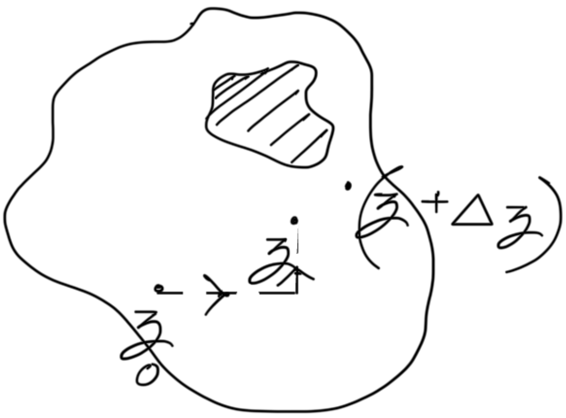
Morera's Theorem

Let f be a continuous function over domain D. If $\int_{\partial R} f(z)\,dz = 0$ for every axis parallel rectangle R, then $f(z)$ is analytic on D.

Proof

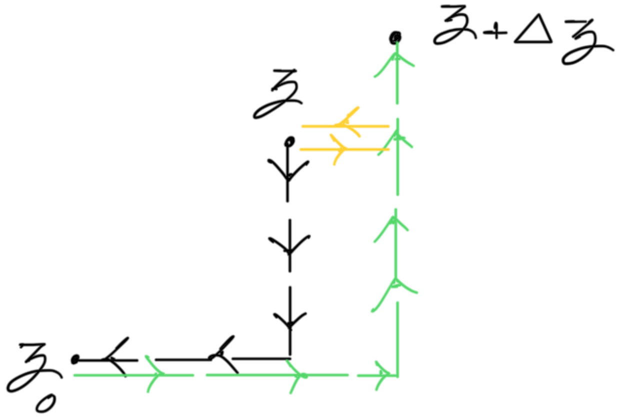


Let $F(z) = \int_{z_0}^{z} f(w)\,dw$ along the path shown in the diagram

$$\Rightarrow F(z+\Delta z) = \int_{z_0}^{z+\Delta z} f(w)\,dw$$

$$\Rightarrow F(z+\Delta z) - F(z) = \left(\int_{z_0} - \int_{z_0} \right) f(\omega)\, d\omega.$$

$$= \int_{z}^{z+\Delta z} f(\omega)\, d\omega$$



$$\Rightarrow \frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} \left\{ f(\omega) - f(z) + f(z) \right\} d\omega$$

$$= f(z) + \frac{1}{\Delta z} \int_{z}^{z+\Delta z} \left\{ f(\omega) - f(z) \right\} d\omega.$$

$$\Rightarrow \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| \leq \frac{1}{|\Delta z|} \cdot \varepsilon \cdot 2|\Delta z|$$

Since the function $f(z)$ is continuous, $f(\omega) - f(z)$ has some bounding value ε and the path between z and $(z+\Delta z)$ has the bounding value $2|\Delta z|$. So bounding value of

$$\int_{z}^{z+\Delta z} \{f(w)-f(z)\}\,dw = \varepsilon \cdot 2|\Delta z|.$$

$$\Rightarrow \lim_{\Delta z \to 0} \left| \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) \right| = 0.$$

as $\varepsilon \to 0$ when $\Delta z \to 0$.

$$\Rightarrow \frac{dF}{dz} = f(z)$$

Thus $F(z)$ is differentiable on D. Its also continuous since f itself is continuous. Which means $F(z)$ is analytic on D.
It is infinitely differentiable on D.
Thus $f(z)$ is also differentiable on D, inferring that $f(z)$ is also analytic on D.

Suppose $f(z,t)$ is a continuous function on D and $a \leq t \leq b$. It is analytic on D for every t. Let $F(z) = \int_a^b f(z,t)\,dt$.

$F(z)$ will also be analytic on D.

Proof: As $f(z+\Delta z, t) \to f(z,t)$ when $\Delta z \to 0$, $F(z+\Delta z) \to F(z)$ and $F(z)$ is thus continuous.

Let R be any axis-parallel rectangle in D. Since $f(z,t)$ is analytic in $a \leq t \leq b$,

$$\int_{\partial R} f(z,t)\,dz = 0 \text{ in that range.}$$

$$\Rightarrow \int_a^b \left[\int_{\partial R} f(z,t)\,dz \right] dt = 0$$

$$\Rightarrow \int_{\partial R} \left[\int_a^b f(z,t)\,dt \right] dz = 0$$

$$\Rightarrow \int_{\partial R} F(z,t)\, dz = 0$$

By Moriras Theorem, $F(z,t)$ is thus analytic on D (where $a \leq t \leq b$).