Signals & Systems ELC 321

Mid Term Examination 1

Solution Guide

Problem 1

$$y[n] = A\,x[\alpha n + \beta] + B$$

(a) Time transformation

- Time scaling through α
- Time shifting through β
- Time reversal for $\alpha < 0$

5 Marks (For any 5).

Amplitude transformation

- Amplitude scaling through A
- Amplitude reversal for $A < 0$
- Amplitude shifting for $B \neq 0$.

(b) Given

$$y[n] = A\,x[\alpha n + \beta] + B \quad \text{—(1)}$$

rewriting (1) gives

$$A\,x[\alpha n + \beta] = y[n] - B$$

$$x[\alpha n + \beta] = \frac{1}{A}\left(y[n] - B\right) \quad \text{—(2)}$$

10 MARKS

Defining $m = \alpha n + \beta$ (which implies $n = \frac{1}{\alpha}(m - \beta)$

Substituting into (2) gives

$$x[m] = \frac{1}{A}\left(y\left[\tfrac{1}{\alpha}(m-\beta)\right] - B\right) = \frac{1}{A}\,y\left[\frac{m-\beta}{\alpha}\right] - \frac{B}{A}.$$

Hence we may write

$$\boxed{x[n] = \frac{1}{A}\,y\left[\frac{n-\beta}{\alpha}\right] - \frac{B}{A}}$$

(c) $y[n] = 0.5x[-n+1] + 2$

Comparing with equation 1 gives

$A = 0.5$, $B = 2$, $\alpha = -1$, and $\beta = 1$

Therefore, from (b)

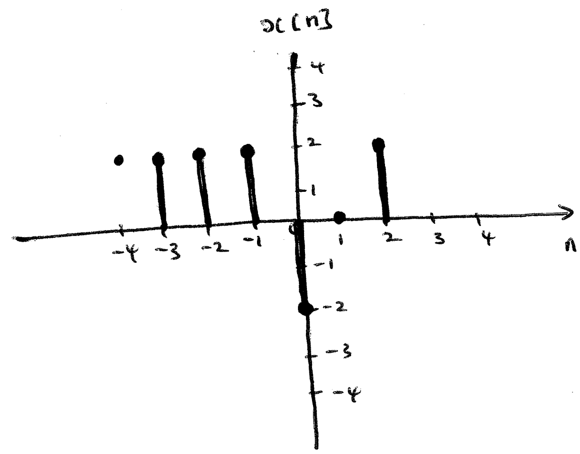
$$x[n] = \frac{1}{0.5} y\left[\frac{n-1}{-1}\right] - \frac{2}{0.5}$$

$$x[n] = 2y[-n+1] - 4$$

First, we do the amplitude transformation i.e

$$x[m] = 2y[m] - 4 \quad (\text{setting } m = -n+1)$$

m	y[m]	x[m]
-3	3	2
-2	3	2
-1	3	2
0	1	-2
1	2	0
2	3	2



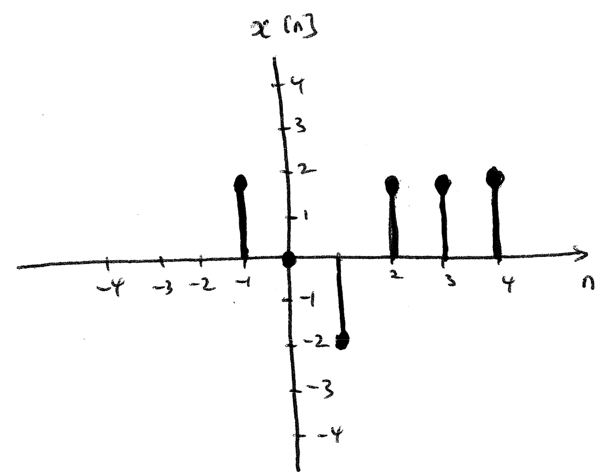
Now, the time transformation

Since $m = -n+1$

we have, $n = -m+1$

10 MARKS

m	x[m]	n
-3	2	4
-2	2	3
-1	2	2
0	-2	1
1	0	0
2	2	-1



Problem 2

$$x(t) = \sin(\pi t)$$

$$x[n] = x(nT) = \sin(\pi nT)$$

(a) 5 MARKS.

$$x[n] = \sin(n\pi T)$$

$$x[-n] = \sin(-n\pi T) = -\sin(n\pi T)$$

Since $x[n] = -x[-n]$, the sequence $x[n] = \sin(n\pi T)$ is a

<u>ODD</u> signal

(b) Sampling period $T = 0.5\,s$

The period T_0 of the continuous-time signal $x(t)$ is obtained as follows:

$$x(t) = \sin \pi t \equiv \sin \omega t$$ 10 MARKS.

Comparing gives;

$$\pi = \omega \quad \text{or} \quad \pi = 2\pi f_0 \quad \text{or} \quad \pi = \frac{2\pi}{T_0}$$

It follows that $\boxed{T_0 = 2\,s}$

To check whether $x[n]$ is periodic, the ratio $\frac{T}{T_0}$ must be rational:

$$\frac{T}{T_0} = \frac{0.5}{2} = \frac{1}{4}$$ → This is a ratio of integers and hence

$x[n] = \sin(n\pi T)$ is <u>Periodic</u>. (5 marks)

Using $\frac{T}{T_0} = \frac{k}{N} \Rightarrow NT = kT_0 \Rightarrow 4T = T_0$

This implies there are 4 samples in one period of $x(t)$.

$\boxed{N = 4}$ (5 marks)

c)

Given $x(t) = \sin(\pi t)$

Substituting into

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\left[\delta(\tau+t) - \delta(\tau-t)\right] d\tau$$

gives

$$y(t) = \int_{-\infty}^{\infty} \sin(\pi\tau)\left[\delta(\tau+t) - \delta(\tau-t)\right] d\tau$$

$$= \int_{-\infty}^{\infty} \sin(\pi\tau)\,\delta(\tau+t)\,d\tau - \int_{-\infty}^{\infty} \sin(\pi\tau)\,\delta(\tau-t)\,d\tau$$

Using sifting property of $\delta(t)$, we have

10 MARKS

$$y(t) = \sin(-\pi t) - \sin(\pi t)$$

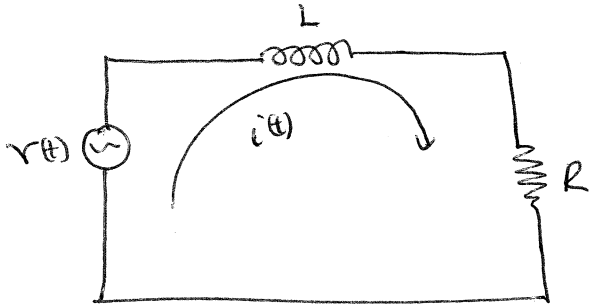
Since $\sin(\pi t)$ is odd, we have

$$y(t) = -\sin(\pi t) - \sin(\pi t) = -2\sin(\pi t).$$

Evaluatuating for $t = 5$ gives

$$y(5) = -2\sin(5\pi).$$

Problem 4



(a) Applying KVL to the above circuit gives

5 MARKS.

$$\boxed{v(t) = L\frac{d\,i(t)}{dt} + i(t)R} \quad \text{—— (1)}$$

— This is the differential equation description of the system.

(b) To draw the simulation diagram, we re-write equation (1)

as

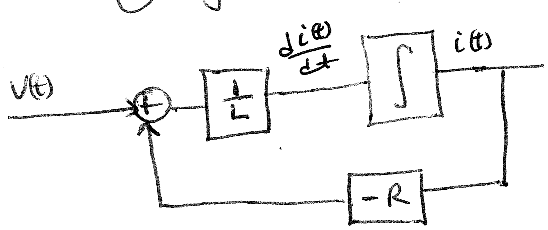
$$L\frac{d\,i(t)}{dt} = v(t) - i(t)R \quad \text{—— (2)}$$

or

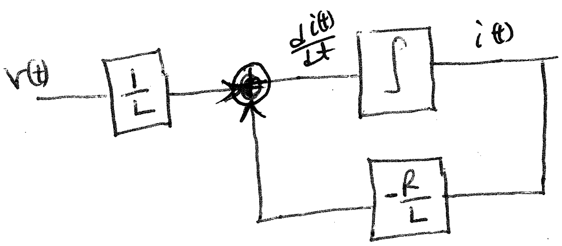
$$\frac{d\,i(t)}{dt} = \frac{1}{L}v(t) - \frac{R}{L}i(t) \quad \text{—— (3)}$$

5 MARKS

(2) gives



(3) gives



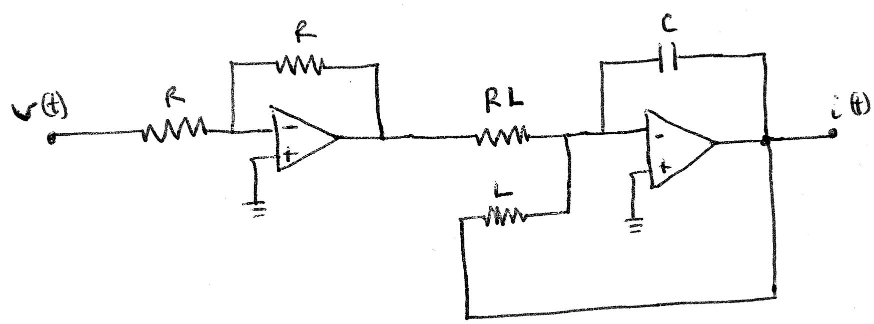
(c) The differential equation

$$\frac{d\,i(t)}{dt} = -\frac{R}{L} i(t) + \frac{1}{L} v(t)$$

is made up of
(1) a summer
(2) an inverter, and
(3) an integrator.

The analog computer implementation can be achieved using

|← inverter →|← summer |← integrator →|



5 MARKS

choose $RC = 1$.

(d) using the method of undetermined coefficients.

Step 1: complementary solution: $i_c(t) = Ae^{-st}$

$$L\frac{d\,i(t)}{dt} + R\,i(t) = v(t) \quad \text{—— (1)}$$

Substituting for $i(t) = i_c(t) = Ae^{-st}$ in the homogenous equation

$$L\frac{d\,i_c(t)}{dt} + R\,i_c(t) = 0 \quad \text{gives}$$

$$-ALse^{-st} + ARe^{-st} = 0 \Rightarrow (-Ls + R)Ae^{-st} = 0$$

This implies that $s = R/L$

Hence the complementary solution is $i_c(t) = Ae^{-\frac{R}{L}t}$.

Step 2: particular solution: $i_p(t) = B$.

Substituting $i(t) = i_p(t)$ in (1) gives

$$RB = u(t) \quad \left\{ \begin{array}{l} v(t) = u(t) \text{ and} \\ \frac{d\,i_p(t)}{dt} = \frac{d(B)}{dt} = 0 \end{array} \right.$$

we have for $t > 0$

$$RB = 1 \Rightarrow B = \frac{1}{R}$$

<u>Step 3</u>: General solution $i(t) = i_c(t) + i_p(t)$

$$i(t) = Ae^{-\frac{Rt}{L}} + \frac{1}{R} \qquad (2)$$

checking for initial condition.

using $i(0) = 0$ in (2) gives

$$0 = Ae^{0} + \frac{1}{R}$$

10 MARKS

$$\Rightarrow A = -\frac{1}{R}$$

Hence (2) becomes

$$i(t) = -\frac{1}{R}e^{-\frac{Rt}{L}} + \frac{1}{R}$$

$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{Rt}{L}}\right); \ t > 0$$