Equivalence Relations

An equivalence relation on a set S is a relation between the elements of S, represented using $a \sim b$, where $a, b \in S$. It satisfies the following properties:

(i) $a \sim a \; \forall \, a \in S$

(ii) $a \sim b$

$\Rightarrow b \sim a \; \forall \, a, b \in S$

(iii) $a \sim b$, $b \sim c$

$\Rightarrow a \sim c \; \forall \, a, b, c \in S$.

Equivalence relations give equivalence classes. For any $a \in S$, the equivalence class of a

$$\boxed{[a] = \{ b \in S \mid a \sim b \}}$$

Let G be a group and H be a subgroup of G. We will define an equivalence relation on G: $a \sim b$ if $a^{-1}b \in H$.

$a \sim a \Rightarrow a^{-1}a = e \in H$.

$$a \sim b \Rightarrow a^{-1}b \in H$$

$$\Rightarrow (a^{-1}b)^{-1} \in H$$

$$\Rightarrow b^{-1}(a^{-1})^{-1} \in H$$

$$\Rightarrow b^{-1}a \in H$$

$$\Rightarrow b \sim a.$$

$a \sim b$ and $b \sim c$.

$\Rightarrow a^{-1}b \in H$ and $b^{-1}c \in H$

$$\Rightarrow (a^{-1}b)(b^{-1}c) \in H$$

$$\Rightarrow a^{-1}(bb^{-1})c \in H$$

$$\Rightarrow a^{-1}c \in H$$

$$\Rightarrow a \sim c$$

Now $[a] = \{b \in G \mid a \sim b\}$

$$= \{b \in G \mid a^{-1}b = h \in H\}$$

$$= \{b \in G \mid b = ah,\ h \in H\}$$

$$= \{ah \mid h \in H\}$$

$$= aH.$$

Let S be a set and $\sim$ be an equivalence relation on S. The equivalence classes of elements of S, partition it. S is a disjoint union of equivalence classes.

Proof: $a \in [a]$ since $a \sim a$, where $a \in S$.

$$S = \bigcup_{a \in S} [a]$$

Now if $a \neq b$, then $[a] \cap [b] = \phi$, $\forall$ $a, b \in S$.

Let $c = [a] \cap [b] \neq \phi$

$\Rightarrow a \sim c$ and $b \sim c$

$\Rightarrow a \sim b$

$\Rightarrow a \in [b]$ and $b \in [a]$

Thus any element $d \sim a$

$\Rightarrow d \sim b$ since $a \sim b$.

and vice versa. So $[a] = [b]$.

Cosets

We defined an equivalence relation on group G: $a \sim b$ if $a^{-1}b \in H$ where $a, b \in G$ and H is a subgroup of G.

The subsets, $[a] = aH$ are called the left cosets of H. They partition G.

$n(aH) = |H|$

Proof / Consider the map $H \longrightarrow aH$.

$\Rightarrow h \longmapsto ah$

By definition of aH, every element of aH is mapped to only 1 element of H. The map is thus injective (one to one)

Its also surjective (onto). The map is thus bijective.

$\Rightarrow n(aH) = |H|$

Now $H = eH$ is a left coset of H.

$G = eH \cup a_1 H \cup a_2 H \cdots$ n times

$\Rightarrow |G| = |eH| + |a_1 H| + \cdots$

$\Rightarrow |G| = |H| + |H| + \cdots$

$\Rightarrow |G| = n|H|$ where n is the total number of left cosets of H. It is called the index of H in G and represented by

$n = [G : H]$.

$$\boxed{|G| = [G : H]\,|H|}$$

$|H|$ thus must divide $|G|$. This is called the Lagrange's Theorem.

Quotient Groups

The set of all left cosets of H is represented by G/H.

$$G/H = \{aH \mid a \in G\}$$

Let H be a normal subgroup of G. Consider $ah \in aH$.

$$ah = ah(a^{-1}a)$$

$$= (aha^{-1})a$$

$$= h'a.$$ Since H is a normal subgroup

of G, $h' = (a h a^{-1}) \in H$. $\Rightarrow a h \in Ha$ which is called the right coset of H. $\Rightarrow$ $aH \subset Ha$

Similarly we can show that $Ha \subset aH$. $\Rightarrow \boxed{aH = Ha}$ $\forall a \in G$. Also $(aH)(bH) = (ab)H$.

Proof

$$(aH)(bH) = a(Hb)H$$

$$= a(bH)H$$

$$= (ab)(HH) = (ab)H.$$

Here $HH = \{hh' \mid h, h' \in H\} \subseteq H$ since $hh' \in H$. And $h \cdot e \in HH$ $\forall h \in H$. $\Rightarrow H \subseteq HH \Rightarrow H = HH$!

G/H is thus a group when H is a normal subgroup of G. The associated binary operation is $(aH)(bH) = (ab)H$. eH is an identity element.

$(aH)(eH) = (ae)H = aH$.

Speaking about inverses

$$
\begin{aligned}
(aH)(a^{-1}H) &= a(Ha^{-1})H \\
&= a(a^{-1}H)H \\
&= (aa^{-1})H = eH.
\end{aligned}
$$

Thus $a^{-1}H$ is the inverse of aH.

$(aH)^{-1} = a^{-1}H$.

G/H is called the quotient group.

Notice that $\varphi : G \longrightarrow G/H$ is a group homomorphism.

$$(ab)H = (aH)(bH)$$

$$\Rightarrow \varphi(ab) = \varphi(a)\varphi(b).$$

$$\ker(\varphi) = \{a \in G \mid aH = eH\}$$

$$= \{a \in G \mid aH = H\}$$

$$\Rightarrow \boxed{\ker(\varphi) = H} \quad (\text{since } HH = H).$$

$$\operatorname{img}(\varphi) = \{\varphi(a) \mid a \in G\}$$

$$= \{aH \mid a \in G\}$$

$$\Rightarrow \boxed{\operatorname{img}(\varphi) = G/H}$$

Lets look at an example.

$G = (\mathbb{Z}, +)$ Since $\mathbb{Z}$ is an Abelian Group, any subgroup of $\mathbb{Z}$ is normal. We will choose $H = 2\mathbb{Z}$.

All left cosets of $2\mathbb{Z}$ in $\mathbb{Z}$, $G/H = \mathbb{Z}/2\mathbb{Z}$

$$= \{m + 2\mathbb{Z} \mid m \in \mathbb{Z}\}$$

$$= \{2\mathbb{Z}, 2\mathbb{Z} + 1\}$$

Identity element.