Zeroes of an Analytic Function

Let f be an analytic function on D and $f(z_0) = 0$, $z_0 \in D$. We say that z_0 is a zero of f of order n if $\lim_{z \to z_0} \frac{f(z)}{(z-z_0)^{n-1}} = 0$ and

$$\lim_{z \to z_0} \frac{f(z)}{(z-z_0)^{n}} \neq 0.$$

Any analytic function f on D is either 0 everywhere on D or the zeroes of f are isolated.

Proof: Let $f(z_0) = 0$ for some $z_0 \in D$. In the neighbourhood of z_0,

$$f(z) = \sum_{k \geq 0} a_k (z-z_0)^k \text{ where } |z-z_0| < r.$$

(1) $\quad a_k = 0 \ \forall \ k \geq 0.$

Let U be the set of all points in D, such that for every $z \in U$, the power series expansion of f around z is 0. We will pick any point $z_1 \in U$. Let $z_2 \in D$, $|z_1 - z_2| \leq \epsilon < r$.

Now $f(z)\Big|_{z=z_2} = 0$.

$\Rightarrow f^{(m)}(z)\Big|_{z=z_2} = 0$ in that circle $|z_1 - z_2| < r$

$\Rightarrow a_K = 0 \ \forall K \geq 0$ at $z = z_2$.

$\Rightarrow z_2 \in U$.

Thus around z_1, in $|z - z_1| < r$, all the points $\in U$.

(2) Lets choose any random point $z_3 \in D \setminus U$ and $a_K \neq 0$ for some K. We will pick a point z_4 from the neighbourhood of z_3.

$$f(z_4) = \sum a_K (z - z_3)^K \Big|_{z=z_4}$$

The first nonzero term, $b_K(z_4 - z_3)^K$ will dominate the sum of the remaining terms. So $f(z_4) \neq 0$.

$\Rightarrow D \setminus \mathcal{U}$ is an open set

$\Rightarrow D = \mathcal{U} \underset{\text{disjoint}}{\cap} (D \setminus \mathcal{U})$

$\Rightarrow \mathcal{U} = \phi$ or D since D is connected.

When $\mathcal{U} = D$, $f(z) = 0$ everywhere on D. And when $\mathcal{U} = \phi$, the zeroes of f are isolated.