Degree of Field Extension

Suppose α is algebraic over F. Then $\operatorname{Ker} \varphi_\alpha = (f(x))$. If we enforce $f(x)$ to be monic, then $f(x)$ becomes unique. It is called the irreducible polynomial of α over F. The degree of α over F is the degree of $f(x)$.

Let degree of α over $F = 1$

$\Rightarrow$ degree of $f(x) = 1$

$\Rightarrow$ $x - \alpha = f(x) \in F[x]$

$\Rightarrow$ $\alpha \in F$.

⊙ Let $\begin{array}{c} K \\ | \\ F \end{array}$ be a field extension. Then K is a vector space over F.

An F-vector space is an Abelian Group with scalar multiplication operation defined as $F \times V \longrightarrow V$

$$\Rightarrow (\lambda, v) \longmapsto \lambda \cdot v.$$

$\lambda \in F$ is called a scalar.

The degree of K over F is the dimension of K as an F-vector space. It is represented by $\dim_F K$ or $[K:F]$.

$$\boxed{\dim_F (F(\alpha)) = \text{Degree of } \alpha \text{ over } K}.$$

Proof: Let $f(x) \in F[x]$ be the irreducible polynomial of α over F. Its degree be $n = \deg_F(\alpha)$.

We know that $F(\alpha) = F[\alpha]$.

Let $g(\alpha) \in F(\alpha)$.

$\Rightarrow g(\alpha) = b_m \alpha^m + b_{m-1} \alpha^{m-1} + \cdots b_0$ where

$b_m, b_{m-1}, \ldots, b_0 \in F$.

If $m \leq (n-1)$, then $g(\alpha)$ is spanned by $\{1, \alpha, \ldots, \alpha^{n-1}\}$ over F.

But if $m > (n-1)$, we need to show that $\alpha^n, \alpha^{n+1}, \ldots$ are spanned by $\{1, \alpha, \ldots, \alpha^{n-1}\}$ over F.

$$f(\alpha) = 0$$

$$\Rightarrow \alpha^n + a_{n-1}\alpha^{n-1} + \cdots = 0 \text{ —— } (1).$$

$$\Rightarrow \alpha^n = -a_{n-1}\alpha^{n-1} - \cdots$$

$$\in F\text{-span of } \{1, \alpha, \ldots, \alpha^{n-1}\}.$$

Multiplying (1) by α,

$$\alpha^{n+1} + a_{n-1}\alpha^n + \cdots = 0$$

$$\Rightarrow \alpha^{n+1} = -a_{n-1}\alpha^n - a_{n-2}\alpha^{n-1} - \cdots$$

$$\in F\text{-span of } \{1, \alpha, \ldots, \alpha^{n-1}\}$$

Same goes for any other α^m.

We also need to show that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is linearly independent over F. Let

$h(\alpha) = a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$

$\Rightarrow$ α is root of polynomial $h(x)$ whose degree $< n$.

$\Rightarrow$ $h(x) = 0$ since $\deg_F(\alpha) = n$

$\Rightarrow$ $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is linearly independent. It is thus a basis of $F(\alpha)$.

$\Rightarrow$ $\dim_F \{F(\alpha)\} = n$.

⊙ Let $\begin{array}{c} K \\ | \\ F \end{array}$ and $\alpha \in K$. α is algebraic over F only if $\dim_F \{F(\alpha)\} < \infty$.

Proof: If α is algebraic over F, then $\dim_F \{F(\alpha)\} = \deg_F \alpha < \infty$.

Conversely, if $\dim_F \{F(\alpha)\} < \infty$

$\Rightarrow$ $\{1, \alpha, \alpha^2, \ldots,\}$ infinite set is linearly dependent. Here $1, \alpha, \ldots \in F(\alpha)$.

$\Rightarrow$ There exists a nontrivial relation such that $a_0 + a_1 \alpha + \cdots = 0$ where all $a_n \in F$ and $\neq 0$.

$\Rightarrow$ α is algebraic over F.

$\odot$ Let
$$\begin{array}{c} K \\ | \\ L \\ | \\ F \end{array}$$
Then $\boxed{[K:F] = [K:L][L:F]}$

Proof Suppose $[K:L]$ or $[L:F] = \infty$

If $[K:L] = \infty$

$\Rightarrow$ $\exists$ an infinite sized linearly independent set in K as an L-vector space.

$\Rightarrow$ $\exists$ an infinite sized linearly independent set in K as an F-vector space, since some $a_n \in L$ are also in F.

$\Rightarrow$ $[K:F] = \infty$

And if $[L:F] = \infty$, its trivial that $[K:F] = \infty$. Thus we need both $[K:L] = m$ and $[L:F] = n < \infty$.

$$\begin{tikzcd} K \arrow[d, dash, "m"] \arrow[dd, bend right=60, dash, "mn"'] \\ L \arrow[d, dash, "n"] \\ F \end{tikzcd}$$

The basis for K as an L-vector space be

$1, \alpha, \dots, \alpha_m \in L$ and for L as an F-vector space be $1, \beta, \dots, \beta_n \in F$. We need to prove that $\{\alpha_i \beta_j\}_{\substack{1 < i < m \\ 1 < j < n}}$ forms the basis of K as an F vector space.

Let $\gamma \in K$.

$\Rightarrow \gamma = \sum_{i=1}^{m} a_i \alpha_i$ for some $a_i \in L$.

$\Rightarrow \gamma = \sum_{i=1}^{m} \left\{ \left(\sum_{j=1}^{n} b_{ji} \beta_j \right) \alpha_i \right\}$ for some $b_{ji} \in F$.

$$= \sum_{i,j} \left\{ (\alpha_i \beta_j) b_{ij} \right\}$$

If $\sum_{i,j} \left\{ (\alpha_i \beta_j) b_{ji} \right\} = 0$

$\Rightarrow \sum_{j} \beta_j b_{ji} = 0$ since α_i are linearly independent, as they are the basis in

K as an L-vector space

K as an L-vector space.

$\Rightarrow b_{ij} = 0 \ \forall i, j$ since β_j are linearly independent, as they are the basis in L as an F-vector space.

$\Rightarrow \{\alpha_i \beta_j\}$ is thus the basis of K as an F-vector space.

$$\left| \{\alpha_i \beta_j\} \right| = mn$$

$$\Rightarrow [K : F] = mn.$$