configuration graph model

introduction to *network analysis* (*ina*)

Lovro Šubelj
University of Ljubljana
spring 2019/20

configuration *model*

- random graphs *Poisson distribution* $p_k \simeq \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}$ [ER59]
- real networks *power-law degree distribution* $p_k \sim k^{-\gamma}$ [BA99]
- *configuration model* random graph for *arbitrary* $\{k\}$ [NSW01]

assume *undirected* G from now on

Mark Newman

Steven Strogatz

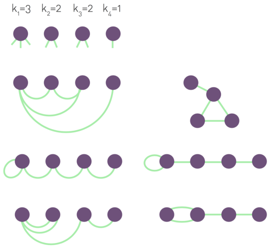
Duncan Watts

configuration $G(\{k\})$ *model*

- $G(\{k\})$ *configuration model* [NSW01]
- *randomly link* m *stub pairs* between n nodes
- *computationally convenient* and *analytically tractable*

graphical $k_1, k_2 \ldots k_n$ $\qquad m = \frac{1}{2}\sum_i k_i$

```
input  sequence {k}
output  graph G
  G ← n nodes with {k} stubs
  while G has node stubs do
      link random node stub pair
  end while
  return G
```



configuration *probability*

— *probability of self-loop* p_i on i

$$p_i = m\frac{\binom{k_i}{2}}{\binom{2m}{2}} \approx \frac{k_i(k_i-1)}{4m}$$

— *probability of link* p_{ij} between i and j

$$p_{ij} = m\frac{k_i k_j}{\binom{2m}{2}} = k_i\frac{k_j}{2m-1} \approx \frac{k_i k_j}{2m}$$

— thus *number of multilinks* and *self-loops* is

$$\left[\frac{\langle k^2\rangle - \langle k\rangle}{\sqrt{2}\langle k\rangle}\right]^2 \qquad \sum_i p_i = \sum_i \frac{k_i(k_i-1)}{2n\langle k\rangle} = \frac{\langle k^2\rangle - \langle k\rangle}{2\langle k\rangle}$$

configuration *neighbors*

- *neighbor degree distribution* p_k is *not* p_k
  - n_k is *number of degree-k* nodes thus $n_k = np_k$

$$\{neighbor\ p_k\} = n_k \frac{k}{2m-1} \approx \frac{kp_k}{\langle k \rangle}$$

- *average neighbor degree* $\langle k \rangle$ is *not* $\langle k \rangle$

$$\frac{\langle k^2 \rangle}{\langle k \rangle} - \langle k \rangle = \frac{\langle k^2 \rangle - \langle k \rangle^2}{\langle k \rangle} = \frac{\sigma_k^2}{\langle k \rangle} > 0$$

$$\langle neighbor\ k \rangle \approx \sum_k k \frac{kp_k}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle} > \langle k \rangle$$

- $\frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle}{\langle k \rangle} = \langle k \rangle + 1$ even for *Poisson graph* [ER59]

network *neighbors*

- *friendship paradox* $\langle neighbor\ k \rangle > \langle k \rangle$ [Fel91] in real networks
- $\langle neighbor\ k \rangle$ *well estimated* by $\frac{\langle k^2 \rangle}{\langle k \rangle}$ whereas $\langle k \rangle \ll \frac{\langle k^2 \rangle}{\langle k \rangle}$

network	n	$\langle k \rangle \ll$	$\langle$**neighbor** $k\rangle$	$\approx \frac{\langle k^2 \rangle}{\langle k \rangle}$
Southern women [DGG41]	32	5.56	7.57	7.02
Karate club [Zac77]	34	4.59	9.61	7.77
American football [GN02]	115	10.71	10.78	10.79
Java dependencies [ŠB11]	1368	16.20	207.52	140.53
Facebook circles [ML12]	4039	43.69	105.55	106.57
Physics collaboration [New01]	36 458	9.42	21.65	27.88
Enron e-mails [LLDM09]	36 692	20.04	472.86	280.16
Internet map [HJJ+03]	75 885	9.42	1853.73	1461.54
Actors collaboration [BA99]	382 219	78.69	282.72	417.69
Physics citation [ŠFB14]	438 943	21.56	78.38	77.72
Patent citation [HJT01]	3 774 768	8.75	17.15	21.33
Facebook snowball [Fer12]	8 217 272	3.06	308.52	157.06

configuration *clustering*

— (*neighbor*) *excess degree distribution* q_k defined as

excess degree is remaining neighbor degree or neighbor degree-1

$$q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}$$

— then *network clustering coefficient* C [NSW01] is

$\sum_{k_i,k_j} q_{k_i} q_{k_j} \frac{k_i k_j}{2m} = \frac{1}{2m} \left[\sum_k k q_k\right]^2 = \frac{1}{2m\langle k \rangle^2} \left[\sum_k k(k+1)p_{k+1}\right]^2 = \frac{1}{n\langle k \rangle^3} \left[\sum_k (k-1)kp_k\right]^2$

$$C = \sum_{k_i,k_j} q_{k_i} q_{k_j} p_{ij} \approx \frac{\left[\langle k^2 \rangle - \langle k \rangle\right]^2}{n\langle k \rangle^3}$$

network *clustering*

- *average clustering coefficient* $\langle C\rangle$ [WS98] of real networks
- *neither* $G(n,p)$ [ER59] *nor* $G(\{k\})$ [NSW01] *explain* $\langle C\rangle \gg 0$

network	n	$\langle C\rangle$	$\gg \frac{[\langle k^2\rangle - \langle k\rangle]^2}{n\langle k\rangle^3}$	$\gg \frac{\langle k\rangle}{n-1}$
Southern women [DGG41]	32	0.000	0.204	0.179
Karate club [Zac77]	34	0.571	0.294	0.139
American football [GN02]	115	0.403	0.078	0.094
Java dependencies [ŠB11]	1368	0.497	0.879	0.012
Facebook circles [ML12]	4039	0.606	0.063	0.011
Physics collaboration [New01]	36 458	0.657	0.002	0.000
Enron e-mails [LLDM09]	36 692	0.497	0.106	0.001
Internet map [HJJ+03]	75 885	0.160	2.985	0.000
Actors collaboration [BA99]	382 219	0.780	0.006	0.000
Physics citation [ŠFB14]	438 943	0.227	0.001	0.000
Patent citation [HJT01]	3 774 768	0.076	0.000	0.000
Facebook snowball [Fer12]	8 217 272	0.019	0.001	0.000

configuration *references*

A.-L. Barabási and R. Albert.
Emergence of scaling in random networks.
Science, 286(5439):509–512, 1999.

A.-L. Barabási.
Network Science.
Cambridge University Press, Cambridge, 2016.

A. Davis, B. B. Gardner, and M. R. Gardner.
Deep South.
Chicago University Press, Chicago, 1941.

David Easley and Jon Kleinberg.
Networks, Crowds, and Markets: Reasoning About a Highly Connected World.
Cambridge University Press, Cambridge, 2010.

P. Erdős and A. Rényi.
On random graphs I.
Publ. Math. Debrecen, 6:290–297, 1959.

Scott. L. Feld.
Why your friends have more friends than you do.
Am. J. Sociol., 96(6):1464–1477, 1991.

Stefano Ferretti.
On the degree distribution of faulty peer-to-peer overlays.
ICST Transactions on Complex Systems, 2012.

M. Girvan and M. E. J Newman.
Community structure in social and biological networks.
P. Natl. Acad. Sci. USA, 99(12):7821–7826, 2002.

configuration *references*

M Hoerdt, M Jaeger, A James, D Magoni, J Maillard, D Malka, and P Merindol.
Internet {IP}v4 overlay map produced by network cartographer (nec), 2003.

B. H. Hall, A. B. Jaffe, and M. Tratjenberg.
The NBER patent citation data file: Lessons, insights and methodological tools.
Technical report, National Bureau of Economic Research, 2001.

Jure Leskovec, Kevin J Lang, Anirban Dasgupta, and Michael W Mahoney.
Community structure in large networks: Natural cluster sizes and the absence of large well-defined clusters.
Internet Math., 6(1):29–123, 2009.

Seth A. Myers and Jure Leskovec.
Clash of the contagions: Cooperation and competition in information diffusion.
In *Proceedings of the IEEE International Conference on Data Mining*, 2012.

M. E. J. Newman.
The structure of scientific collaboration networks.
P. Natl. Acad. Sci. USA, 98(2):404–409, 2001.

Mark E. J. Newman.
Networks: An Introduction.
Oxford University Press, Oxford, 2010.

M. E. J. Newman, S. H. Strogatz, and D. J. Watts.
Random graphs with arbitrary degree distributions and their applications.
Phys. Rev. E, 64(2):026118, 2001.

Lovro Šubelj and Marko Bajec.
Community structure of complex software systems: Analysis and applications.
Physica A, 390(16):2968–2975, 2011.

configuration *references*

Lovro Šubelj, Dalibor Fiala, and Marko Bajec.
Network-based statistical comparison of citation topology of bibliographic databases.
Sci. Rep., 4:6496, 2014.

D. J. Watts and S. H. Strogatz.
Collective dynamics of 'small-world' networks.
Nature, 393(6684):440–442, 1998.

Wayne W. Zachary.
An information flow model for conflict and fission in small groups.
J. Anthropol. Res., 33(4):452–473, 1977.