Discrete Mathematics

Set and Function

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Sets

- A *set* is an unordered collection of objects.
  - E.g., the students in this class, the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set.
  - A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of A
  - If a is not a member of A, write $a \notin A$

Defining a Set: Roster Method

- $S = \{a,b,c,d\}$
  - Order does not mean any thing: $S = \{a,b,c,d\} = \{b,c,a,d\}$

- Each distinct object is either a member or not; listing more than o nce does not change the set.

  $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $S = \{a,b,c,d, \ldots, z\}$

Examples

- Set of all vowels in the English alphabet:

$$V = \{a,e,i,o,u\}$$

- Set of all odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

- Set of all positive integers less than 100:

$$S = \{1,2,3,\ldots\ldots,99\}$$

- Set of all integers less than 0:

$$S = \{\ldots, -3,-2,-1\}$$

Conventional Notions of Important Sets

$\mathbf{N}$ = *natural numbers* = {1,2,3, …}

$\mathbf{Z}$ = *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z}^{+}$ = *positive integers* = {1,2,3,…..}

$\mathbf{R}$ = set of *real numbers*

$\mathbf{R}^{+}$ = set of *positive real numbers*

$\mathbf{C}$ = set of *complex numbers*.

$\mathbf{Q}$ = set of rational numbers

Defining a Set with Set-Builder Notation

- To specify the properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$

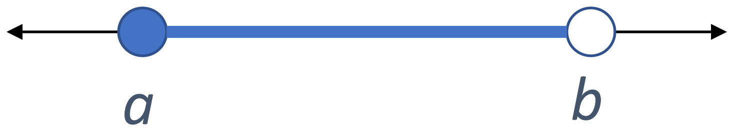
- A predicate may be used: $S = \{x \mid \mathrm{P}(x)\}$
  - Example: $S = \{x \mid \mathrm{Prime}(x)\}$
  - Example: Positive rational numbers:

    $\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q \text{ for } p \in \mathbb{Z}^{+} \text{ and } q \in \mathbb{Z}^{+}\}$

Defining a Set with Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

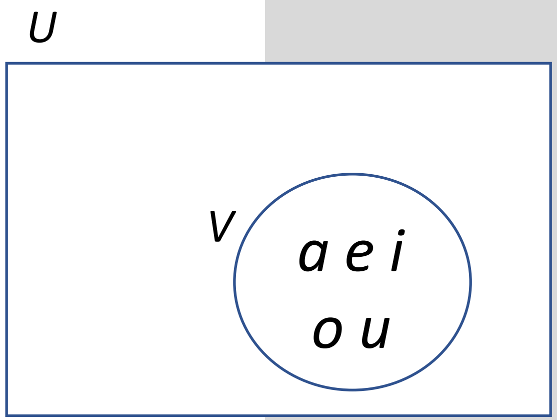
E.g., [a,b)



closed interval	[a,b]
open interval	(a,b)

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated
  - Contents depend on the context



- The empty set is the set with no elements.
  - denoted as $\emptyset$, or $\{\}$

Some things to remember

- Sets can be elements of sets.

$$\{\{1,2,3\},\ a, \{b,c\}\}$$

$$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$$

- The empty set is different from a set containing the empty set.

$$\emptyset \neq \{\ \emptyset\ \}$$

Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a non-negative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\varnothing| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\varnothing\}| = 1$
5. The set of integers is infinite.

Russell's Paradox

- Let *R* be the set of all sets each of which is not a member of itself.
  - $R = \{S \mid S \notin S\}$
  - A paradox results from trying to answer the question "Is *R* a member of itself?"

- Related Paradox:
  - Henry is a barber who shaves every man if and only if the man does not shave himself. A paradox results from trying to answer the question "Does Henry shave himself?"

Bertrand Russell (1872-1970) Cambridge, UK Nobel Prize Winner

Set Equality

Definition: Two sets are *equal* if and only if they have the same ele ments.

- Therefore if A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B)$$

- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$ iff $\forall x\ (x \in A \leftrightarrow x \in B)$

- Using logical equivalences we have that $A = B$ iff $\forall x\ [(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$

- This is equivalent to

$$A \subseteq B \qquad \text{and} \qquad B \subseteq A$$

Subsets

Definition: The set *A* is a *subset* of *B*, if and only if every element of *A* is also an element of *B*.

- The notation $A \subseteq B$ is used to indicate that *A* is a subset of the set *B*.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.

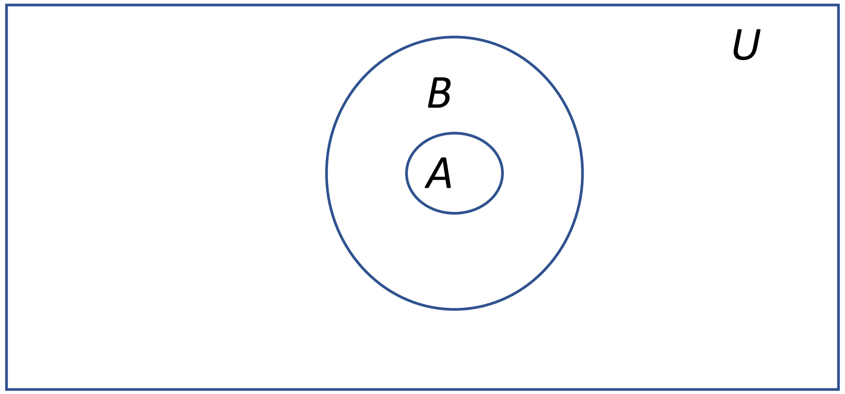
1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set *S*.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set *S*.

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$ or $A \subsetneq B$

If $A \subset B$, then

$$\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$



Power Sets

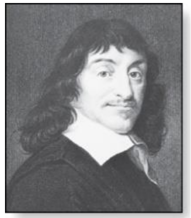
Definition: The set of all subsets of a set *A*, denoted P**(***A***)**, is calle d the *power set* of *A*.

Example: If A = {a,b} then

$$\mathcal{P}(A) = \{\varnothing, \{a\},\{b\},\{a,b\}\}$$

- If a set has *n* elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Cartesian Product



René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

- **c.f. Relation**
  - A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B.
  - Will be covered in depth in Chapter 9.

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered *n*-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots\ n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)}

Tuples

- The *n-tuple* $(a_1, a_2, \ldots.., a_n)$ is the ordered collection of objects, whic h has a_1 as its first element and a_2 as its second element and so o n until a_n as its last element.
  - Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
  - The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

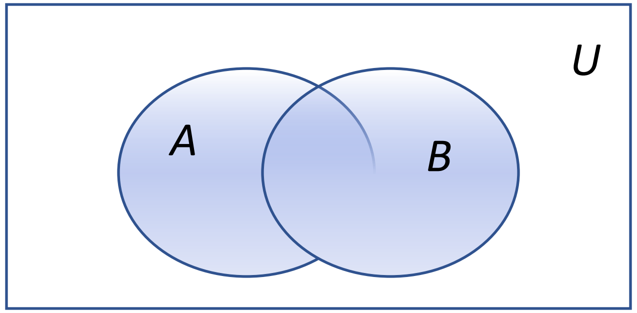
Union

- **Definition**: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set: $\{x|x \in A \vee x \in B\}$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

  Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

- **Definition**: The *intersection* of sets A and B, denoted by $A \cap B$, is

$$\{x | x \in A \wedge x \in B\}$$

  - Note if the intersection is empty, then A and B are said to be *disjoint*.

- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

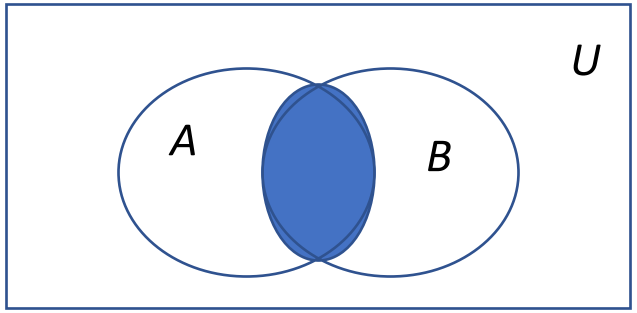
  Solution: $\{3\}$

- **Example:** What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

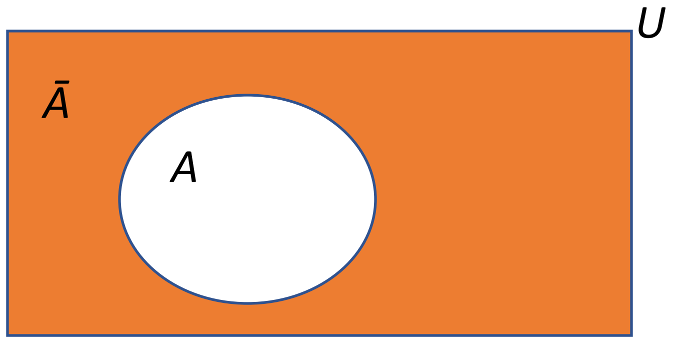
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

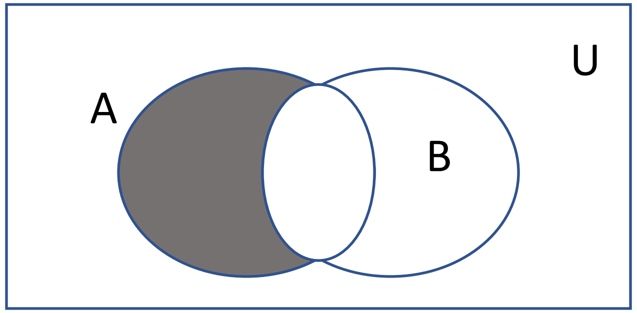
Venn Diagram for Complement



Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

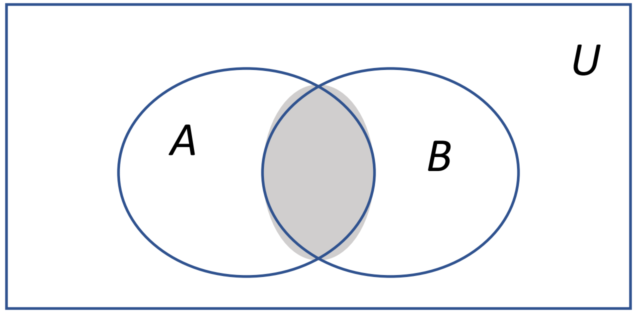


Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

- **Example**: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
- We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by is the set

$$A \oplus B$$

$$(A - B) \cup (B - A)$$

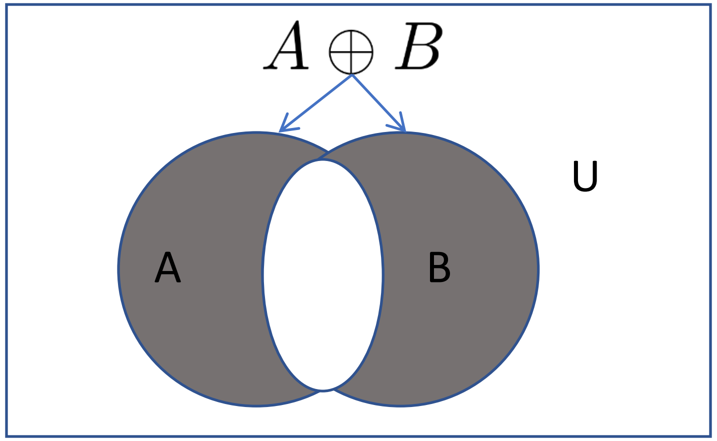
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is:

- **Solution**: $\{1,2,3,6,7,8\}$



Venn Diagram

Set Identities

- Identity laws
- Domination laws
- Idempotent laws
- Complementation law

$$A \cup \emptyset = A \qquad A \cap U = A$$

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

$$A \cup A = A \qquad A \cap A = A$$

$$\overline{(\overline{A})} = A$$

Continued on next slide →

Set Identities

- Commutative laws $A \cup B = B \cup A$ $A \cap B = B \cap A$
- Associative laws $$A \cup (B \cup C) = (A \cup B) \cup C$$ $$A \cap (B \cap C) = (A \cap B) \cap C$$
- Distributive laws $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$ $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Set Identities

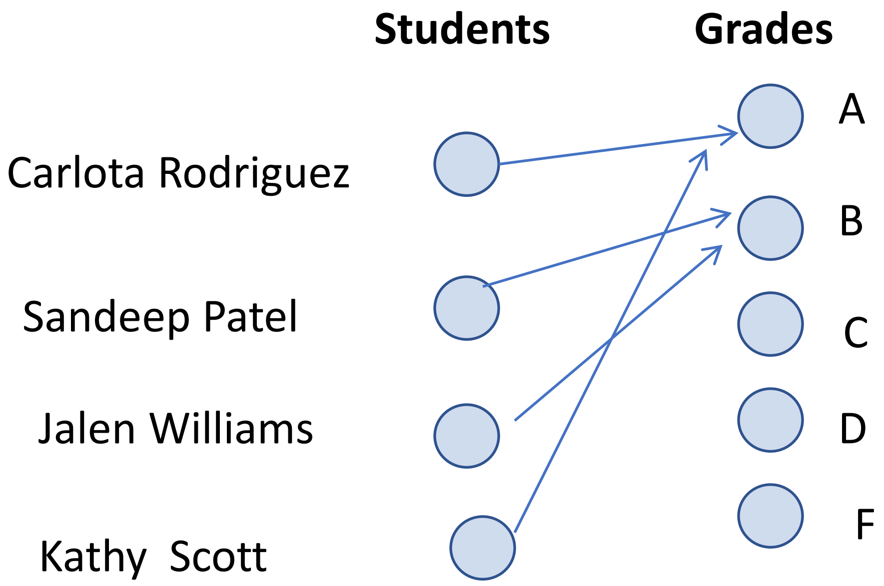
- De Morgan's laws $\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- Absorption laws $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$

- Complement laws $A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$

Functions

- Let A and B be nonempty sets.
- A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- Functions are sometimes called *mappings* or *transformations*.



Functions

- A function $f: A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

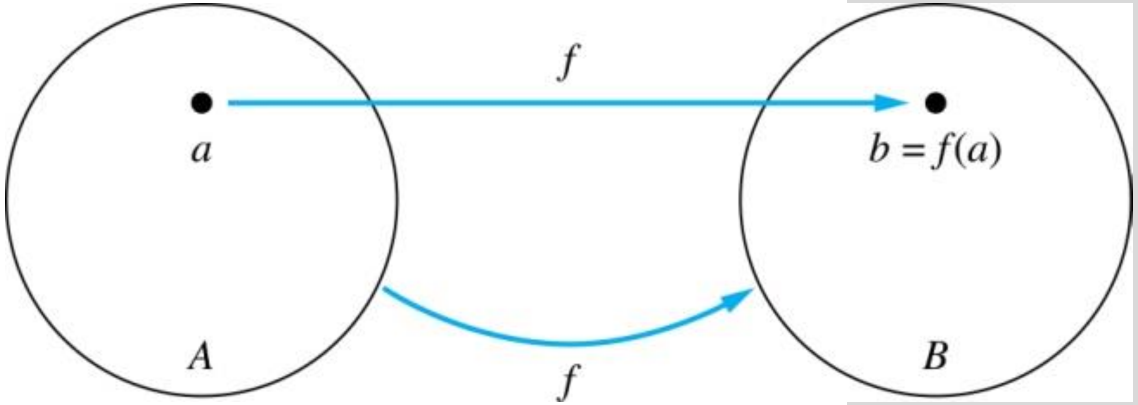
$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2)] \rightarrow y_1 = y_2]$$

Functions

Given a function $f: A \to B$:

- We say *f maps A* to *B or*

  f is a *mapping* from *A* to *B*.
- *A* is called the *domain* of *f*.
- *B* is called the *codomain* of *f*.
- If $f(a) = b$,
  - then *b* is called the *image* of *a* under *f*.
  - *a* is called the *preimage* of *b*.



Questions

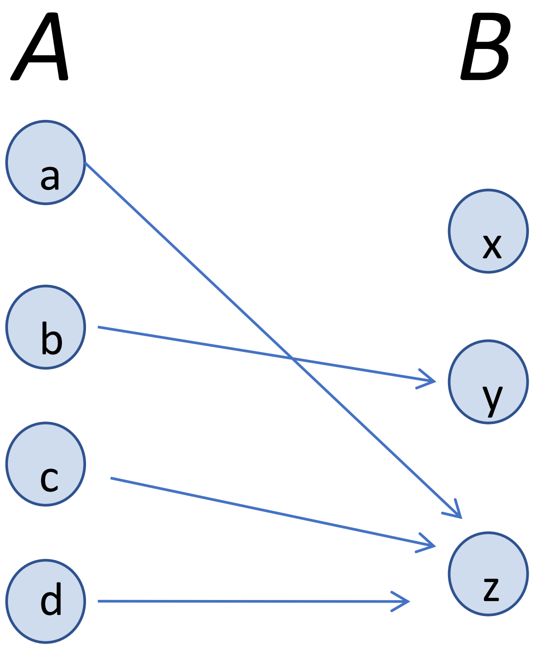
f(a) = ? z

The image of d is ? z

The domain of f is ? *A*

The codomain of f is ? *B*

The preimage of y is ? b



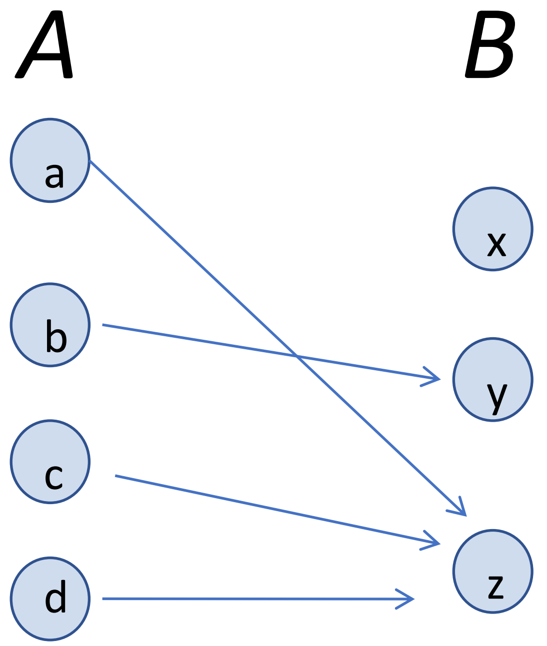
Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

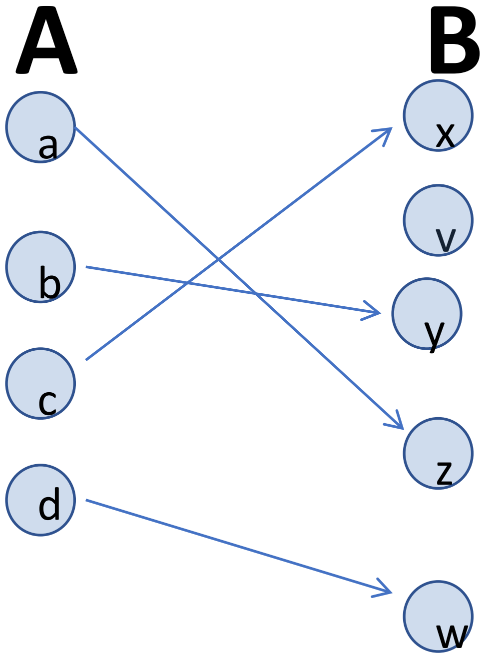
f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}



Injections

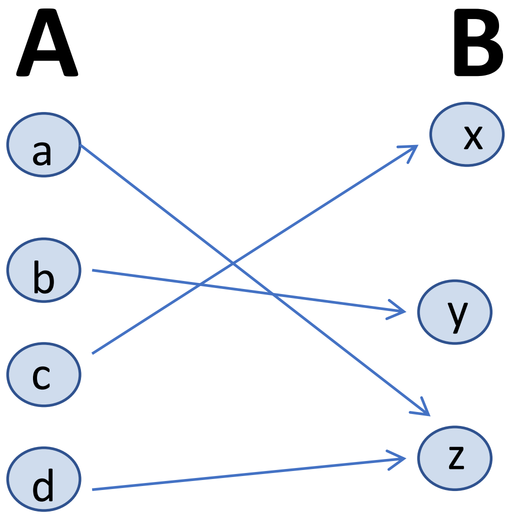
Definition: A function f is said to be *one-to-one* , or *injective*, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.



Surjections

A function $f: A \rightarrow B$ is called *onto* or *surjective* iff for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

A function f is called a *surjection* if it is **onto**.



Example

Example 1: for $f : \{a,b,c,d\} \to \{1,2,3\}$, $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

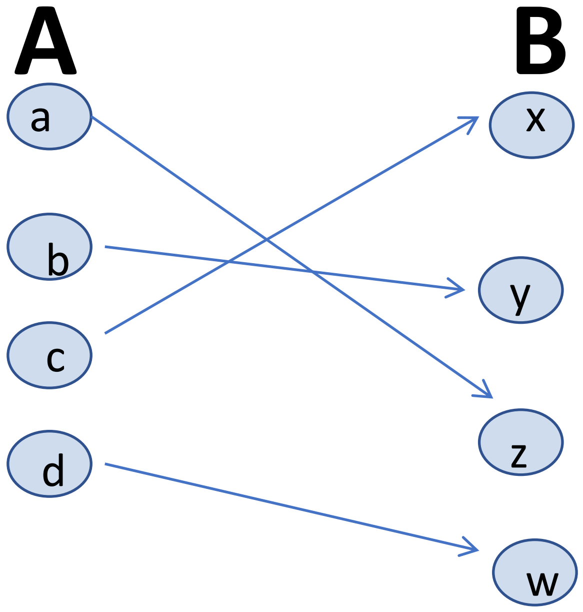
Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain.

Example 2: Is the function $f(x) = x^2$ from the set of integers onto?

Solution: No, f is not onto since there is no integer x with $x^2 = -1$, for example.

Bijections

A function f is a ***one-to-one correspondence***, or a *bijection*, if it is b oth one-to-one and onto (surjective and injective).

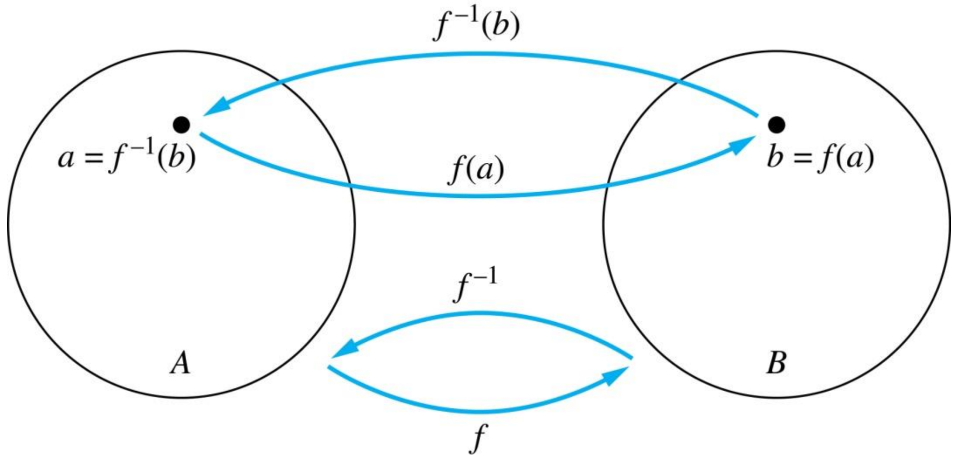


Inverse Functions

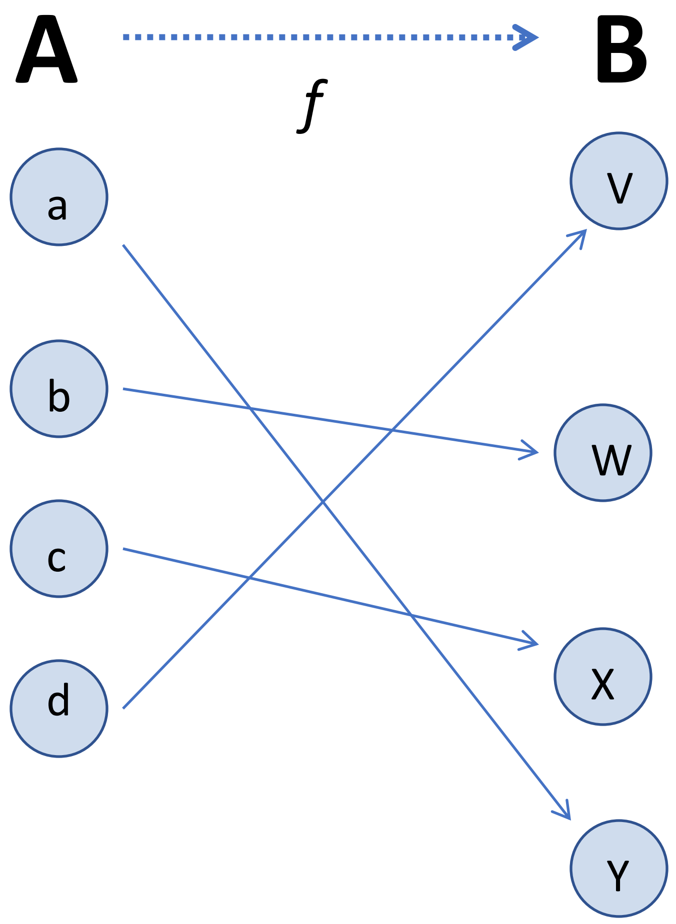
Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted, is the function from *B* to *A* defined as

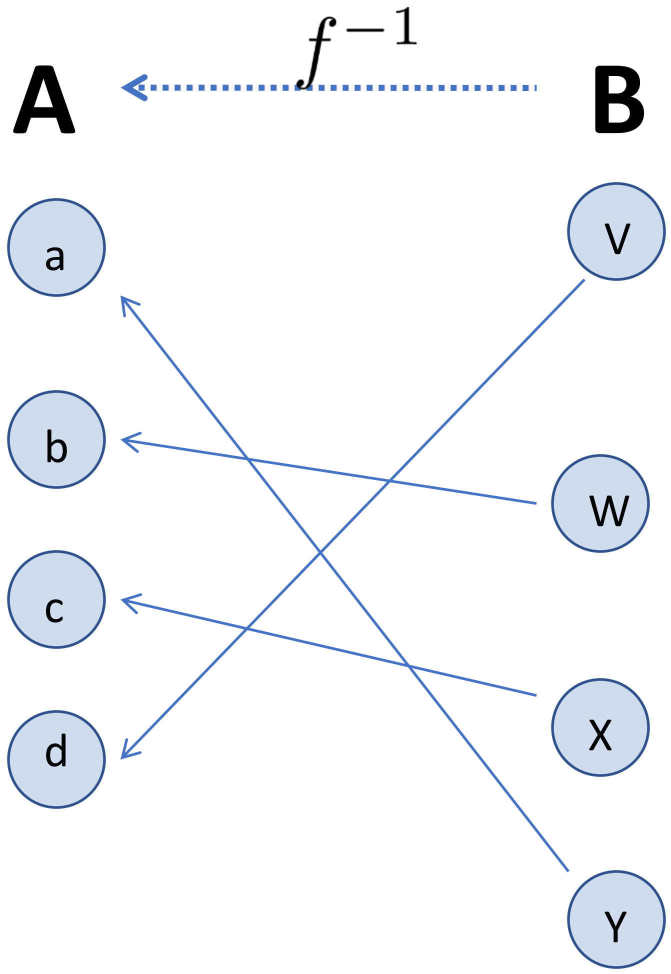
No inverse exists unless *f* is a bijection. Why?

$$f^{-1}(y) = x \text{ iff } f(x) = y$$



Inverse Functions





Questions

Example 2: Let f: $\mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

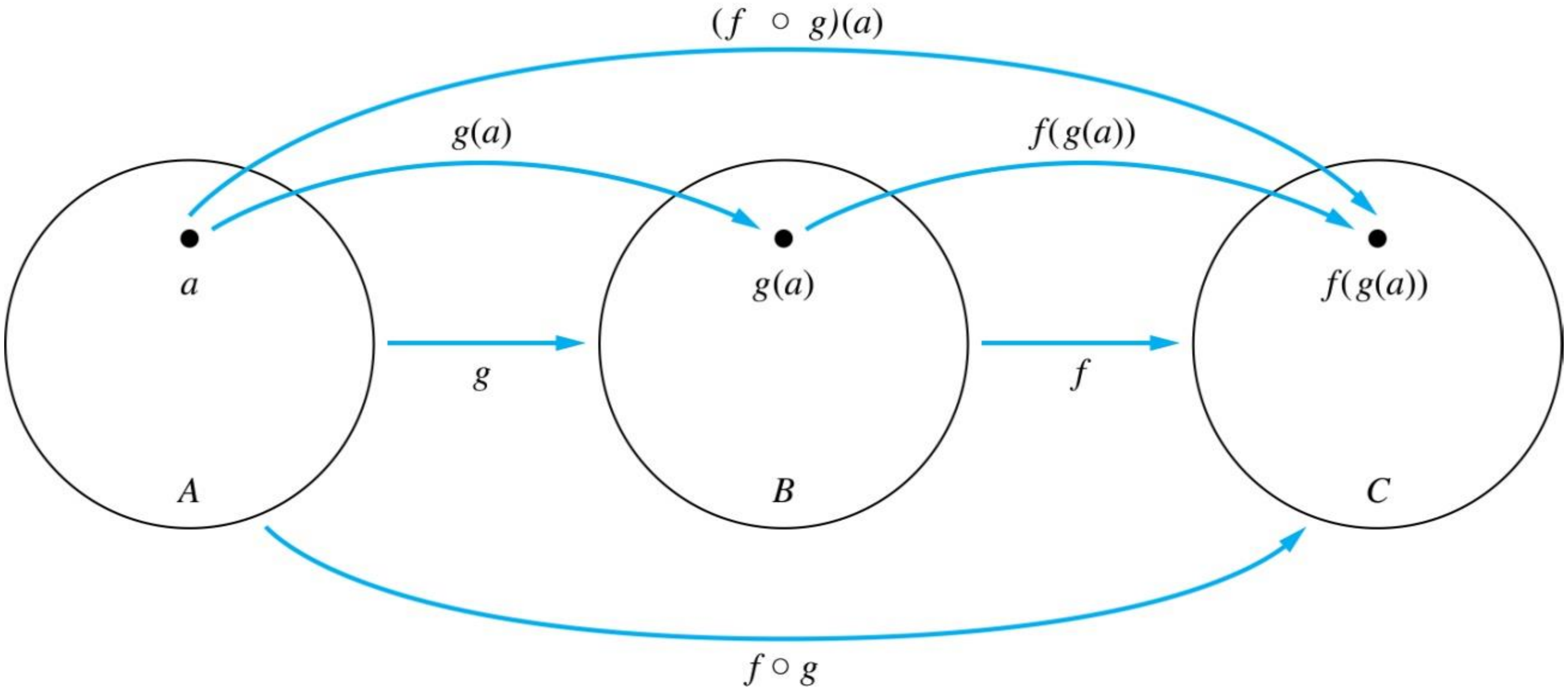
Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Example 3: Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

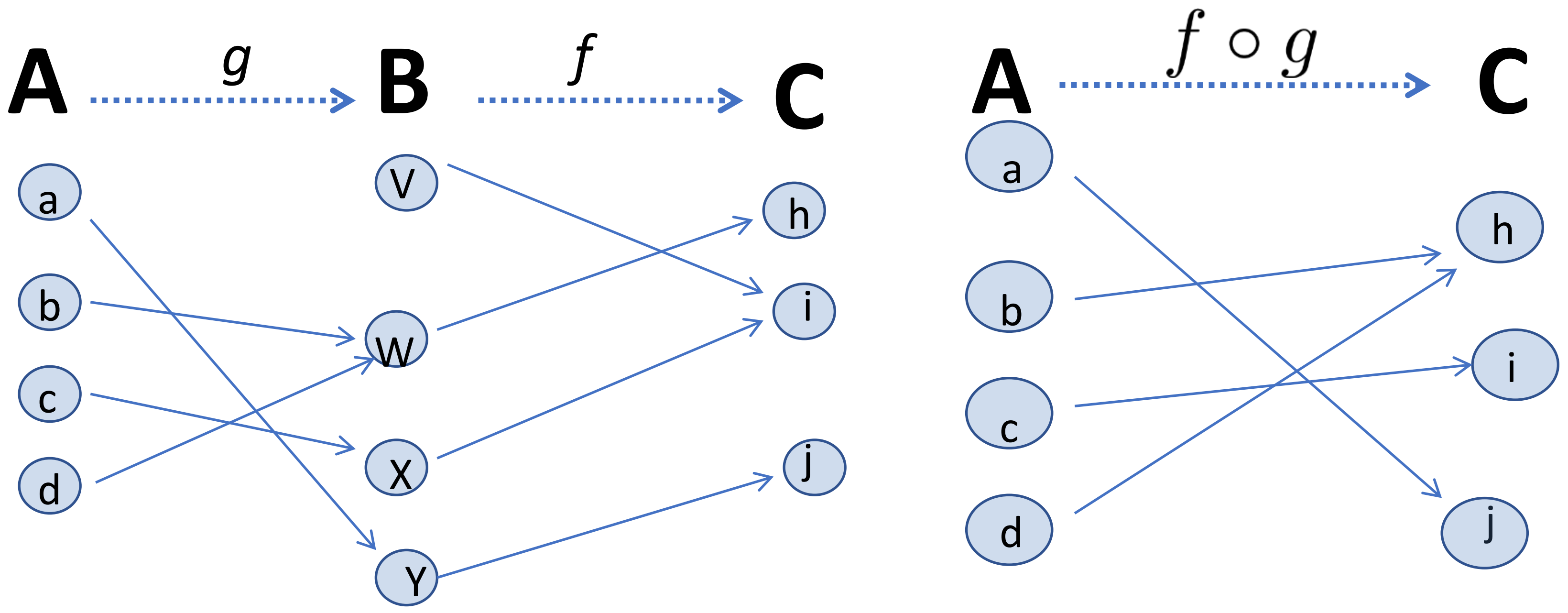
Solution: The function f is not invertible because it is not one-to-one.

Composition

- **Definition**: Let *f*: *B* → *C*, *g*: *A* → *B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by $f \circ g(x) = f(g(x))$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

and $f(g(x)) = (2x + 1)^2$

$g(f(x)) = 2x^2 + 1$

Composition Questions

Example 2: Let g be a function from $\{a,b,c\}$ to itself s.t.
$g(a) = b$, $g(b) = c$, and $g(c) = a$.

Let f be a function from $\{a,b,c\}$ to $\{1,2,3\}$ s.t.
$f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

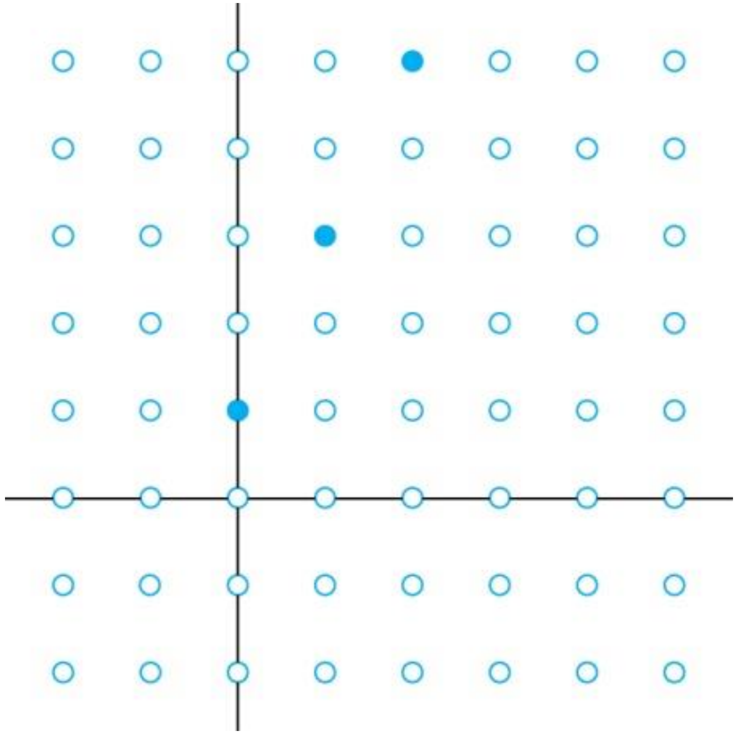
Solution: The composition $f \circ g$ is defined by

$$f \circ g\ (a) = f(g(a)) = f(b) = 2.$$
$$f \circ g\ (b) = f(g(b)) = f(c) = 1.$$
$$f \circ g\ (c) = f(g(c)) = f(a) = 3.$$

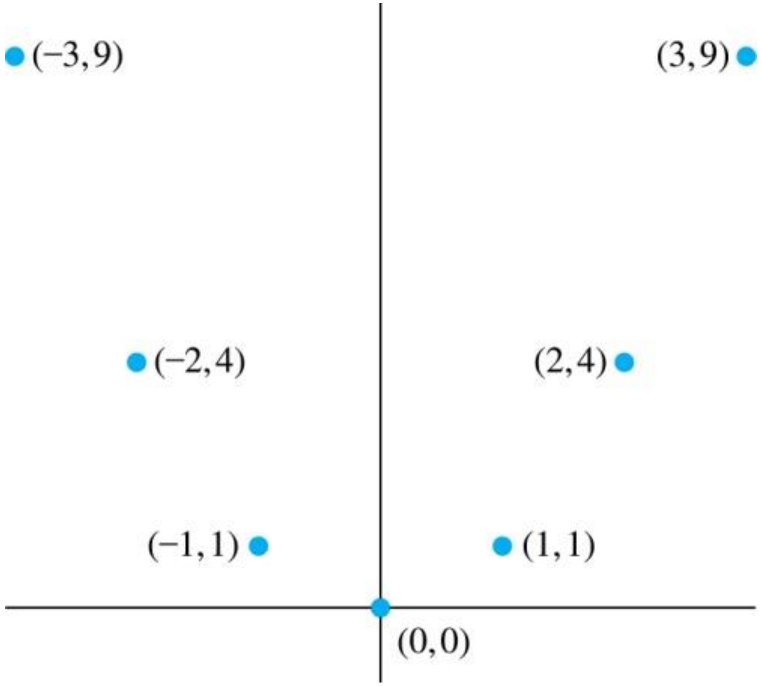
Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Graphs of Functions

- Let *f* be a function from the set *A* to the set *B*. The *graph* of the fun ction *f* is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$
from Z to Z



Graph of $f(x) = x^2$
from Z to Z

Partial Functions

A *partial function f* from a set A to a set B, *denoted* $f : A \multimap\!\!\longrightarrow B$ is an assignment to each element a in a subset of A on a unique element b in B.

- *The subset of A is called the domain of definition of f*
- f is *undefined* for elements in A that are not in the domain of definition of f.
- When the domain of definition of f equals A, we say that f is a *total function*.

Example: f: $\mathbf{N} \rightarrow \mathbf{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbf{Z}$ to $\mathbf{R}$ where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.