1. Homework

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Question 1

Take the derivative of the function $f(x) = (x + x^2)^3 - x$.

$$\begin{aligned}\frac{df}{dx} &= 3(x + x^2)^2(2x + 1) - 1\\ &= 3(x^2 + 2x^3 + x^4)(2x + 1) - 1\\ &= 3(2x^3 + x^2 + 4x^4 + 2x^3 + 2x^5 + x^4) - 1\\ &= 6x^5 + 15x^4 + 12x^3 + 3x^2 - 1\end{aligned}$$

Question 2

Linearize the following function at $x0 = 3$: $f(x) = x^2 + 2x^3$. Plot the original function and its linear approaximation for the values $x \in [-5, 5]$.

The linerization of a single variable function around x_0 can be written as:

$$f(x_0 + \delta) \approx f(x_0) + \frac{df}{dx}(x_0)\delta$$

For $x_0 = 3$ we obtain:

$$f(x_0 = 3) = 3^2 + 2 \cdot 3^3 = 63$$

and

$$\frac{df}{dx}(x_0 = 3) = 2 \cdot 3 + 6 \cdot 3^2 = 60$$

The linearization around $x_0 = 3$ can now be written as:

$$f(3 + \delta) = 63 + 60 \cdot \delta$$

The equation of the tangent line can be found by solving $63 = 60 \cdot x_0 + n$ for n. The equation of the tangent line is therefore:

$$g(x) = 60x - 117$$

Figure 1 shows the plot of the linearization.

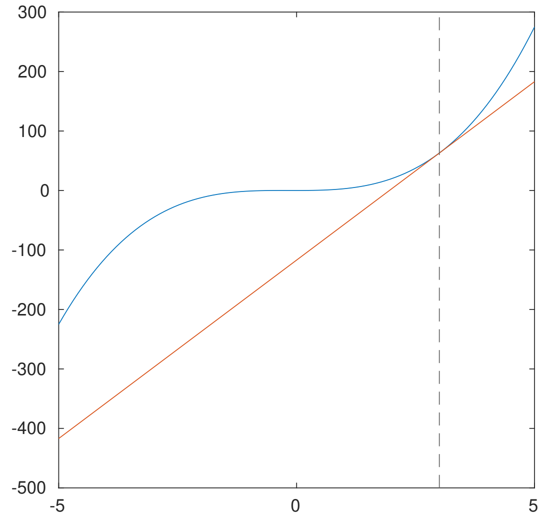

Figure 1: Linearization of the function f at $x_0 = 3$.

Question 3

Linearize the function $f(x_1(p), x_2(p)) = 4x_1^2(p) + x_2^4(p)$ with parameters of x_1 and x_2 defined as $x_1(p) = p_1$ and $x_2(p) = p_1 + 3p_2$.

The linearization around $\vec{p_0}$ can be written as:

$$f(x_1(\vec{p_0} + \vec{\delta}), x_2(\vec{p_0} + \vec{\delta})) \approx f(x_1(\vec{p_0}), x_2(\vec{p_0})) + \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_2}{\partial p_1} \\ \frac{\partial x_1}{\partial p_2} & \frac{\partial x_2}{\partial p_2} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

This can be rewritten as:

$$\begin{aligned} f(x_1(\vec{p_0} + \vec{\delta}), x_2(\vec{p_0} + \vec{\delta})) \approx f(x_1(\vec{p_0}), x_2(\vec{p_0})) &+ \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial p_1}\delta_1 + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial p_1}\delta_1 + \\ &+ \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial p_2}\delta_2 + \frac{\partial f}{\partial x_2}\frac{\partial x_1}{\partial p_2}\delta_2 \end{aligned}$$

Computing the partial derivatives yields:

$$f(x_1(\vec{p_0} + \vec{\delta}), x_2(\vec{p_0} + \vec{\delta})) \approx 4p_1^2 + (p_1 + 3p_2)^4 + 8p_1\delta_1 + 4(p_1 + 3p_2)^3\delta_1 + 12(p_1 + 3p_2)^3\delta_2$$

Question 4

Assume a vector $x \in R^d$ and a positive definite matrix $A \in R^{d \times d}$. Compute the gradient of function $f(x) = x^t A x \cdot 3$.

The gradient can be computed by differentiating the expression with respect to the vector $\vec{x}$ using the product rule:

$$\frac{df(\vec{x})}{d\vec{x}} = 3((A\vec{x})^T \frac{\partial \vec{x}}{\partial \vec{x}} + \vec{x}^T \frac{\partial A\vec{x}}{\vec{x}}) = 3(\vec{x}^T A^T + \vec{x}^T A) = 3\vec{x}^T (A^T + A) = 6\vec{x}^T A = \nabla f^T$$

or by using the identity $\frac{\partial \det(x^T Ax)}{\partial x} = 2\det(x^T Ax)Ax(x^T Ax)^{-1}$ by observing that $x^T Ax$ is a scalar. Let $c = x^T Ax$. The gradient can now be computed as:

$$\nabla f = 3(2cAxc^{-1}) = 6Ax$$