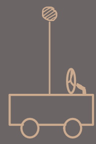
References:
Control Systems- Norman Nise, Linear Systems- T. Kailath
Feedback Control of Dynamic Systems- Franklin, Powell

EE302

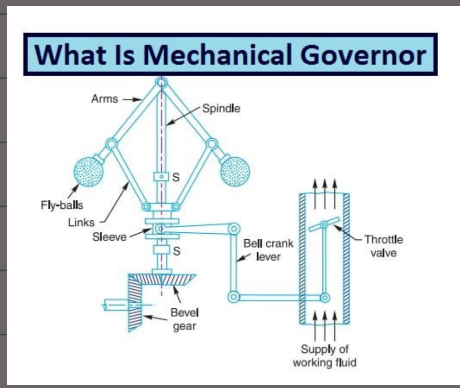


Steps to become Control Guru:

1. Model the system
2. Set performance metrics
3. Check if system meets criteria (Analysis)
4. If not- Decide the type of controller you need
5. Synthesize/Design Controller

Not in this case
6. Carry out simulations
7. Implement controller on system to carry out experiments

Early Control System: In steam engine



Linearity:

if

$u_1 \longrightarrow \boxed{G} \longrightarrow G(u_1) = y_1$

$u_2 \longrightarrow \boxed{G} \longrightarrow G(u_2) = y_2$

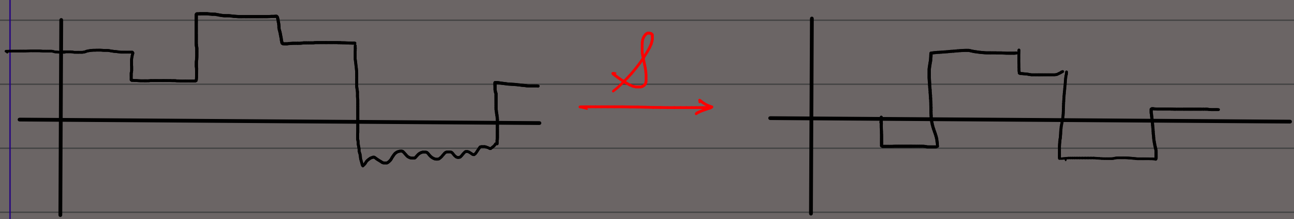
then

$\alpha_1 u_1 + \alpha_2 u_2 \longrightarrow \boxed{G} \longrightarrow G(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 y_1 + \alpha_2 y_2$

If this holds true $\forall \alpha_1, \alpha_2 \in \mathbb{R}$, the system is linear

Note: $y = 5u$, $u = \begin{bmatrix} a \\ 0 \end{bmatrix}$ OR $\begin{bmatrix} 0 \\ b \end{bmatrix}$ $\longrightarrow$ Doesn't follow superposition $\therefore$ Not linear

Q// Is the following linear?
Output only tracks the jump discontinuities by keeping the quantum of jumps



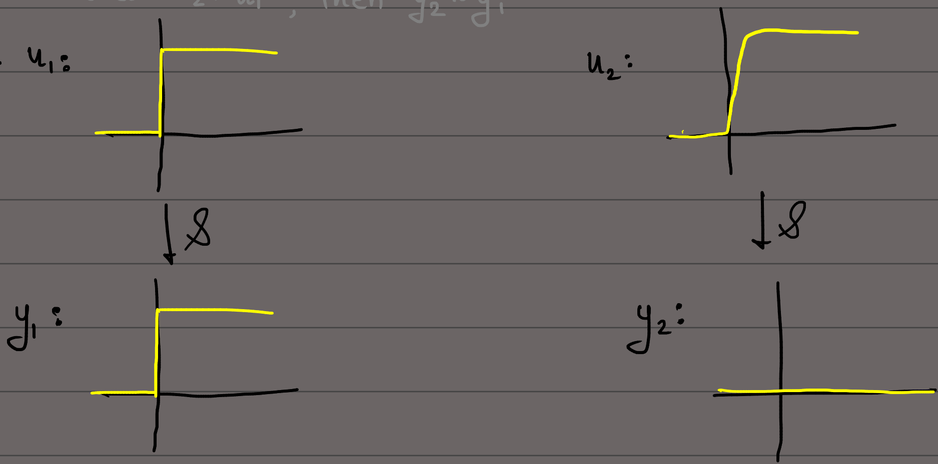
→ Yes. It follows additivity as well as homogeneity.

However : It violates a principle that is usually followed by linear systems →

if $u_1 \xrightarrow{S} y_1$

and $u_2 \approx u_1$, then $y_2 \approx y_1$

e.g.



Convergence :

Let us say that $\hat{u} = \sum_{i=1}^{\infty} \alpha_i u_i$, and we know that $u_i \xrightarrow{S} y_i$ (basis of the system).

Then linearity may make you think that $\hat{y} = \sum_{i=1}^{\infty} \alpha_i y_i$ BUT unless convergence is ensured, we cannot state that.

Static v/s Dynamic Systems

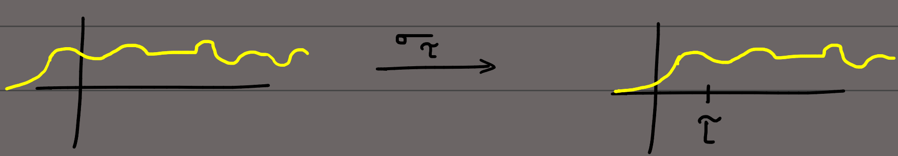
Static Systems → Memoryless, Non-Causal e.g. Ohm's Law $v(t) = R\, i(t)$

Dynamic Systems → Has Memory, Can be Causal e.g. Inductors (Current), Capacitors (Charge)

"No system is static - if transient behaviour / dynamics decay out really quickly, we can approximate a system as static".

Time invariant v/s Time varying

Let us first define shift operator : σ_τ



$$u(t) \longrightarrow \boxed{\sigma_\tau} \xrightarrow{u(t-\tau)} \boxed{G} \longrightarrow$$

$$u(t) \longrightarrow \boxed{G} \xrightarrow{G\{u(t)\}} \boxed{\sigma_\tau} \longrightarrow$$

Same Output

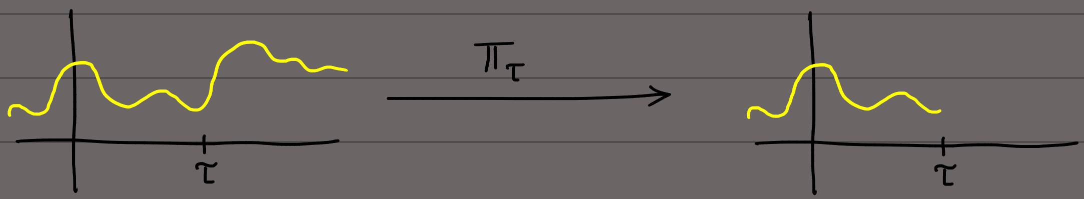
In general, such compositions are not commutative.

If they are – it is time invariant.

Thus mathematically, $\forall \tau$ & $\forall u$, $G\{\sigma_\tau(u(t))\} = \sigma_\tau\left(G\{u(t)\}\right)$ ensures time invariance.

Causal v/s Non-Causal

Let us define another operator: Truncation operator (Π_τ)



Thus mathematically, $\forall u(t)$, $\forall \tau$, if $\Pi_\tau\left[G\left\{\Pi_\tau(u(t))\right\}\right] = \Pi_\tau\left\{G(u(t))\right\}$ $\boxed{\text{OR}}$ $\Pi_\tau G \Pi_\tau = \Pi_\tau G$

Basically truncating input doesn't change anything for all time till $t = \tau$.

$$\longrightarrow \boxed{\Pi_\tau} \longrightarrow \boxed{G} \longrightarrow \boxed{\Pi_\tau} \longrightarrow$$

$$\longrightarrow \boxed{G} \longrightarrow \boxed{\Pi_\tau} \longrightarrow$$

Same output.

Will Time invariance help us?

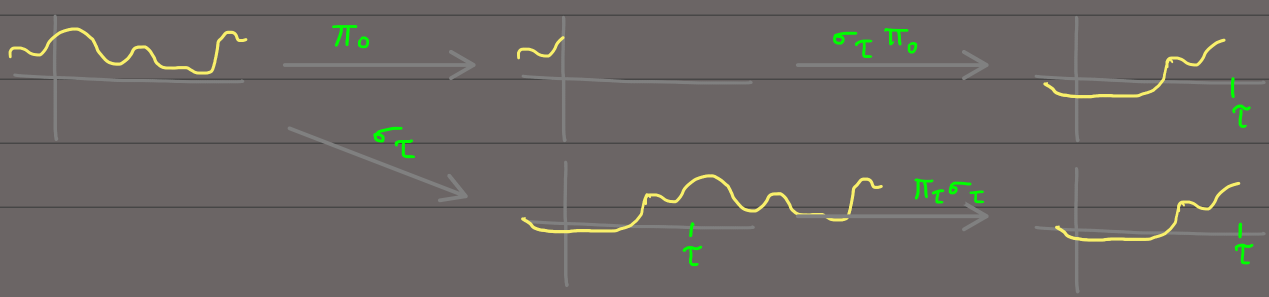
Let $\Pi_\tau\left[G\left\{\Pi_\tau(u(t))\right\}\right] = \Pi_\tau\left\{G(u(t))\right\}$ be true for $\tau = 0$.

$$\therefore \Pi_0 G \Pi_0 = \Pi_0 G$$

$\downarrow$ (we haven't used shift invariance yet)

$$\sigma_\tau \Pi_0 G \Pi_0 = \sigma_\tau \Pi_0 G$$

Claim: $\sigma_\tau \Pi_0 = \Pi_\tau \sigma_\tau$



Thus $\pi_\tau \sigma_\tau G \pi_0 = \pi_\tau \sigma_\tau G \qquad \forall \tau$

$\downarrow$ Time invariance

$$\pi_\tau G \sigma_\tau \pi_0 = \pi_\tau G \sigma_\tau \qquad \forall \tau$$

$\downarrow$ $\sigma_\tau \pi_0 = \pi_\tau \sigma_\tau$

$$\pi_\tau G \pi_\tau \sigma_\tau = \pi_\tau G \sigma_\tau \qquad \forall \tau$$

$\downarrow$ σ_τ is invertible

$$\pi_\tau G \pi_\tau = \pi_\tau G$$

Thus just proving for one causality at a specific time is enough for causality $\forall \tau$ provided system is time-invariant.

Now prove time invariance for -

1. $y(t) = u(3t)$
2. $y(t) = \int_0^t \sqrt{u(\tau)}\, d\tau$
3. $y(t) = \int_{t-3}^{t+3} u^2(t)\, dt$

$\longrightarrow$ 1. Study the property by verifying $\sigma_\tau G = G \sigma_\tau$.

$$\left.\begin{array}{l} Gu(t) = u(3t) \longrightarrow \sigma_\tau G u = u(3t-\tau) \\ \sigma_\tau u(t) = u(t-3) \longrightarrow G\sigma_\tau u = u(3(t-\tau)) \end{array}\right] \text{Not equal} \therefore \text{Time varying}$$

2. $\sigma_\tau G = \sigma_\tau G \rightarrow Gu(t) = \int_0^t \sqrt{u(m)}\, dm \longrightarrow \sigma_\tau G u(t) = \int_0^{t-\tau} \sqrt{u(m)}\, dm$

$\sigma_\tau u(t) = u(m-\tau) \longrightarrow G\sigma_\tau u(t) = \int_0^t \sqrt{u(m-\tau)}\, dm$

$$= \int_{-\tau}^{t-\tau} \sqrt{u(m)}\, dm$$

Time varying

3. $Gu(t) = \int_{t-3}^{t+3} u^2(m)\, dm \longrightarrow \sigma_\tau G u = \int_{t-3-\tau}^{t+3-\tau} u^2(m)\, dm$

$\sigma_\tau(u(t)) = u(m-\tau) \longrightarrow G\sigma_\tau u = \int_{t-3}^{t+3} u^2(m-\tau)\, dm = \int_{t-3-\tau}^{t+3-\tau} u^2(m)\, dm$

Time invariant.

Is truncation operator commutative? → If so : $\pi_\tau G = G\pi_\tau$
↳ This means that if input ceases to exist, so does the output ⇒ Thus G has to be static
(Doubt: Will it be linear too? Dk.)

Discrete time v/s Continuous time

Ya know what it is

Deterministic v/s Stochastic

↳ Repeatedly same input will give us the same exact output at some other instant.

All systems are stochastic to some extent.

Stochastic → Same input – Different outputs. Mostly because noise in the system we can't predict – we can only characterise its μ, σ^2 etc...

Different from Time invariance : We simply have same intial conditions but noise cannot be taken care of.

- For linear, dynamic, causal, ..., systems, our differential equations always look like –

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

Notice how the input can only get derivatives upto the highest derivative of output.

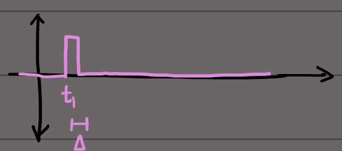
Reason : If the higher derivative of the input is higher order than that of output, you would lose causality. Now why does Causality get violated – there is a very muddled approach. Let it be.

How to prove – **Linearity :** If y_1, y_2 are solutions for u_1, u_2 respectively, we can see that $\hat{y} = \alpha_1 y_1 + \alpha_2 y_2$ is the soln for $\hat{u} = \alpha_1 u_1 + \alpha_2 u_2$

Dynamic : It's a differential eqn ∴ Definitely won't die off $\bar{\omega}$ input = 0

Causal : Depends on how we choose soln from the bases.

Pulse



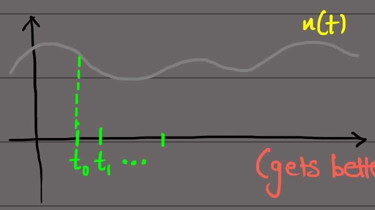
∴ Pulse ⇒ $1(t-t_1) - 1(t-t_1-\Delta)$

↓ Laplace

$$\mathcal{L}(\text{Pulse}) = e^{-st_1} - e^{-s(t_1+\Delta)}$$

Now area of pulse is Δ. So to give it area of 1, we give it an amplitude of $1/\Delta$.

$\therefore\ \delta(t)$: Dirac Delta $= \lim_{\Delta \to 0} \frac{1}{\Delta}\left(1(t) - 1(t-\Delta)\right) \xrightarrow{\mathcal{L}} \mathcal{L}(\delta(t)) = \frac{1 - e^{-s\Delta}}{s\Delta} = 1$



$$\tilde{u}(t) = \sum_{k=0}^{\infty} u(t_k)\underbrace{\left[1(t-t_k) - 1(t-t_k-\Delta)\right]}_{\delta_\Delta(t-t_k) \times \Delta}$$

approximation (gets better and better $\bar{w}$ $\Delta \to 0$)

[here $\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$]

$$= \Delta \sum_{k=0}^{\infty} u(t_k)\ \delta_\Delta(t-t_k)$$

If $y(t) = G(u(t))$ and $\tilde{y}(t) = G(\tilde{u}(t)) \to$ (We will use $\tilde{y}$ to find y by making $\tilde{u} \approx u$; thus we need smoothness to say that).

$$\tilde{y}(t) = G\left(\sum_{k=0}^{\infty} u(t_k)\,\delta_\Delta(t-t_k)\Delta\right)$$ (use Linearity)

$$= \sum_{k=0}^{\infty} G(\delta_\Delta(t-t_k)) \cdot u(t_k)\Delta$$ (pass through $\lim_{\Delta \to 0}$ + smoothness $\therefore$ $\tilde{u} \to u \Rightarrow \tilde{y} \to y$)

$$y(t) = \lim_{\Delta \to 0} \sum_{k=0}^{\infty} G\left(\delta_\Delta(t-t_k)\right)\Delta\, u(t_k)$$

(Implicitly, we also took time invariance in this whole ordeal because $G(\delta(t-\tau))$ is a function of $(t-\tau)$ and not both t & τ)

$$= \int_0^{\infty} G\left(\delta(t-\tau)\right) u(\tau)\, d\tau$$

$$= \int_0^{\infty} \underbrace{g(t-\tau)}_{\text{Impulse response}} u(\tau)\, d\tau$$

Laplace Transform

$$\mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = F(s)$$

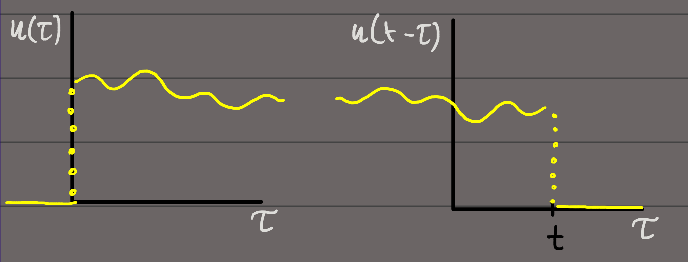
0^-] why is this important? Find out soon.

Let's consider $y(t) = \int_{0^-}^{\infty} g(t-\tau)\, u(\tau) dt$ (using causality) $\rightarrow$

$$= \int_{0^-}^{t} g(t-\tau)\, u(\tau)\, d\tau$$

$$= \int_{0^-}^{t} g(\tau)\, u(t-\tau)\, d\tau$$

Let us put another assumption: System and i/p were initially relaxed. (i.e. 0 before $t=0$)



Thus $u(t-\tau)=0 \ \forall \tau > t \rightarrow$

$$\therefore \int_{0^-}^{t} g(\tau)\,u(t-\tau)\,d\tau = \int_{0^-}^{\infty} g(\tau)\,u(t-\tau)\,d\tau$$

$$\therefore Y(s) = \mathcal{L}\left[\int_{0^-}^{\infty} g(\tau)\,u(t-\tau)\,d\tau\right] = \int_{0^-}^{\infty}\int_{0^-}^{\infty} g(\tau)\,u(t-\tau)\,e^{-st}\,d\tau\,dt$$

$$= \int_{0^-}^{\infty} e^{-s\tau} g(\tau) \int_{0^-}^{\infty} u(t-\tau)\,e^{-s(t-\tau)}\,dt\,d\tau \qquad [\text{Let } \gamma = t-\tau]$$

$$= \int_{0^-}^{\infty} e^{-s\tau} g(\tau) \int_{-\tau}^{\infty} u(\gamma)\,e^{-s\gamma}\,d\gamma\,d\tau \qquad [u(\alpha)=0 \ \forall \alpha<0]$$

$$= \int_{0^-}^{\infty} e^{-s\tau} g(\tau) \int_{0^-}^{\infty} u(\gamma)\,e^{-s\gamma}\,d\gamma\,d\tau$$

$$= \int_{0^-}^{\infty} e^{-s\tau} g(\tau)\,d\tau \cdot U(s) = G(s)\cdot U(s)$$

$\because g(t)=0 \ \forall t<0$

$\therefore Y(s) = G(s)\cdot U(s)$ [for causal $g(t)$ $\bar{w}$ $u(t)=0 \ \forall t<0$]

Some properties:

- Linearity $\rightarrow \mathcal{L}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{L}(f_1) + \alpha_2 \mathcal{L}(f_2)$
- Delayed $\rightarrow \mathcal{L}(f(t-t_0)) = e^{-st_0}\cdot \mathcal{L}(f(t)) = \mathcal{L}(\sigma_{t_0}(f(t)))$
- Derivative $\rightarrow \mathcal{L}\left(\frac{d}{dt} f(t)\right) = sF(s) - f(0^-)$
  - $f(0^+)$ if we defined $\mathcal{L}: \int_{0^+}^{\infty}$
- Integral $\rightarrow \mathcal{L}\left(\int_0^t f(t)\,dt\right) = \frac{F(s)}{s}$

IVT: $f(0^+) = \lim_{s\to\infty} sF(s)$ Even if you defined $\mathcal{L}: \int_{0^+}^{\infty}$, you will still only be able to find $f(0^+)$

Proof: $\lim_{s\to\infty} \mathcal{L}\left(\frac{d}{dt} f\right) = \int_{0^-}^{0^+} e^{-st} \frac{df}{dt}\, dt + \int_{0^+}^{\infty} e^{-st} \frac{df}{dt}\, dt = \lim_{s\to\infty} sF(s) - f(0^-)$

$$\lim_{s\to\infty} sF(s) - f(0^-) = \int_{0^-}^{0^+} 1 \cdot \frac{df}{dt}\, dt + \int_{0^+}^{\infty} 0 \cdot \frac{df}{dt}\, dt$$

$$\lim_{s\to\infty} sF(s) - \cancel{f(0^-)} = f(0^+) - \cancel{f(0^-)}$$

- **Convolution and product:** $(f * g)(t) := \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$, $\mathfrak{L}(f * g) = F(s)G(s)$
- **Dirac delta:** $\delta * f = f$ and $\mathfrak{L}(\delta) = 1$
- **IVT:** $f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$ **(provided LHS exists, i.e. no impulses/their derivatives at $t = 0$.)**
- **FVT:** $f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ **(provided LHS exists, i.e. f neither diverges, nor oscillates)**
- **Time multiplication** $\mathfrak{L}(tf(t)) = -\frac{d}{ds}F(s)$
- **Complex shift:** $\mathfrak{L}(e^{at}f(t)) = F(s-a)$
- **Time scaling:** $\mathfrak{L}(f(\frac{t}{a})) = aF(as)$

- $\mathfrak{L}(1) = \frac{1}{s}$ **(note: functions are only on $[0, \infty)$)**
- $\mathfrak{L}(t) = \frac{1}{s^2}$
- $\mathfrak{L}(e^{at}) = \dfrac{1}{s-a}$
- $\mathfrak{L}(\sin(\omega t)) = \dfrac{\omega}{s^2+\omega^2}$ **and** $L(\cos(\omega t)) = \dfrac{s}{s^2+\omega^2}$ **(Use IVT to be sure of which is of which.)**
- $\mathfrak{L}(e^{at}\sin(\omega t)) = \dfrac{\omega}{(s-a)^2+\omega^2}$

Systems

Why is the 0^- used instead of 0^+?

Take $\dot{x} + x = \delta(t)$, $x(0^-) = 1$. Obtain $x(t)\ \forall t \geq 0$

Let us consider $x(0^+)$: $\dot{x} + x = 0$

But we don't know this value

$\hookrightarrow x(t) = e^{-t} \cdot x(0^+)$

So how do we continue? Well RHS has an impulse, so should LHS. Now if $x(t)$ is also $\delta(t)$ then $\dot{x}$ would be a doublet ∴ Not feasible ∴ $x(t) \Rightarrow$ step $\rightarrow \dot{x} = \delta(t)$

We needed to make this observatn to solve. Solutn didn't come naturally.

So that means $x(0^+) - x(0^-) = 1 \longrightarrow x(0^-) = 1 \longrightarrow x(0^+) = 1+1 = 2$

Now $\mathcal{L}_+$: $\int_{0^+}^{\infty} e^{-st} f(t)\,dt \Rightarrow$ called unilateral Laplace transform

$\mathcal{L}(\delta(t)) = 1$ $\boxed{\text{BUT}}$ $\mathcal{L}_+(\delta(t)) = 0$

If we used $\mathcal{L}_+$ for solving this $\rightarrow$ $sX(s) - \underbrace{x(0^+)}_{\text{we don't know this}} + X(s) = 0$

∴ Again we are stuck.

∴ Same issue as classical differential method.

Now if we used $\mathcal{L}(s) \longrightarrow sX(s) - x(0^-) + X(s) = 1 \longrightarrow X(s) = \dfrac{x(0^-)+1}{s+1}$

$$\boxed{x(t) = 2e^{-t} \cdot 1(t) \quad \forall t \geq 0}$$

And thus we use 0^- in $\mathcal{L}$ as we usually know initial conditions.

Also if you somehow knew x(0+) in some hypothetical case, then using Unilateral Laplace Transform would give the right result as we don't need to be able to predict anything about the x(0-).

Now if you have x(0-) and are forced to use the unilateral one, you can only work if you make the assumption that there is no discontinuity at x=0 thereby allowing you to use x(0+)=x(0-)

Rational functions:

$G(s) = \dfrac{N(s)}{D(s)}$ ← Polynomials

$\deg(N) = n$, $\deg(M) = m$

$n < m$: proper

$n = m$: biproper

$n > m$: improper

Take → $\dddot{y}+a_2\ddot{y}+a_1\dot{y}+a_0y=b_1\dot{u}+b_0u$, $y(0^-)=\alpha$, $\dot{y}(0^-)=\beta$, $\ddot{y}(0^-)=\gamma$

↓

① Transfer function (assume initial conditions (at 0^-) as 0)

$$\Rightarrow G(s)=\frac{b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}$$

② Solution of a linear ODE (constant coefficients) = $\underbrace{Z.S.\ sol^n}_{\text{zero state solution}}+\underbrace{Z.I.\ sol^n}_{\text{zero input sol}^n}$

Obtain $U(s)$

③ $\mathcal{L}^{-1}\{U(s).G(s)\}=y(t)$: Zero State Solution

④ Now calculate the zero input solution → $u\equiv 0$

$$\therefore\ s^3Y(s)-s^2y(0^-)-s\dot{y}(0^-)-\ddot{y}(0^-)+a_2(s^2Y(s)-sy(0^-)-\dot{y}(0^-))$$
$$+a_1(sY(s)-y(0^-))+a_0Y(s)=0$$: Zero Input Solution

Zeros of Transfer function: $\{z\in\mathbb{C}\ ;\ N(z)=0\}$

Poles of Transfer function: $\{z\in\mathbb{C}\ ;\ D(z)=0\}$

where $G(s)=\frac{N(s)}{D(s)}$

e.g. $\ddot{y}+3\dot{y}+2y=\dot{u}$, $y(0^-)=0$, $\dot{y}(0^-)=-5$, $u(t)=10e^{-3t}\,1(t)$

→ $\{s^2Y(s)-sy(0^-)-\dot{y}(0^-)\}+3\{sY(s)-y(0^-)\}+2Y(s)=\frac{10s}{s+3}$ ($u(0^-)=0$ as step function)

$$Y(s)\{s^2+3s+2\}+5=\frac{10s}{s+3}$$

↓

$$Y(s)=\underbrace{\frac{-5}{(s+1)(s+2)}}_{\text{Zero input solution}}+\underbrace{\frac{10s}{(s+1)(s+2)(s+3)}}_{\text{Zero State solution}}$$

Notice how this is just $G(s)\cdot U(s)$

$$\frac{s}{(s+1)(s+2)}\cdot\frac{10}{s+3}$$

$$Y(s) = \left[\frac{a}{s+2} + \frac{b}{s+3}\right] + \left[\frac{c}{s+1} + \frac{d}{s+2} + \frac{e}{s+3}\right]$$

How to easily find a, b, ... e easily?

$$\frac{N(s)}{(s-p_1)^{r_1}(s-p_2)^{r_2}\cdots(s-p_k)^{r_k}} = \left(\frac{a_{11}}{s-p_1} + \frac{a_{12}}{(s-p_1)^2} + \cdots + \frac{a_{1r_1}}{(s-p_1)^{r_1}}\right) + \left(\frac{a_{21}}{(s-p_2)} + \cdots\right)$$

$$\therefore N(s) = \left\{\left(\frac{a_{11}}{s-p_1} + \frac{a_{12}}{(s-p_1)^2} + \cdots + \frac{a_{1r_1}}{(s-p_1)^{r_1}}\right) + \left(\frac{a_{21}}{(s-p_2)} + \cdots\right)\right\} (s-p_1)^{r_1}(s-p_2)^{r_2}\cdots(s-p_k)^{r_k}$$

$\downarrow$

To find a_{ir_i}, calculate just $N(p_i)$.

What if poles are $\mathbb{C}$ for a rational system? Wouldn't coefficients be $\mathbb{C}$ as well?

$$\frac{ps+q}{(s+a)^2+b^2} = \frac{k_1+jk_2}{s-p_1} + \frac{k_3+jk_4}{s-p_1^*}$$ $\longrightarrow$ We might be scared that we have 4 unknowns, but $k_1 = k_3$ and $k_2 = -k_4$ (else solution would $\in \mathbb{C}$)

$$(ps+q) = (k_1+jk_2)(s-p_1^*) + (k_3+jk_4)(s-p_1)$$

$$p, q, a, b \in \mathbb{R} \therefore p = k_1 + k_2 \Rightarrow k_2 = -k_4$$

Solve a few qs from Norman Nise

$$Y(s) = \left[\frac{-5}{s+1} + \frac{5}{s+2}\right] + \left[\frac{-15}{s+3} - \frac{5}{s+1} + \frac{20}{s+2}\right]$$

$10s = c(s^2+3s+2)$
$+d(s^2+5s+6)$
$+e(s^2+4s+3)$

$$y(t) = \underbrace{\left\{-5e^{-t} + 5e^{-2t}\right\}}_{\text{zero input}} + \underbrace{\left\{-15e^{-3t} - 5e^{-t} + 20e^{-2t}\right\}}_{\text{zero state}} \quad \forall t \geq 0$$

$y(0) = 0 + 0$ ✓

$\dot{y}(0) = 5 - 10 + 45 + 5 - 40 \neq -5$ Why doesn't this match our initial condition of $\dot{y}(0^-) = -5$?

Well the zero state solution is Fucking us up. It considers the output only when $u(t)$ is applied. At $t = 0^-$, there was no input applied thus the zero input solution works.

zero solution$'(0) = 5 - 10 = -5$ ✓ matches $\bar{w}$ $y(0^-)$.

Note : We can only solve causal, relaxed systems ∴ $u(0^-) = 0$ is compulsory else we can't solve those systems $\bar{w}$ this.

Now do you remember natural and forced response methods we had learnt in 113? The natural response is derived from just the poles → Thus it would be the terms we get on adding up e^{-t}, e^{-2t} terms above ∴ $y_{Natural}(t) = -10e^{-t} + 25e^{-2t}$

$$\therefore y_{forced} = -15e^{-3t}.$$

- $y(t) = \int_{0^-}^{t} g(\tau)u(t-\tau)\,d\tau$. Take $u(\tau) = e^{\alpha\tau}\cdot 1(t)$

$$= \int_{0^-}^{\infty} g(\tau)e^{\alpha t}\cdot e^{-\alpha\tau}\,d\tau = e^{\alpha t}\int_{0^-}^{\infty} e^{-\alpha\tau} g(\tau)\,d\tau = G(\alpha)\cdot e^{\alpha t}$$

$$u(t) = e^{\alpha t}\cdot 1(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t) = G(\alpha)e^{\alpha t}\cdot 1(t)$$

e.g. $G(s) = \dfrac{2se^{-5s} + 3se^{-2s} + s}{(s+1)(s+2)(s+3)} = e^{-5s}\left\{\dfrac{s}{(s+1)(s+2)(s+3)}\right\} + e^{-2s}\left\{\dfrac{3s}{(s+1)(s+2)(s+3)}\right\} + \dfrac{s}{(s+1)(s+2)(s+3)}$

If $Y(s) = G(s)\,U(s) \longrightarrow y(t) = \mathcal{L}^{-1}\{G(s).U(s)\}$

If $G(s)$ → biproper and $u(t)$ → bounded i.e. $U(s)$ → proper, then $G(s)U(s)$ = proper ∴ roots can be $e^{p_i t}$ $\boxed{\text{OR}}$ If repeated, $\dfrac{t^{k-1}}{k!}e^{p_i t}$ $\left[\text{i.e. } G(s)U(s) \text{ contains } \dfrac{1}{(s-p_i)^{r_i}}\right]$ $k \in [0, r_i]$

$$G(s).U(s) = \frac{N(s)}{D(s)}.$$

Dependence on roots of D(s)

- How would repeated roots look in the above diagram?

→ Polynomials are less "powerful" in deciding trajectory for higher t in comparison to exponentials. However, in the lower t (or transient time), it can help create maximas/minimas in linear combination $\bar{w}$ normal exponentials (i.e. $c_1 e^{p_i t} + c_2 t e^{p_i t} \xrightarrow{d/dt} (\underbrace{p_i c_1 + c_2 + c_2 p_i}_{\text{no roots if } c_2 = 0.}) e^{p_i t} = 0$)

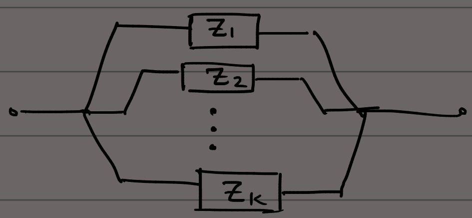
Thus for repeated poles, qualitatively there is no difference at $t >>$ $\boxed{\text{HOWEVER}}$ roots on imaginary axis blow up $\bar{w}$ repeated roots. (as there is no exponential to dominate)

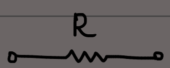
Analysis of electrical ckts

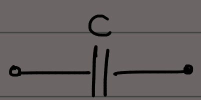
$Z_{eq} = \sum Z_k$; $Y_{eq} = \dfrac{1}{\sum 1/Y_k}$

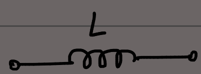
↑ conductance ↓

$Z_{eq} = \dfrac{1}{\sum 1/Z_k}$; $Y_{eq} = \sum Y_k$







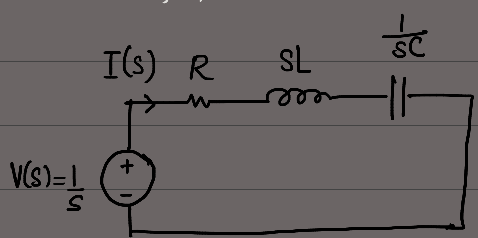


$V = iR \quad\downarrow\quad V(s) = I(s)R$

$i = c\dfrac{dV}{dt} \quad\downarrow\quad V(s) = \dfrac{I(s)}{sC}$

$V = L\dfrac{di}{dt} \quad\downarrow\quad V(s) = sL\,I(s)$

- How would R, C, L in series behave to unit step voltage applied at $t=0$?



$\longrightarrow I(s)\left[R + sL + \dfrac{1}{sC}\right] = \dfrac{1}{s} \longrightarrow I(s) = \dfrac{1}{s^2L + sR + 1/C}$

$= \dfrac{1/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$

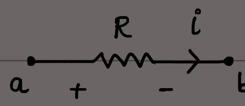
$$I(s) = \frac{1/L}{\left(s+\frac{R}{2L}\right)^2 + \frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}{\left(s+\frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} \cdot \frac{1}{L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

$$= \underbrace{\frac{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}{\left(s+\frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}}_{\omega/s^2+\omega^2} \cdot \frac{1}{\sqrt{\frac{L}{C} - \frac{R^2}{4}}}$$

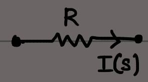
$$\therefore i(t) = \frac{1}{\sqrt{\frac{L}{C} - \frac{R^2}{4}}} e^{-\frac{R}{2L}t} \sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t\right)$$

This method is good enough for no initial conditions. But it is zero state solution.

How to factor in the zero input solution?

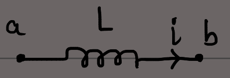


$\Longleftrightarrow$



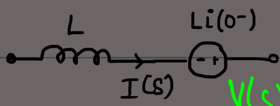
$V(s) = I(s)\,R$

We don't need to factor in intial conditions as ideal resistance change V_{ab}, i_R instantaneously.



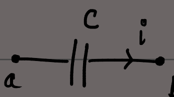
$\Longleftrightarrow$

$v = L\frac{di}{dt}$



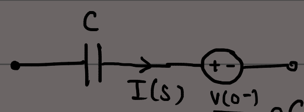
← You can use this (due to this intuition from here) to simplify your logic

$V(s) = sLI(s) - Li(0^-)$



$\Longleftrightarrow$

$C\frac{dv}{dt} = i$

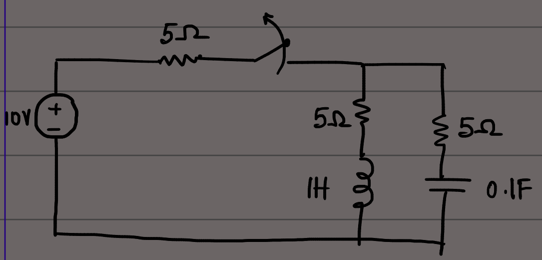


$sCV(s) - Cv(0^-) = I(s)$

$\downarrow$

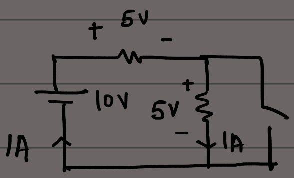
$V(s) = \frac{1}{sC} I(s) + \frac{1}{s} v(0^-)$

Q11

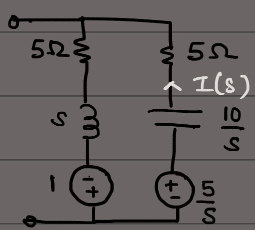


At $t=0$, the switch is opened. Characterise the ckt.

→ At $t=0^-$, we have steady state →



$\therefore i_L(0^-) = 1A$, $v_C(0^-) = 5V$ ⟶

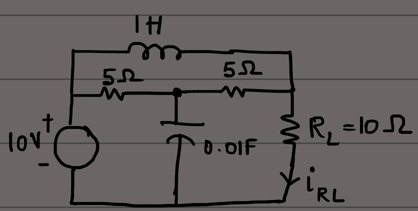


$\therefore I(s)\left(5 + 5 + \frac{10}{s} + s\right) = \frac{5}{s} + 1$

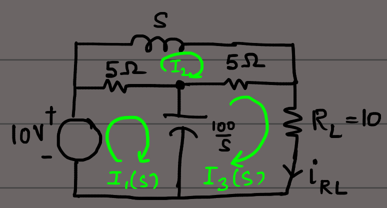
$\downarrow$

$$I(s) = \frac{s+5}{s^2 + 10s + 10}$$

Q11



Find equation between $V_{in}(s)$ and $I_{R_L}(s)$



KVL Equations

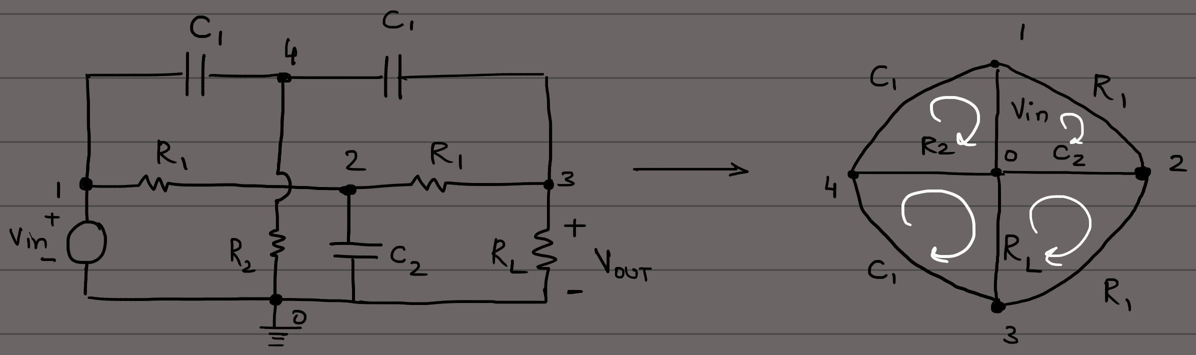
$$\begin{bmatrix} V(s) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \left(\frac{100}{s}+5\right) & -5 & -\frac{100}{s} \\ -5 & (10+s) & -5 \\ -\frac{100}{s} & -5 & \left(15+\frac{100}{s}\right) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix}$$

$\downarrow$

Use Cramer's Rule

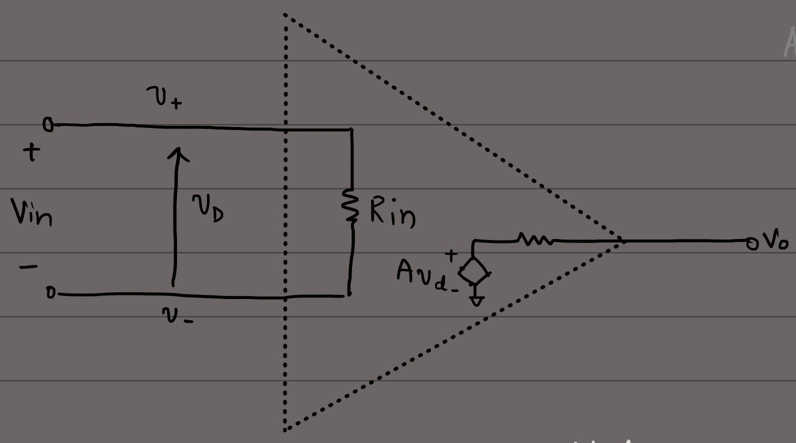
$\boxed{\text{OR}}$ You can make KCL equations like usual as well.

Let us simplify the twin-T ckt below :



We already discussed that passive elements can only create ckts $\overline{w}$ proper transfer functions. (because V_{OUT} cannot have deg(N) > deg(D). Try any such ckt but you'll never get an improper transfer function of voltages).

Thus active elements such as op-amps are needed to make some desired ODEs.

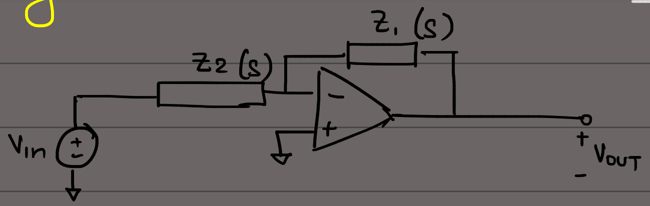


Assumptions:

$A \to \infty$, $R_{in} \to \infty$, $R_o \to 0$

We assume $A_v \longrightarrow \infty$ $\therefore v_o = A_v(v_+ - v_-) \xRightarrow[\text{shorting}]{\text{virtual}} v_+ = v_-$

Inverting mode

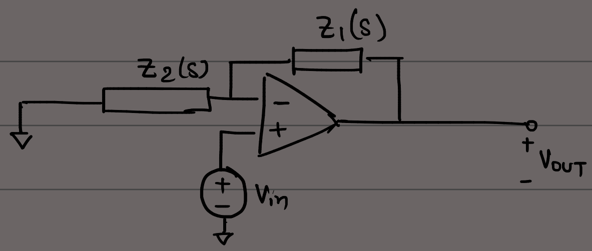


$$\frac{V_i - 0}{Z_2} + \frac{V_o - 0}{Z_1} = 0$$

$$\downarrow$$

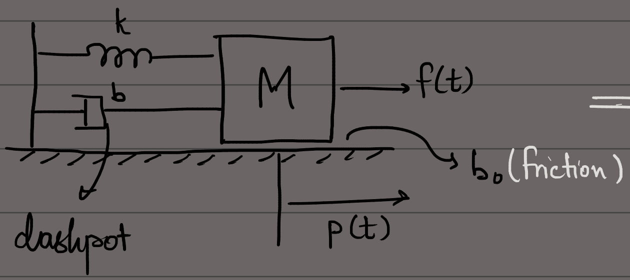
$$\frac{V_o}{V_i} = -\frac{Z_1}{Z_2}(s)$$

Non inverting mode

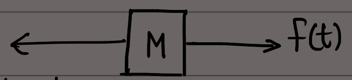


$$\frac{V_o}{V_i} = \frac{Z_2(s) + Z_1(s)}{Z_2(s)}$$

Translational Mechanical Systems



$\Longrightarrow$



$M\ddot{p} + kp + (b+b_o)\dot{p}$

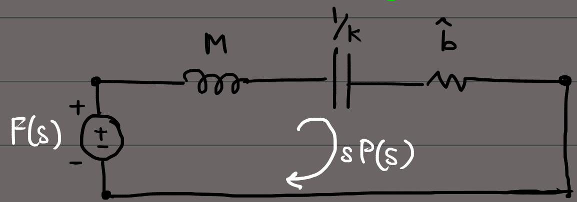
$$M\ddot{p}(t) + (b+b_o)\dot{p}(t) + kp(t) = f(t)$$

$$(Ms^2 + \hat{b}s + k)\,P(s) = F(s)$$

$$sP(s)\left[Ms + \hat{b} + \frac{k}{s}\right] = F(s)$$

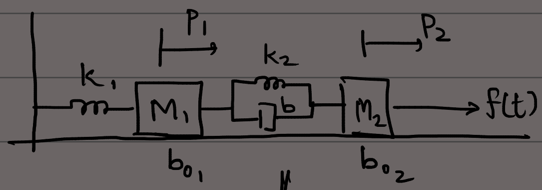
$\downarrow$ elec

$$I(s)\left[sL + R + \frac{1}{sC}\right] = V(s)$$



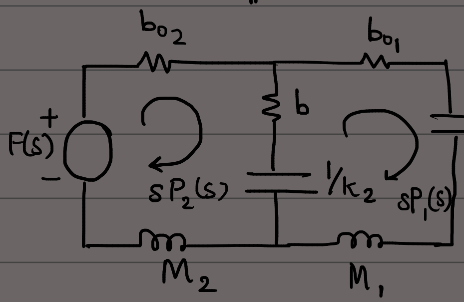
Series equivalent

How was studying this useful? we will now be able to study multiple masses easily.



series equivalent

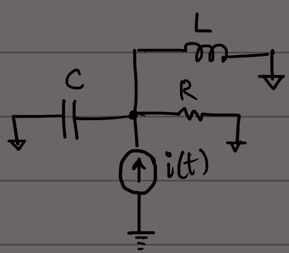
(from a mechanical system, we can always create an electrical system but an electrical system cannot definitely be converted.
e.g. inductor at a shared edge cannot be translated to a mass exactly.
Google search inerter if you wanna know more) http://bit.ly/3GFTPeZ



- Why did we call it series equivalent? Does that imply existence of parallel equivalent?

Parallel equivalent

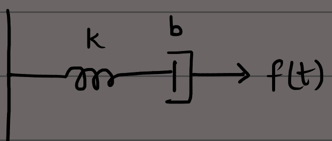
We use that to conserve the physical perspective of mechanical systems. Notice how we made dashpot and spring in series although in reality they are in parallel. Thus we now think of F(s) as current.



$$I(s) = V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right)$$

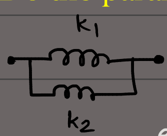
$$\Downarrow$$

$$F(s) = sP(s)\left(\hat{b} + \frac{k}{s} + Ms\right)$$



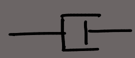
$$F(s)\left(\frac{1}{k} + \frac{1}{bs}\right) = P(s) \longrightarrow \frac{P(s)}{F(s)} = \left(\frac{1}{k} + \frac{1}{bs}\right).$$

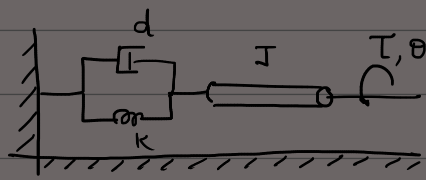
Do the parallel equivalent bullshit for the earlier difficult example as well.

(Like  here $\to$ ΔP is same for both $\therefore$ like voltage for parallel $\therefore$ Parallel eq. is nice)

But for masses,  ,  are both same despite being diff configurations. Infact ΔP across m isn't allowed (due to rigidity) thus this is one setback of m.

Rotational Mechanical Systems

 $J\ddot{\theta} = \tau$  $d\dot{\theta} = \tau$  $k\theta = \tau$

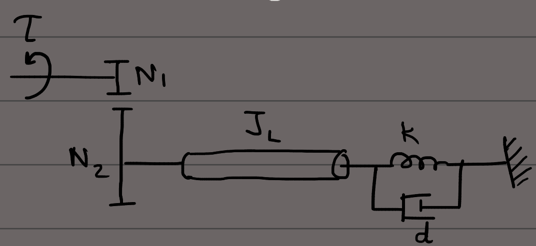


$$\tau(s) = \left(Js + d + \frac{k}{s}\right) s\,\theta(s)$$

Series eq. $V(s) = I(s)\left(Ls + R + \frac{1}{sC}\right)$

Parallel eq. $I(s) = V(s)\left(\frac{1}{sL} + \frac{1}{R} + sC\right)$

Now if we actually joined a gear of teeth N_1 to the end of the rod and tether a gear of teeth N_2 to it w̄ another system.



Now since tethering is needed, the tooth widths are the same. $\therefore \frac{N_1}{N_2} = \frac{r_1}{r_2}$. To ensure velocity at pt of contact is same $\rightarrow \frac{r_1}{r_2} = \frac{w_2}{w_1} \longrightarrow \frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{w_2}{w_1}$

Now power conservation $\therefore$ $\tau_1 w_1 = \tau_2 w_2 \longrightarrow \frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{w_2}{w_1} = \frac{\tau_1}{\tau_2}$

Thus this is a transformer in electrical domain.

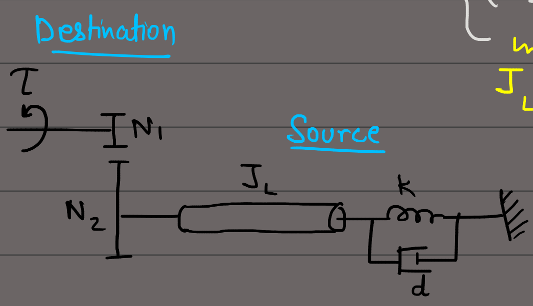
Let us say I want a transfer fxn b/w $\tau_1(s)$ and $\theta_1(s)$ above.

Then $\tau_2(s) = \frac{N_2}{N_1}\tau_1(s)$

$$\tau_2(s) = (J_L s^2 + k + ds)\,\theta_2(s)$$

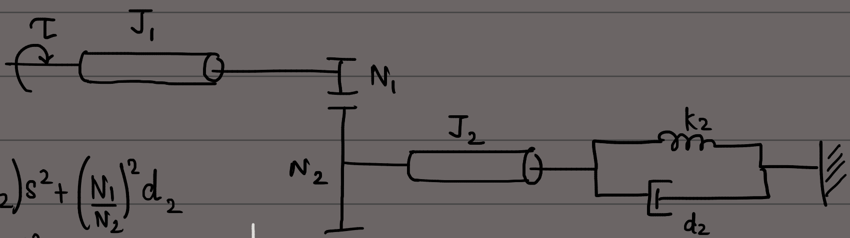
$$\frac{N_2}{N_1}\tau_1(s) = (J_L s^2 + k + ds)\,\frac{N_1}{N_2}\,\theta_1(s)$$

$$\therefore\ T_1(\theta) = \left\{ \underbrace{(J_L)\left(\frac{N_1}{N_2}\right)^2}_{J_{L,eq.}} s^2 + \underbrace{d\left(\frac{N_1}{N_2}\right)^2}_{d_{eq.}} s + \underbrace{k\left(\frac{N_1}{N_2}\right)^2}_{k_{eq.}} \right\} \theta_1(s)$$



$$\text{Hence } Z(s) = \left(\frac{N_{destination}}{N_{source}}\right)^2$$

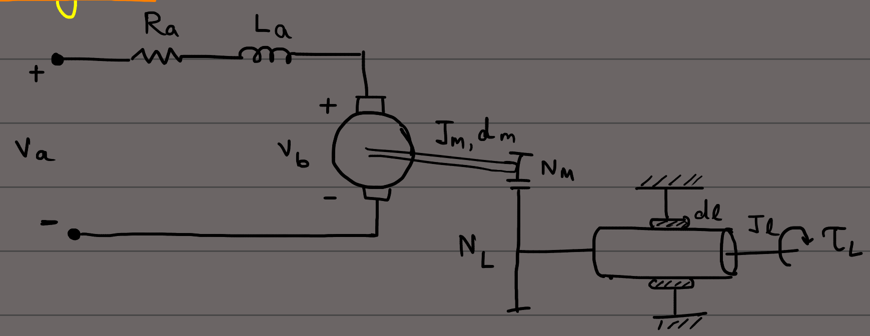
∴



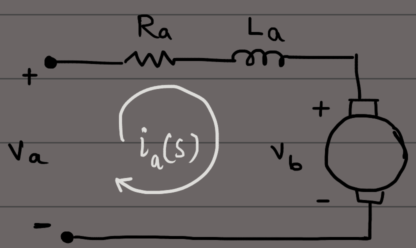
$$T(s) = \left\{ \left(J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2\right)s^2 + \left(\frac{N_1}{N_2}\right)^2 d_2 + \left(\frac{N_1}{N_2}\right)^2 k_2 \right\} \theta_2(s)$$

It's better to read a bit from Norman Nise. Prof said something about parallel dashpots.

Electro-Mechanical System (Armature Control of DC Motor)



We want a transfer function that relates $V_a(s)$ to $\theta_L(s)$ OR $\Omega_L(s)$ $(w(t) \rightarrow \Omega(s))$



$$V_a(s) = I_a(s)\left[R_a + sL_a\right] + V_b(s) \quad \text{—— (1)}$$

$$V_b(s) = k_b\,\Omega_M(s) \quad (\text{basically } v_b(t) \propto w(t)) \quad \text{—— (2)}$$

$$T_L(s) = \left\{ \left(J_L + J_M\left(\frac{N_L}{N_M}\right)^2\right)s^2 + \left(d_L + d_M\left(\frac{N_L}{N_M}\right)^2\right)s \right\} \theta_L(s) \quad \text{—— (3)}$$

$$\tau_M \propto \phi_{armature} \cdot \underbrace{\phi_{field}}_{\text{constant}} \qquad \therefore \tau_M \propto i_a \text{ (reasonable approximation)}$$

$$\therefore \tau_M(s) = k_a \cdot I_a(s) \longrightarrow T_L(s) = k_a' \cdot I_a(s) \quad \text{--- (4)}$$

$$(1),(2),(4): \; V_a(s) = (R_a + sL_a) \cdot \frac{T_L(s)}{k_a'} + k_b \cdot s\theta_L(s) \left(\frac{N_L}{N_M}\right)$$

$$\text{Now use (4): } V_a(s) = \frac{(R_a + sL_a)}{k_a'} \left\{ \left(J_L + J_M\left(\frac{N_L}{N_M}\right)^2\right)s^2 + \left(d_L + d_m\left(\frac{N_L}{N_M}\right)^2\right)s \right\} \theta_L(s) + k_b s \frac{N_L}{N_M} \theta_L(s)$$

$$\therefore V_a(s) = \left[\frac{(R_a + sL_a)}{K_a'} \left\{ \left(J_L + J_M\left(\frac{N_L}{N_M}\right)^2\right)s + \left(d_L + d_M\left(\frac{N_L}{N_M}\right)^2\right) \right\} + k_b \frac{N_L}{N_M} \right] \Omega(s)$$

$$\frac{\Omega(s)}{V_a(s)} = \frac{1}{\left[\frac{(R_a + sL_a)}{K_a'} \left\{ \left(J_L + J_M\left(\frac{N_L}{N_M}\right)^2\right)s + \left(d_L + d_M\left(\frac{N_L}{N_M}\right)^2\right) \right\} + k_b \frac{N_L}{N_M} \right]}$$

Now we know that in DC Machines, $L_a \ll R_a$ $\therefore$ We can write $R_a\left(1 + \frac{sL_a}{R_a}\right) \longrightarrow R_a$

Thus this decreases degree of denominator (Thus we take assumptth of fast decay of dynamics)

$$\frac{\Omega(s)}{V_a(s)} = \frac{1}{\frac{R_a}{K_a'} \left\{ (J_{eq.})s + (d_{eq.}) \right\} + k_b \frac{N_L}{N_M}}$$

$$= \frac{1}{\frac{J_{eq} \cdot R_a}{K_a'} \left\{ s + \frac{d_{eq.}}{J_{eq.}} + \frac{k_b (N_L/N_M) \cdot K_a'}{J_{eq} \cdot R} \right\}}$$

$$\frac{\theta_L(s)}{V_a(s)} = \frac{1}{\frac{J_{eq} \cdot R_a}{K_a'} \left\{ s + \frac{d_{eq.}}{J_{eq.}} + \frac{k_b (N_L/N_M) \cdot K_a'}{J_{eq} \cdot R} \right\} \cdot s} \quad \longleftarrow \text{2 poles}$$

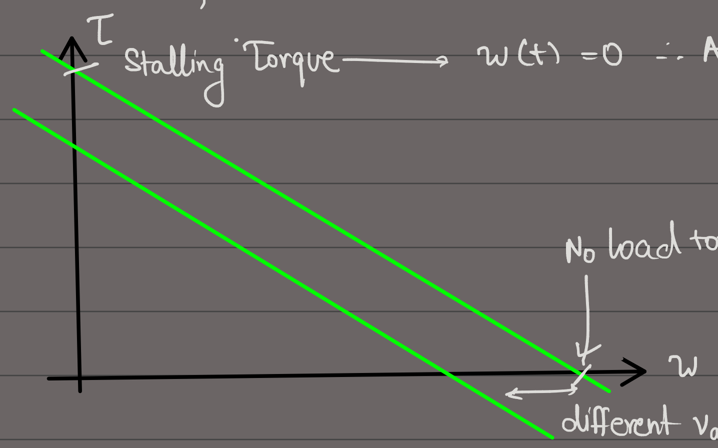
But now what are all of these constants? How do we find them?

(d_{eq}, J_{eq} can be found mechanically. How about K_a, k_b?)

Now go back: $v_a(t) = v_b(t) + i_a(t) R_a$ (we ignore L_a abhi se cause too small)

$$v_a(t) = k_b w(t) + R_a \frac{T_m(t)}{K_a}$$

∴ For a fixed v_a, the T-W curve looks like →



Stalling Torque → $w(t) = 0$ ∴ At this pt. $K_a = \frac{T_m(\text{stalling}) \cdot R_a}{K_a \cdot V_a}$

No load torque ∴ $T_m(t) = 0$ → $k_b = \frac{v_a}{T_{\text{no-load}}}$

we found them !!1!

Useful responses

① Step response ② Ramp response ③ Parabolic response

Good for studying steady state error

Transient Response: It's important to study thus we use 2 models –

① First order systems ② Second order systems.

1 First order

e.g. $G(s) = \frac{a}{s+a} \longrightarrow Y(s) = \frac{1}{s} \cdot \frac{a}{s+a}$ ← We used this numerator to normalize the DC gain.

step response

Assume $a > 0$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{a}{s+a} = 1$$ ∴ G(s) has a DC gain equal to unity.

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot G(s)\right\} = \mathcal{L}^{-1}\left\{\frac{s+a-s}{s(s+a)}\right\} = (1-e^{-at})1(t)$$

Time constant: Defined as time taken from 0–63% of steady state value

$$t_c = \frac{1}{a} \text{ seconds} \longrightarrow \boxed{+}$$

Rise time: 10–90% of steady state value

Settling time: When the system reaches $\bar{w}$ in 2% of steadystate value

$$\left.\begin{aligned} 1-e^{-at_1} &= 0.1 \\ 1-e^{-at_9} &= 0.9 \end{aligned}\right] \to t_9 - t_1 = \frac{\ln 9}{a} = \frac{2.2}{a} = t_{rise} \longrightarrow \oplus$$

$$1-e^{-at_s} = 0.98 \to t_{settling} = \frac{\ln 50}{a} \approx \frac{4}{a} \longrightarrow \oplus$$

We just play w̄ a to make first order systems more or less dynamic.

thus if we want to decrease the settling time/ rise time, we just increase a

First order systems don't take any thing away in exchange for static-ity (?)

2nd order systems

$$G(s) = \frac{\overset{\text{fundamental freq.}}{\omega_n^2}}{s^2 + 2\underset{\text{damping ratio}}{\zeta}\omega_n s + \omega_n^2}$$

$\zeta > 1$: Poles: $p_1, p_2 = \dfrac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2\zeta^2 - 4\omega_n^2}}{2}$ → distinct, $\mathbb{R}$ and LHP (over damped system)

$\therefore G(s) = \dfrac{p_1 \cdot p_2}{(s-p_1)(s-p_2)}$ ← since $p_1 \cdot p_2 = \omega_n^2$

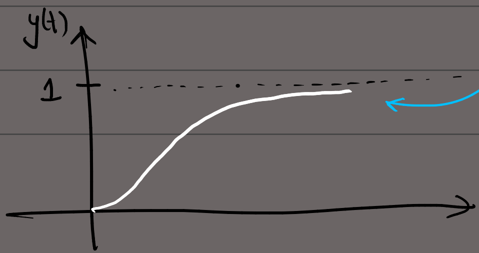
Step response: $\mathcal{L}^{-1}\left\{\dfrac{1}{s} \cdot \dfrac{p_1 p_2}{(s-p_1)(s-p_2)}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{a}{s} + \dfrac{b}{s-p_1} + \dfrac{c}{s-p_2}\right\} \Rightarrow a = 1,\ b = \dfrac{p_2}{p_1 - p_2},\ c = \dfrac{p_1}{p_2 - p_1}$

$= \left(1 + \dfrac{1}{p_1 - p_2}\left(p_2 e^{p_1 t} - p_1 e^{p_2 t}\right)\right) 1(t)$

No time constant → Only defined for first order states.

We can't study the rise times/ settling time for a 2nd order system however we can approximate the system as first order if we had $p_1 \gg p_2$ or viceversa.



How do we know if a system is purely first order or an approximatn of second order?

→ Look at slope at 0. If 0, probably second order.

$\zeta = 1 \Rightarrow$ Critically damped $\qquad G(s) = \dfrac{p^2}{(s-p)^2}$

Step response: $\mathcal{L}^{-1}\left\{\dfrac{p^2}{s(s-p)^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{1}{s-p} + \dfrac{p}{(s-p)^2}\right\}$

$$= 1(t)\left(1 - e^{pt} + pt\,e^{pt}\right)$$

$\zeta < 1 \Rightarrow$ Underdamped

$$p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad \therefore |p_1| = |p_2| = \omega_n$$

"parlance"

"languishing"

$$\frac{1}{s}G(s) = \frac{1}{s}\cdot\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{A}{s} + \frac{Bs+C}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{s} - \frac{(s+2\zeta\omega_n)}{(s+\zeta\omega_n)^2+(\omega\sqrt{1-\zeta^2})^2}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+(\omega\sqrt{1-\zeta^2})^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega\sqrt{1-\zeta^2})^2}$$

"I'm pulling out"

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+(\omega\sqrt{1-\zeta^2})^2} - \frac{1}{\sqrt{1-\zeta^2}}\quad\frac{\zeta\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+(\omega\sqrt{1-\zeta^2})^2}$$

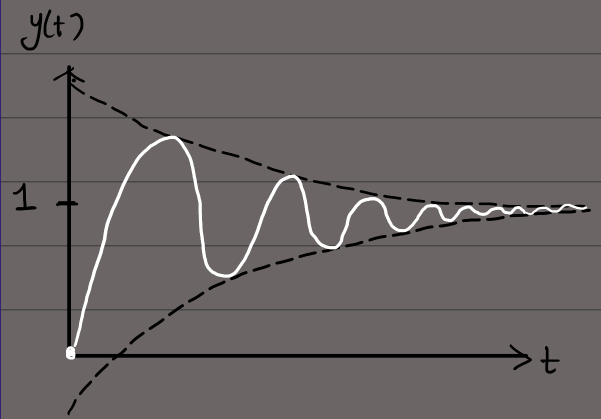
$$\omega_d := \omega_n\sqrt{1-\zeta^2}$$

$$\therefore \mathcal{L}^{-1}(sG(s)) = 1(t)\left[1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t\right]$$

neper freq.

$$= 1(t)\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\cos(\omega_d t - \phi)\right)\right)$$

$\hookrightarrow \sin^{-1}(\zeta)$



Now here, the idea of rise time/ settling time make less sense.

$\therefore$ Thus we will define settling time: Time after which the output can never leave the 2 % band about steady state.

We will also redefine rise time: Time taken to reach 100% of steady state for the first time.

We also define % o.s. $\equiv \mu_p \equiv$ percentage over-shoot $= \dfrac{y_{max} - y_{steady\text{-}state}}{y_{steady\text{-}state}} \times 100$

(hence we have and will use normalised transfer functions).

Peak time : $t_p := t$ s.t. $y = y_{max}$.

Now we will use a few approximations

→ Settling time : We will overestimate the value of $t_{settling}$ – We say that $t_s \sim$ time for envelope to enter the 2% range.

$$\therefore\ 1 - \frac{e^{-\zeta \omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.98 \longrightarrow t_s = \frac{-1}{\zeta \omega_n} \ln\left(\frac{\sqrt{1-\zeta^2}}{50}\right)$$

$\ln\left(\frac{\sqrt{1-\zeta^2}}{50}\right) \approx 4\ \forall \zeta \in (0.1, 0.85)$

$$t_s \approx \frac{4}{\zeta \omega_n}$$

Notice ⇒ Sum of roots of the transfer function is inversely $\propto$ to t_s.

Thus the 2 cases have wildly different unit responses yet have the same t_s.

→ Rise time : $y_{ss} = 1$ ∴ For the value to $y(t)$ to be 1, $\cos\left(\omega_d t - \sin^{-1}(\zeta)\right) = 0$

$$\omega_n t_r \times \sqrt{1-\zeta^2} = \sin^{-1}(\zeta) + \pi/2$$

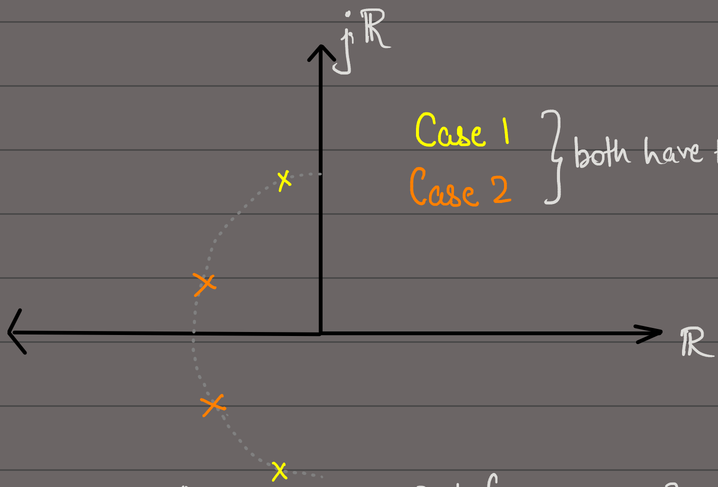
$$\hookrightarrow t_r = \frac{\sin^{-1}(\zeta) + \pi/2}{\omega_n \sqrt{1-\zeta^2}}$$

Now we make the approximation → $\sin^{-1}(\zeta) \approx \zeta$

approximat^n # 2

$$t_r \approx \frac{\zeta + \pi/2}{\omega_n\sqrt{1-\zeta^2}} \approx \frac{\pi}{2\omega_n} \approx \frac{1.6}{\omega_n}$$

(it is actually $\frac{1.8}{\omega_n}$ as we are doing some compensat^n of the earlier approximations)



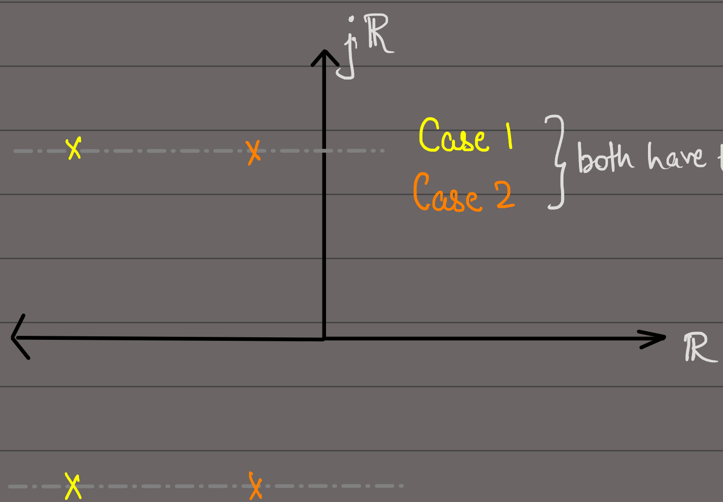
Case 1
Case 2 } both have the same rise time.

→ Peak time: $\dot{y}(t) = 0 \longrightarrow \mathcal{L}^{-1}\left\{ s \cdot \frac{1}{s} G(s) \right\} = 0$

$$\therefore 0 = \frac{\omega_n}{\sqrt{1-\zeta^2}} \mathcal{L}^{-1}\left\{ \frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2} \right\}$$

$$e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t\right) = 0$$

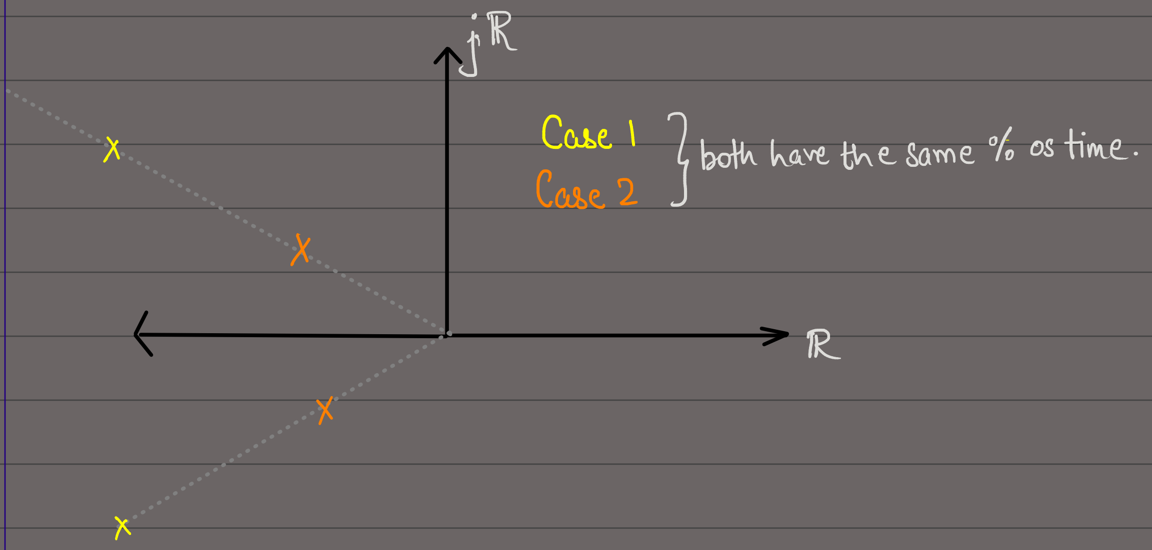
$$\therefore t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$



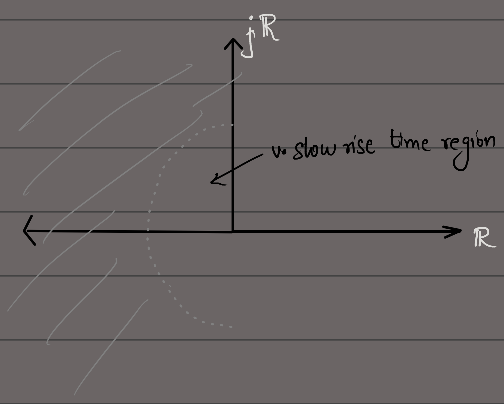
Case 1
Case 2 } both have the same peak time.

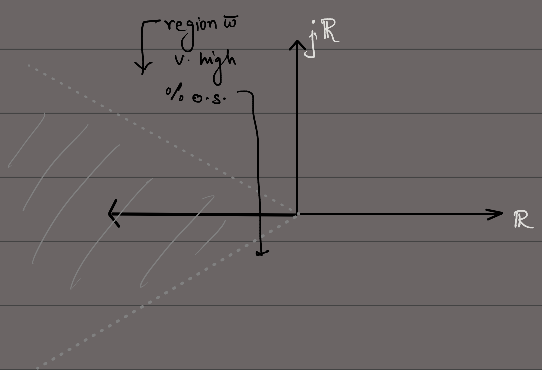
→ % o.s : $\%\,os = 100\left(1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\cos\left(\omega_d t_p - \phi\right)\right)$

$$= 100\left(1 - \frac{e^{-\zeta\pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}}\left(\cos\left(\pi - \cos^{-1}\left(\sqrt{1-\zeta^2}\right)\right)\right)\right)$$

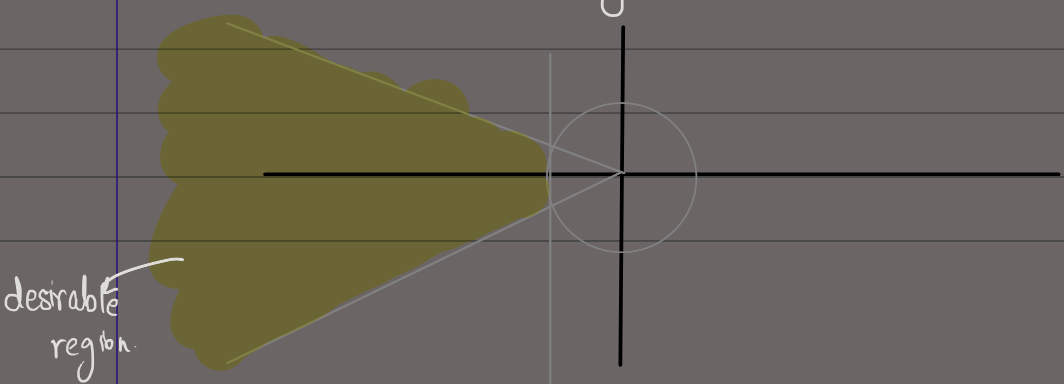


So now we will analyse a few things- say a person comes to you and says the machine you have given is very sluggish in reaching the rise time. It is also important for %os to lie below a specification.





If user wants a faster settling time too →

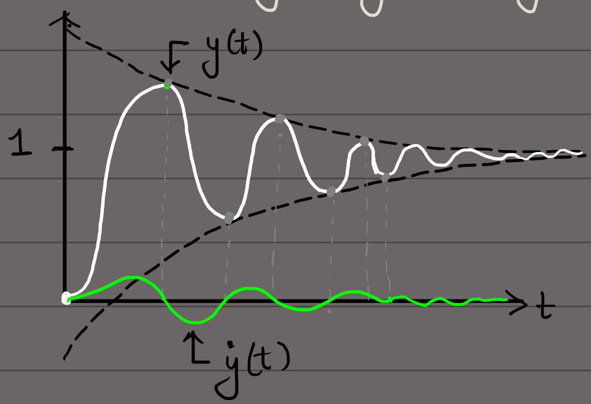


Zeros

Effects of additional poles/zeros

$$G_1(s) = \frac{\omega_n^2\,(1+T_z s)}{(s^2+2\zeta\omega_n s+\omega_n^2)} \xrightarrow{\mathcal{L}^{-1}} y_1(t)$$

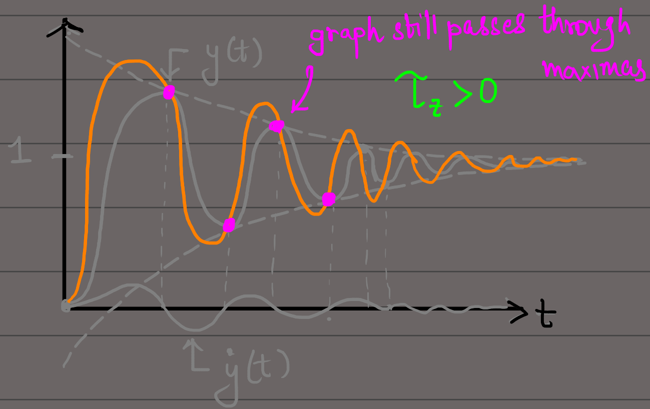
$$\therefore\ y_1(t) = y(t) + T_z\,\dot{y}(t)$$

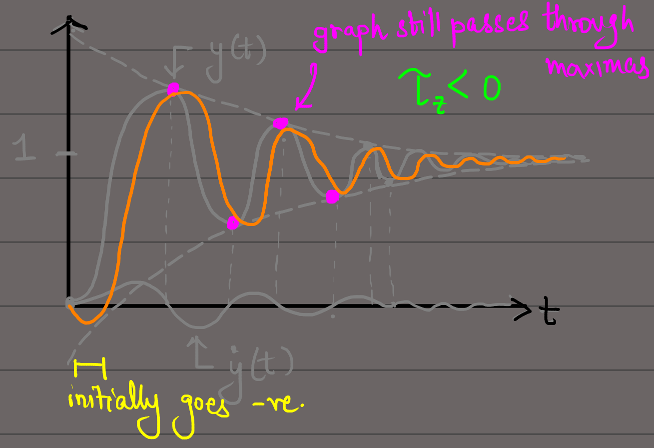


If T_z is +ve $\Rightarrow$ t_r is reduced

If T_z is -ve $\Rightarrow$ RHP zero thus leads to a -ve response initially before going +ve.

$\Rightarrow$ t_r is increased





Poles

$$G_1(s) = \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)(1+T_p s)}$$

I have no fucking clue what is happening.

Padé approximation

$$P(m,n) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_{m+1}}{s^n + b_1 s^{n-1} + \cdots + b_n}$$

$$\approx e^{-sT}$$

$$P(1,1) = \frac{a_0 s + a_1}{s+b_1} = \frac{1}{b_1}(a_0 s + a_1)\left(1+\frac{s}{b_1}\right)^{-1} = 1 - sT + \frac{s^2T^2}{2} - \frac{s^3T^3}{3} + \ldots$$

Quiz-1

hence we say that the role of a RHP zero is the same as a delay in a very approximate sense.