

FREQUENCY-DOMAIN ANALYSIS OF DIGITAL FILTERS

- Frequency response of LTI systems
- Frequency response of LTI systems, FIR example
- Frequency response of LTI systems, IIR example
- Frequency response of LTI systems, discrete case
- FIR filters implemented as IIR filters, integer multiplier filters
- Laboratory work
- (Frequency response of LTI systems, example)
- (FIR filters implemented as IIR filters, integer multiplier filters)
- (Examples)

Frequency response of LTI systems

- **Transfer function, $H(z)$** , is complex function of a complex variable over entire z plane, $h[n]$ (impulse response) \rightarrow Z transform $\rightarrow H(z)$
- **Frequency Response, $H(\omega)$** , is **Transfer Function, $H(z)$** , evaluated on the unit circle

$$H(\omega) = H(z) \big|_{z=e^{j\omega} \text{ or } |z|=1} = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}, \quad \omega = 2\pi f$$

- $H(\omega)$ - **Frequency response** of the system (in the frequency domain)
- $|H(\omega)|$ - Amplitude (magnitude) response of the system
- $\Theta(\omega)$ - Phase response of the system

$$H(\omega) = H_R(\omega) + j H_I(\omega) \quad (\text{DTFT})$$

$$H(\omega) = |H(\omega)| \cdot e^{j\theta(\omega)} \quad (\text{Polar notation})$$

$$|H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)}$$

$$\theta(\omega) = \angle H(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right)$$

Frequency response of LTI systems, FIR example

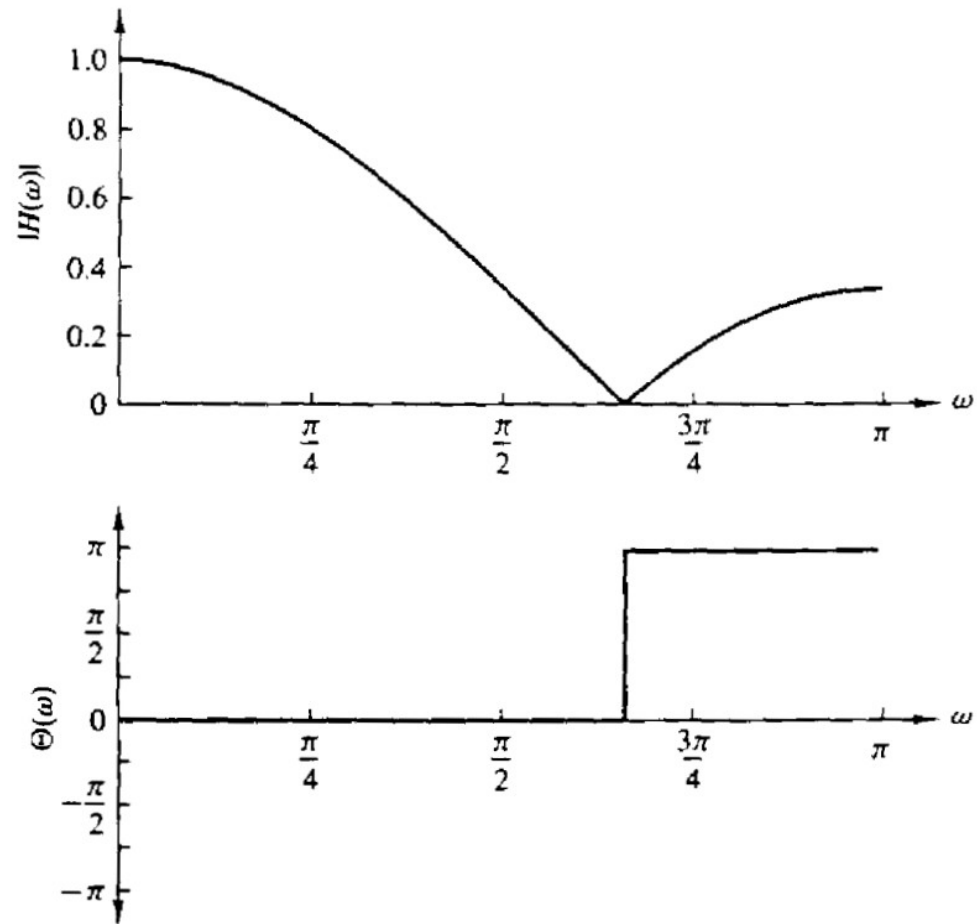
$$y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$$

$$h(n) = \left\{ \frac{1}{3}, \underset{\uparrow}{\frac{1}{3}}, \frac{1}{3} \right\}$$

$$H(\omega) = \frac{1}{3}(e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3}(1 + 2\cos\omega)$$

$$|H(\omega)| = \frac{1}{3}|1 + 2\cos\omega|$$

$$\Theta(\omega) = \begin{cases} 0, & 0 \leq \omega \leq 2\pi/3 \\ \pi, & 2\pi/3 \leq \omega < \pi \end{cases}$$



(Proakis, Manolakis)

Frequency response of LTI systems, IIR example

- Example** $y[n] = 0.8y[n-1] + x[n]$

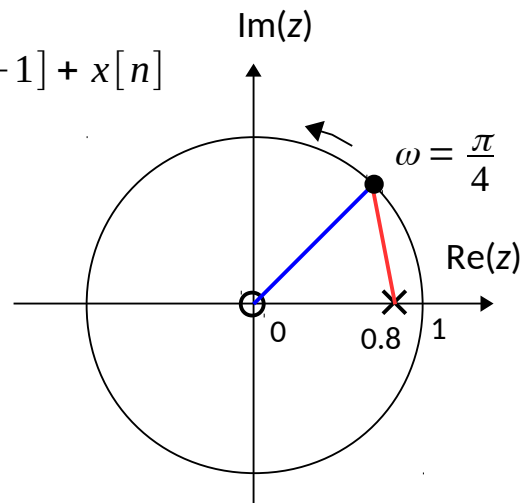
$$H(z) = \frac{1}{1 - 0.8z^{-1}} = \frac{z}{z - 0.8}$$

$$z_1 = 0, \quad p_1 = 0.8$$

$$H(\omega) = \frac{e^{j\omega}}{e^{j\omega} - 0.8}$$

$$|H(\omega)| = \frac{V_1(\omega)}{U_1(\omega)} = \frac{|e^{j\omega}|}{|e^{j\omega} - 0.8|}$$

$$|H(\omega)| = \frac{1}{\sqrt{1.64 - 1.6 \cos \omega}}$$



$$|H(0)| = \frac{1}{0.2} = 5$$

$$|H(\frac{\pi}{4})| = \frac{1}{0.71} = 1.4$$

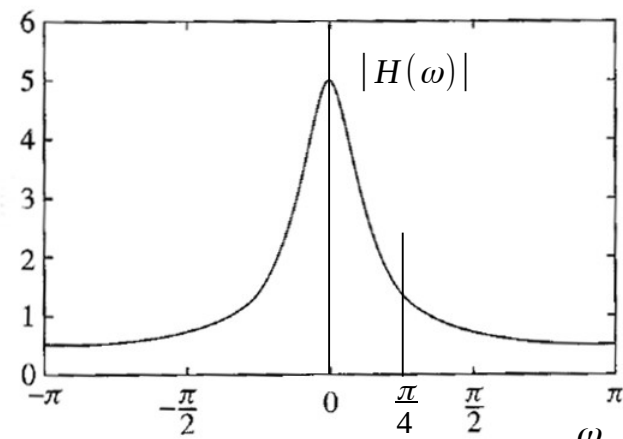
$$\theta(\omega) = \Theta_1(\omega) - \Phi_1(\omega)$$

$$\theta(0) = 0 - 0 = 0$$

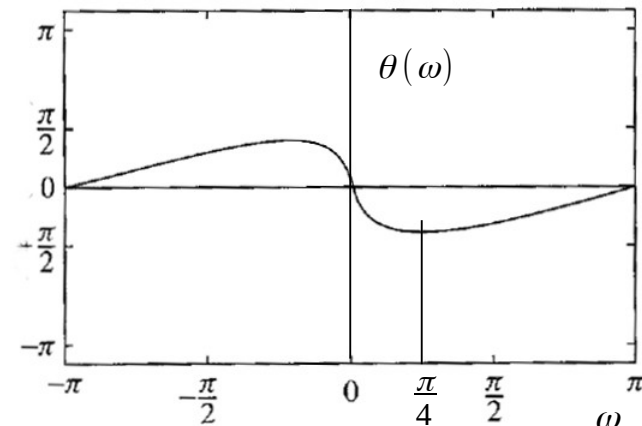
$$\theta(\omega) = \omega - \arctan \frac{\sin \omega}{\cos \omega - 0.8}$$

$$\theta(\frac{\pi}{4}) = \frac{\pi}{4} - 1.7 = -0.91$$

Amplitude response



Phase response



Frequency response of LTI systems, discrete case

- The Z-transform is equivalent to DTFT transform on the unit circle in the Z plane

$$z = e^{j\omega} = e^{j2\pi f} = e^{j2\pi F/F_s}$$

- Discrete case, DFT (in terms of sampling frequency):

$$z_k = e^{j\omega_k} = e^{j2\pi f_k} = e^{j2\pi F_k/F_s}$$

$$\omega_k = 2\pi f_k = 2\pi F_k/F_s$$

$$2\pi F_k/F_s = 2\pi k/N, \quad k = 0, 1, 2, \dots, N-1$$

$$F_k = (k/N) \cdot F_s \quad f_k = k/N$$

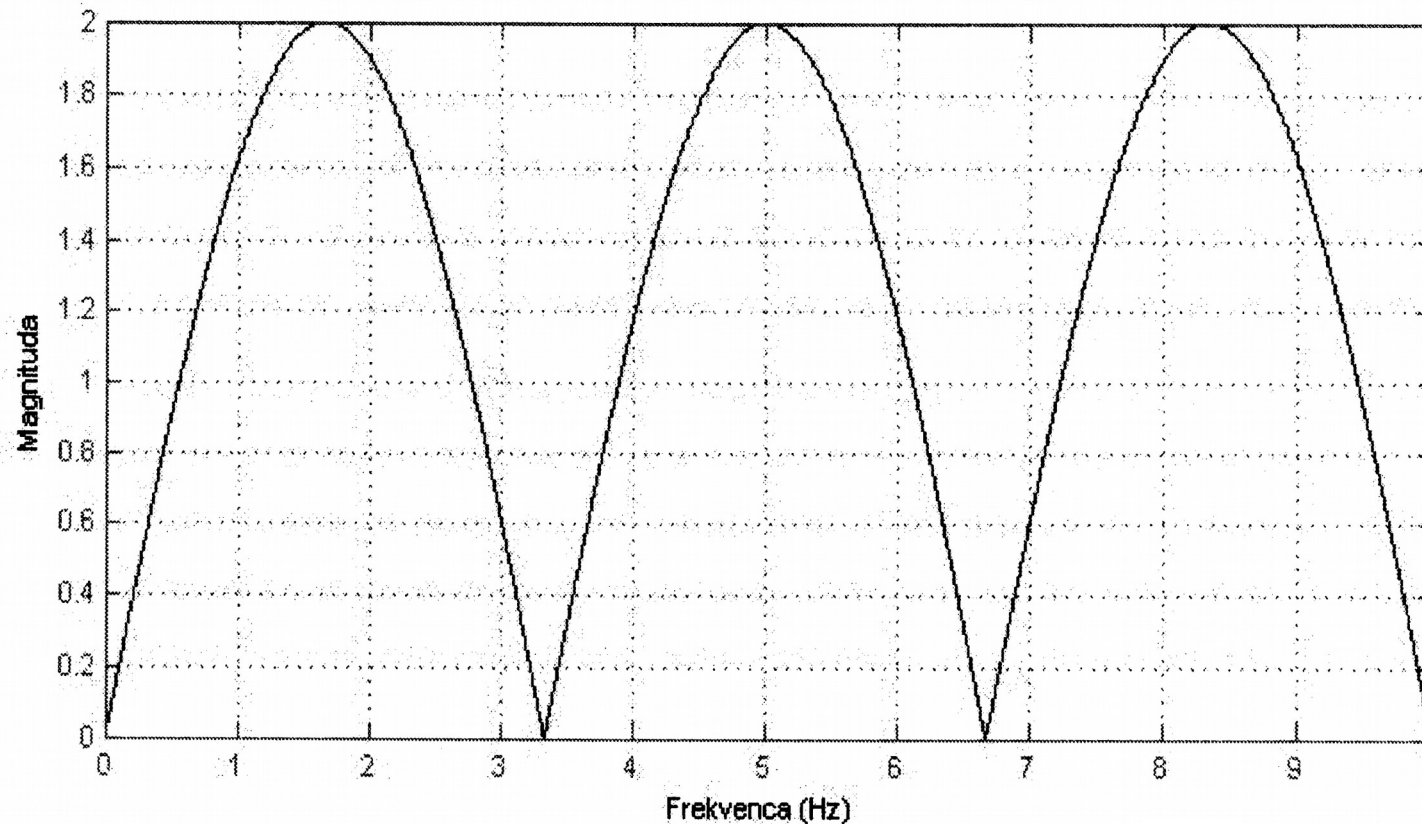
FIR filters implemented as IIR filters, integer multiplier filters

- Amplitude response characteristic of a differentiator ($F_s = 20$ smp/sec)

$$H(z) = (1 - z^{-6})$$

$$|H(\omega)|$$

$$\omega_k = 2\pi(k/N)F_s, \quad F_k = (k/N)F_s$$



FIR filters implemented as IIR filters, integer multiplier filters

- Example, recall moving average

$$M = 8$$

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]$$

$$h[n] = \begin{cases} \frac{1}{M}, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

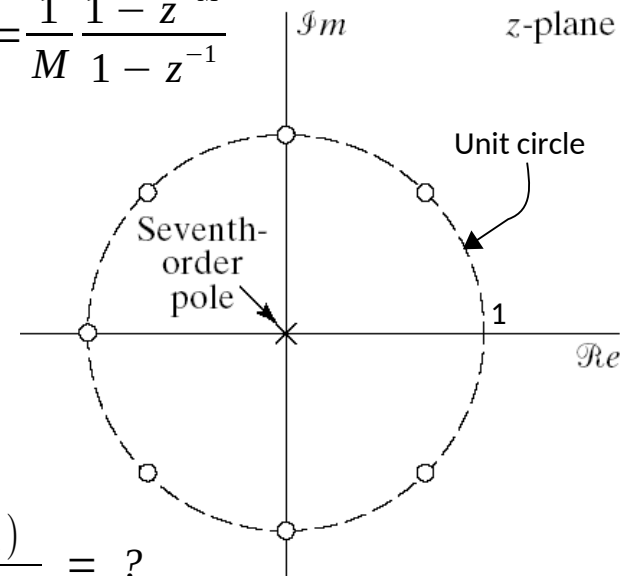
- The transfer function is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \frac{1}{M} \sum_{n=0}^{M-1} z^{-n} = \frac{1}{M} (1 + z^{-1} + \dots + z^{-(M-1)}) = \frac{1}{M} \frac{1 - z^{-M}}{1 - z^{-1}}$$

- The zeros, $z_{(k+1)}$, can be written as

$$z_{(k+1)} = a e^{j2\pi k / M}, \quad k = 0, 1, \dots, M-1$$

- For $k = 0$ we have a zero at $z_1 = 1$
- The zero cancels the pole at $p_1 = 1$



$$H(z) = \frac{1}{M} \frac{1 - z^{-M}}{1 - z^{-1}} = \frac{1}{M} \frac{z^M}{z^M} \frac{(1 - z^{-M})}{(1 - z^{-1})} = \frac{1}{M} \frac{(z^M - 1)}{z^{M-1}(z - 1)} = ?$$

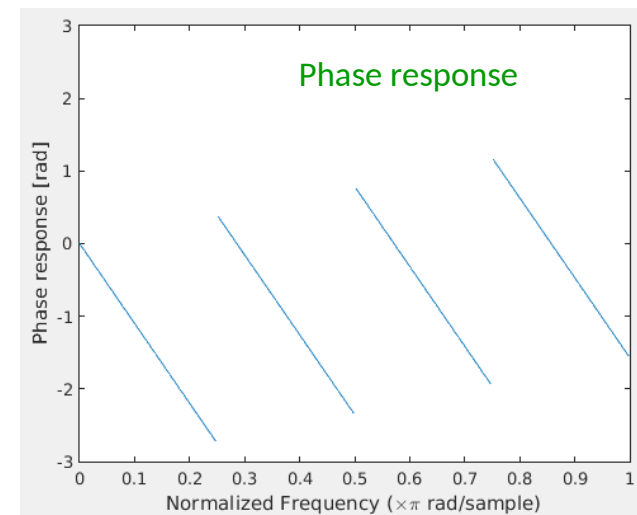
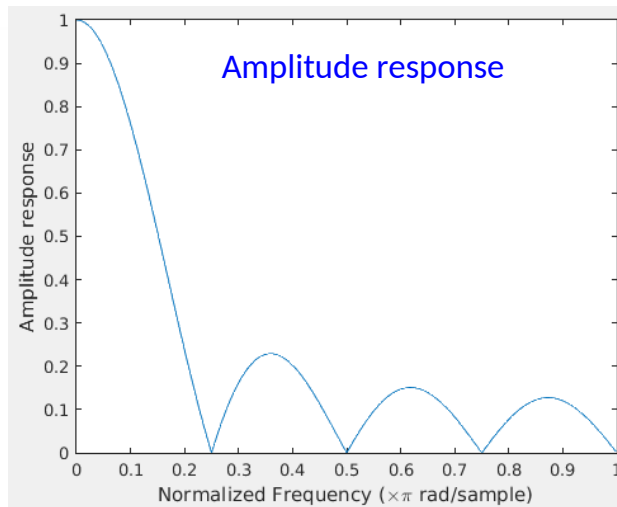
FIR filters implemented as IIR filters, integer multiplier filters

- Example, recall moving average

$M = 8$

$$H(\omega) = |H(\omega)| \cdot e^{j\theta(\omega)}$$

$$H(\omega) = \frac{1}{M} e^{-j\omega \frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$



$$|H(\omega)| = \left| \frac{1}{M} \right| \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right|$$

$$\theta(\omega) = -\frac{(M-1)}{2} \omega + \pi r$$

FIR filters implemented as IIR filters, integer multiplier filters

- Example, recall moving average

$M = 8, M = 1000 ?$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \frac{1}{M} \sum_{n=0}^{M-1} z^{-n} = \frac{1}{M} (1 + z^{-1} + \dots + z^{-M+1}) = \frac{1}{M} \frac{1 - z^{-M}}{1 - z^{-1}}$$

The output, $y[n]$

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]$$

$$y[n] = y[n-1] + \frac{1}{M} (x[n] - x[n-M])$$

$$y'[n] = y'[n-1] + (x[n] - x[n-M])$$

$$y[n] \leftarrow y'[n] \cdot \frac{1}{M}$$

FIR filters implemented as IIR filters, integer multiplier filters

- $M \rightarrow m$

$$H(z) = \frac{(1 - z^{-M})}{(1 - z^{-1})} \quad (M \rightarrow m) \rightarrow H(z) = \frac{(1 - z^{-m})}{(1 - z^{-1})}$$

- Introducing another parameter, M

$$H(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M}$$

What are the **amplitude** and **phase response characteristics** of the following moving average filter? What are the delays of output signal? How do m and M influence the characteristics and delays?

$$H(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M} = e^{-j\omega T(\frac{m}{2} - \frac{1}{2})M} \cdot \left(\frac{\sin(m/2 \omega T)}{\sin(1/2 \omega T)} \right)^M$$

Laboratory work

- Study the **frequency, amplitude and phase response characteristics** for the following filters. What are the delays of output signals? How do ***l, m, n, N, M, ak, bk*** and ***ck*** influence the characteristics and delays?

$$H(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M}$$

$$H(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M} \quad a_k, b_k \in \mathbb{Z}$$

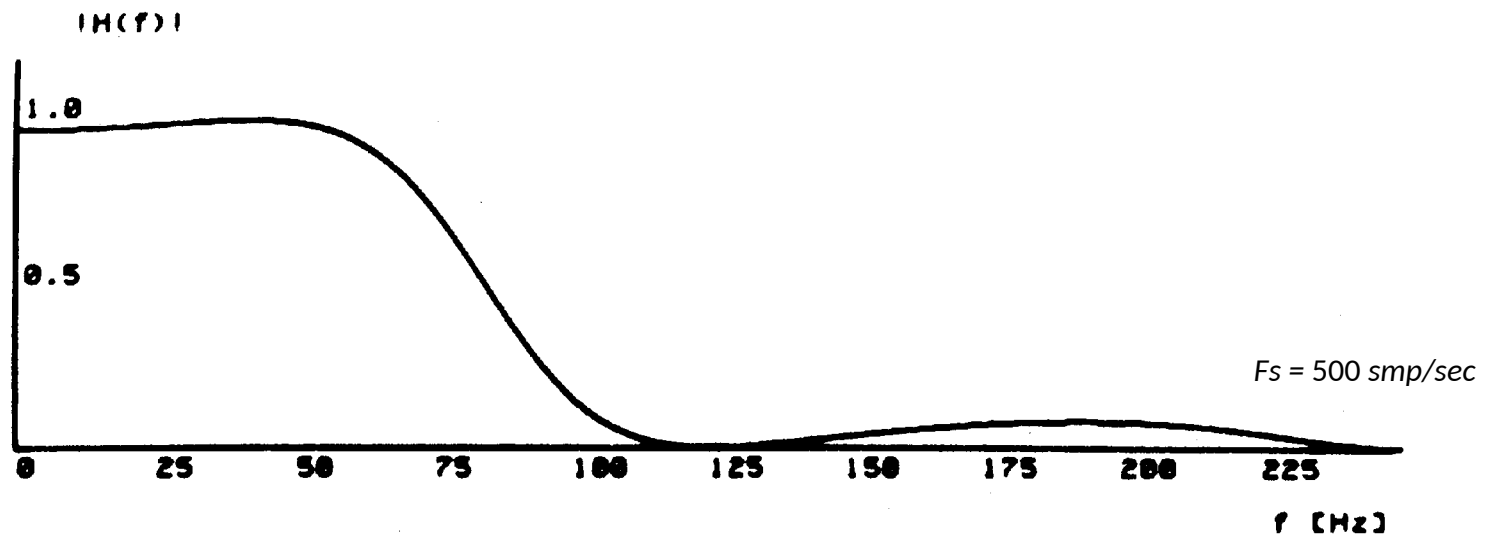
$$H(z) = (1 + c_k z^{-l})^N \quad c_k \in \mathbb{R}$$

$$H(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M} \cdot \frac{1}{(1 + c_k z^{-l})^N}$$

Laboratory work

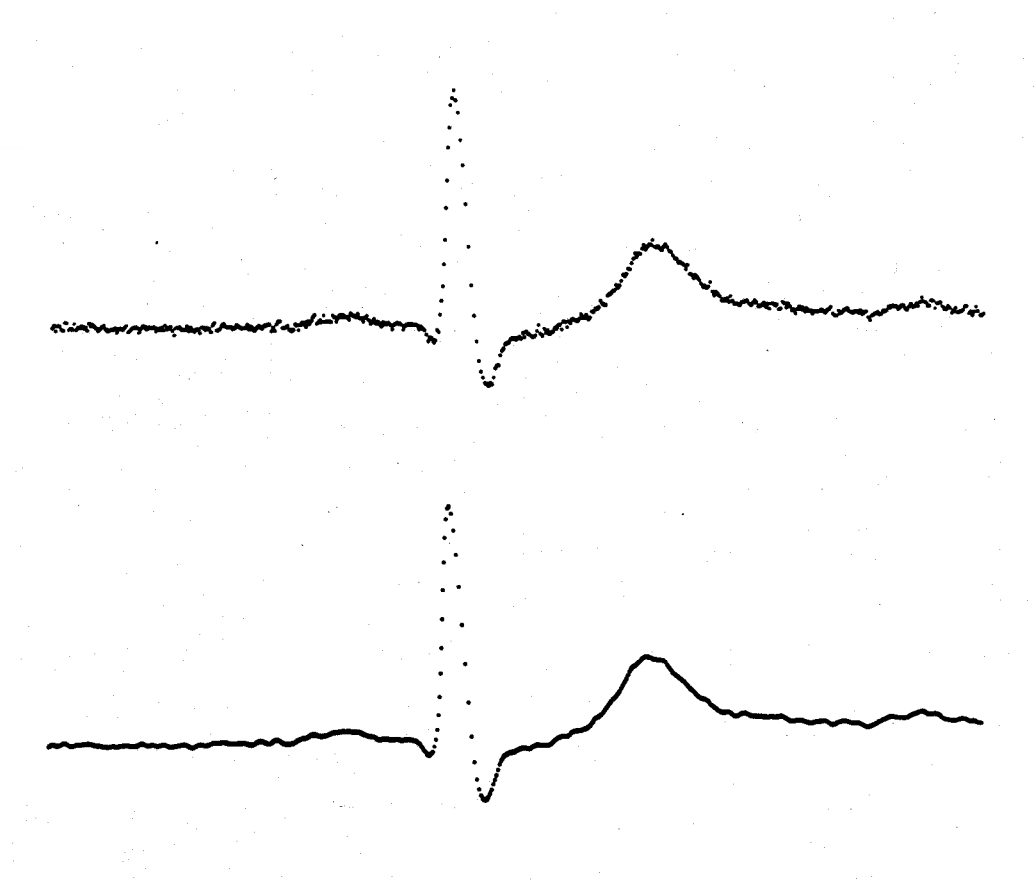
- Low-pass filter ($m = 4$, $l = 3$, $M = 2$, $c_k = 0.25$, $N = 2$)

$$H(z) = H_L(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M} \cdot \frac{1}{(1 + c_k z^{-l})^N}$$



Laboratory work

- Original signal and signal after low-pass filtering



(Frequency response of LTI systems, example)

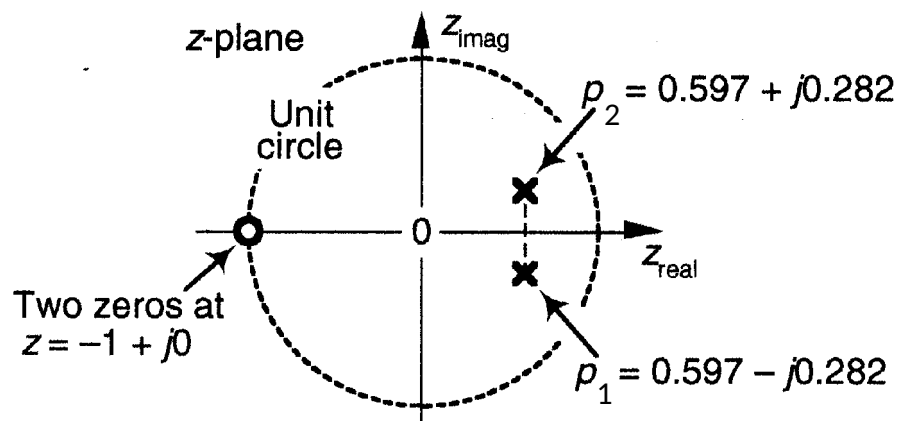
- Transfer function, $H(z)$, is complex function of a complex variable over entire z plane
- **Frequency Response** is Transfer Function, $H(z)$, evaluated on the unit circle

$$H(\omega) = H(z) \big|_{z=e^{j\omega} \text{ or } |z|=1}$$

- Example

$$H(z) = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = \frac{(z+1)(z+1)}{(z-0.597+j0.282)(z-0.597-j0.282)}$$

$$H(z) = \frac{(1+2z^{-1}+z^{-2})}{(1-1.194z^{-1}+0.4359z^{-2})}$$

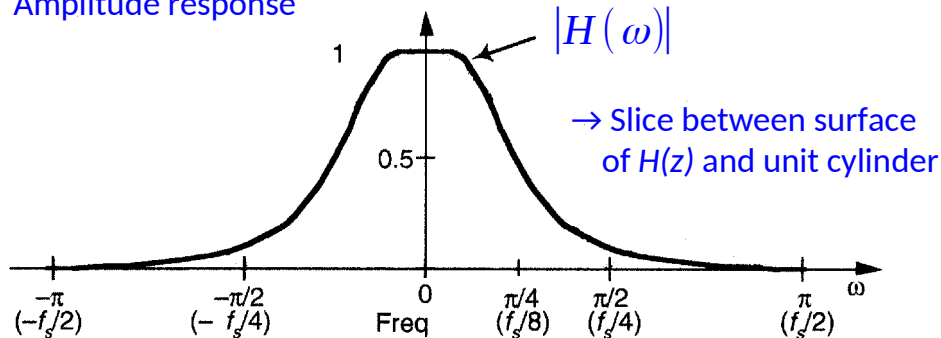


(Frequency response of LTI systems, example)

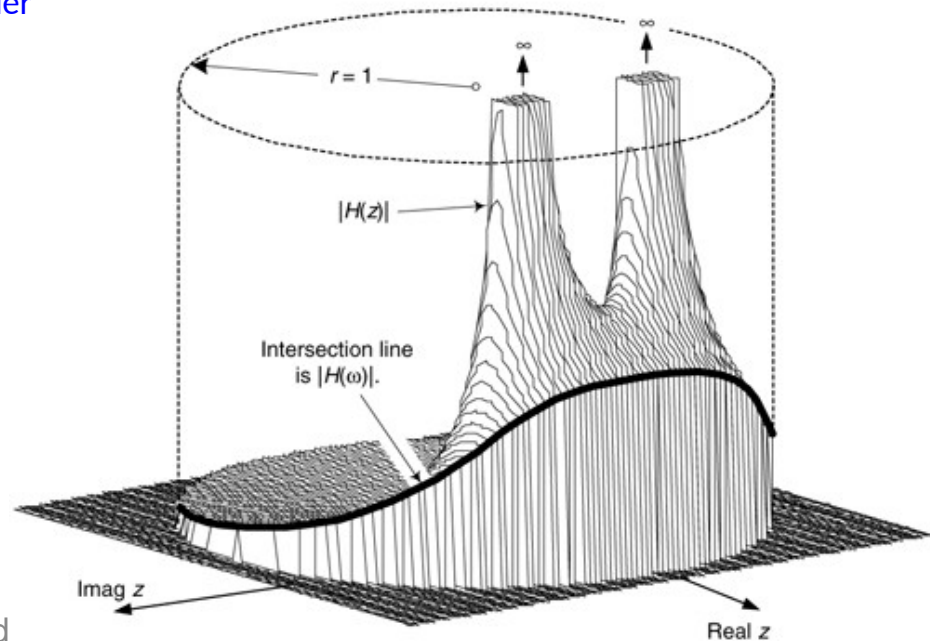
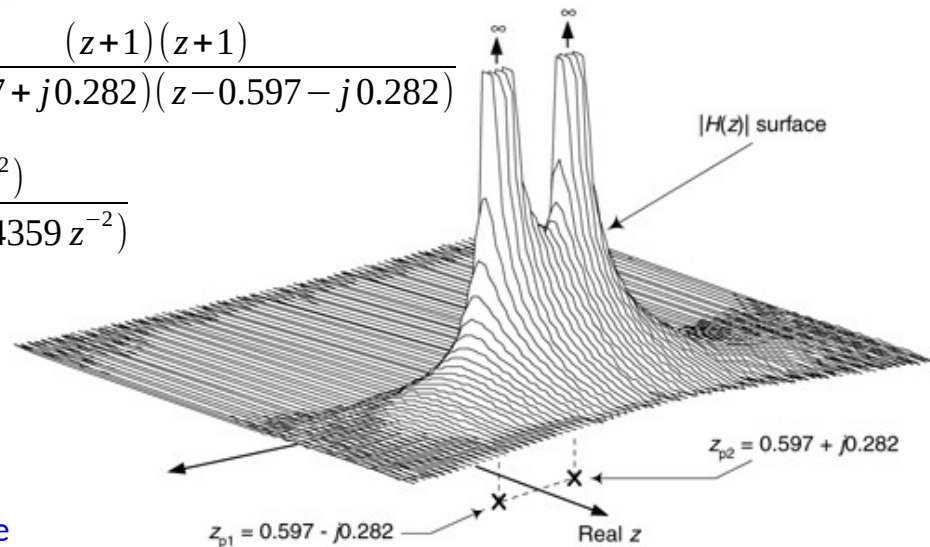
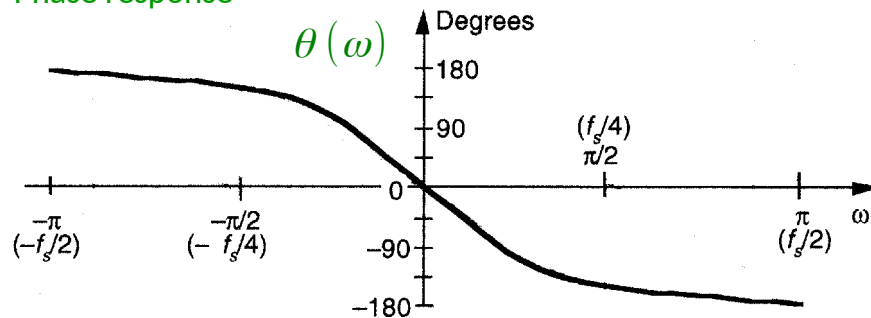
$$H(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} = \frac{(z + 1)(z + 1)}{(z - 0.597 + j0.282)(z - 0.597 - j0.282)}$$

$$H(z) = \frac{(1 + 2z^{-1} + z^{-2})}{(1 - 1.194z^{-1} + 0.4359z^{-2})}$$

Amplitude response

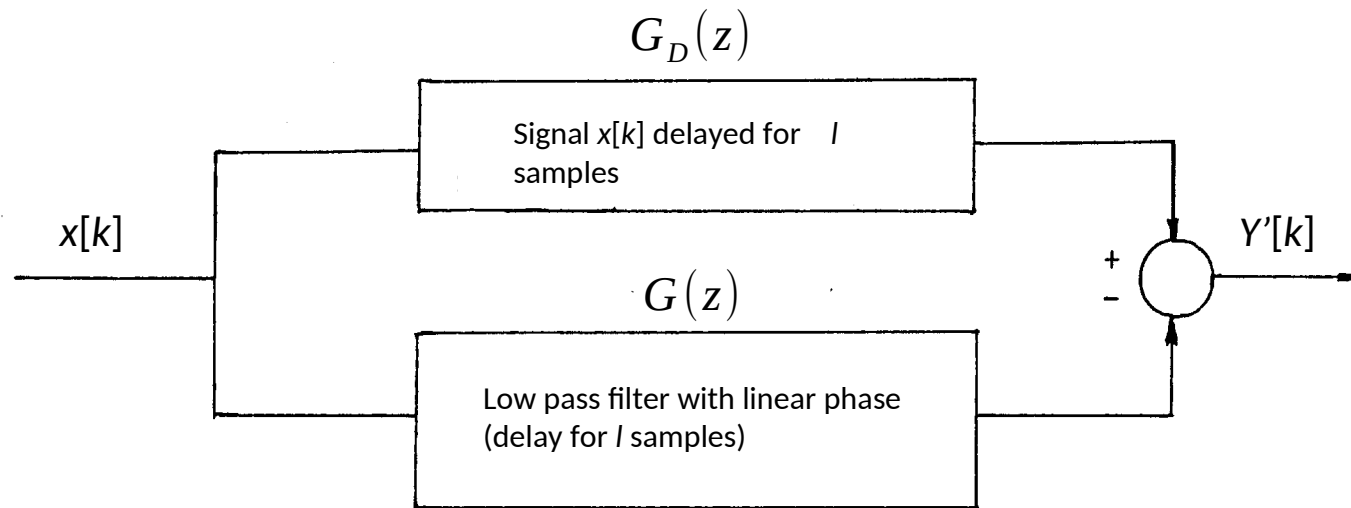


Phase response



(FIR filters implemented as IIR filters, integer multiplier filters)

- How to obtain band-stop characteristic using band-pass filter (or, vice versa)?



$$G(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M}$$

$$H(z) = G_D(z) - G(z) = \left(\frac{m}{n}\right)^M \cdot z^{-j\omega\left(\frac{m}{2} \cdot M - \frac{n}{2} \cdot M\right)} - G(z)$$

(FIR filters implemented as IIR filters, integer multiplier filters)

- High-pass filtering using low-pass filter, $H_{LP}(z)$

$$H_{LP}(z) = \frac{(1 - z^{-344})^2}{(1 - z^{-1})^2}$$

$$G(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M}$$

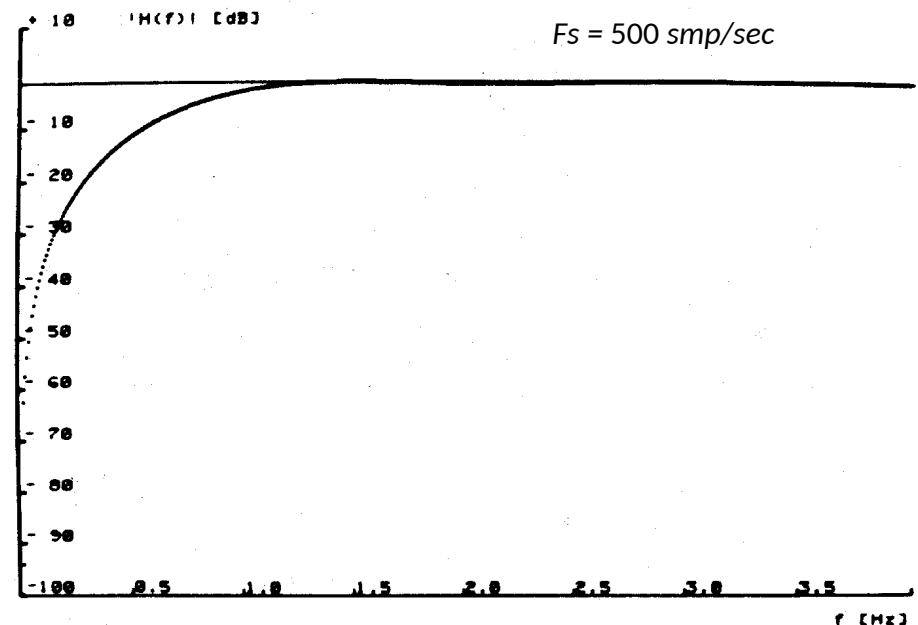
$a_k = -1, b_k = -1, m = 344, n = 1$ in $M = 2$

$$y(k) = 2 \cdot y(k - 1) - y(k - 2) + x(k) - 2 \cdot x(k - 344) + x(k - 688)$$

$$y'(k) = k_v \cdot x(k - 343) - y(k)$$

$$y'(k) \leftarrow y'(k) / k_v$$

$$k_v = 344^2$$



(FIR filters implemented as IIR filters, integer multiplier filters)

- High-pass and 50, 100, 150, 200, 250 Hz notch filtering using combined band-pass filter, i.e., low-pass and 50, 100, 150, 200, 250 Hz band-pass filter, $H_{L,50}(z)$

$$G(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M}$$

$$H_{L,50}(z) = \frac{(1 - z^{-330})^2}{(1 - z^{-10})^2}$$

$$a_k = -1, b_k = -1, m = 330, n = 10 \text{ in } M = 2$$

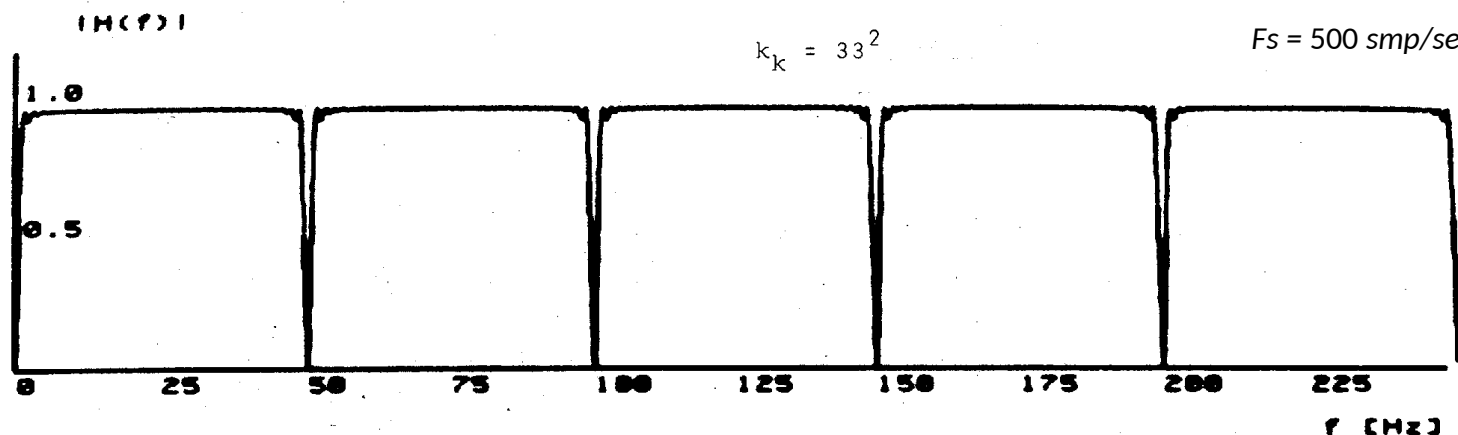
$$y(k) = 2 \cdot y(k - 10) - y(k - 20) + x(k) - 2 \cdot x(k - 330) + x(k - 660)$$

$$y'(k) = k_k x(k - 320) - y(k)$$

$$y'(k) \leftarrow y'(k)/k_k$$

$$k_k = 33^2$$

$$F_s = 500 \text{ smp/sec}$$



(FIR filters implemented as IIR filters, integer multiplier filters)

- 50 Hz notch filtering using 50, 150, 250 Hz band-pass filter, $H_{BP}(z)$

$$H_{BP}(z) = \frac{(1 + z^{-185})^2}{(1 + z^{-5})^2}$$

$$G(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M}$$

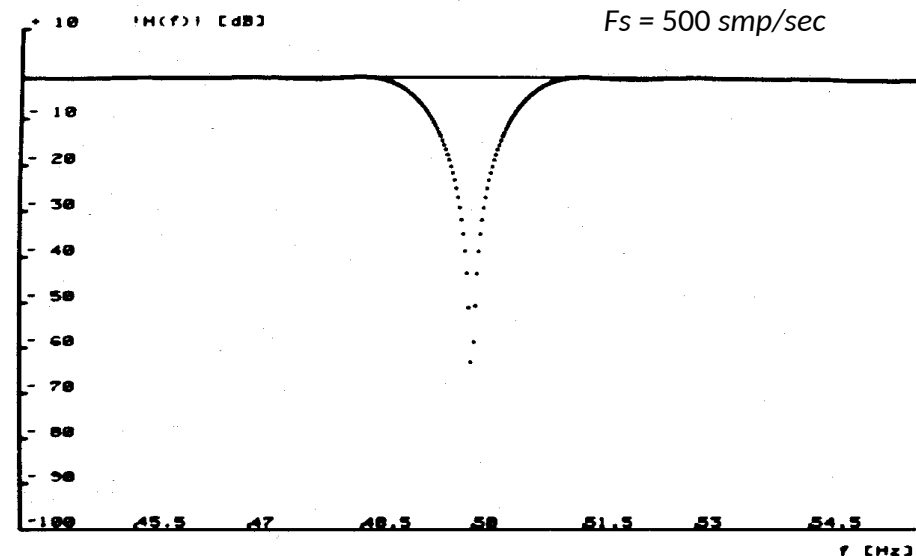
$a_k = 1, b_k = 1, m = 185, n = 5$ in $M = 2$

$$y(k) = -2y(k - 5) - y(k - 10) + x(k) + 2x(k - 185) + x(k - 370)$$

$$y'(k) = k_{50} \cdot x(k - 180) - y(k)$$

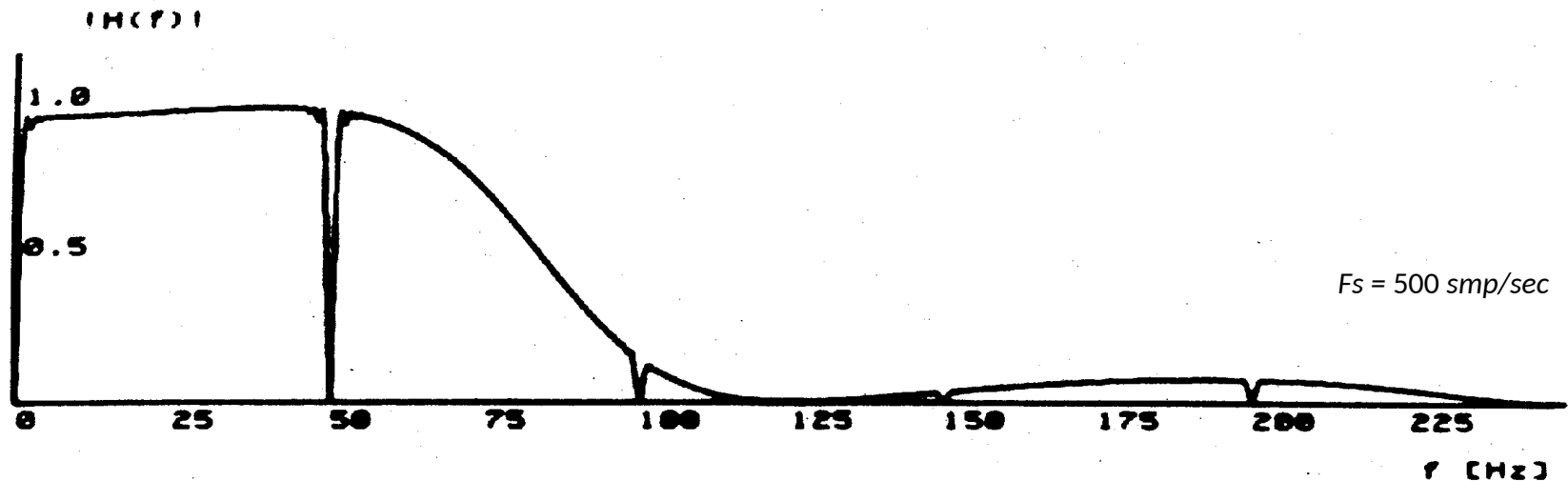
$$y'(k) \leftarrow y'(k) / k_{50}$$

$$k_{50} = 37^2$$



(Examples)

- Amplitude response of low-pass filter, $H_L(z)$, and, high-pass and 50, 100, 150, 200, 250 Hz notch filter, $H_{L,50}(z)$



(Examples)

- Original signal and signal after using $HL(z)$ and $HL_{50}(z)$

