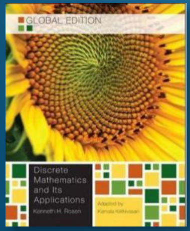
Relations

Chapter 9



- Taken from the instructor's resource of *Discrete Mathematics and Its Applications*, 7/e
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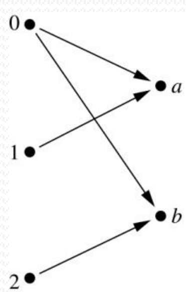
Binary Relations

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a), (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

Binary Relation on a Set

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $\mathrm{R} = \{(a,b) \mid a \text{ divides } b\}$ are

  (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution:
Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$.
Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs

(1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1, R_3, R_4 , and R_6: (1,2) is in R_1 and R_6: (2,1) is in R_2, R_5, and R_6: (1, −1) is in R_2, R_3, and R_6 : (2,2) is in R_1, R_3, and R_4.

Reflexive Relations

R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$.

- R is reflexive if and only if $\forall x \in A\ ((x,x) \in R)$
- If $A = \emptyset$ then the empty relation is reflexive vacuously.

reflexive relations:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

non reflexive relations :

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

Symmetric Relations

R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.

- $\forall x \in A\ (\ \forall y \in A\ (\ (x,y) \in R \longrightarrow (y,x) \in R)\ )$

Example

symmetric:

$R_3 = \{(a,b) \mid a = b$ or $a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),

$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

- R on a set A is **antisymmetric** when $(a,b) \in R$ and $(b,a) \in R$ only if $a = b$ for all $a \in A$, $b \in A$
- **Example**
  - antisymmetric:

    $R_1 = \{(a,b) \mid a \leq b\}$,

    $R_2 = \{(a,b) \mid a > b\}$,

    $R_4 = \{(a,b) \mid a = b\}$,

    $R_5 = \{(a,b) \mid a = b + 1\}$.
  - not antisymmetric:

    $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

    (note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

    $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

Transitive Relations

- R on a set A is **transitive** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$.
  - $\forall x \, \forall y \, \forall z \, [(x,y) \in R \wedge (y,z) \in \mathrm{R} \longrightarrow (x,z) \in R\,]$

- **Examples of transitive relations:**

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

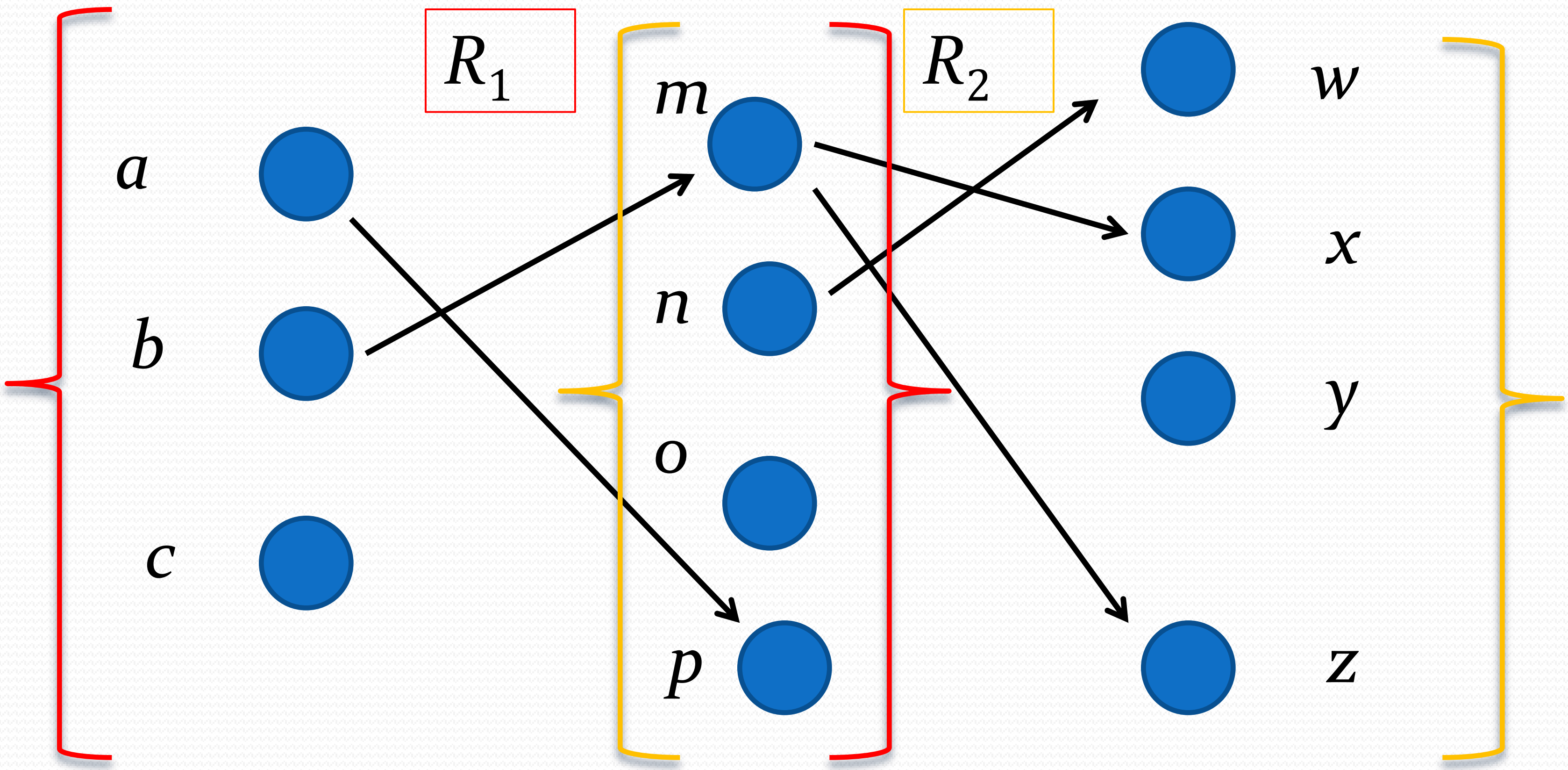
Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

The *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_2 \circ R_1 = \{(b,x),(b,z)\}$$

Powers of a Relation

Definition: Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

(*see the slides for Section* 9.3 *for further insights*)

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem 1: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3 \ldots$.

(*see the text for a proof via mathematical induction*)

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- A relation $R \subseteq A \times B$ is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Example

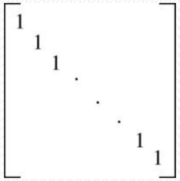
Example 1: Suppose that A = {1,2,3} and B = {1,2}. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because R = {(2,1), (3,1),(3,2)}, the matrix is

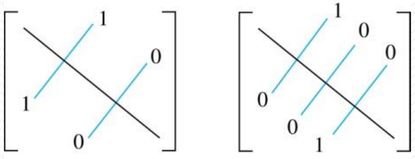
$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Matrices of Relations on Sets

- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.



- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric (b) Antisymmetric

Example of a Relation on a Set

Example 3: Suppose that the relation *R* on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

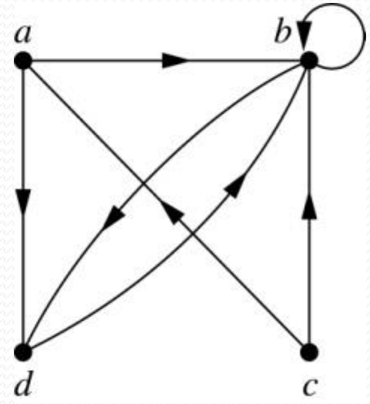
Is *R* reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements are equal to 1, *R* is reflexive. Because M_R is symmetric, *R* is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

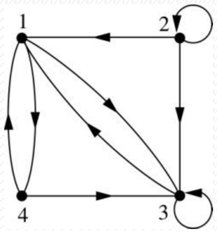
- A *directed graph*, or *digraph*, consists of a set *V* of *vertices* (or *nodes*) together with a set *E* of ordered pairs of elements of *V* called *edges* (or *arcs*). The vertex *a* is called the *initial vertex* of the edge (*a*,*b*), and the vertex *b* is called the *terminal vertex* of this edge.
  - An edge of the form (*a*,*a*) is called a *loop*.

Example 7: A drawing of the directed graph with vertices *a*, *b*, *c*, and *d*, and edges (*a*, *b*), (*a*, *d*), (*b*, *b*), (*b*, *d*), (*c*, a), (*c*, *b*), and (*d*, *b*) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?



Solution: The ordered pairs in the relation are (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3)

Determining which Properties a Relation has from its Digraph

- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x).
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z).

Equivalence Relations

- A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

Suppose that R is the relation on the set of strings of English letters such that $(a, b) \in R$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution

- *Reflexivity*: as $l(a) = l(a)$, it follows $(a, a) \in R$ for every string a.
- *Symmetry*:$(b, a) \in R$ if $(a, b) \in R$ because $l(a) = l(b)$, $l(b) = l(a)$
- *Transitivity*: $(a, c) \in R$ if $(a, b) \in R \land (b, c) \in R$ because, if $l(a) = l(b)$ and $l(b) = l(c)$ then $l(a) = l(a)$

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\,m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)\,m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not symmetric. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.
When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s|(a,s) \in R\}$.

- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a+2m, a+2m, \ldots\}$. For example,

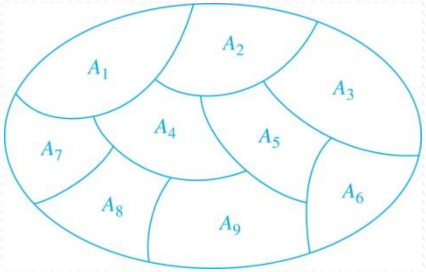
$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S.$



A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

.

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

Partial Orderings

Definition 1: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Partial Orderings (*continued*)

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation (|) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

- $(\mathbf{Z}^+, |)$ is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements a and b of a poset $(S, \preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain.*

Definition 4: $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Lexicographic Order

Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

- This definition can be easily extended to a lexicographic ordering on strings (*see text*).

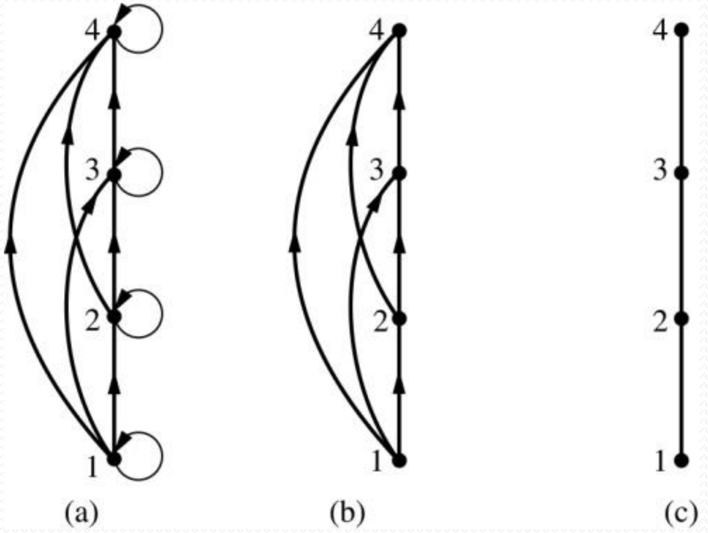
Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

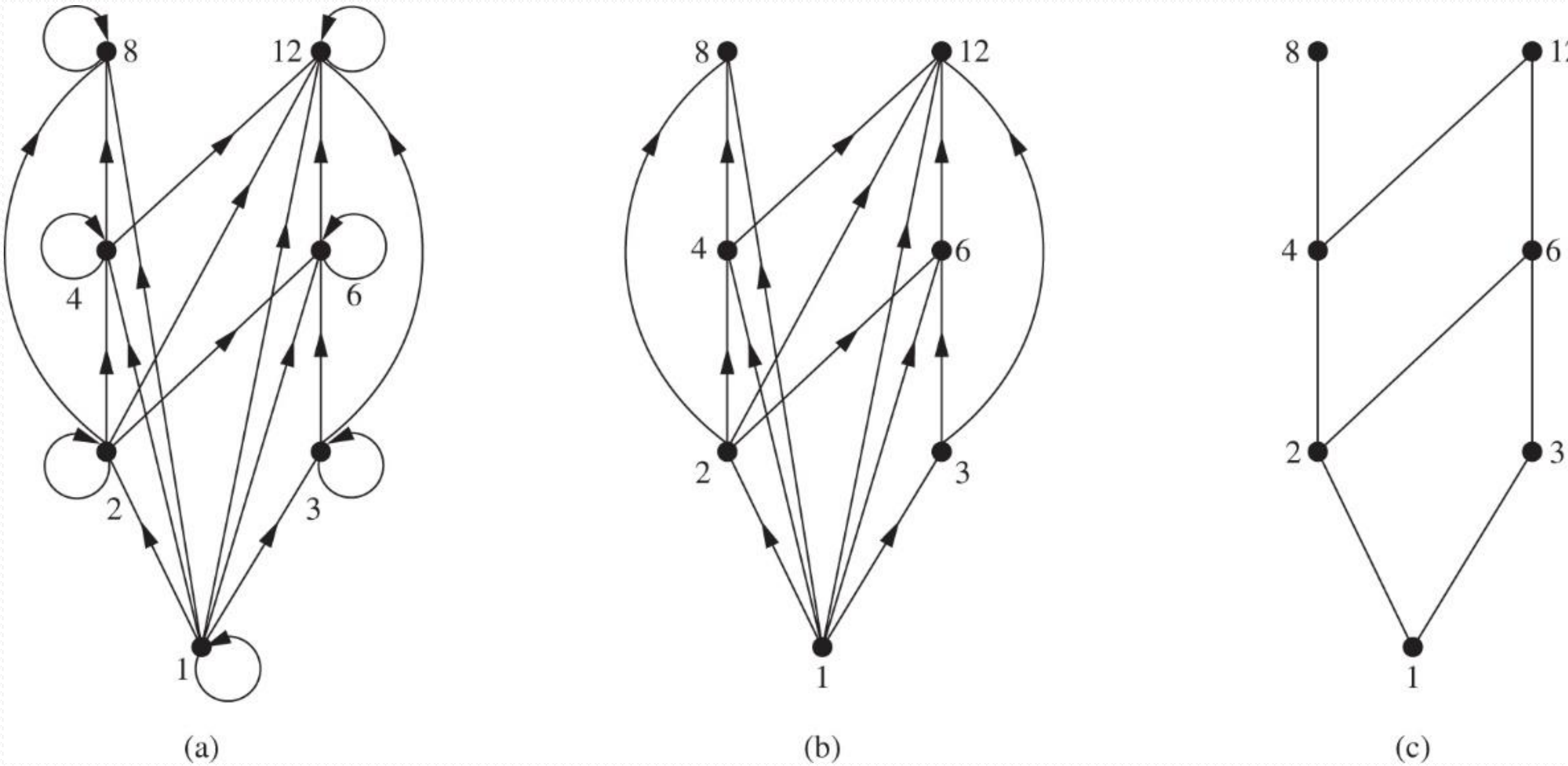
Hasse Diagrams

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

- The loops due to the reflexive property are deleted.
- The edges due to the transitive property are deleted
- Arrows are omitted because of antisymmetry



Example

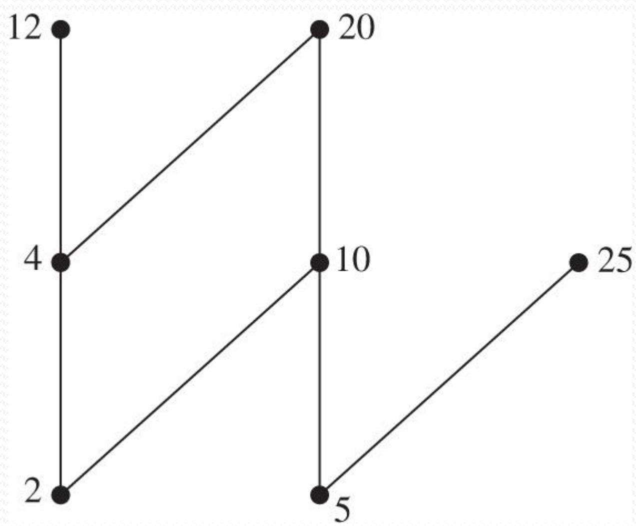


Procedure for Constructing a Hasse Diagram

- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

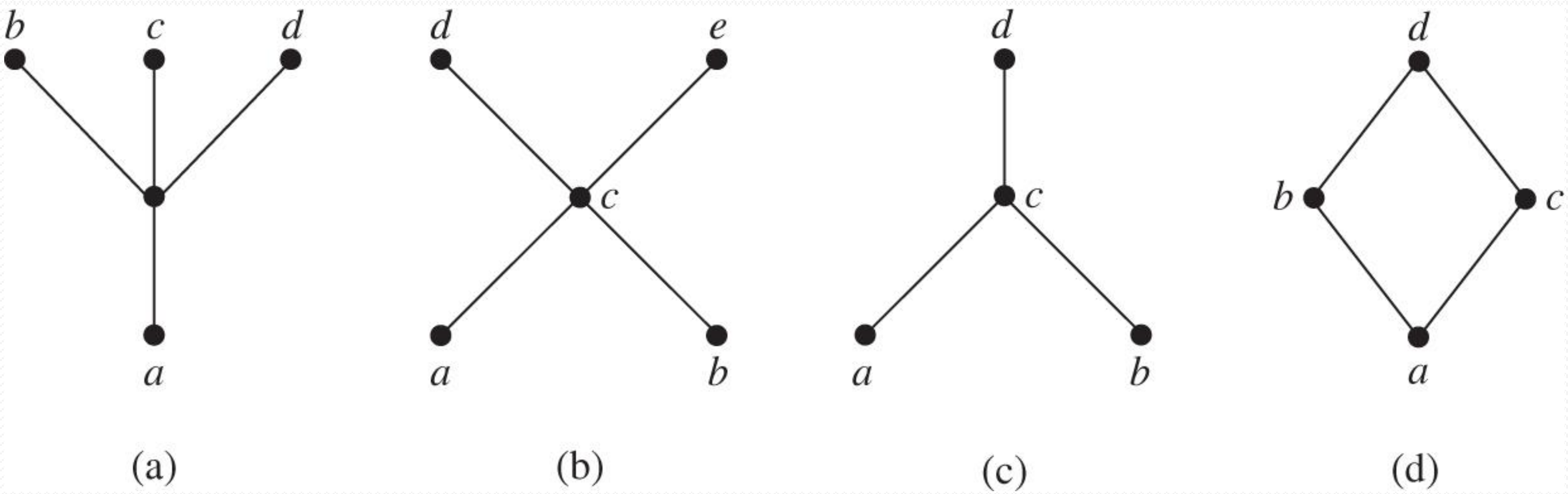
Maximal and Minimal Elements

- For a poset (S, $\preccurlyeq$), $a \in S$ is **maximal** if there exists no $b \in S$ such that $a \prec b$
- For a poset (S, $\preccurlyeq$), $a \in S$ is **minimal** if there exists no $b \in S$ such that $b \prec a$
- Examples



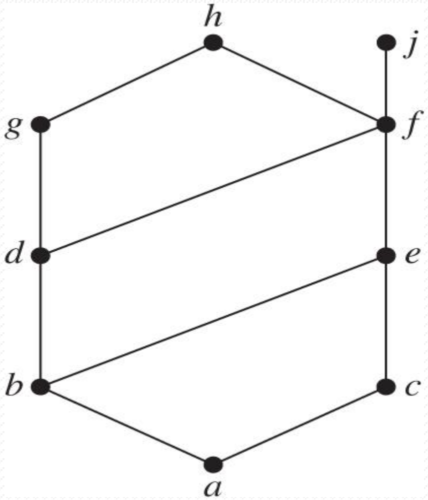
Greatest and Least Elements

- For a poset (S, ≼), $a \in S$ is **greatest** if for all $b \in S,\ b \preccurlyeq a$
- For a poset (S, ≼), $a \in S$ is **least** if for all $b \in S,\ a \preccurlyeq b$
- The greatest/least element is unique if there exists



Upper and Lower Bound of Subset

- For a poset (S, ≼), $a \in S$ is **upper bound** of elements $S' \subseteq S$ if for all $s' \in S'$, $s' \preccurlyeq a$
  - there may exist multiple upper bounds for a given set of elements
- The element x is the **least upper bound** of a set of elements A (i.e., lub(A)) if x is the upper bound of A and is less than every other upper bound of A
  - the least upper bound is unique if it exists for a set of elements
- Example
  - upper bounds of {a, b, c} ?
  - upper bound of {j, h}?
  - lower bound of {a, c, d, f} ?



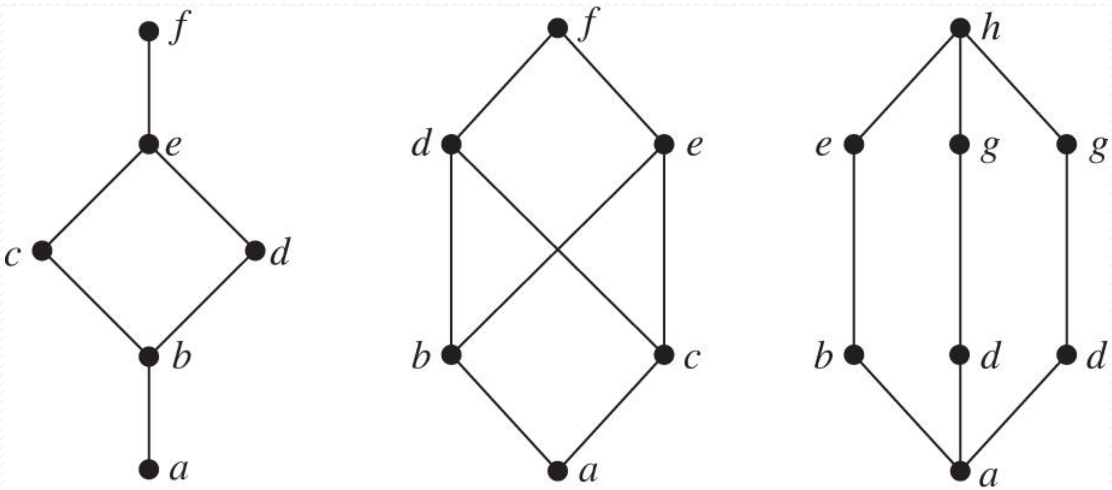
Example

Find the greatest lower bound and the least upper bound of $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$ if there exists, in $(\mathbf{Z}^+, |)$

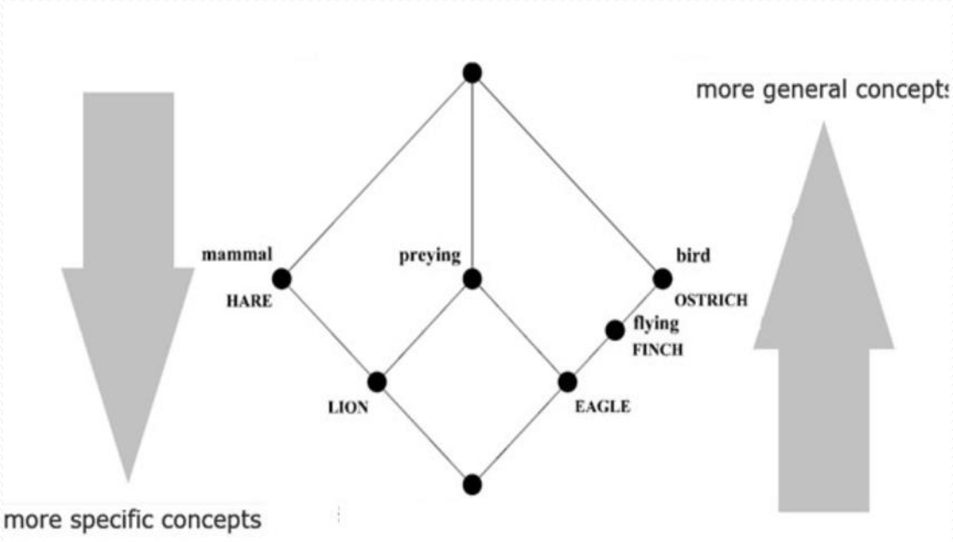
- $a \mid b$ if b is divisible by a

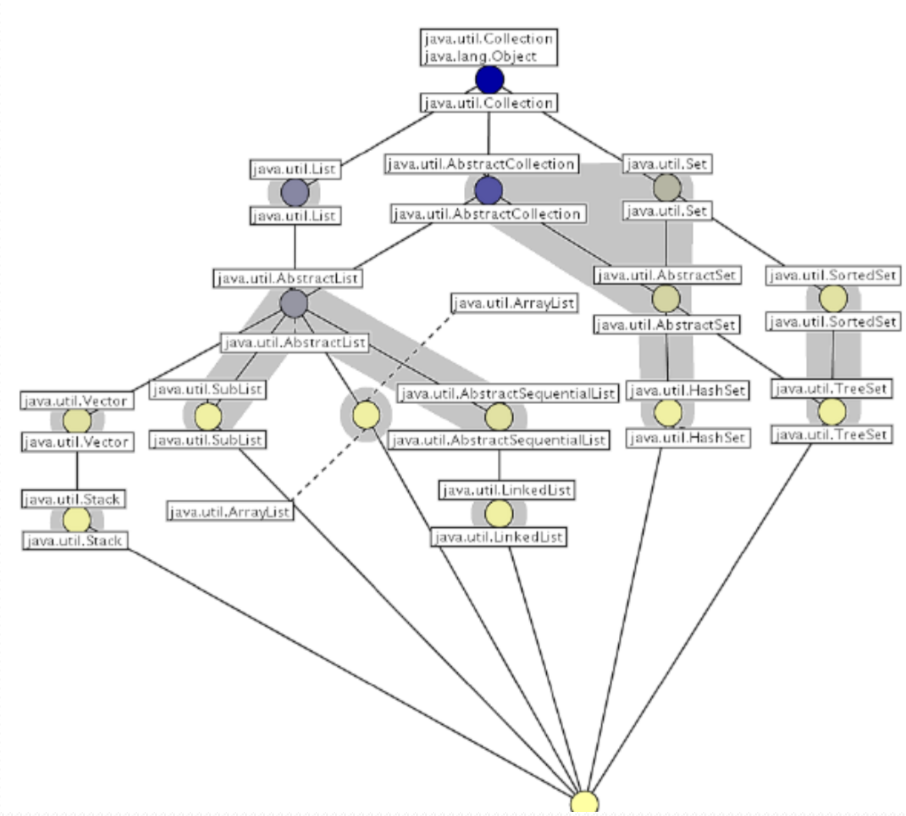
Lattices

- A poset is a **lattice** when there exists a least upper bound and a greast lower bound for every pair of elements
  - used in many applications such as information flow and typing
- Examples
  - Is the poset ($\mathbf{Z}^+$, |) a lattice?
  - which one in the figure is a poset?



Lattice as Knowledge Representation





Topological Sorting

- A total ordering $(S, \lessdot)$ is compatible with a poset $(S, \preccurlyeq)$ when $a \lessdot b$ if $a \preccurlyeq b$
- The topological sorting is to find a compatible total ordering from a given partially ordered set
  - take a minimal element at a time until no element remains

Input: a poset $(S, \preccurlyeq)$

$k \leftarrow 1$

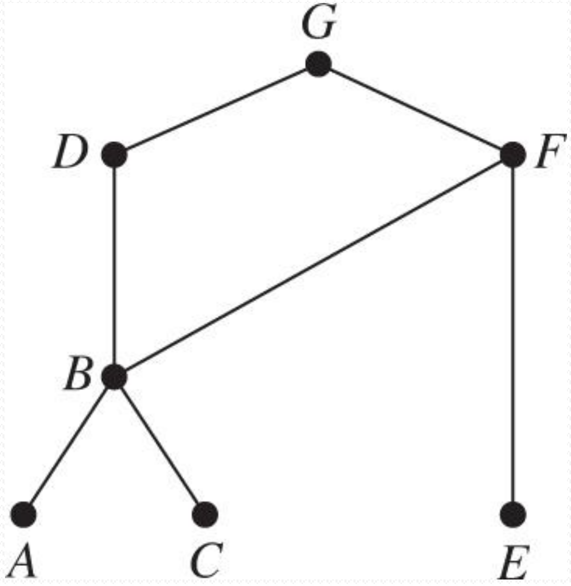
while $S \neq \emptyset$

  $a_k \leftarrow$ a minimal element of S

  $S \leftarrow S \setminus \{a_k\}$

  $k \leftarrow k + 1$

return $< a_1, a_2, \ldots, a_{|S\}} >$



Example

A development project equires completion of Tasks A to G where some of these have dependencies such that some tasks can be started only after other tasks are finished.

If the dependency is given as a partial ordering among the tasks, how can we find an order in which all tasks can be carried out to complete the project?