



THE Z TRANSFORM

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Introduction

- The Z-transform is a generalization of the DTFT
- The Z-transform can be used to characterize the response of linear, time-invariant filters to complex exponential signals

Definition

- The Discrete-Time Fourier Transform (DTFT) definition

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- The Z Transform (ZT) is generalization of the DTFT

$$ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

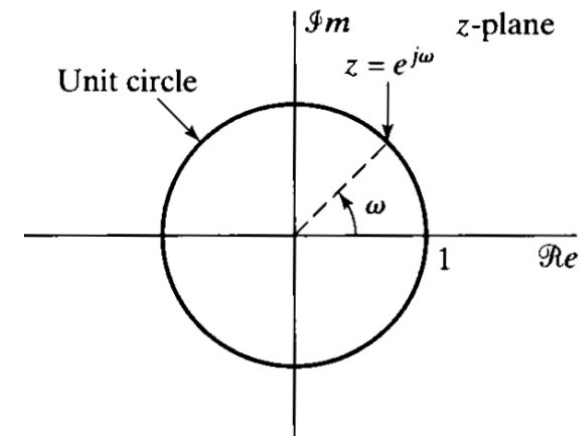
z is complex variable $z = r \cdot e^{j\omega}$

- If $r = 1$, $z = e^{j\omega} \rightarrow$ DTFT

- The DTFT is the Z transform on the unit circle

$$X(f) = X(z) \Big|_{z=e^{j\omega} \text{ or } |z|=1}$$

- Compare the ZT to DTFT definition



Definition

- For LTI systems described by a unit-sample response $h[n]$, $H(z)$ is also referred to as the *system function* or *transfer characteristic*

$$H(z) \triangleq \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

- The system function is a generalization of the *frequency response*, since the DTFT is a special case of the Z-transform, that is:

$$H(f) = H(z)|_{z=e^{j2\pi f}}$$



Properties

Property	Time Domain	z -Domain
Notation	$x(n)$	$X(z)$
	$x_1(n)$	$X_1(z)$
	$x_2(n)$	$X_2(z)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$
Time shifting	$x(n - k)$	$z^{-k} X(z)$
Scaling in the z -domain	$a^n x(n)$	$X(a^{-1} z)$
Time reversal	$x(-n)$	$X(z^{-1})$
Conjugation	$x^*(n)$	$X^*(z^*)$
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$

(Proakis, Manolakis)



Some common Z-transform pairs

	Signal, $x(n)$	z -Transform, $X(z)$
1	$\delta(n)$	1
2	$u(n)$	$\frac{1}{1 - z^{-1}}$
3	$a^n u(n)$	$\frac{1}{1 - az^{-1}}$
4	$na^n u(n)$	$\frac{az^{-1}}{(1 - az^{-1})^2}$
5	$-a^n u(-n - 1)$	$\frac{1}{1 - az^{-1}}$
6	$-na^n u(-n - 1)$	$\frac{az^{-1}}{(1 - az^{-1})^2}$

(Proakis, Manolakis)



Examples of Z transform

- The relation between a signal $h[n]$ and its Z-transform $H(z)$: $h[n] \longleftrightarrow H(z)$

- Gain:

$$G\delta[n] \longleftrightarrow G$$

In particular, the Z-transform of the unit sample $\delta[n]$ is the constant 1.

- Delay by n_0 samples:

$$\delta[n - n_0] \longleftrightarrow z^{-n_0}$$

This explains why the notation z^{-1} is often used to designate a unit delay.

- Rectangular (“boxcar”) filter of length N :

$$R_N[n] \triangleq u[n] - u[n - N] \longleftrightarrow \sum_{n=0}^{N-1} z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}}$$

This filter has $N - 1$ zeroes equally spaced on the unit circle, except for $z = 1$ where the zero is cancelled by a pole.

- First-order recursive lowpass filter $y[n] = ay[n - 1] + x[n]$:

$$a^n u[n] \longleftrightarrow \frac{1}{1 - az^{-1}} \quad \text{for } |z| > |a|$$

Properties

- The most important property of Z-transforms is the *convolution theorem*
$$x_1[n] * x_2[n] \leftrightarrow X_1(z) \cdot X_2(z)$$
- * Because it replaces the complicated convolution operation by a simpler multiplication, this theorem is:
 - (1) useful for computing the responses of digital filters to signals given by analytic expressions
 - (2) for finding the unit-sample response of cascade combinations of filters

The system function of LTI systems

- The **output** (response) $y[n]$ of an LTI system to an input sequence $x[n]$ can be obtained by computing the convolution of $x[n]$ with the unit-sample response of the system $h[n]$ (Note that $h[n]$ is the response of the system to unit-sample signal $\delta[n]$)

The **convolution property** allows us to express this relationship in the Z-domain as:

$$Y(z) = H(z) \cdot X(z)$$

where $Y(z)$ is the Z-transform of the output sequence $y[n]$,

$X(z)$ is the Z-transform of the input sequence $x[n]$, and

$H(z)$ is the Z-transform of the unit-sample response $h[n]$.

We then determine **output** (response) $y[n]$ by evaluating the inverse Z-transform of $Y(z)$.

- Alternatively, if we know $x[n]$ and we observe the output $y[n]$ of the system, we can determine the **unit-sample response** by first solving for $H(z)$ from the relation:

$$H(z) = Y(z) / X(z)$$

and then evaluating the inverse Z-transform of $H(z)$ to get $h[n]$,

where $H(z)$ represents the Z-domain characterization of the system and the **unit-sample response** $h[n]$ is the corresponding time-domain characterization of the system.

- The transform $H(z)$ is called the **system function** or **transfer characteristic** of the system

Finding response of LTI systems (example)

• The system characterized by diff. eq.: $\longrightarrow y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$

• The input signal: $\longrightarrow x(n) = \delta(n) - \frac{1}{3}\delta(n-1).$

• The system function: $\longrightarrow H(z) = \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$
Poles at $z_1 = 1/2$ and $z_2 = 1/3$
$$= \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

• The Z transform of the input signal: $\longrightarrow X(z) = 1 - \frac{1}{3}z^{-1}$

$$Y(z) = H(z)X(z)$$

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

• The response (output) of the system: $\longrightarrow y(n) = (\frac{1}{2})^n u(n)$

Finding unit-sample response of LTI systems (example)

- The system characterized by difference equation: $\longrightarrow y(n) = 2.5y(n-1) - y(n-2) + x(n) - 5x(n-1) + 6x(n-2)$

- The system function: $\longrightarrow H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$
Poles at $z_1 = 2$ and $z_2 = 1/2$
$$= \frac{1 - 5z^{-1} + 6z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}$$

The zeros occur at $z = 2$ and $z = 3$

The zero at $z = 2$ cancels the pole at $z = 2$

Finding unit-sample response of LTI systems (example)

- Consequently, $H(Z)$ reduces to: \longrightarrow

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}} = \frac{z - 3}{z - \frac{1}{2}}$$

$$= 1 - \frac{2.5z^{-1}}{1 - \frac{1}{2}z^{-1}}$$
- The unit-sample (impulse) response: \longrightarrow $h(n) = \delta(n) - 2.5\left(\frac{1}{2}\right)^{n-1}u(n-1)$
- The system is characterized by: \longrightarrow $y(n) = \frac{1}{2}y(n-1) + x(n) - 3x(n-1)$

Z-transform of filters defined by a difference equation

- Digital filters described by linear, constant-coefficient difference equations (LCCDEs):

$$y[n] = \sum_{k=1}^K a_k y[n-k] + \sum_{m=0}^M b_m x[n-m]$$

- The Z-transform yields:

$$Y(z) = -\sum_{k=1}^K a_k Y(z) z^{-k} + \sum_{m=0}^M b_m X(z) z^{-m}$$

$$Y(z) \left(1 + \sum_{k=1}^K a_k z^{-k} \right) = X(z) \sum_{m=0}^M b_m z^{-m}$$

- Transfer (system) function*

The Z transform of LCCDE LTI systems is described by fraction of two polynomials

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^M b_m z^{-m}}{1 + \sum_{k=1}^K a_k z^{-k}} = \frac{b_0}{a_0} z^{(K-M)} \frac{\prod_{m=1}^M (z - z_m)}{\prod_{k=1}^K (z - p_k)}$$

Z-transform of filters defined by a difference equation

$$y[n] = \sum_{k=1}^K a_k y[n-k] + \sum_{m=0}^M b_m x[n-m]$$

- The Z-transform of a digital filter defined by a finite difference equation is a rational function in z
- Since any reasonably well-behaved function of z can be approximated by a rational function, any LTI filter can be approximated by a finite difference equation that can be implemented

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^M b_m z^{-m}}{1 + \sum_{k=1}^K a_k z^{-k}} = \frac{b_0}{a_0} z^{(K-M)} \frac{\prod_{m=1}^M (z - z_m)}{\prod_{k=1}^K (z - p_k)}$$

- The K complex roots of the denominator of $H(z)$ are called the poles of the filter
- The M complex roots of the numerator are called zeros of the filter
- For FIR filters, the denominator is unity, so that the system function has only zeros.
- For purely recursive filters, the numerator is the constant b_0 , so that the system function has only poles (IIR, AR; otherwise IIR, ARMA)
- This explains the terms *all-zero* and *all-pole* to designate FIR filters and purely-recursive filters

(Definition)

- Let consider the response of a filter with unit-sample response $h[n]$ to the complex exponential z^n , where z is an arbitrary complex number:

$$y[n] = h[n] * z^n = \sum_{m=-\infty}^{\infty} h[m]z^{n-m}$$

$$y[n] = z^n \sum_{m=-\infty}^{\infty} h[m]z^{-m} = x[n]H(z)$$

$$H(z) \triangleq \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

- For LTI systems described by a unit-sample response $h[n]$, $H(z)$ is also referred to as the **system function**. The system function is a generalization of the frequency response, since the DTFT is a special case of the Z-transform, that is:

$$H(f) = H(z)|_{z=e^{j2\pi f}}$$