Noetherian Rings

Let R be an Integral Domain. We will construct a field K containing R, such that $\boxed{K = \left\{ \frac{a}{b} \;\middle|\; a, b \in R,\ b \neq 0 \right\}}$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Since $b, d \neq 0$ and R is an Integral Domain, $bd \neq 0$.

K is called the field of fractions of R.
It is represented by $ff(R)$.

$$\frac{a}{b} = \frac{c}{d} \text{ if } ad = bc.$$

Let I be an ideal of a ring R. I is finitely generated if there exists finitely many $a_1, \dots, a_n \in I$ such that $I = (a_1, \dots, a_n)$. Any element of I can be written as $a = b_1 a_1 + \cdots + b_n a_n$, $b_i \in \mathbb{Z}$.

R is called a Noetherian Ring if every ideal of R is finitely generated.

Let R be a ring. An ascending chain of ideals in R, is the collection of ideals $I_1 \subseteq I_2 \subseteq \cdots$. The chain stabilizes if there exists an n such that $I_n = I_{n+1} = \cdots$ (or $I_m = I_n \ \forall\, m \geq n$).

Any ring R is Noetherian only if every ascending chain of ideals in R stabilizes.

Proof/ Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in R.

Suppose R is a Noetherian Ring.

$$I = \bigcup_{n \geq 1} I_n = I_1 \cup I_2 \cup \cdots$$

It is trivial that $0 \in I$.

Let $a, b \in I \Rightarrow a \in I_n$ and $b \in I_m$,

where $n \leq m$.

$\Rightarrow I_n \subseteq I_m$. So $a \in I_n \subseteq I_m$

$b \in I_m$

$\Rightarrow (a+b) \in I_m \subseteq I$.

$(I, +)$ is thus a subgroup of R. And it is trivial that if $r \in R$ and $a \in I$, then $ra \in I$. I is thus an ideal in R

$\Rightarrow I$ is finitely generated.

$I = (a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \in I$.

Here $a_1 \in I_{n_1}, \ldots\ a_i \in I_{n_i}$.

If $n \geq \max(n_1, n_2, \ldots, n_i)$, then any

$a_i \in I_n \subseteq I$.

$\Rightarrow I = I_n$.

Thus the ascending chain of ideals stabilizes.

Now, we will assume that any ascending chain of ideals stabilize and then try to prove the converse.

Let $I \subset R$ be an ideal of R.

$I_1 = (x_1)$ where $x_1 \in I$.

$I_2 = (x_1, x_2)$ where $x_2 \in I \setminus (x_1) = I \setminus I_1$

$\vdots$

$I_n = (x_1, x_2 \dots, x_n)$ where $x_n = I \setminus I_{n-1}$.

$I_1 \subseteq I_2 \dots \subseteq I_n \subseteq I_{n+1} \dots$

$I = I_n = I_{n+1} = \dots$ and thus R is a Noetherian Ring.

① Let R be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Then R is not Noetherian.

~~Proof~~ $I_n = \{ f \in R \mid f(x) = 0 \ \forall x \geq n \}$ where $n \geq 1$.

$I_1 \subseteq I_2 \subseteq \dots$ We need to show that this chain doesn't stabilize.

Suppose $f_n(x) = \begin{cases} x - n, & x \leq n \\ 0 & \text{otherwise} \end{cases}$

$f_n(x) \in I_n$ but $\notin I_{n-1}$ $\forall n$.

$\Rightarrow I_{n-1} \neq I_n \ \forall n$

$\Rightarrow$ The ascending chain of ideals never stabilizes.

$\Rightarrow R$ is thus not Noetherian.

We can also deduce that a subring of a Noetherian Ring is not always Noetherian.

② Let I be an ideal of the Noetherian Ring R. Then R/I is also a Noetherian Ring.