

Discrete Mathematics

Cardinality

Shin Hong

Cardinality

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- The *cardinality* of a set A is equal to the cardinality of a set B , denoted $|A| = |B|$, if and only if there is a one-to-one correspondence (i.e., bijection) from A to B .
- If there is a one-to-one function from A to B but no bijection, the cardinality of A is less than or the cardinality of B and we write $|A| < |B|$.

Cardinality of Finite Sets

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- A set S is finite with cardinality $n \in \mathbb{N}^0$ if there is a bijection from the subset of non-negative integers $\{0, 1, \dots, n-1\}$ to S

Let's Think About Infinite Sets

1. A set S is infinite if one of its subsets is an infinite set
2. Every subset of a finite set is finite.
3. If $f: S \rightarrow T$ be an injection and S is infinite, T is infinite.
4. If S is infinite, the power set of S is infinite.
5. If S and T are infinite sets, $S \cup T$ is infinite.
6. If S is infinite and $T \neq \emptyset$, $S \times T$ is infinite.
7. If S is infinite and $T \neq \emptyset$, the set of functions from T to S is infinite.

Cardinality of Infinite Set

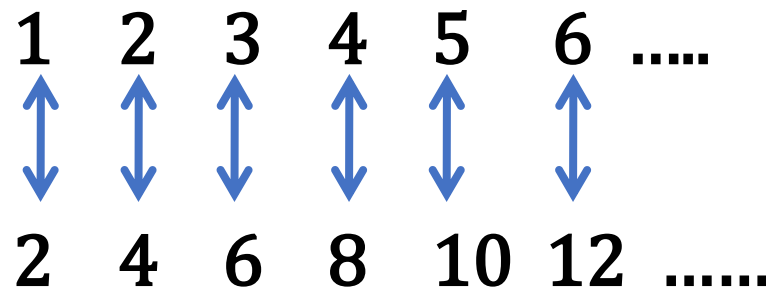
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- A set is *countable* when
 - the set is finite, or
 - the set has the same cardinality as the set of positive integers
- When an infinite set is countable (calling it *countably infinite*), its cardinality is denoted as \aleph_0 (i.e., aleph null)
- A set that is not countable is *uncountable*

Showing that a Set is Countable

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- An infinite set is countable iff there is a way to list the elements in a sequence with indexes of positive integers
 - there must exist a function $f: \mathbb{N} \rightarrow S = \{a_1, a_2, \dots\}$ such that $a_1 = f(1)$, $a_2 = f(2), \dots, a_n = f(n), \dots$
- Ex. Show that the set of positive even integers E is countable
Let $f(x) = 2x$. Then f is a bijection from \mathbb{N} to E .



Showing that a Set is Countable

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- Ex. Show that the set of integers \mathbb{Z} is countable.

\mathbb{Z} can be listed as a sequence:

0, 1, -1, 2, -2, 3, -3,

This sequence can be defined by a bijection f from \mathbb{N} to \mathbb{Z} :

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

The Positive Rational Numbers are Countable

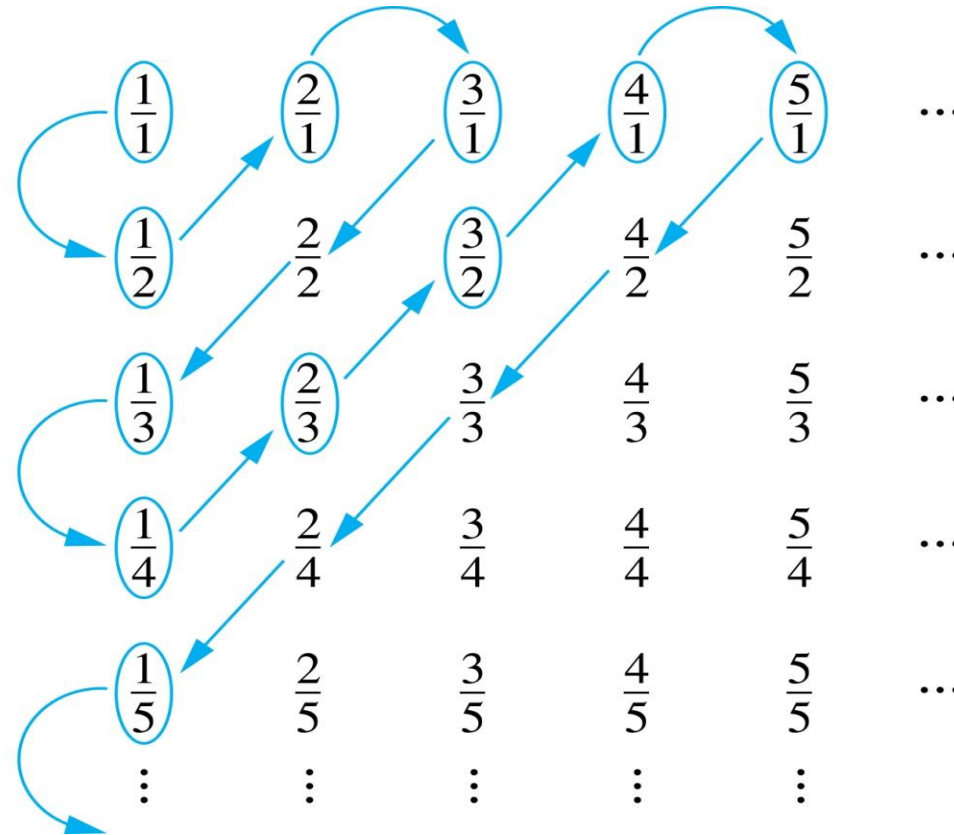
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- Theorem. the set of all positive rational numbers is countable.

Constructing a sequence

List p/q with $p + q = 2$ first, and then list p/q with $p + q = 3$, and so on

1, $\frac{1}{2}$, 2, 3, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$,



Enumeration

- An enumeration of a set S is a surjective function f from an initial segment of \mathbb{N} to S .
 - f is a string where every element of S appears at least once
 - f is an enumeration without repetitions if f is bijective
 - f is an enumeration with repetitions if it is not injective
 - Example
 - $S = \{\alpha, \beta, \gamma, \delta\}$
 - $\langle \alpha, \gamma, \beta, \beta, \delta, \alpha \rangle$ is an enumeration with repetition
 - $\langle \gamma, \alpha, \delta, \beta \rangle$ is an enumeration without repetition
- A set S is countable iff there is an enumeration of S

Strings

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- The set of strings over a finite alphabet A is countably infinite.
- Proof.
 - Show that the strings can be listed in a sequence:
 - All the strings of length 0 in alphabetical order.
 - Then all the strings of length 1 in lexicographic (as in a dictionary) order.
 - Then all the strings of length 2 in lexicographic order.
 - And so on.
 - This implies a bijection from \mathbf{N} to S and hence it is countably infinite.

Every Java programs is a string, thus countable

The set of all Java programs is countable.

- Proof

- Let S be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:
 - Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
 - If the compiler says YES, this is a syntactically correct Java program, we add the program to the list
 - We move on to the next string
- In this way we construct an implied bijection from \mathbb{N} to the set of Java programs. Hence, the set of Java programs is countable.

Uncountable Set

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- Theorem. \mathbb{R} is uncountable.
- Proof (proof by contradiction)
 - Suppose that \mathbb{R} is countable. Then, the set of all real numbers in $[0, 1)$ is countable, and the elements can be listed with positive integer indexes as follow:

$$r_1 = 0.d_{11}d_{12}d_{13}d_{14} \dots \quad d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$r_2 = 0.d_{21}d_{22}d_{23}d_{24} \dots$$

$$r_3 = 0.d_{31}d_{32}d_{33}d_{34} \dots$$

$$r_4 = 0.d_{41}d_{42}d_{43}d_{44} \dots$$

$$\vdots$$

$$r_i = 0.d_{i1}d_{i2} \dots d_{ii} \dots$$

- There exists a real number $r' = 0.d'_1d'_2d'_3 \dots d'_i \dots$ such that
 $d'_i = 4$ iff $d_{ii} \neq 4$ and $d'_i = 5$ iff $d_{ii} = 4$
- Thus, $\forall i \in \mathbb{N} (r' \neq r_i)$. Consequently, this conclusion is a contradiction.

Cardinalities of the Uncountable

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- A set S is of cardinality c if there is a bijection from the set of real numbers in $[0, 1]$ to S .
 - $\aleph_0 < c$
 - c.f. the set of real numbers in $[0, 1]$ is called a continuum
- The continuum hypothesis claims that there exists no set A such that $\aleph_0 < |A| < c$
 - Note that, for an infinite set A , $\aleph_0 \leq |A|$
- For a set S , $|S| < |\mathcal{P}(S)|$.
 - $\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$

Uncountable Set Example

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- Let S be a finite alphabet and Σ^* the set of all strings over Σ . Then $\mathcal{P}(\Sigma^*)$ is uncountable.
- Proof using the Cantor's diagonalization
 - Let $\langle s_0, s_1, s_2, \dots \rangle$ be an enumeration of strings in Σ^* .
 - Suppose that $\langle A_0, A_1, \dots \rangle$ is an enumeration of $\mathcal{P}(\Sigma^*)$, such that A_i represents a subset of strings Σ^* as a bit vector $\langle a_{i0}, a_{i1}, \dots \rangle$
 - Think about $A' \in \mathcal{P}(\Sigma^*)$ such that $s_i \in A'$ iff $s_i \notin A_i$

	s_0	s_1	s_2	...
A_0	a_{00}	a_{01}	a_{02}	...
A_1	a_{10}	a_{11}	a_{12}	...
A_2	a_{20}	a_{21}	a_{22}	...
\vdots	\vdots	\vdots	\vdots	