⊙ Let $\sigma_1, \dots, \sigma_n : K \to L$ be distinct field homomorphisms and F be their fixed field. Then $\boxed{[K:F] \geq n}$.

$$\begin{array}{c} K \\ | \;\geq n. \\ F \end{array}$$

~~Proof~~ Let $r = [K:F]$. Suppose $r < n$.

We will choose a basis $\alpha_1, \dots, \alpha_r \in K$ of K as an F vector space.

Now consider the following homogeneous system of linear equations with coefficients in L

$$\underbrace{\begin{bmatrix} \sigma_1(\alpha_1) & \cdots & \sigma_n(\alpha_1) \\ \sigma_1(\alpha_2) & \cdots & \sigma_n(\alpha_2) \\ \vdots & & \\ \sigma_1(\alpha_r) & \cdots & \sigma_n(\alpha_r) \end{bmatrix}}_{A_{r\times n}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0.$$

or in short $A_{r\times n} X_{n\times 1} = 0$.

Since $r < n$, the system has a non trivial solution. $\beta \dots \beta \in L$ Atleast one β is

$\beta_1, \ldots, \beta_n$

non zero.

$$\sigma_1(\alpha_1)\cdot\beta_1 + \cdots + \sigma_n(\alpha_1)\cdot\beta_n = 0 \quad \text{———} (1)$$

$$\vdots$$

$$\sigma_1(\alpha_r)\cdot\beta_1 + \cdots + \sigma_n(\alpha_r)\beta_n = 0.$$

Let any $\alpha \in K$.

$$\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r, \quad a_i \in F.$$

Multiplying (1) by $\sigma_1(a_1)$,

$$\sigma_1(a_1)\cdot\sigma_1(\alpha_1)\beta_1 + \cdots + \sigma_1(a_1)\cdot\sigma_n(\alpha_1)\beta_n = 0.$$

$$\Rightarrow \sigma_1(a_1)\cdot\sigma_1(\alpha_1)\beta_1 + \cdots + \sigma_n(a_1)\cdot\sigma_1(\alpha_1)\beta_n = 0$$

since $a_1 \in F$ which is a fixed field.

$$\Rightarrow \sigma_1(a_1\alpha_1)\beta_1 + \cdots + \sigma_n(a_1\alpha_1)\beta_n = 0.$$

Similarly we can get

$$\sigma_1(a_r\alpha_r)\cdot\beta_1 + \cdots + \sigma_n(a_r\alpha_r)\beta_n = 0.$$

Adding all those equations,

$$\sigma_1(a_1\alpha_1 + \cdots + a_r\alpha_r)\beta_1 + \cdots = 0$$

$$\Rightarrow \sigma_1(\alpha)\beta_1 + \cdots + \sigma_n(\alpha)\beta_n = 0 \quad \forall\, \alpha \in K$$

$$\Rightarrow \left(\beta_1\sigma_1 + \cdots + \beta_n\sigma_n\right)(\alpha) = 0 \quad \forall\, \alpha \in K$$

$$\Rightarrow \beta_1\sigma_1 + \cdots + \beta_n\sigma_n \equiv 0.$$

This violates the independence of $\sigma_1, \ldots, \sigma_n$.

Thus $[K : F] \geq n$