Zariski Topology

A subset of $\mathbb{K}^n$ is called an algebraic set if it is of the form $Z(S)$.

① $\boxed{\mathbb{K}^n = Z(\{0\})}$

because every point in $\mathbb{K}^n$ satisfies the zero polynomial.

② $\boxed{\phi = Z\left(\mathbb{K}\left(x_1, \dots, x_n\right)\right)}$ since $\mathbb{K}(x_1, \dots, x_n)$ contains the 1 polynomial which can never be 0.

③ $\boxed{Z(S_1) \cup \dots \cup Z(S_m) = Z\left(S_1 \dots S_m\right)}$

where $Z\left(S_1 \dots S_m\right) = \{f_1 \dots f_m \mid f_i \in S_i\}$

Its obvious that if $(\lambda_1, \dots, \lambda_n) \in Z(S_i)$

$\Rightarrow (\lambda_1, \dots, \lambda_n) \in Z\left(S_1 \dots S_m\right)$

Also $Z(S_i) \subset Z\left(S_1 \dots S_m\right)$

$\Rightarrow Z(S_1) \cup \dots Z(S_m) \subset Z(S_1 \dots S_m)$

Conversely if $(\lambda_1, \dots, \lambda_m) \subset Z(S_1 \dots S_m)$

Now if $(\lambda_1, \dots, \lambda_m) \notin Z(S_i) \; \forall i$, then

$\exists g_i \in S_i \; \forall i$ such that $g_i(\lambda_1, \dots, \lambda_m) \neq 0$.

$\Rightarrow g_1 \dots g_m \subset Z(S_1 \dots S_m)$ but $(g_1 \dots g_m)(\lambda_1 \dots \lambda_m) \neq 0$

which is a contradiction.

$\Rightarrow \bigcup_i Z(S_i) = Z(S_1 \dots S_m)$.

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$$\boxed{\bigcap_\alpha Z(S_\alpha) = Z\left(\bigcup_\alpha S_\alpha\right)}$$

Let $(\lambda_1, \dots, \lambda_m) \in \bigcap_\alpha Z(S_\alpha)$

$\Rightarrow (\lambda_1, \dots, \lambda_m) \in Z\left(\bigcup_\alpha S_\alpha\right)$

The above properties tell us that $\mathbb{K}^n$ becomes a topological set if the algebraic sets are closed. The topology of $\mathbb{K}^n$ is called the Zariski topology. Any $f \in \mathbb{K}[x_1, \dots, x_n]$ gives a mapping from $\mathbb{A}^n_{\mathbb{K}}$ to $\mathbb{A}^1_{\mathbb{K}}$.

We will denote the ideal generated by S by (S). (S) is thus the smallest ideal generated by S.

$$\boxed{\bigcap_{\substack{\text{ideal } J \\ J \subset S}} J = (S)}$$

We will focus our study on

$$\begin{tikzcd} \begin{array}{c} Z(S) = Z((S)) \\ \subset \mathbb{A}^n_{\mathbb{K}} \end{array} \arrow[r, bend left] & \begin{array}{c} (S) \xrightarrow{} \text{finite.} \\ \subset \mathbb{K}[x_1, \ldots, x_n] \end{array} \arrow[l, bend left] \end{tikzcd}$$

A commutative ring containing 1, is Noetherian if every ideal is finitely generated. And if R is Noetherian, then so is $R[x_1, \ldots, x_n]$. This is called Hilbert's Basis theorem / Emmy Noether's theorem.

$\Rightarrow$ If $Z(S) \neq \phi$, then even if S is infinite, $(S) = (f_1, \ldots, f_n)$ for some $f_i \in \mathbb{K}[x_1, \ldots, x_n]$.

Any field $\mathbb{F}$ is always Noetherian since it has 2 ideals which are finitely generated :

(i) 0 ideal

(ii) full ideal generated by 1.

$\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian

$\Rightarrow$ (S) is finitely generated regardless of S being finite or infinite.