Matematično modeliranje

3. poglavje

Dinamično modeliranje: diferencialne enačbe, sistemi diferencialnih enačb

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2015/2016

4. Differential equations and dynamic models

A differential equation (a DE) is an equation relating the independent variable (or variables), the dependent variable and its derivatives.

Ordinary differential equation, ODE is an equation for an unknown function of one independent variable

$$F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0$$

Partial differential equation, PDE for an unknown function of several independent variables

$$F(x, y, u_x, u_y, u_{xx}, \ldots) = 0,$$

We will not consider these, from now on DE means an ODE.

The *order* of the differencialn equation is the order of the highest derivative.

Differential equations are used for modelling a *deterministic process*: a law relating a certain quantity depending on some independent variable (for example time) with its rate of change, and higher derivatives

Examples

Newton's law of cooling: $\dot{T} = k(T - T_\infty)$
where $T(t)$ is the temperature of a homogeneous body (can of beer) at time t, T_0 is the initial temperature at time $t_0 = 0$, T_∞ is the temperature of the environment, k is a constant (heat transfer coefficient)

Radioaktive decay: $\dot{y}(t) = -ky(t), \quad k = \frac{\log 2}{t_{1/2}}$
where $y(t)$ is the remaining quantity of a radioactive isotope at time t, $t_{1/2}$ is the *half-life* and k is the *decay constant*.

Used for mydefcarbon dating: (Willard Libby, 1949, Nobel prize for chemistry in 1960) dating organic remains on the basis of the ratio between the unstable isotope C^{14} and the stable isotope C^{12}.

Oscillators $\ddot{x} + \omega x = 0$ (simple harmonic oscillator)

The function $x(t)$ is a *solution* of

$$F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}) = 0,$$

on the interval I if it satisfies the identity

$$F(t, x(t), \dot{x}(t), \ddot{x}(t), \ldots, x^{(n)}(t)) = 0$$

for all $t \in I$.

Solving (analytically) a differential equation is typically very difficult, usually impossible.

Use simplifications, numerical approximations, ...

4.1. First order ODE's

We will (mostly) consider first order ODE's in the form

$$y' = f(x, y)$$

Typically such an equation has

- a *general solution*: a one-parametric family of solutions $y = y(x, C)$,
- *particular solutions*: a specific function fro the general solution, that satisfies the *initial condition* $y(t_0) = y_0$.
- sometimes a *singular solution*: a specific solution that is not part of the general solution.

A *separable equation* is of the form

$$y' = f(x)g(y)$$

This can be solved by using $y = \dfrac{dy}{dx}$, *separating variables* and integrating:

$$\frac{dy}{g(y)} = f(x)\,dx$$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C$$

Example: *population growth*

$y(t)$ size of a population (bacteria, trees, hares, ...) at time t.

- Exponential growth $\dot{y} = ky$
- Growth with constraints: $\dot{y} = ky(1 - y/y_{max})$ (logistic law), where y_{max} is the capacity of the environment, i.e. maximal population size that it stiull supports
- General model $\dot{y} = k(y, t)f(y)$, not a separable equation, analytically solvable only in very specific cases.

Example: information spreading
$y(t)$ part of a closed group that at time t knows a certain piece of information.
Consider two possible models:

- spreading through an external source: the rate of change is linear with respect to the uninformed part $\dot{y} = k(1 - y)$,
- spreading through "word of mouth"the rate of change is linear with respect to te number of encounters between informed and uninformed members $\dot{y} = ky(1 - y)$ *logistic law again*.

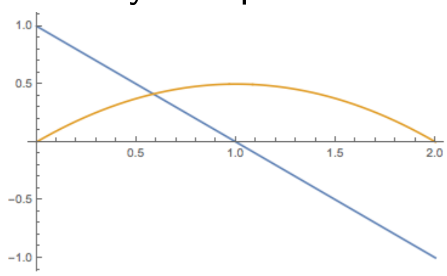
The solution $y(t)$ depends also on the initial value $y_0 = y(t_0)$ at time $t = t_0$.

A ball of mass 1kg is thrown vertically into the air with initial velocity $v_0 = 25$ m/s. We follow its trajectory

- assuming no air friction,
- assuming *linear law of friction*: $F_u = -kv$.

Question: *What will take longer: going up or falling down?*

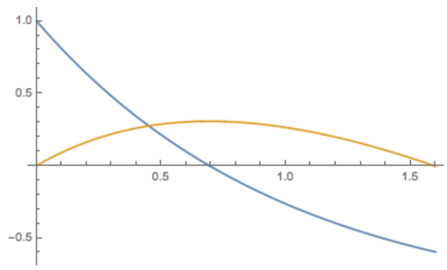
Equation | Velocity and position | Solution



$ma = -F_g$

$\dot{v} = -10$

$v(t) = -t + 25$
$x(t) = -t^2/2 + 25t$



$ma = -F_u - F_g$

$\dot{v} = -v - 10$

$v(t) = 10(1 - e^{-t})$
$x(t) = \dots$

A *first order linear DE* is of the form

$$y' + f(x)y = g(x)$$

The equation is *homogeneous* if $g(x) = 0$ and *nonhomogeneous* if $g(x) \neq 0$.

The general solution is a sum $y(x) = y_p(x) + Cy_h(x)$, where Cy_h is he general solution to the homogeneous equation, and y_p is any particulkar solution of the nonhomogeneous equation.

The particular solution y_p can be obtained by *variation of the constant*, that is, by substituting the constant C is the homogenenous solution by an unknown function $C(x)$ which is then determined from the equation.

Example: $x^2y' + xy = 1$, $y(1) = 2$

Look at the geomtric information given by an equation $y' = f(x, y)$.

At each point (x, y) from the domain of the function f the equation

$$y' = f(x, y)$$

gives the value $f(x, y)$ of the coefficient of the tangent to the solution $y(x)$ through this specific point, that is, the direction in which the solution passes through the point.

Altogether, these directions form the *directional field* of the equation.
A solution of the equation is represented by a curve $y = y(x)$ that follows the given directions at every point x, that is, the coefficient of the tangent at every point $(x, y(x))$ corresponds to the value $f(x, y(x))$.

The general solution to the equation is a famil ty of curves that each follows the given directions.

Directional fields and solutions of

$y' = ky$ $\qquad\qquad$ $y' = ky(1-y)$