① Let $\varphi : R \rightarrow R'$ be a ring homomorphism. φ is injective only if $\mathrm{Ker}(\varphi) = \{0\}$.

Proof/ Remember, φ is injective if
$\varphi(a) = \varphi(b)$

$\Rightarrow a = b$ for any $a, b \in R$.

Suppose φ is injective. We know that
$\varphi(0_R) = 0_{R'}$

Let $a \in \mathrm{Ker}(\varphi)$. $\Rightarrow \varphi(a) = 0$

$\Rightarrow \varphi(a) = \varphi(0)$

$\Rightarrow a = 0$ since φ is injective

$\Rightarrow \mathrm{Ker}(\varphi) = \{0\}$.

Going the other way around, let $\mathrm{Ker}(\varphi) = \{0\}$
and $\varphi(a) = \varphi(b)$

$\Rightarrow \varphi(a - b) = 0 = \varphi(0)$

$\Rightarrow a - b = 0$

$\Rightarrow a = b$ which means φ is injective

f, which means f is injective.

② Let I and J be ideals in R. Then the following are also ideals:

$I \cap J = \{ a \in R \mid a \in I \text{ and } J \}$.

$I + J = \{ a + b \mid a \in I \text{ and } b \in J \}$.

Proof/ Let $(a+b)$ and $(c+d) \in (I+J)$.

$(a+b) + (c+d) = (a+c) + (b+d) \in (I+J)$

since $(a+c) \in I$ and $(b+d) \in J$.

For any $r \in R$, $r(a+b) = ra + rb \in (I+J)$

since $ra \in I$ and $rb \in J$.

Thus $(I+J)$ is an ideal. It is trivial that $(I \cup J) \subseteq (I+J)$, since

$\forall a \in I$, $a + 0 \in (I+J)$ and similarly

$\forall b \in J$, $b + 0 \in (I+J)$.

$IJ = \{ \sum a_i b_i \mid a_i \in I \text{ and } b_i \in J \}$ is also an

$\left\{ \sum_{\text{finite}} a_i x_i \mid x_i \in I \text{ and } a_i \in \right\}$

ideal.

③ Let $\varphi : R \to R'$ be a ring homomorphism. If I' is an ideal of R', then $\varphi^{-1}(I') = I$ is an ideal in R.

Proof

$a, b \in \varphi^{-1}(I') = I$

$\Rightarrow \varphi(a)$ and $\varphi(b) \in I'$

$\Rightarrow \varphi(a+b) \in I'$

$\Rightarrow (a+b) \in \varphi^{-1}(I') = I.$

Thus $(\varphi^{-1}(I'), +)$ is a subgroup of R.

For $r \in R$, $\varphi(r) \in R'$.

$\Rightarrow \varphi(r) \cdot a \in I'$ since $a \in$ ideal $\varphi^{-1}(I')$

$\Rightarrow \varphi(ra) \in I'$

$\Rightarrow ra \in \varphi^{-1}(I').$

Thus $I = \varphi^{-1}(I')$ is an ideal of R.

④ Generally, $\varphi(I)$ isn't an ideal of R'. It is when φ is onto.

Proof

Let $\varphi(a), \varphi(b) \in \varphi(I)$.

$\Rightarrow \varphi(a) + \varphi(b) = \varphi(a+b) \in \varphi(I)$

Now, let $r' \in R'$. If φ is surjective, then there must exist $r \in R$ such that

$\exists\, r \in R$ such that $\varphi(r) = r'$.

$r' \cdot \varphi(a) = \varphi(r) \cdot \varphi(a) = \varphi(ra) \in \varphi(I)$

$\Rightarrow \varphi(I)$ is an ideal in R, if φ is surjective

⑤ $a \in R$ is called nilpotent if $a^n = 0$ for some $n \geq 1$.

$I = \{a \in R \mid a \text{ is nilpotent}\}$ is an ideal.

Proof

Let $a, b \in I$ such that $a^m = b^n = 0$ for $m, n \geq 1$.

Since the distributive property holds for a ring, applying Binomial Theorem,

$$(a+b)^K = a^K + K \cdot a^{K-1} \cdot b + \binom{K}{2} a^{K-2} \cdot b^2 + \cdots$$

$= 0$ since for each term, either $a^m = 0$ or $b^n = 0$ if $K \geq (m+n)$.

$$\Rightarrow (a+b)^{m+n} = 0$$

$\Rightarrow (a+b) \in I$.

Thus $(I, +)$ is a subgroup.

Let $r \in R$.

$$(ra)^m = r^m \cdot a^m = 0$$

$\Rightarrow ra \in I$. I thus forms an ideal and is called the nilradical of R.