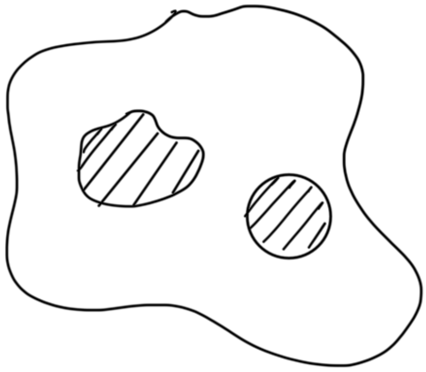
A domain D on $\mathbb{C}$ is an open set such that any 2 points in D can be connected by a curve within D. The domain can have holes in it but we are interested in only those domains which have finite number of piecewise differentiable boundaries.



A function f is analytic on domain D, if it is differentiable everywhere in D.

Cauchy Riemann Equations

Consider the function $F : \mathbb{R}^2 \longrightarrow \mathbb{C}$

$F(x, y) = f(x+iy)$ where $f : \mathbb{C} \rightarrow \mathbb{C}$.

$$f'(x+iy) = \lim_{(\alpha,\beta)\to(0,0)} \frac{F(x+\alpha, y+\beta) - F(x,y)}{\quad}$$

$$(\alpha,\beta)\to(0,0) \qquad \alpha+\beta i$$

$$= \lim_{\alpha\to 0,\,\beta=0} \frac{F(x+\alpha,y)-F(x,y)}{\alpha}$$

$$= \lim_{\alpha=0,\,\beta\to 0} \frac{F(x,y+\beta)-F(x,y)}{i\beta}$$

$$\Rightarrow \frac{\partial F}{\partial x} = \frac{1}{i}\cdot\frac{\partial F}{\partial y}$$

Now let $F(x,y) = u(x,y) + i\cdot v(x,y)$. So,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\cdot\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ and } \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

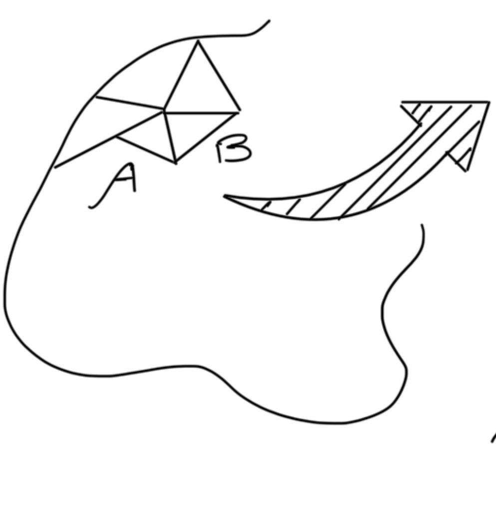
Thus complex differentiable functions must satisfy the above equations.

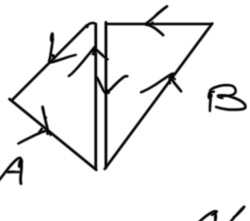
Cauchy Theorem

Let D be a bounded domain and f be an analytic function on D. ∂D be the boundary of D. f must smoothly extend to ∂D. Then

$$\boxed{\int_{\partial D} f(z)\, dz = 0}$$

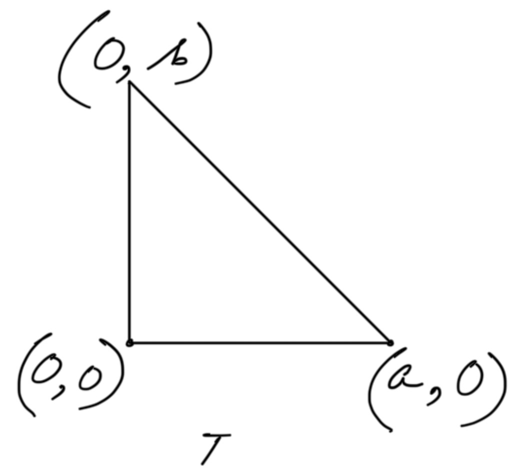
Proof: We will break D into finitely many triangles.





Notice that along the intersection point of A and B, we evaluate $\int f(z)\, dz$ twice – once clockwise and once anti clockwise, cancelling each other out.

Now for any boundary triangle T,



$$\int_{\partial T} f(z)\,dz$$

$$= \int_{\partial T} (u+iv)(dx+i\cdot dy)$$

$$= \int_{\partial T} (u\,dx - v\,dy) + i\int_{\partial T} (u\cdot dy + v\cdot dx)$$

Lets focus on the first integral

$$\int_{\partial T} (u\cdot dx - v\cdot dy)$$

$$= \int_{(0,0)}^{(a,0)} u(x,0)\,dx + \left[\int_{(a,0)}^{(0,b)} \left\{u(x,y)\,dx - v(x,y)\,dy\right\}\right]$$

$$+ \int_{(0,b)}^{(0,0)} \left\{-v(0,y)\right\} dy$$

a

$$= \int_0^a \left\{ u(x,0) - u(x, mx+c) \right\} dx$$

$$+ \int_0^b \left\{ v(0,y) - v\left(\frac{y-c}{m}, y\right) \right\} dy.$$

$$= \int_0^a \left\{ -\int_0^{mx+c} \frac{\partial u}{\partial y}\, dy \right\} dx + \int_0^b \left\{ -\int_0^{\frac{y-c}{m}} \frac{\partial v}{\partial x}\, dx \right\} dy$$

$$= -\int_0^a\int_0^{mx+c} \frac{\partial u}{\partial y}\, dy \cdot dx - \int_0^a\int_0^{mx+c} \frac{\partial v}{\partial x}\, dy \cdot dx$$

$$= -\int_0^a\int_0^{mx+c} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) dy \cdot dx$$

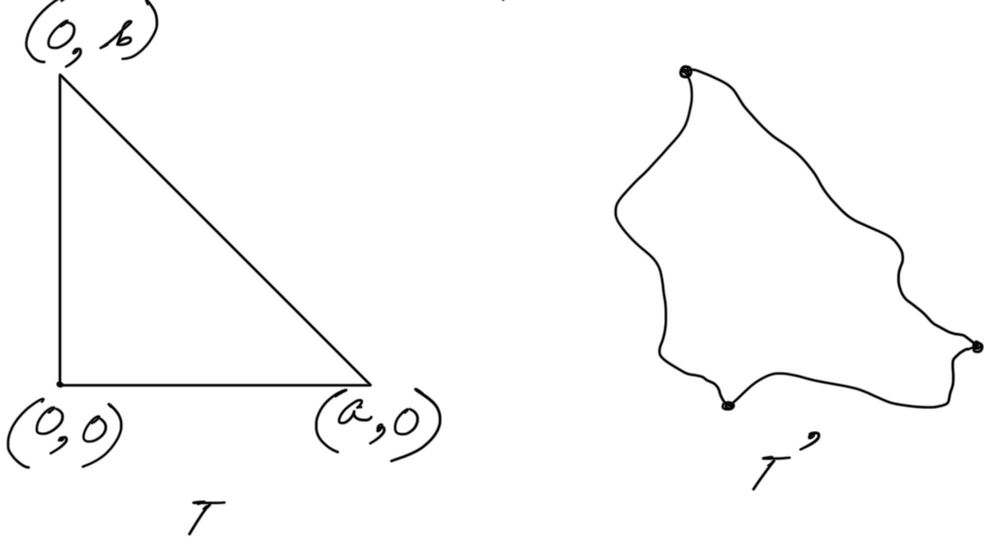
$$= 0.$$

Similarly, we can show that $\int_{\delta T} (u\,dx + v\,dy) = 0$

Thus $\int_{\delta T} f(z)\,dz = 0$. But in this integral,

we are considering $f(z)$ to be on the bound ary of the triangle.

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{a-\varepsilon} \int_{\varepsilon}^{(mx+c-\varepsilon)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) dy \cdot dx$$



According to Real Analysis, there exists a smooth map (τ, σ) from T to T'. Now,

$$\int_{\partial T'} (u \cdot dx - v \cdot dy)$$

$$= \int_{\partial T} \left\{ u(\tau,\sigma) \left(\frac{\partial \tau}{\partial \alpha} \cdot d\alpha + \frac{\partial \tau}{\partial \beta} \cdot d\beta \right) - v(\tau,\sigma) \left(\frac{\partial \sigma}{\partial \alpha} \cdot d\alpha + \frac{\partial \sigma}{\partial \beta} \cdot d\beta \right) \right\}$$

Here we have done the coordinate

transformation $x \longrightarrow \tau(\alpha, \beta)$

$y \longrightarrow \sigma(\alpha, \beta)$

$$= \int_{\partial T} \left[\left\{ u(\tau, \sigma) \cdot \tau_\alpha - v(\tau, \sigma) \cdot \sigma_\alpha \right\} d\alpha \right.$$

$$\left. + \left\{ u(\tau, \sigma) \cdot \tau_\beta - v(\tau, \sigma) \cdot \sigma_\beta \right\} d\beta \right]$$

If we can show that

$$\frac{\partial}{\partial \beta} \left(u\, \tau_\alpha - v\, \sigma_\alpha \right) = \frac{\partial}{\partial \alpha} \left(u\, \tau_\beta - v\, \sigma_\beta \right), \text{ then}$$

$$\int_{\partial T'} \left(u \cdot dx - v \cdot dy \right) = 0.$$

$$\frac{\partial}{\partial \beta} \left(u\, \tau_\alpha - v\, \sigma_\alpha \right) - \frac{\partial}{\partial \alpha} \left(u\, \tau_\beta - v\, \sigma_\beta \right)$$

$$= \left(u_\sigma + v_\tau \right) \left(\tau_\alpha\, \sigma_\beta - \sigma_\alpha\, \tau_\beta \right)$$

$$= \left\{ \frac{\partial}{\partial \sigma} u(\tau, \sigma) + \frac{\partial}{\partial \tau} v(\tau, \sigma) \right\} \left(\tau_\alpha \sigma_\beta - \sigma_\alpha\, \tau_\beta \right)$$

$$= 0$$

Similarly, $\displaystyle\int_{\partial T'} (u \cdot dy + v \cdot dx) = 0$

$$\Rightarrow \int_{\partial T'} f(z)\, dz = 0.$$

$$\Rightarrow \sum_{T'} \left[\int_{\partial T'} f(z)\, dz \right] = 0$$

$$\Rightarrow \int_{\partial D} f(z) \cdot dz = 0$$