ITP 20002-01 Discrete Mathematics, 2018 Fall

Final Exam: Problems

Write an answer to each problem on the corresponding box in your answer sheets.
Note that some problems are related with each other: Problems 3&4, Problems 5&6 and Problems 8&9

1. Explain what is mathematical induction, and why it works. (12 points)

2. Use mathematical induction to prove $a^k \equiv b^k \pmod{m}$ with an integer k for $1 < k$. (9 points)

3. Prove that, for every positive integer *n*, there exist at most $\left\lceil \frac{n}{2} \right\rceil$ number of prime numbers each of which does not not exceed *n*. (7 points)

4. Suppose that a set *S* has total *n* + 1 number of positive integers each of which is greater than 1 and less than or equal to 2*n*. Based on the theorem of Problem 3, prove that *S* always has a pair of numbers whose greatest common divisor (GCD) is greater than 1 (i.e., two numbers are not mutually prime). (7 points)

5. Suppose that P is the set of all strings that represent balanced brackets (e.g., “”, “`[]`”, “`[[]]`”, “`[][[]][]`”). Define *P* recursively (6 points)

6. Remind P of Problem 5. Define a recursive function d that returns the maximum depth of nested brackets in a string $s \in P$. For example, d(“`[[]][]`”) = 2 and d(“`[][[[]]][]`”) = 3 (7 points)

7. Suppose that you are flipping an ordinary coin 10 times while recording the results in sequence as a binary string: 1 for head and 0 for tail. What is the probability that the string is a palindrome (i.e., the sequence that reads the same backward as forward)? (5 points)

8. Professor Hong is grouping 10 students to make 4 teams such that each team has 2 to 3 students. How many different team compositions are possible? (7 points)

9. Professor Hong finds Tom and Jerry in the 10 students of Problem 8. We want to make teams such that Tom and Jerry do not belong to the same team. Then, how many different team compositions are possible? (6 points)

10. Prove the generalized Bayes’ theorem (12 points)

11. Suppose that R is a binary relation on a set *A*. Give a predicate formula that determines whether R is a lattice, or not (12 points).

12. Suppose that $\preccurlyeq_1$ is a partial ordering on a set A_1, and $\preccurlyeq_2$ is a partial ordering on another set A_2. Let’s define $\preccurlyeq \subseteq A_1 \times A_2$ such that $(a_1, b_1) \preccurlyeq (a_2, b_2)$ if and only if $a_1 \preccurlyeq_1 a_2$ or $b_1 \preccurlyeq_2 b_2$. Prove (or disprove) that $\preccurlyeq$ is a partial ordering (10 points)

Bonus. List all names of your team members in alphabetic order, for each programming assignment. (up to 4 points)