Power Series

The power series is the sum $\boxed{\sum_{K \geq 0} a_K (z - z_0)^K}$

where $a_K, z_0 \in \mathbb{C}$. Infinitely many a_K are non zero.

→ $\sum_{K \geq 0} a_K (z - z_0)^K$ is absolute convergent in

$$|z - z_0| < r, \text{ if } \sum_{K \geq 0} |a_K| \cdot |z - z_0|^K < \infty$$

→ And it is said to be uniformly convergent in $|z - z_0| < r$, if every $m \geq 0$,

$$\sum_{K \geq 0} a_K (z - z_0)^K - \sum_{0 \leq K \leq m} a_K \cdot (z - z_0)^K \leq \varepsilon_m$$

and $\lim\limits_{m \to 0} \varepsilon_m = 0$.

Let R be the largest real for which the sequence $\{|a_K| r^K\}$ converges to 0. R is some sequence $\{|a_K| r^K\}$ converges to 0. R is

then called the radius of convergence of

$$\sum_{K \geq 0} a_K (z - z_0)^K = f(z).$$

f is uniformly convergent in $|z - z_0| \leq r < R$

Proof Let $r < s < R$. By the definition of R, $\{|a_K| s^K\}$ converges to 0.

$$\left| \sum_{K \geq 0} a_K (z - z_0)^K - \sum_{0 \leq K \leq m} a_K (z - z_0)^K \right|$$

$$= \left| \sum_{K > m} a_K (z - z_0)^K \right| \leq \sum_{K > m} |a_K| \cdot |z - z_0|^K$$

$$\leq \sum_{K > m} |a_K| r^K$$

$$= \sum_{K > m} |a_K| s^K \left(\frac{r}{s}\right)^K \leq c \sum_{K > m} \left(\frac{r}{s}\right)^K$$

$m+1$

$$= C \cdot \left(\frac{r}{s}\right)^{m+1} \frac{1}{1 - r/s}$$

$$= c' \cdot \left(\frac{r}{s}\right)^{m+1}.$$

C here is the bounding value of $\sum_{k > m} |a_k| s^k$

Outside the radius of convergence (R), f diverges.

~~Proof~~ Let $f(z) = \sum_{k \geq 0} a_k z^k$ where

$|z| = r$ and $R < s < r$. By definition of R, $f(z)$ doesn't converge to 0.

Thus $|a_k| s^k \geq \varepsilon > 0$

Now $\sum_{k > m} |a_k| r^k$

$= \sum_{k > m} |a_k| \cdot s^k \left(\frac{r}{s}\right)^k$ will shoot up to infinity because of this.

Note that we didn't prove anything regarding the behaviour of f at $r = R$. It depends on f, whether it'll converge or diverge at $r = R$.

Lets study the behaviour of f in $|z - z_0| < R$.

f is analytic in $|z - z_0| < R$.

Proof

Let $P_m(z) = \sum_{0 \le k \le m} a_k z^k$. Since it is a polynomial made off z, it is analytic.

For every m, $P_m(z)$ is continuous since $\left|P_m(z + \Delta z) - P_m(z)\right| \le \delta_m$ which is close to 0.

$$\left|f(z + \Delta z) - f(z) - \{P_m(z + \Delta z) - P_m(z)\}\right|$$

$$= \left|\{f(z + \Delta z) - P_m(z + \Delta z)\} - \{f(z) - P_m(z)\}\right|$$

$\le 2\varepsilon_m$ where $\lim_{m \to \infty} \varepsilon_m = 0$.

$m \to 0$

$\Rightarrow \left| f(z+\Delta z) - f(z) \right| \leq 2\varepsilon_m + \delta_m$ which tends to 0 as $m \to 0$. Thus $f(z)$ is continuous.

Let T be any rectangle in $|z - z_0| < R$.

Using Cauchy's theorem $\int_{\partial T} P_m(z)\, dz = 0$.

Now $\int_{\partial T} f(z)\, dz$

$$= \int_{\partial T} \{ f(z) - P_m(z) \} dz$$

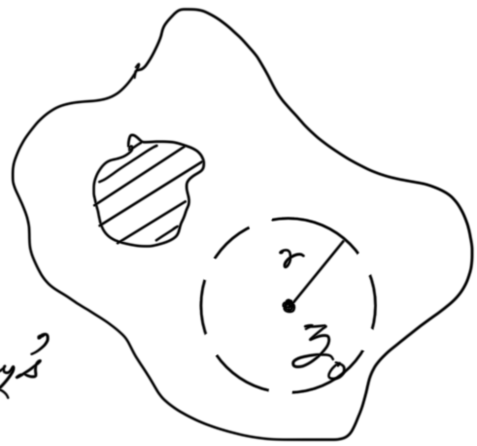
$$\Rightarrow \left| \int_{\partial T} f(z) \cdot dz \right| \leq \int_{\partial T} \left| f(z) - P_m(z) \right| dz \leq c\,\varepsilon_m$$

As $m \to 0$, $\int_{\partial T} \left| f(z) - P_m(z) \right| dz = 0$. So

$\int_{\partial T} f(z)\, dz = 0$. By Moore's Theorem, $f(z)$ is thus analytic.

Let f be an analytic function in D. Then for any r such that the disk $|z-z_0| \leq r$ lies inside D, f can be expressed as

$$f(z) = \sum_{k \geq 0} a_k (z-z_0)^k.$$



Proof: For any z such that $|z-z_0| < r$, using Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z}\, dw$$

$$= \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)-(z-z_0)}\, dw$$

$$= \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z_0} \cdot \frac{1}{\left(1 - \frac{z-z_0}{\quad}\right)}\, dw$$

$$\left(\qquad \omega - z_0 \right)$$

$$= \frac{1}{2\pi i} \cdot \int_{|\omega - z_0| = r} \frac{f(\omega)}{\omega - z_0} \cdot \sum_{k \geq 0} \left(\frac{z - z_0}{\omega - z_0} \right)^k d\omega$$

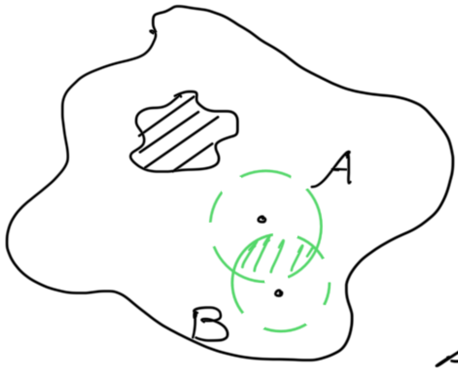
$$= \frac{1}{2\pi i} \cdot \int_{|\omega - z_0| = r} \sum_{k \geq 0} \frac{f(\omega)}{(\omega - z_0)^{k+1}} \cdot d\omega \cdot (z - z_0)^k$$

$$= \frac{1}{2\pi i} \sum_{k \geq 0} \left[\int_{|\omega - z_0| = r} \frac{f(\omega)}{(\omega - z_0)^{k+1}} d\omega \right] (z - z_0)^k$$

Comparing with $\sum_{k \geq 0} a_k z^k$, f thus can be expressed as a power series in $|z - z_0| < r$.

Here $\boxed{a_k = \frac{1}{2\pi i} \int_{|\omega - z_0| = r} \frac{f(\omega)}{(\omega - z_0)^{k+1}} d\omega}$

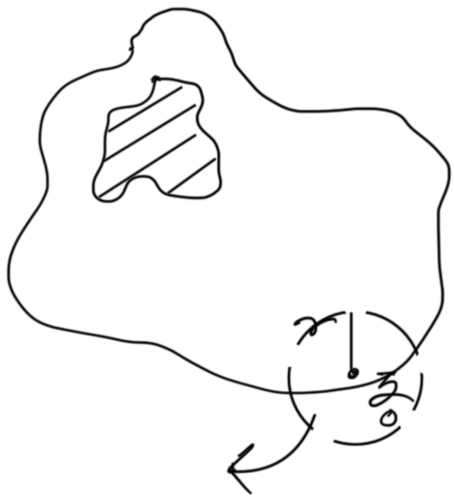
In fact by considering such circles, we can express f as a Power series at any point in D. Now consider this situation:



f can be expressed as 2 different power series in the intersecting region of A and B.

In that region, the 2 power series have the same value.

Another thing to note, is that to some extent we can predict whether f is analytic outside D or not.



If $r < R$, then f is analytic.