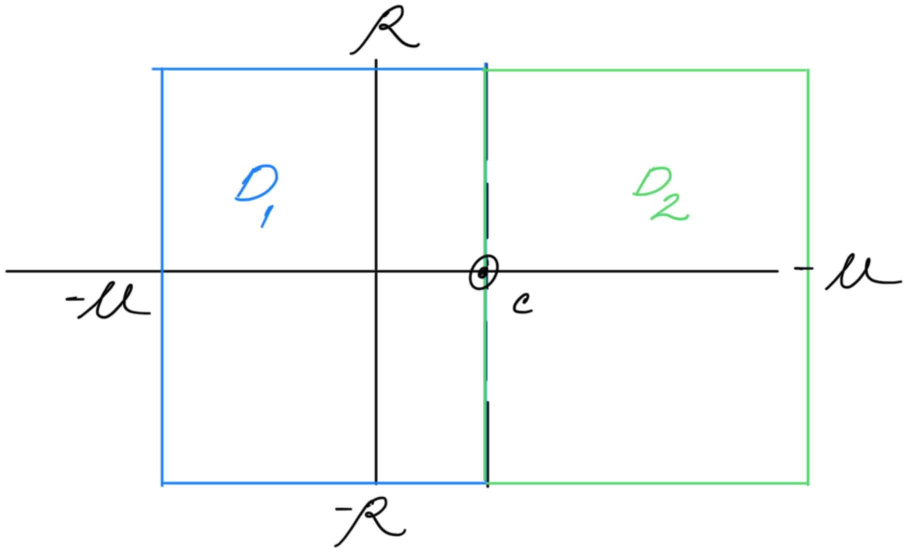
$$\boxed{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\,ds = \delta(x)}, \quad c > 0 \text{ where}$$

$$\delta(x) = \begin{cases} 0, & 0 \leq x < 1 \text{ and } \frac{1}{2} \text{ at } x = 1. \\ 1 & \text{otherwise} \end{cases}$$

Proof



$I = \int_{\partial D_1} \frac{x^s}{s}\,ds = 2\pi i$. Because in D, $\frac{x^s}{s}$ is analytic everywhere except at $s = 0$. At $s = 0$, $\frac{x^s}{s}$ has a pole of order 1.

$$\operatorname{Res}_{s=0}\left(\frac{x^s}{s}\right) = \lim_{s \to 0}\left(\frac{x^s}{s} \cdot s\right) = 1.$$

Let $I_1 = \int^{-U+iR} \frac{x^s}{s}\,ds$

$$\Rightarrow |I_2| = \left| \int_{c+iR}^{-u+iR} \frac{x^s}{s}\, ds \right| \leq \int_{c+iR}^{-u+iR} \left|\frac{x^s}{s}\right| ds.$$

$$\Rightarrow |I_2| \leq \int_{c+iR}^{-u+iR} \frac{\left|e^{s\cdot \ln x}\right|}{R}\, ds$$

$$\Rightarrow |I_2| \leq \frac{1}{R}\left|\int_{c}^{-u} e^{t\cdot \ln x}\, dt\right|$$

$$\Rightarrow |I_2| \leq \frac{1}{R}\cdot\left|\left[\frac{e^{t \ln x}}{\ln x}\right]_{c}^{-u}\right|$$

$$\Rightarrow |I_2| \leq \frac{1}{R}\cdot\left(\frac{x^{c}}{\ln x} - \frac{x^{-u}}{\ln x}\right)$$

The same goes for $I_4 = \int_{-u+iR}^{c+iR} \frac{x^s}{\ln x}\, ds.$

$$I_3 = \int_{-u+iR}^{-u-iR} \frac{x^s}{\ln n}\, ds$$

$$\Rightarrow |I_3| \leq \left| \int_{-u+iR} \frac{|{\cdot}|}{u} \, ds \right|$$

$$\Rightarrow |I_3| \leq \frac{1}{u} \cdot \left| \int_{-u+iR}^{-u-iR} x^{-u} \, ds \right|$$

$$\Rightarrow |I_3| \leq \frac{2R}{u} \cdot x^{-u}$$

Now $I = I_1 + I_2 + I_3 + I_4$

$$\Rightarrow 2\pi i = I_1 + O\left(\frac{2}{R}\left(\frac{x^c}{\ln x} + \frac{x^{-u}}{\ln x} \right) + \frac{2R}{u} x^{-u} \right)$$

$$\Rightarrow I_1 = 2\pi i - O\left(\frac{x^c}{R \ln x} \right)$$ by letting $u \to \infty$ and $x > 1$.

$$\Rightarrow \lim_{R \to \infty} I_1 = 2\pi i$$

$$\Rightarrow \lim_{R \to \infty} \int_{c-iR}^{c+iR} \frac{x^s}{s} \, ds = 2\pi i \text{ when } x > 1$$

In D_2, $\frac{x^s}{s}$ is completely analytic. So,

$$I = \int_{\partial D_2} \frac{x^s}{s}\, ds = 0.$$

$$I_2 = \int_{c+iR}^{U+iR} \frac{x^s}{s}\, ds$$

$$\Rightarrow |I_2| \leq \left| \int_{c+iR}^{U+iR} \frac{|e^{s \ln x}|}{R}\, ds \right|$$

$$\Rightarrow |I_2| \leq \frac{1}{R} \left| \int_{c}^{U} e^{t \ln x}\, ds \right|$$

$$\Rightarrow |I_2| \leq \frac{1}{R} \left(\frac{x^U}{\ln x} - \frac{x^c}{\ln x} \right)$$

Same goes for I_4.

$$I_3 = \int_{U+iR}^{U-iR} \frac{x^s}{s}\, ds$$

$$\Rightarrow |I_3| \leq \left| \int_{U+iR}^{U-iR} \frac{x^U}{U}\, ds \right|$$

$$\Rightarrow |I_3| \leq \frac{2R}{U} x^{U}$$

$$I = \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds + O\left(\frac{2R}{U} x^{U} + \frac{2}{R}\left(\frac{x^{U}}{\ln x} + \frac{x^{c}}{\ln x}\right)\right)$$

$$\Rightarrow I = \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds + O\left(\frac{x^c}{R \ln x}\right)$$ by letting

$U \to 0$ and when $x < 1$.

$$\Rightarrow \lim_{R\to\infty} \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds = 0 \quad \text{when } x < 1.$$

$$|e^{z \ln x}| = |e^{(a+ib)\ln x}|$$

$$= |e^{a \ln x} \cdot e^{ib \ln x}|$$

$$\leq |x^{a}| \cdot |e^{ib \ln x}|$$

$$= |x^{a}|$$

$$\Rightarrow |(a+ib) \ln x| < |a|$$

$$\left| e^{ib \ln x} \right| = 1.$$