First Isomorphism Theorem

Let $\varphi : R \to R'$ be a ring homomorphism.
And $S = \varphi(R) \subseteq R'$. Then,

(1) S is a subring of R'.

(2) $R/_{Ker(\varphi)} \xrightarrow{\sim} S$

(3) There is a bijective correspondance

$$\left\{ \begin{array}{c} \text{Ideals in } R \\ \text{containing } Ker(\varphi) \end{array} \right\} \longleftrightarrow \{ \text{Ideals in } S \}$$

$\Rightarrow \quad Ker(\varphi) \subseteq J \longmapsto \varphi(J)$

and $\varphi^{-1}(J') \longleftarrow J'$

Proof (1) $\varphi(1_R) = 1_{R'}$. So S contains $1_{R'}$.

$\varphi(a+b) = \varphi(a) + \varphi(b)$ which means S is closed under addition. Similarly it is closed under multiplication since

$\varphi(ab) = \varphi(a) \cdot \varphi(b)$

Thus S is a subring of R'.

(2) Let $\psi : \frac{R}{\mathrm{Ker}\,\varphi} \longrightarrow S$ be a function

such that $\psi\left(a + \mathrm{Ker}(\varphi)\right) = \varphi(a)$, $a \in R$.

We will first show that, ψ is well defined. Let $a + \mathrm{Ker}(\varphi) = b + \mathrm{Ker}(\varphi)$

$$\Rightarrow (a-b) \in \mathrm{Ker}(\varphi)$$

$$\Rightarrow \varphi(a-b) = 0$$

$$\Rightarrow \varphi(a) - \varphi(b) = 0$$

$$\Rightarrow \varphi(a) = \varphi(b).$$

So ψ is well defined.

$$\psi\left((a + \mathrm{Ker}\,\varphi) + (b + \mathrm{Ker}\,\varphi)\right)$$

$$= \psi\left((a+b) + \mathrm{Ker}\,\varphi\right)$$

$$= \varphi(a+b) = \varphi(a) + \varphi(b)$$

$$= \psi(a + \mathrm{Ker}\,\varphi) + \psi(b + \mathrm{Ker}\,\varphi)$$

And $\psi\left((a + \mathrm{Ker}\,\varphi)(b + \mathrm{Ker}\,\varphi)\right)$

$= \psi(ab + \operatorname{Ker} \varphi)$

$= \varphi(ab) = \varphi(a) \cdot \varphi(b)$

$= \psi(a + \operatorname{Ker} \varphi) \cdot \psi(b + \operatorname{Ker} \varphi).$

$\psi(1 + \operatorname{Ker} \varphi) = \varphi(1) = 1$

So ψ is a ring homomorphism.

Now, suppose $\psi(a + \operatorname{Ker} \varphi) = \psi(b + \operatorname{Ker} \varphi)$

$\Rightarrow \varphi(a) = \varphi(b)$

$\Rightarrow \varphi(a - b) = 0$

$\Rightarrow (a - b) \in \operatorname{Ker} \varphi$

$\Rightarrow a + \operatorname{Ker} \varphi = b + \operatorname{Ker} \varphi.$

Thus ψ is one to one.

For any $s \in S = \varphi(R)$

$\Rightarrow \exists\ a \in R$ such that $\varphi(a) = s$

Now $\psi(a + \operatorname{Ker} \varphi) = \varphi(a) = s.$

So every element of $\frac{R}{\operatorname{Ker} \varphi}$ is mapped to S. ψ is thus bijective.

$\Rightarrow \varphi$ is a ring isomorphism.

(3) Let $A = \left\{ \begin{array}{l} \text{Ideals in } R \\ \text{containing } \operatorname{Ker}(\varphi) \end{array} \right\}$ and

$B = \{\text{Ideals in } S\}$. We need to show that $A \overset{f}{\longmapsto} B$ and $A \underset{g}{\longleftarrow\!\!\!\mid} B$.

Now if J is an ideal in R, then $\varphi(J)$ is an ideal in S. Thus f is well defined.

Now if $J' \in B$

$\Rightarrow J'$ is an ideal of S

$\Rightarrow \varphi^{-1}(J')$ is an ideal of R.

and $0_S \in J' \Rightarrow \varphi^{-1}(0_S) \subseteq \varphi^{-1}(J')$

$\Rightarrow \operatorname{Ker}(\varphi) \subseteq \varphi^{-1}(J')$

Thus g is well defined. To show that there is a bijective mapping between A and B, we will prove that f and g are inverses.

$a \in J \Rightarrow \varphi(a) \in \varphi(J)$

$a \in J \Rightarrow \varphi(a) \in \varphi(J)$

$\Rightarrow a \in \varphi^{-1}(\varphi(J))$

And if $a' \in \varphi^{-1}(\varphi(J)) \Rightarrow \varphi(a') \in \varphi(J)$.

$\exists\, b \in J$ such that $\varphi(b) = \varphi(a')$

$\Rightarrow \varphi(b) - \varphi(a') = 0$

$\Rightarrow \varphi(b - a') = 0$

$\Rightarrow b - a' \in Ker(\varphi)$

$\Rightarrow b - a' \in J$ since $J \in A$

$\Rightarrow a' \in J$.

Thus $\varphi^{-1}(\varphi(J)) = J \in A$.

$\Rightarrow g \circ f = 1_A$.

To prove that $f \circ g = 1_B$, we will show that $\varphi(\varphi^{-1}(J')) = J' \in B$.

$a \in \varphi(\varphi^{-1}(J'))$

$\Rightarrow a = \varphi(b)$ where $b \in \varphi^{-1}(J')$

$\Rightarrow a = \varphi(b) \in J'$

And $a' \in J' \subseteq S = \varphi(R)$

$\exists\, b \in R$ such that $a' = \varphi(b)$

$\Rightarrow b \in \varphi^{-1}(J')$

$\Rightarrow a = \varphi(b) \in \varphi(\varphi^{-1}(J'))$.