Field Extensions

A field F is a ring such that $(F \setminus \{0\}, \cdot)$ is a group.

Let K be a field. A subfield F of K is a subset such that

(1) F is a subgroup of $(K, +)$

(2) $F \setminus \{0\}$ is a subgroup of $(K \setminus \{0\}, \cdot)$.

We say that K is a field extension of F and denote this using K/F or $\begin{matrix} K \\ | \\ F \end{matrix}$.

We will only consider fields for which $0 \neq 1$. So $\min |F| = 2$.

Let $\begin{matrix} K \\ | \\ F \end{matrix}$ and $\alpha \in K$. We will be constructing a field $F(\alpha)$ such that $\begin{matrix} K \\ | \\ F(\alpha) \\ | \\ F \end{matrix}$.

$F(\alpha)$ is called the intermediate field.

It should be the smallest subfield

of K containing F and α.

$F(\alpha)$ contains all polynomials of the form $a_n \alpha^n + \cdots + a_1 \alpha + a_0$, $a_n, \ldots a_0 \in F$, as well as all ratios of those polynomials, which are of the form $\dfrac{f(\alpha)}{g(\alpha)}$ where $g(\alpha) \neq 0$.

Notice that $\left\{ \dfrac{f(\alpha)}{g(\alpha)} \,\middle|\, f(x), g(x) \in F[x],\ g(\alpha) \neq 0 \right\}$ is a field

$$\Rightarrow \boxed{F(\alpha) = \left\{ \dfrac{f(\alpha)}{g(\alpha)} \,\middle|\, f(x), g(x) \in F[x],\ g(\alpha) \neq 0 \right\}}$$

We say α is algebraic over F if there exists a polynomial $\underset{\neq 0}{f(x)} \in F[x]$ such that $f(\alpha) = 0$. α is the root of $f(x)$.

Otherwise, α is transcendental over F.

Consider the ring homomorphism $\varphi_\alpha : F[x] \to K$

$\varphi_\alpha(f(x)) = f(\alpha)$. Now

$$\mathrm{Ker}(\varphi_\alpha) = \{ f(x) \in F[x] \mid f(\alpha) = 0 \}$$

If α is algebraic over F, then $\mathrm{Ker}(\varphi_\alpha) \neq \{0\}$.

$\mathrm{Im}(\varphi_\alpha) = \{ f(\alpha) \mid f(x) \in F[x] \} = F[\alpha]$. Keep in mind that $F[\alpha]$ and $F(\alpha)$ are not the same. $F(\alpha)$ is bigger than $F[\alpha]$.

By the first Ring isomorphism theorem, there exists a ring isomorphism

$$\overline{\varphi_\alpha} : \frac{F[x]}{\mathrm{Ker}(\varphi_\alpha)} \xrightarrow{\sim} F[\alpha]$$

Notice that $F[x]$ is a Principal Ideal.

Suppose $\mathrm{Ker}\,\varphi_\alpha = (f(x))$, $f(x) \in F[x]$ and $f(x) \neq 0$. This means all polynomials which have α as the root, are multiples of $f(x)$.

$f(x)$ is irreducible.

$$\varphi_\alpha : \frac{F[x]}{\mathrm{Ker}\,\varphi_\alpha} \xrightarrow{\sim} F[\alpha]$$

$$\Rightarrow \varphi_\alpha : \frac{F[x]}{(f(x))} \xrightarrow{\sim} F[\alpha]$$

$\Rightarrow \dfrac{F[x]}{(f(x))}$ is an Integral Domain since it is the subring of $F[\alpha]$ which itself is ID.

$\Rightarrow (f(x))$ is a Prime ideal. $f(x)$ is thus irreducible.

$\Rightarrow (f(x))$ is a Maximal Ideal.

$\Rightarrow \dfrac{F[x]}{(f(x))}$ is a field.

$\Rightarrow$ Since $\dfrac{F[x]}{(f(x))} \xrightarrow{\sim} F[\alpha]$, $F[\alpha]$ is also a field.

$\Rightarrow$ $\boxed{F[\alpha] = F(\alpha)}$ if α is algebraic over F.

$F[\alpha] \subseteq F(\alpha)$ in general, since $F(\alpha)$ also contains ratios of polynomials.

② Suppose α is transcendental over F.

$\Rightarrow \operatorname{Ker} \varphi_\alpha = \{0\}$.

$\Rightarrow \varphi_\alpha$ is injective

$\Rightarrow F[x]$ is isomorphic to $F[\alpha]$ as rings. α then behaves like the variable x. It doesn't have any inverse in $F[\alpha]$, since x doesn't have one in $F[x]$. $F[\alpha]$ is thus not a field.