Counting Primes

We want to find a function which can tell us the number of primes in a given interval $[1, x]$. Gauss made a conjecture that there exists such a function $\boxed{\pi(x) \approx \frac{x}{\ln x}}$

$$\pi(x) = \sum_{n=1}^{x} p(n) \quad \text{where } p(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \pi(x) = \sum_{n=1}^{\infty} p(n) \cdot \delta\left(\frac{x}{n}\right) \quad \text{where}$$

$$\delta(m) = \begin{cases} 0 & \text{if } 0 < m < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$\Rightarrow \pi(x) = \sum_{n=1}^{\infty} \left[p(n) \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s} \, ds \right]$$

$$\lfloor \qquad\qquad\qquad \rfloor$$

Since $\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s}\,ds = \frac{1}{2}$ at $\frac{x}{n} = 1$,

we have done a variable transformation from x to $\left(x+\frac{1}{2}\right)$.

$$\Rightarrow \pi(x) = \sum_{n=1}^{\infty} p(n) \cdot \left[\frac{1}{2\pi i} \cdot \int_{c-iR}^{c+iR} \frac{(x/n)^s}{s}\,ds + O\left(\frac{x^c}{R\ln x}\right)\right]$$

$$\Rightarrow \pi(x) = \frac{1}{2\pi i}\int_{c-iR}^{c+iR} \left\{\sum_{n=1}^{\infty} \frac{p(n)}{n^s}\right\} \frac{x^s}{s}\,ds$$

$$+ \sum_{n=1}^{\infty}\left[p(n) \cdot O\left(\frac{x^c}{R\ln x}\right)\right]$$

Here $\displaystyle\sum_{n=1}^{\infty} \frac{p(n)}{n^s}$ is uniformly convergent for $c > 1$.

$$\left|\sum_{n=1}^{\infty} \frac{p(n)}{n^s} - \sum_{n=1}^{m-1} \frac{p(n)}{n^s}\right|$$

$$= \left|\sum_{n=m}^{\infty} \frac{p(n)}{n^s}\right| \leq \sum_{n=m}^{\infty} \frac{1}{n^c} \leq \int_m^{\infty} \frac{dt}{t^c}$$

$$= \left[t^{-c+1}\right]^{\infty}$$

$$\left[\frac{\cdots}{1-c}\right]_m$$

$$= 0 - \frac{m^{1-c}}{1-c}, \text{ if } c > 1$$

$$= \frac{1}{(c-1)\, m^{1-c}} \quad (c > 1)$$

$$\lim_{m\to\infty} \frac{1}{(c-1)\, m^{1-c}} = 0.$$

Now let $$S_m = \int_{c-iR}^{c+iR} \left\{ \sum_{n=1}^{m} \frac{f(n)}{n^s} \right\} \frac{x^s}{s}\, ds$$

$$= \sum_{n=1}^{m} \left\{ \int_{c-iR}^{c+iR} \frac{f(n)}{n^s} \cdot \frac{x^s}{s}\, ds \right\}$$

$$\left| S_\infty - S_m \right| = \left| \int_{c-iR}^{c+iR} \left\{ \sum_{n=m+1}^{\infty} \frac{f(n)}{n^s} \right\} \frac{x^s}{s}\, ds \right|$$

$$\leq O\left(\frac{1}{m^{c-1}} \cdot \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds \right)$$

$$= O\left(\frac{1}{m^{c-1}} \right).$$

$$\Rightarrow \lim_{m\to\infty} \left| S_\infty - S_m \right| = 0$$

$$\Rightarrow \int_{c-iR}^{c+iR} \left\{ \sum_{n=1}^{\infty} \frac{p(n)}{n^s} \right\} \frac{x^s}{s}\, ds = \sum_{n=1}^{\infty} \left\{ \int_{c-iR}^{c+iR} \frac{p(n)}{n^s} \cdot \frac{x^s}{s}\, ds \right\}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{p(n)}{n^s}$ is not a very nice function to analyse. So we will not continue with this approach.