ITP 20002-03 Discrete Mathematics, 2021 Fall

Test #1

September 17, 2021

You are given five problems with total 100 points to be solved in 75 minutes.

Write an answer of each of the following questions on the corresponding answer box in the answer sheet.

1. A *clause* is a propositional statement that consists of a disjunction of one or multiple propositional variables and/or their negations. For example, p is a clause, and $(p \vee q)$ is a clause, and also $(\neg p \vee q \vee r)$ is a clause. We call that a propositional statement is in conjunctive normal form (CNF) if it is a conjunction of one or multiple clauses. For example, $p \wedge (\neg r \vee \neg p) \wedge (q \vee r \vee p)$ is a propositional statement in CNF.

   Find a propositional statement in CNF, which is equivalent to $(\neg p \rightarrow q) \rightarrow (q \rightarrow \neg p)$. (20 points)

2. Prove that, for two positive integers x and y, xy is even if either x or y is even. (20 points)

3. Suppose that we have a new quantifier $\forall!\, x\, P(x)$ which holds if and only if $P(x)$ holds for all cases of x except at most one case. Give a definition of $\forall!\, x\, P(x)$ as a predicate statement with the existential quantifier and/or the universal quantifier. (20 points)

4. Prove that $\sqrt[3]{3}$ is irrational. (20 points)

5. Prove that $\exists x(\neg Q(x) \wedge P(x))$ if $\forall x(Q(x) \rightarrow R(x))$ and $\exists x(P(x) \wedge \neg R(x))$ using the following inference rules. (20 points)

Rule of Inference	*Name*
$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$	Modus ponens
$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$	Modus tollens
$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	Hypothetical syllogism
$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	Disjunctive syllogism
$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$	Addition
$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$	Simplification

Rule of Inference	*Name*
$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$	Conjunction
$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$	Resolution
$\begin{array}{l} \forall x P(x) \\ \hline \therefore P(c) \end{array}$	Universal Instantiation
$\begin{array}{l} P(c) \text{ for an arbitrary } c \\ \hline \therefore \forall x P(x) \end{array}$	Universal generalization
$\begin{array}{l} \exists x P(x) \\ \hline \therefore P(c) \text{ for some element } c \end{array}$	Existential Instantiation
$\begin{array}{l} P(c) \text{ for some element } c \\ \hline \therefore \exists x P(x) \end{array}$	Existential generalization

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