3. Homework

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Question 1

Using the Euler-Lagrange equations show that the given functional is minimized at u and v that satisfy the given equations.

The cost function being integrated over the pixels domain is written as:

$$E_c(u, v, x, y) = (I_x u + I_y v + I_t)^2 + \alpha(u_x^2 + u_y^2 + v_x^2 + v_y^2)$$

To write the first Euler-Lagrange equation, we must compute the following derivatives:

$$\frac{\partial L}{\partial u} = 2(I_x u + I_y v + I_t) \cdot I_x$$

$$\frac{\partial L}{\partial u_x} = 2\alpha u_x$$

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} = 2\alpha u_{xx}$$

$$\frac{\partial L}{\partial u_y} = 2\alpha u_y$$

$$\frac{\partial}{\partial y}\frac{\partial L}{\partial u_y} = 2\alpha u_{yy}$$

Using this, we can write the first Euler-Lagrange equation:

$$\begin{aligned} 2(I_x u + I_y v + I_t) \cdot I_x - 2\alpha u_{xx} - 2\alpha u_{yy} &= 0 \\ I_x(I_x u + I_y v + I_t) - \alpha(u_{xx} + u_{yy}) &= 0 \end{aligned}$$

By performing similar differentiations required to form the second Euler-Lagrange equation, we get the equation:

$$\begin{aligned} 2(I_x u + I_y v + I_t) \cdot I_y - 2\alpha v_{xx} - 2\alpha v_{yy} &= 0 \\ I_y(I_x u + I_y v + I_t) - \alpha(v_{xx} + v_{yy}) &= 0 \end{aligned}$$

Question 2

Using the definition $\Delta u = \overline{u} - u$, show that the Horn Schunck equations can be rewritten into the given matrix form.

The Horner-Schunck equations are written as:

$$I_x(I_x u + I_y v + I_t) - \alpha \Delta u = 0$$
$$I_y(I_x u + I_y v + I_t) - \alpha \Delta v = 0$$

Substitution $\overline{u} - u$ for Δu we can rewrite the equations as:

$$I_x^2 u + I_x I_y v + I_x I_t - \alpha \overline{u} + \alpha u = 0$$
$$I_x I_y u + I_y^2 v + I_y I_t - \alpha \overline{v} + \alpha v = 0$$

$$u(I_x^2 + \alpha) + I_x I_y v = \alpha \overline{u} - I_x I_t$$
$$I_x I_y u + v(I_y^2 + \alpha) = \alpha \overline{v} - I_y I_t$$

This system of equations can be written in matrix form as:

$$\begin{bmatrix} (I_x^2 + \alpha) & I_x I_y \\ I_x I_y & (I_y^2 + \alpha) \end{bmatrix} = \begin{bmatrix} \alpha \overline{u} - I_x I_t \\ \alpha \overline{v} - I_y I_t \end{bmatrix}$$

Which answers the question.