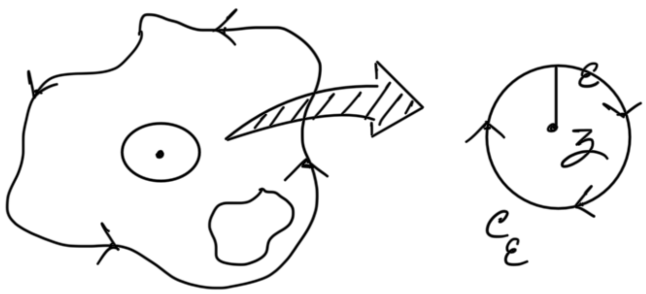
Cauchy's Integral Formula

$$\boxed{f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\omega)}{\omega - z}\, d\omega} \quad \forall z \in D.$$

Proof

Notice that the function $\dfrac{f(z)}{\omega - z}$ is not analytic at $\omega = z$.



We will define a new domain $D_\varepsilon = D / C_\varepsilon$

$$\int_{\partial D_\varepsilon} \frac{f(\omega)}{\omega - z}\, d\omega = \left(\int_{\partial D} - \int_{\partial C_\varepsilon}\right) \frac{f(\omega)}{\omega - z}\, d\omega = 0$$

The traversal of a boundary must be done in that direction for which the points inside the domain will be on your left side.

$$\Rightarrow \int_{\partial D} \frac{f(\omega)}{\omega - z}\, d\omega = \int_{\partial C_\varepsilon} \frac{f(\omega)}{\omega - z}\, d\omega$$

$\omega = z + \varepsilon e^{i\theta}$. So,

$$\int_{\partial D} \frac{f(\omega)}{\omega - z}\, d\omega = \int_0^{2\pi} \frac{f\left(z + e^{i\theta}\varepsilon\right)}{\varepsilon e^{i\theta}} \left(\varepsilon i e^{i\theta} d\theta\right)$$

$$= i\int_0^{2\pi} f\left(z + e^{i\theta}\varepsilon\right) d\theta$$

$$= i\int_0^{2\pi} f(z) + \left\{ f\left(z + e^{i\theta}\varepsilon\right) - f(z) \right\} d\theta$$

$$= 2\pi i \cdot f(z) + i\int_0^{2\pi} \left\{ f\left(z + e^{i\theta}\varepsilon\right) - f(z) \right\} d\theta$$

$\approx 2\pi i \cdot f(z)$ considering $\varepsilon \to 0$.

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\omega)}{\omega - z}\, d\omega.$$

We will also claim that the analytic function f is infinitely differentiable in D.

$$f^{(m)}(z) = \frac{\lfloor m}{2\pi i}\int_{\partial D} \frac{f(\omega)}{(\omega - z)^{m+1}}\, d\omega, \quad m \geq 0$$

Proof/ We already know that,

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\omega)}{\omega - z}\, d\omega.$$ So the

statement holds true for $m=0$. Lets assume that the statement is true for $m-1$. Now for m,

$$f^{(m)}(z) = \lim_{\Delta z \to 0} \frac{f^{(m-1)}(z+\Delta z) - f^{(m-1)}(z)}{\Delta z}$$

where $f^{(m-1)}(z+\Delta z) = \dfrac{\lfloor m-1}{2\pi i}\displaystyle\int_{\partial D} \frac{f(\omega)}{(\omega - z - \Delta z)^{m}}\, d\omega$

$$\Rightarrow f^{(m-1)}(z+\Delta z) - f^{(m-1)}(z)$$

$$= \frac{\underline{|m-1}}{2\pi i}\int_{\partial D} \frac{f(\omega)}{(\omega-z-\Delta z)^m} - \frac{f(\omega)}{(\omega-z)^m}\, d\omega$$

$$= \frac{\underline{|m-1}}{2\pi i}\int_{\partial D} \frac{f(\omega)}{(\omega-z-\Delta z)^m(\omega-z)^m}$$

$$\left\{(\omega-z)^m - (\omega-z-\Delta z)^m\right\} d\omega$$

Here $(\omega-z)^m - (\omega-z-\Delta z)^m$

$$= (\omega-z)^m - \sum_{j=0}^{m} {}^mC_j\, (\omega-z)^{m-j} \cdot (-\Delta z)^j$$

$$= -\left\{ {}^mC_1\, (\omega-z)^{m-1}(-\Delta z) - (\Delta z)^2 \cdot g(\omega-z) \right\}$$

$$= m(\omega-z)^{m-1} \cdot \Delta z + g(\omega-z)\cdot(\Delta z)^2$$

$$\therefore\ f^{(m-1)}(z+\Delta z) - f^{(m-1)}(z)$$

$$= \frac{(m-1)!}{2\pi i} \int_{\partial D} \frac{f(w)\, m \cdot \Delta z}{(w - z - \Delta z)^{m} (w - z)}\, dw + (\Delta z)^{2} g'(z)$$

$$\Rightarrow f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w - z)^{m+1}}\, dw$$