A function f is called meromorphic on domain D, if f is analytic or has a pole at every point in D.

Integration of meromorphic functions

Let $f(z)$ has a pole of order n at $z_0 \in D$.

$$f(z) = \sum_{K=-\infty}^{\infty} a_K (z-z_0)^K$$ which is not valid for every point in D.

Now consider $\displaystyle\int_{|z-z_0|=r} f(z)\,dz$.

$$= \int_{|z-z_0|=r} \left[\sum_{K=-\infty}^{\infty} a_K (z-z_0)^K \right] dz$$

$$= \sum_{K=-\infty}^{\infty} \left[\int_{|z-z_0|=r} (z-z_0)^K \, dz \right] a_K$$

Using Cauchys Theorem, for $K \geq 0$,

$$\int_{|z-z_0|=r} (z-z_0)^K \, dz = 0$$, since $(z-z_0)^K$ is completely analytic in $|z-z_0| \leq r$.

Remember $f(z) = \frac{1}{2\pi i} \int_{\delta D} \frac{f(\omega)}{\omega - z} \, d\omega$.

$$\Rightarrow 2\pi i \cdot f(z) = \int_{\delta D} \frac{f(\omega)}{\omega - z} \, d\omega.$$

When $f(z) = 1$ on D, $\int_{\delta D} \frac{d\omega}{\omega - z} = 2\pi i$.

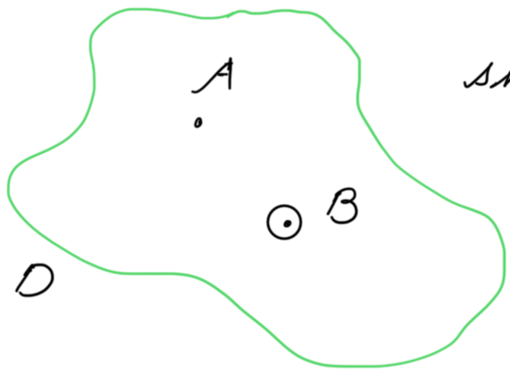
So $$\int_{|z-z_0|=r} \frac{dz}{z-z_0} = 2\pi i$$

$$\Rightarrow \frac{d}{dz} \int_{|z-z_0|=r} \frac{dz}{z-z_0} = 0$$

$$\Rightarrow \int_{|z-z_0|=r} \frac{dz}{(z-z_0)^2} = 0 \qquad \Rightarrow \int_{|z-z_0|=r} \frac{dz}{(z-z_0)^n} = 0 \text{ for } n \leq 1.$$

Thus $\displaystyle\int_{|z-z_0|=r} f(z)\,dz = a_{-1}\cdot 2\pi i$

Now what if there are 2 poles in D.



We will construct a very small disk B around the 2nd pole.

Let $D' = D \setminus B$

$$\int_{\partial D'} f(z) = 2\pi i\cdot a_{-1}.$$

In B, let $f(z) = \sum_{K=-\infty}^{\infty} a'_K (z-z_0)^K.$

$$\int_{\partial B} f(z)\,dz = 2\pi i\cdot a'_{-1}$$

$$\Rightarrow \int_{\partial D} f(z)\,dz = 2\pi i\left(a_{-1} + a'_{-1}\right)$$

We will define $\boxed{\operatorname{Res}_{z_0} f(z) = a_{-1}}$ which

is called the residue of $f(z)$ at the pole z_0.

If there are n poles in the domain D,

then $$\boxed{\int_{\delta D} f(z)\, dz = 2\pi i \cdot \sum_{j=1}^{n} \operatorname{Res}_{z_j}(f)}$$

Calculating residues

Let $f(z)$ has a pole of order 1 in D.

$$f(z) = \sum_{K=-\infty}^{-1} a_K (z - z_0)^K$$

$$\Rightarrow \lim_{z \to z_0} \left\{ f(z) \cdot (z - z_0) \right\} = a_{-1}$$

If order of pole is $n > 1$, then

$$\lim_{z \to z_0} \left\{ f(z) \cdot (z - z_0)^n \right\} = \sum_{K=-n}^{-1} a_K (z - z_0)^{n+K}$$

$$\Rightarrow \boxed{\lim_{z \to z_0} \left\{ f(z) \cdot (z - z_0)^n \right\}^{(n-1)} = \operatorname{Res}_{z_0}(f)}$$