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Exam 2 Corrections

1. Show that $Y_1 = e^{5x}$, $Y_2 = e^{-7x}$ form a fundamental set of solutions of the differential equation $Y'' + 2Y' - 35Y = 0$ on the interval $(-\infty, \infty)$

Solution:

$$W = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = \begin{vmatrix} e^{5x} & e^{-7x} \\ 5e^{5x} & -7e^{-7x} \end{vmatrix} = (e^{5x} \cdot -7e^{-7x}) - (5e^{5x} \cdot e^{-7x})$$

$$= -7e^{-2x} - (5e^{-2x}) = -12e^{-2x} \neq 0$$

$\therefore$ Y_1 & Y_2 are linearly independant.

→ Prove they are solutions of $Y'' + 2Y' - 35Y = 0$

$$Y'' + 2Y' - 35Y = 0, \quad m^2 + 2m - 35 = 0$$

$$b^2 = 4 > -140 = 4 \cdot 1 \cdot (-35) = 4ac, \quad a = 1 \quad b = 2 \quad c = -35$$

$\therefore$ real distinct roots $\quad m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$m = \frac{-2 \pm \sqrt{4 + 140}}{2} = \frac{-2 \pm 12}{2} = -1 \pm 6, \quad m_1 = -7, \quad m_2 = 5$$

$$Y = C_1 e^{5x} + C_2 e^{-7x}$$

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If $Y(x) = \cos(4x)$ is a solution to $Y'' + 16Y = 0$ use the method of reduction of order to find a second solution using the substitution $Y_2 = U(x) \cdot Y_1$

Solution: $Y_2 = u \cdot Y_1 = u \cdot \cos(4x)$

$Y_2' = [u'] \cdot \cos(4x) + u[-\sin(4x) \cdot 4] = u' \cdot \cos(4x) - 4u \cdot \sin(4x)$

$Y_2'' = ([u''] \cdot \cos(4x) + u' \cdot [-4\sin(4x)]) - (4u \cdot [4 \cdot \cos(4x)] + [4u'] \cdot 4 \cdot \sin(4x))$

plug into $Y'' + 16Y = 0$

$0 = Y'' + 16Y = [u'' \cdot \cos(4x) - 8u' \cdot \sin(4x) - 16u \cdot \cos(4x)] + 16[u \cdot \cos(4x)]$

$0 = u'' \cdot \cos(4x) - 8u' \cdot \sin(4x)$; substitute $w = u'$

$0 = w' \cdot \cos(4x) - 8w \cdot \sin(4x)$ } linear first order ode

$$0 = w' - \frac{8w \cdot \sin(4x)}{\cos(4x)} = w' - 8w \cdot \tan(4x)$$ } standard form

Use integrating factor method: $P(x) = -8\tan(4x)$; $e^{\int P(x)dx}$

$\int P(x)dx = \int -8\tan(4x)dx = -8 \cdot \frac{1}{4} \cdot \ln(|\sec(4x)|) + C = -2\ln(|\sec(4x)|) + C$

$e^{\int P(x)dx} = e^{-2 \cdot \ln(|\sec(4x)|)} = e^{-2 \cdot \ln(\frac{1}{\cos(4x)})} = e^{\ln((\frac{1}{\cos(4x)})^{-2})} = (\frac{1}{\cos(4x)})^{-2} = \cos^2(4x)$

distribute: $$w' \cdot \cos^2(4x) - \frac{8w \cdot \sin(4x)}{\cos(4x)} \cdot \cos^2(4x) = 0$$

reverse product rule: $\frac{d}{dx}(w \cdot \cos^2(4x)) = 0$; $\int \frac{d}{dx}(w \cdot \cos^2(4x))dx = \int 0 \, dx$

$w \cdot \cos^2(4x) = C_1$; $w = \frac{C_1}{\cos^2(4x)}$ since $w = u'$, $u = \int w$

$u = \int \frac{C_1}{\cos^2(4x)} dx = C_1 \int \frac{1}{\cos^2(4x)} dx = C_1 \int \sec^2(4x)dx$; sub $u = 4x$ $du = 4dx$

$C_1 \cdot \int \frac{\sec^2(u) \cdot du}{4} = \frac{C_1}{4} \int \sec^2(u)du = (\tan(u) + C_2)\frac{C_1}{4} = (\tan(4x) + C_2)\frac{C_1}{4}$

$= C_3 \cdot \tan(4x) + C_4$

$$Y_2 = (C_3 \cdot \tan(4x) + C_4) \cdot \cos(4x) = C_3 \cdot \sin(4x) + C_4 \cdot \cos(4x)$$

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Exam 2 Corrections

3. a.) use the method of undetermined coefficients to solve IVP
$y''-14y'+49y=5e^{7x}+1$, $y(0)=2$ $y'(0)=11$
b.) Use the reduction of order formula to confirm a second solution to the homogeneous solution.

solution* find y_c: $m^2-14m+49=0$, $(m-7)(m-7)=0$ $m=7$ twice
$y_c=C_1e^{7x}+C_2xe^{7x}$
find y_p: guess $y_p=Ax^2e^{7x}+B$
$y_p'=2Axe^{7x}+7Ax^2e^{7x}$
$y_p''=2Ae^{7x}+14Axe^{7x}+14Axe^{7x}+49Ax^2e^{7x}$
sub: $2Ae^{7x}+28Axe^{7x}+49Ax^2e^{7x}-14(2Axe^{7x}+7Ax^2e^{7x})+49(Ax^2e^{7x}+B)$
$=5e^{7x}+1$
$2Ae^{7x}+49B=5e^{7x}+1$; $2A=5$, $A=\frac{5}{2}$ $B=1/49$
$y_p=\frac{5}{2}x^2e^{7x}+\frac{1}{49}$; $y=C_1e^{7x}+C_2xe^{7x}+\frac{5}{2}x^2e^{7x}+\frac{1}{49}$
$y'=\frac{35}{2}x^2e^{7x}+7C_2xe^{7x}+5xe^{7x}+C_2e^{7x}+7C_1e^{7x}$
for $y(0)=2$: $2=C_1+\frac{1}{49}$, $C_1=\frac{97}{49}$
$y'=7(\frac{97}{49})e^{7x}+C_2\cdot e^{7x}+7C_2\cdot xe^{7x}+2(\frac{5}{2})\cdot xe^{7x}+7(\frac{5}{2})x^2e^{7x}$
for $y'(0)=11$: $11=\frac{97}{7}+C_2$, $C_2=-\frac{20}{7}$
$y=(\frac{97}{49})e^{7x}-(\frac{20}{7})xe^{7x}+\frac{5}{2}x^2e^{7x}+\frac{1}{49}$

b.) $y_1=C_1e^{7x}$; $y_2=y_1\cdot\int\frac{e^{-\int P(x)dx}}{y_1^2}dx$

$y_2=C_1\cdot e^{7x}\cdot\int\frac{e^{-\int -14dx}}{(C_1e^{7x})^2}=C_1e^{7x}\cdot\int\frac{e^{14x}}{(C_1e^{7x})^2}\overset{C_1e^{7x}}{=}\int\frac{e^{14x}}{C_1^2e^{14x}}\overset{C_1e^{7x}}{=}\int\frac{x}{C_1^2}$

$=C_1e^{7x}\cdot\frac{x}{C_1^2}=C_2xe^{7x}$ ✓

4. Use variation of parameters to solve IVP $y'' - 14y' + 49y = 5e^{7x} + 1$

Solution: $y_c = c_1 e^{7x} + c_2 x e^{7x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{7x} & xe^{7x} \\ 7e^{7x} & e^{7x}+7xe^{7x} \end{vmatrix} = e^{7x}(e^{7x} + 7xe^{7x}) - 7xe^{7x}$

$= e^{14x} + 7xe^{14x} - 7xe^{14x} = e^{14x}$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{7x} \\ 5e^{7x}+1 & e^{7x}+7xe^{7x} \end{vmatrix} = -xe^{7x}(5e^{7x}+1) = -5xe^{14x} - xe^{7x}$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} e^{7x} & 0 \\ 7e^{7x} & 5e^{7x}+1 \end{vmatrix} = 5e^{14x} + e^{7x}$

$u_1' = \dfrac{5xe^{14x} + xe^{7x}}{e^{14x}} = 5x + xe^{-7x}$

$u_2' = \dfrac{5e^{14x} + e^{7x}}{e^{14x}} = 5 + e^{-7x}$

$u_1 = \int u_1' = \int 5x + xe^{-7x}$

$= \frac{5}{2}x^2 - \frac{1}{7}xe^{-7x} - \frac{1}{49}e^{-7x}$

$u_2 = \int u_2' = \int 5 + e^{-7x}$

$= 5x - \frac{1}{7}e^{-7x}$

$y_p = \left(\dfrac{5x^2}{2} - \dfrac{1}{7}xe^{-7x} - \dfrac{1}{49}e^{-7x}\right)e^{7x} + \left(5x - \frac{1}{7}e^{-7x}\right)xe^{7x}$

$y = 2e^{7x} - 3xe^{7x} + \left(\dfrac{5x^2}{2} - \dfrac{1}{7}xe^{-7} - \dfrac{1}{49}e^{-7x}\right) + \left(5x - \frac{1}{7}e^{-7x}\right)xe^{7x}$

5. Find inverse laplace transform of $F(s) = \frac{3}{9s^2+4}$

$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$

$\dfrac{3}{9s^2+4} = \dfrac{1}{2} \cdot \dfrac{\frac{2}{3}}{s^2 + (\frac{2}{3})^2}$, $\dfrac{1}{2} \cdot \mathcal{L}^{-1}\left[\dfrac{\frac{2}{3}}{s^2+(\frac{2}{3})^2}\right] = \frac{1}{2}\sin(\frac{2}{3}t)$

6. Use the laplace transform to Solve IVP

$y'' + 16y = e^{5t}$, $y(0) = 0$, $y'(0) = 0$

this is a mathematical model for what spring system?

$\mathcal{L}\{y''\} + 16\mathcal{L}\{y\} = \mathcal{L}\{e^{5t}\}$

$\mathcal{L}\{y''\} = s^2Y(s) - sY(0) - Y'(0) = s^2Y(s)$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{e^{5t}\} = \frac{1}{s-5}$

$s^2Y(s) + 16Y(s) = \frac{1}{s-5}$, $Y(s)(s^2+16) = \frac{1}{s-5}$

$Y(s) = \frac{1}{(s-5)(s^2+16)} = \frac{A}{s-5} + \frac{Bs+C}{s^2+16}$

$A(s^2+16) + Bs^2 + Cs - 5Bs - 5C = 1$

$As^2 + Bs^2 + Cs - 5Bs + 16A - 5C = 1$

s^2: $A + B = 0$, $A = -B$

s: $C - 5B = 0$, $C = 5B = -5A$

1: $16A - 5C = 1$

$16A - 5(-5A) = 1$

$16A + 25A = 1$

$A = 1/41$ $B = -1/41$ $C = -5/41$

$Y(s) = \mathcal{L}^{-1}\left\{\frac{1/41}{s-5}\right\} + \mathcal{L}^{-1}\left\{\frac{-s/41}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{-5/41}{s^2+16}\right\}$

$Y(t) = \frac{e^{5t}}{41} - \frac{\cos(4t)}{41} - \frac{5\sin(4t)}{164}$