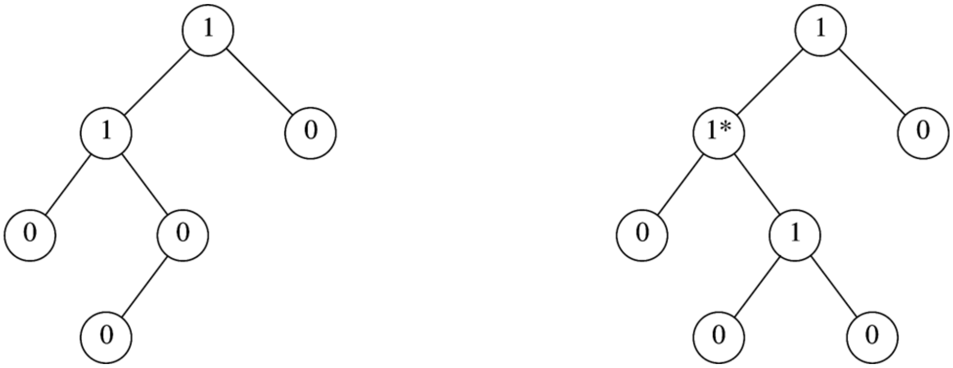
Leftist heap: is a binary tree with the normal heap ordering property, but the tree is not balanced. In fact it attempts to be very unbalanced!

Definition: the **null path length** npl(x) of node x is the length of the shortest path from x to a node without two children.

The null path lengh of any node is 1 more than the minimum of the null path lengths of its children. (let npl(nil)=-1).

Only the tree on the left is leftist.



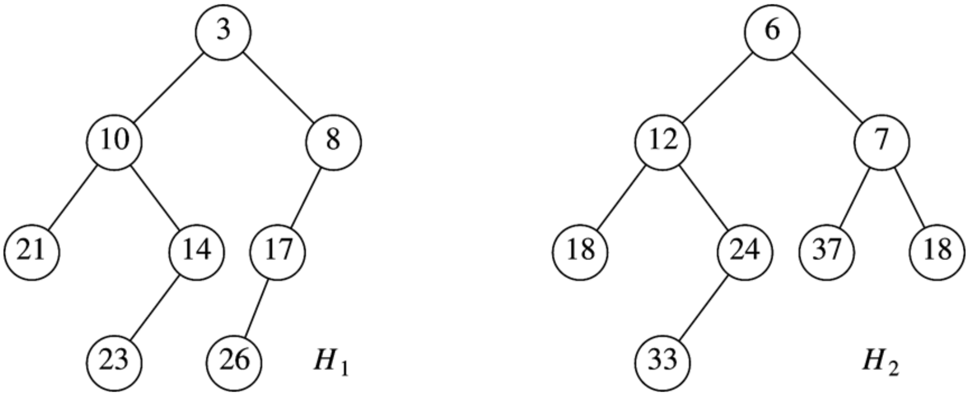
Null path lengths are shown in the nodes.

Definition: the **leftist heap property** is that for every node x in the heap, the null path length of the left child is at least as large as that of the right child.

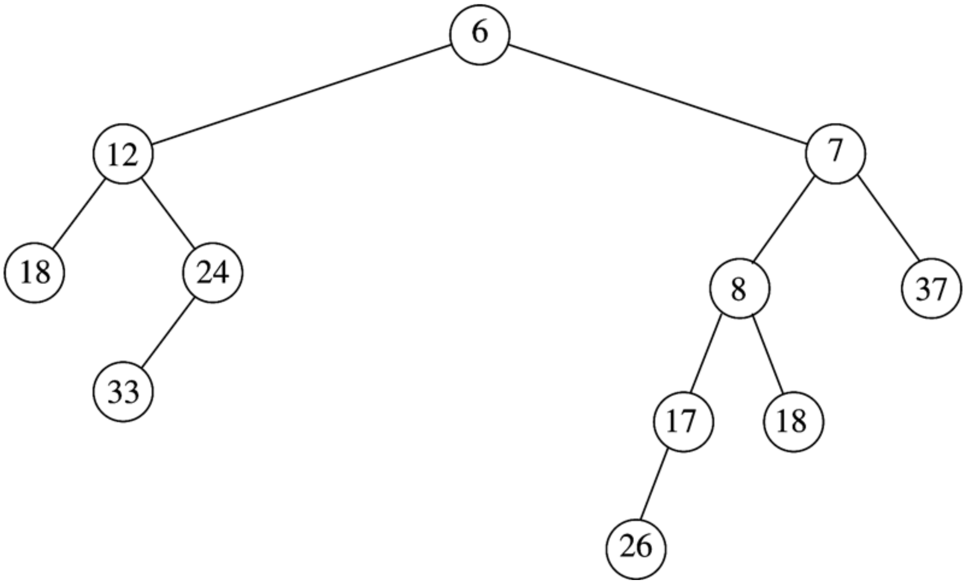
This property biases the tree to get deep towards the left. It may generate very unbalanced trees, which facilitates merging!

It also also means that the right path down a leftist heap is as short as any path in the heap. In fact, the right path in a leftist tree of N nodes contains at most $\lfloor \lg(N+1) \rfloor$ nodes. We perform all the work on this right path, which is guaranteed to be short.

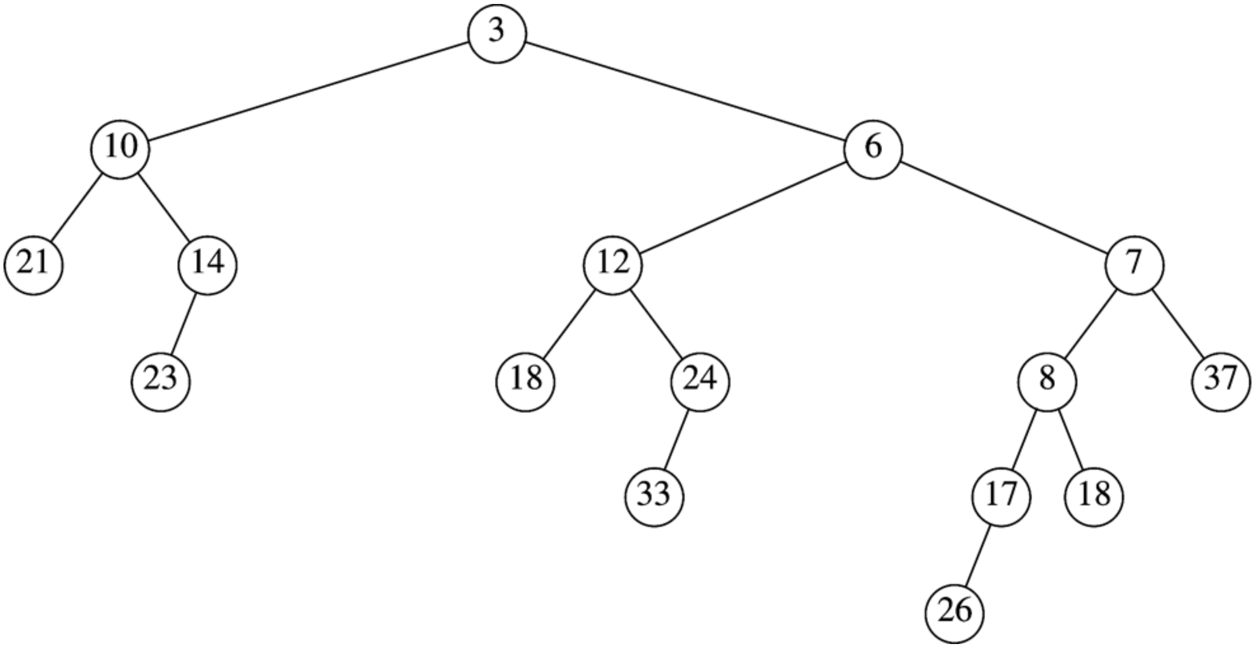
Merging on a leftist heap. (Notice that an insert can be considered as a merge of a one-node heap with a larger heap.)



1. (Magically and recursively) merge the heap with the larger root (6) with the right subheap (rooted at 8) of the heap with the smaller root, creating a leftist heap.



Make this new heap the right child of the root (3) of h1.



The resulting heap is not leftest at the root. (The npl of 3's left subtree is less than the npl of 3's right subtree.) Simply swap the root's left and right children.

Working only on the right paths, the time to merge is the sum of the length of the right paths. Thus, we obtain a O(lgn) bound.

The time for deleteMin is also O(lgN): destroy the root and merge the two heaps.

The time for building a leftist heap is also O(N).

Skew heap: is another *self-adjusting* data structure, like the splay tree.

Skew heaps are binary trees with heap order, but no structural constraint. Unlike leftist heaps, no information is maintained about npl and the right path of a skew heap can be arbitrarily long at any time.

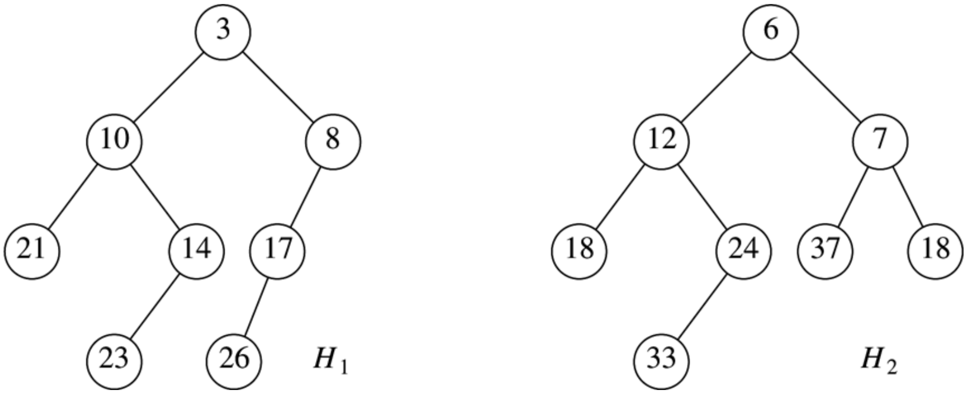
This implies that the worst case running time of all operations is O(N).

However, m consecutive operations take a total worst case time of O(mlgn), an *amortized* cost of O(lgn) per operation.

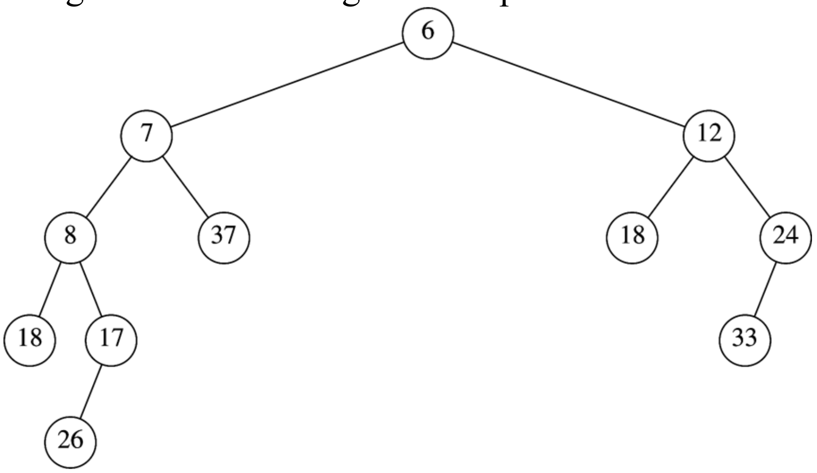
Merging on skew heaps. Merging is performed exactly as it was with leftist heaps, with one exception.

For skew heaps we always swap the left and right children, regardless of whether they satisfy the leftist heap order property (the right paths become the new left path) with the exception that the largest of all the nodes on the right path does not have its children swapped.

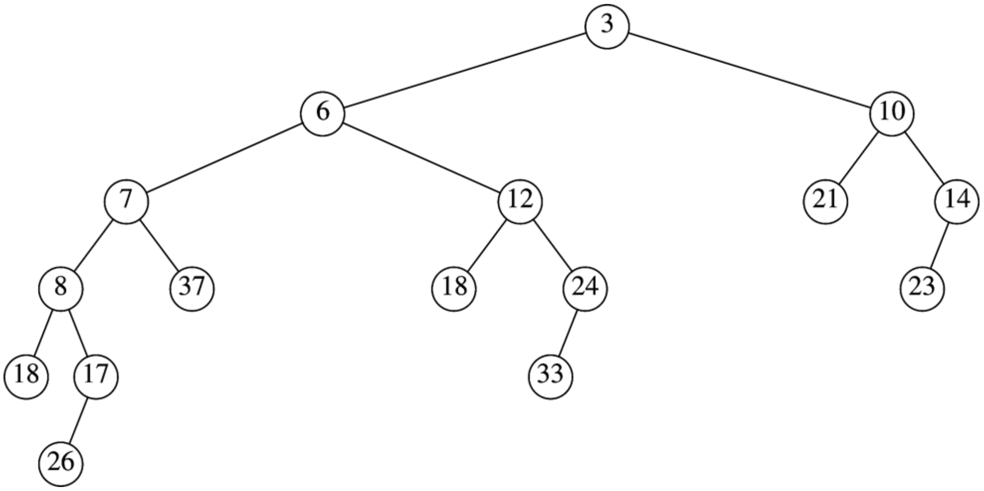
Skew Heaps H1 and H2:



Merge H2 with H1’s right subheap:



Make this heap the left child of H1 and the old left child of H1 becomes the new right child:



The result is a leftist tree.

But if we insert 15 into this heap, the result would not be leftist.

Comparing leftist and skew heaps with binary heaps:

- leftist and skew heaps: merge, insert, and deleteMin in effectively O(lgn) time
- binary heaps support insert in constant average time per operation.

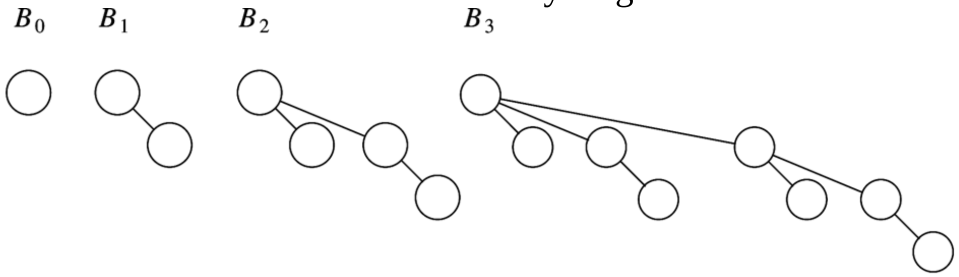
Binomial queues: support merge, insert, and deleteMin in worst case O(lgn) time per operation and insert in constant average time per operation.

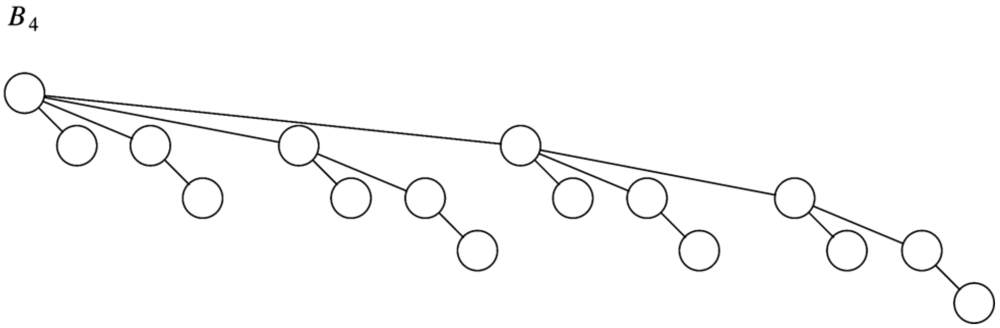
A binomial queue is a collection of heap-ordered **binomial** trees, known as a forest.

Facts about binomial trees:

A binomial tree b_k of height k is formed by attaching a binomial tree b_{k-1} to the root of another binomial tree, b_{k-1}.

There is at most one binomial tree of every height.





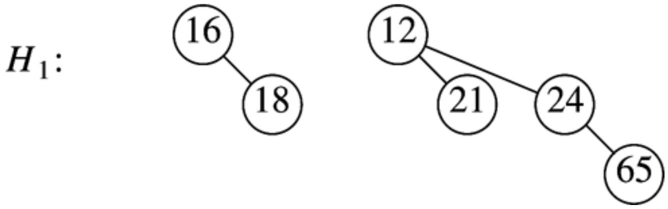
Binomial trees of height k have exactly 2^k nodes.

The number of nodes at depth d is the binomial coefficient () = "k choose d" = k!/(d!(k-d)!)

A priority queue of any size can be represented by a collection of binomial trees.

Example 1: 13_{10} = **1**1**0**1$_2$ ==> 13 can be represented by the forest b_3, b_2, b_0.

Example 2: 6_{10} = **11**0 ==> the forest b_2, b_1.

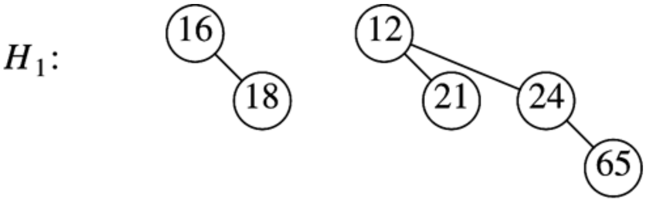


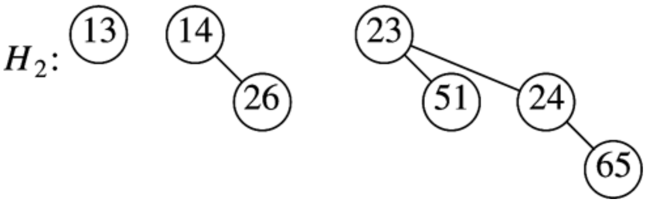
Binomial queue operations:

findMin:

- sequential search of the roots of all the trees ==> O(lgn) time.
- if we maintain information about the minimum root ==> O(1) time.

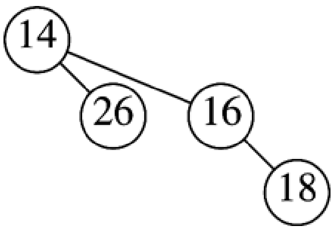
Merge: O(lgn) worst case. (Merging two trees takes constant time and there are O(lgn) binomial trees.)



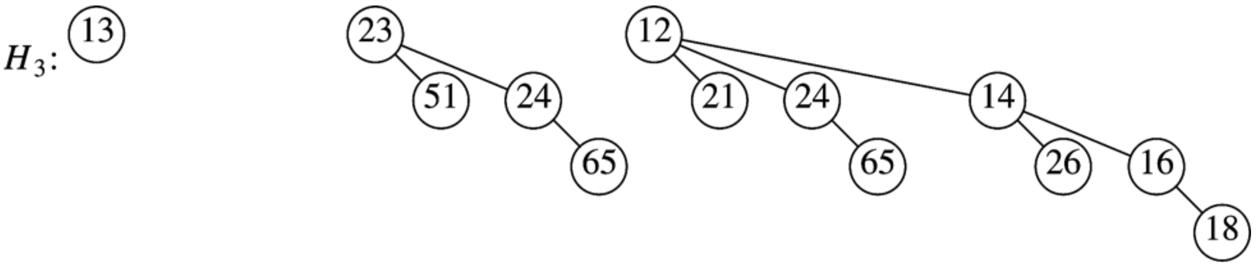


Merge the height 0 trees: there's only 1

Merge the height 1 trees to create a height 2 tree:

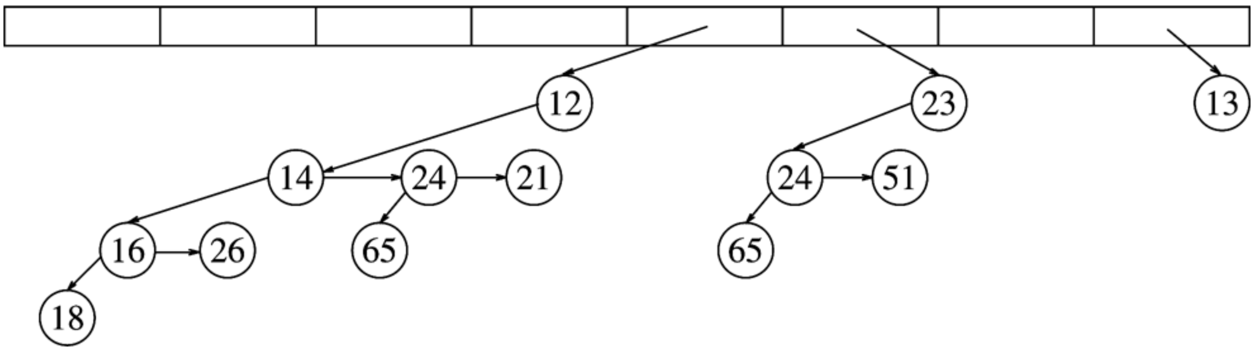


Now there are 3 height 2 trees. Keep one and merge the other 2 into a height 3 tree:



Merge: O(lgn) worst case. (Merging two trees takes constant time and there are O(lgn) binomial trees.)

Implementation of H3 using an array, with trees in descending order:

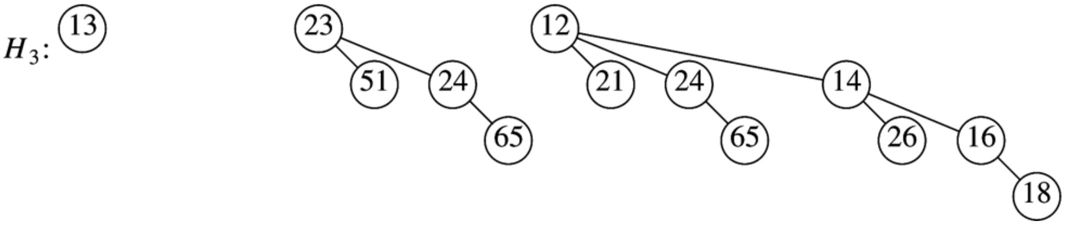


insert: a special case of merging a one-node tree.

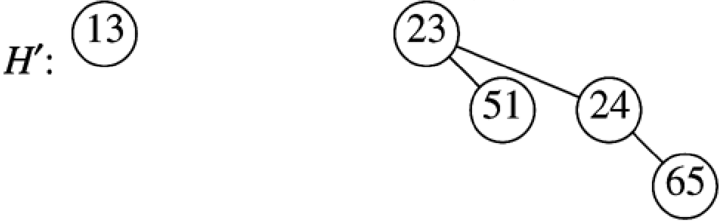
- O(lgN) worst case. O(1) average case. (Stop when you find a missing B_i tree.)
- Perform N inserts on an initially empty tree in O(N) time.

deleteMin: O(lgN) worst case.

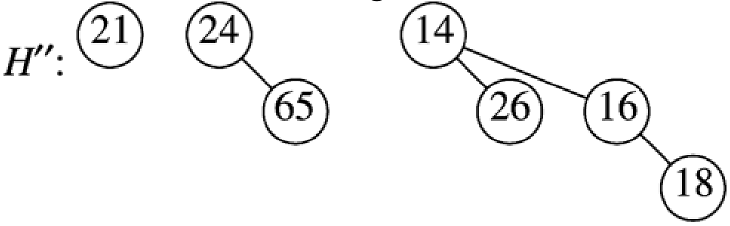
Let H =



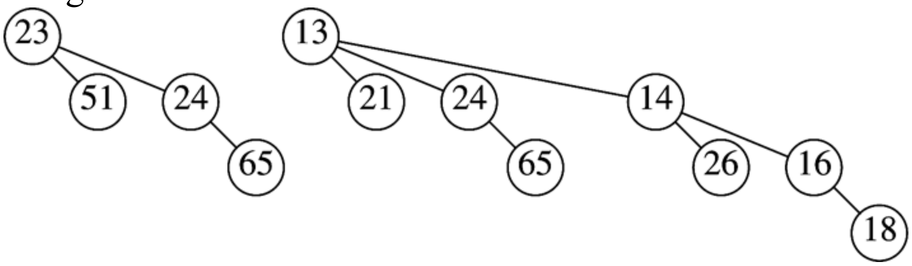
Find tree B_k with smallest root (12). Remove B_k from H to form H’.



Remove root from B_k to get H”



Merge H’ and H”:



Series of O(lgN) operations ⇒ O(lgN).

Splay Trees

Splay trees: instead of maintaining a balanced tree, after a node is accessed, it is pushed closer to the root, and so are other nodes on the path from the root.

Splaying results in a "flatter" tree - the accessed node is moved to the root, and the depths of most nodes on the access path are roughly halved. (If a node takes a long time to access, it will be moved so that it will be accessed quicker next time.)

The splay operations are similar to the AVL rotate operations, but no balancing information need be maintained.

A single operation in a splay tree may take O(n) time.

However, m consecutive operations are guaranteed to take at most O(mlgn) time. We say that a splay tree has O(lgn) **amortized** cost per operation. Over a long sequence of operations, some may take more, some less.

```
Example:                K5
                      /    \
                   K4        F
                  /   \
               K3       E
             /    \
          K2        D
        /    \
      A        K1
             /    \
           B        C
```

Access K1: Perform AVL double rotation with K1, K2 (K1's parent), and K3 (K1's grandparent). Move K1 up:

```
                     K5                              K1
                   /     \                         /       \
                K4          F                  K2            K4
               /   \                          /    \       /     \
            K1       E                      A        B   K3         K5
          /     \                ⇒                      /   \      /   \
      K2          K3                                   C     D    E     F
    /    \      /    \
  A        B   C       D
```

Keep moving K1 up until K1 becomes root. Depth of nodes along access path (from original root to K1) has roughly been halved. Some shallow nodes were pushed down two levels, e.g. K5.

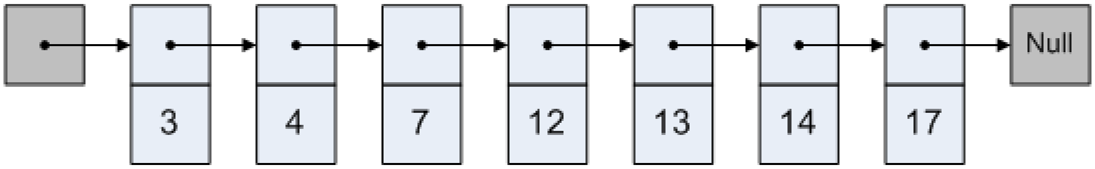
Skip Lists

Operation	*Time Complexity*
Insertion	O(log N)
Removal	O(log N)
Contains	O(log N)
Enumerate in order	O(N)

Also:

- Find the element in the set that is closest to some given value, in O(log N) time.
- Find the k-th largest element in the set, in O(log N) time. Requires a simple augmentation of the the skip list with partial counts.
- Count the number of elements in the set whose values fall into a given range, in O(log N) time. Also requires a simple augmentation of the skip list.

Sorted singly-linked list (O(N)):



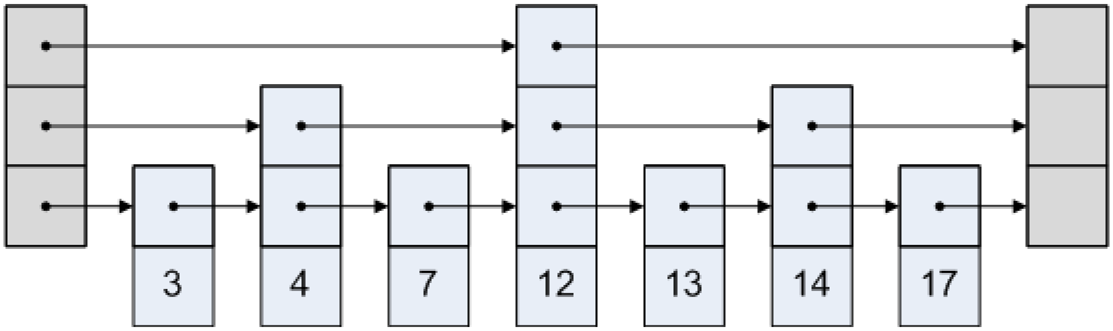
Enumerate in order O(N)

Why so slow? It takes so long to get into its middle, so:

=> add a level-2 list that skips every other node.

add a level-3 list that skips every other node in the level-2 list.

…etc

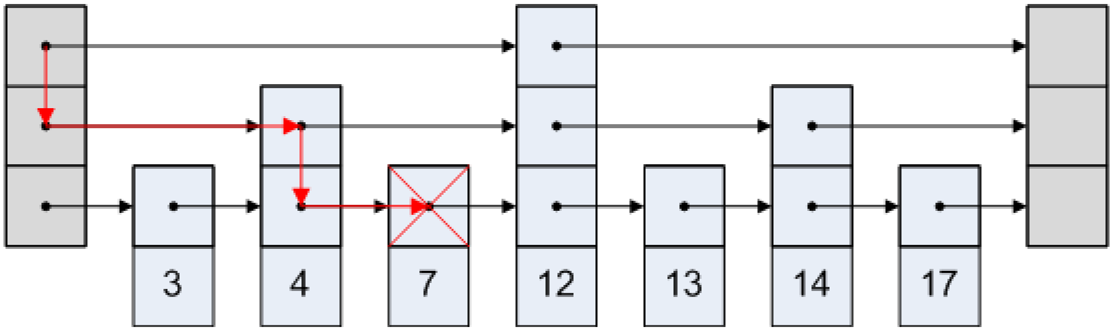


Looks like a binary tree!

Searching is now like a O(lgN) binary search:

- first look in the top-most list - move to the right but don't jump too far.
- If can't move further right on a particular level, drop to the next lower level, which has shorter jumps.

Search for 8:

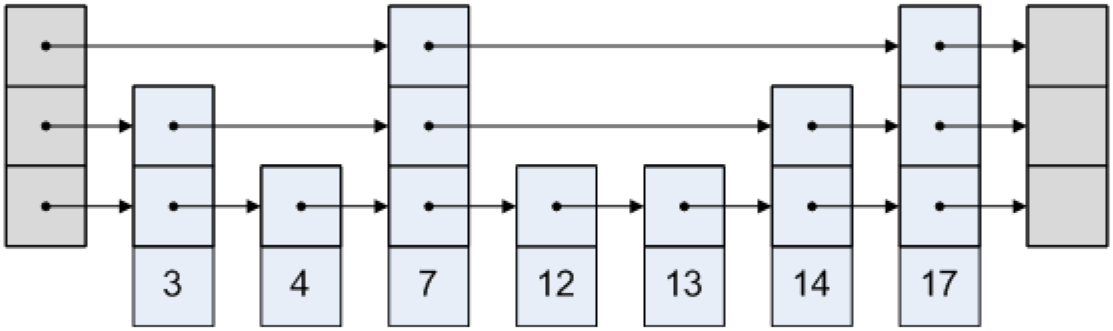


Since we landed on a 7, but we were looking for an 8, that means that 8 is not in the set.

How to implement insertions and removals efficiently?

A probabilistic approach: Instead of ensuring that the level-2 list skips every other node, a skip list is designed in a way that the level-2 list skips one node on average. In some places, it may skip two nodes, and in other places, it may not skip any nodes

Here is an example of what a skip list may look like:



- Insertion: decide how many lists will this node be a part of. With a **probability** of 1/2, make the node a part of the lowest-level list only. With 1/4 probability, the node will be a part of the lowest two lists. With 1/8 probability, the node will be a part of three lists. And so forth. Insert the node at the appropriate position in the lists that it is a part of.
- Deletion: remove the node from all sorted lists that it is a part of.
- Contains: we can use the O(log N) algorithm similar described above on multi-level lists.

And, all of these operations are pretty simple to implement in O(log N) time!