Multivariate Rational Function Interpolation

Input: Modular black box B for rational function $\frac{ff(x_1,\ldots,x_n)}{gg(x_1,\ldots,x_n)}$ over $\mathbb{Z}_p$, where $ff, gg \in \mathbb{Z}_p[x_1,\ldots,x_n]$, $\gcd(ff,gg)=1$, gg is monic, prime p,number of variables n, and list of variables $vars$.
Output: $ff(x_1,\ldots,x_n), gg(x_1,\ldots,x_n)$ or **FAIL**
$\sigma \leftarrow [2,3,\ldots,p_n] \in \mathbb{Z}_p^n$
Pick random vector $\beta = [\beta_2,\ldots,\beta_n] \in \mathbb{Z}_p^{n-1}$
$\Sigma \leftarrow [\ ]$ ▷ indexing starts at 1
$num \leftarrow [\ ]$, $den \leftarrow [\ ]$
$num\_points\_mqrfr \leftarrow 0$
$numerator\_done \leftarrow$ false
$denominator\_done \leftarrow$ false
$j\_init \leftarrow 2$ ▷ index to data structure Σ
$T\_old \leftarrow 0$ ▷ exponent of prime points
$T \leftarrow 4$ ▷ number of points
$\sigma_0 \leftarrow [1,\ldots,1] \in \mathbb{Z}_p^n$
$(f,g) \leftarrow \mathrm{NDSA}(B,\sigma_0,\beta,p,T)$
$num\_points\_mqrfr \leftarrow \deg(f)+\deg(g)+2$
num.append($f_j(\sigma_{01})$ mod p)
den.append($g_j(\sigma_{01})$ mod p)
while true **do**
  for $l \leftarrow T\_old$ to $2T-1$ **do**
    $\Sigma \leftarrow [\sigma_{T\_old},\ldots,\sigma_{2T-1}]$ where, $\sigma_l \leftarrow [2^l,3^l,\ldots,p_n^l]$ mod p
  end for
  for $j \leftarrow j\_init$ to $2T-1$ **do**
    $(f_j,g_j) \leftarrow \mathrm{NDSA}(B,\Sigma_j,\beta,p,num\_points\_mqrfr)$
    if $not(numerator\_done)$ **then**
      num.append($f_j(\Sigma_{j1})$ mod p)
    end if
    if $not(denominator\_done)$ **then**
      den.append($g_j(\Sigma_{j1})$ mod p)
    end if
  end for
  Construct minimum characteristic polynomials using Berlekamp-Massey algorithm
  if $not(denominator\_done)$ **then**
    $\Lambda_{num}(z) \leftarrow$ Berlekamp_Massey(num,p,z) $\in \mathbb{Z}_p[z]$
    $s \leftarrow \deg(\Lambda_{num}(z))$
    Let $roots_{num}$be the distinct roots of $\Lambda_{num}(z) \in \mathbb{Z}_p[z]$
  end if
  if $not(denominator\_done)$ **then**
    $\Lambda_{den}(z) \leftarrow$ Berlekamp_Massey(den,p,z) $\in \mathbb{Z}_p[z]$
    $d \leftarrow \deg(\Lambda_{den}(z))$

Let $roots_{den}$ be the distinct roots of $\Lambda_{den}(z) \in \mathbb{Z}_p[z]$
end if
if $s = |roots_{num}|$ **then**
$numerator\_done \leftarrow$ true
end if
if $d = |roots_{den}|$ **then**
$denominator\_done \leftarrow$ true
end if
if $numerator\_done \wedge denominator\_done$ **then**
break
end if
$T \leftarrow 2T$
$T\_old \leftarrow T$
$j\_init \leftarrow T\_old + 1$
end while
Recover monomials from roots using trial division
$N \leftarrow$ get_monomial$(roots_{num}, \sigma, n, vars)$
$D \leftarrow$ get_monomial$(roots_{den}, \sigma, n, vars)$
if N = FAIL or D = FAIL **then**
return FAIL
end if
Recover coefficients via Zippel Vandermonde solver
$A \leftarrow$ Zippel_Vandermonde_solver$(num, s, roots_{num}, \Lambda_{num}(z), p)$
$B \leftarrow$ Zippel_Vandermonde_solver$(den, d, roots_{den}, \Lambda_{den}(z), p)$
$ff \leftarrow \sum_{m=1}^{s} A_m N_m, \quad gg \leftarrow \sum_{m=1}^{d} B_m D_m$
Let μ be the leading coefficient of gg in grlex order where $x_1 > \cdots > x_n$.
$ff \leftarrow \mu^{-1} ff$ mod p, $gg \leftarrow \mu^{-1} gg$ mod p.
return ff, gg.

Numerator Denominator Separation Algorithm (NDSA)

Input: Modular black box B for rational function $\frac{ff(x_1,\ldots,x_n)}{gg(x_1,\ldots,x_n)}$ over $\mathbb{Z}_p$, prime p, where $ff, gg \in K[x_1,\ldots,x_n]$, $\gcd(ff,gg) = 1$, $\sigma \in \mathbb{Z}_p^n$, $\beta \in \mathbb{Z}_p^{n-1}$, $num\_points \in \mathbb{N}$.
Output: $f(x) = \frac{ff(x,\beta_2(x-\sigma_1)+\sigma_2,\ldots,\beta_n(x-\sigma_1)+\sigma_n)}{c}$, $g(x) = \frac{gg(x,\beta_2(x-\sigma_1)+\sigma_2,\ldots,\beta_n(x-\sigma_1)+\sigma_n)}{c}$, for some scalar $c \in \mathbb{Z}_p^n$.
$t \leftarrow num\_points$
while true **do**
    Pick random vector $\alpha = [\alpha_1,\ldots,\alpha_t] \in \mathbb{Z}_p^t$
    $m(x) \leftarrow \prod_{k=1}^{t}(x-\alpha_k) \in \mathbb{Z}_p[x]$
    $\Phi \leftarrow [\phi(\alpha_1),\ldots,\phi(\alpha_t)] \in \mathbb{Z}_p^{t\times n}$ such that: $\phi(\alpha_k) \leftarrow [\alpha_k, \beta_2(\alpha_k - \sigma_1)+\sigma_2,\ldots,\beta_n(\alpha_k-\sigma_1)+\sigma_n] \mod p \ \forall\ 1 \le k \le t$
    $Y \leftarrow [B(\Phi(\alpha_1),p),\ldots,B(\Phi(\alpha_t),p)] \in \mathbb{Z}_p^t$
    $u(x) \leftarrow \text{Interpolate}(\alpha, Y, x) \mod p$
    $(f(x), g(x), deg\_q) \leftarrow \text{MQRFR}(m,u) \mod p$
    if $deg\_q > 1$ **then**
        break
    else
        $t \leftarrow 2t$
    end if
end while
return f, g