ENG 342: Advanced Engineering Math II

Quiz #6

November 29, 2016

Problem 1 [5 pts]

Consider a thin rectangular plate with vertical edges $x = 0$ and $x = \pi$ insulated, and horizontal edges $y = 0$ and $y = 2\pi$ maintained at 0 and $f(x)$, respectively.

(a) Set up the boundary-value problem for the steady-state temperature $u(x, y)$ at any given point (x, y) on the plate. Draw a diagram to represent the situation. [1.5 pts]

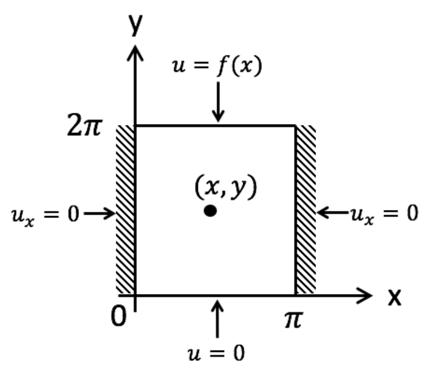
From Laplace's Equation, the boundary-value problem is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < \pi,\ 0 < y < 2\pi$$

$$u_x(0, y) = 0,\ u_x(\pi, y) = 0 \qquad 0 < y < 2\pi$$

$$u(x, 0) = 0,\ u(x, 2\pi) = f(x) \qquad 0 < x < \pi$$

Here is a diagram representing this situation:



(b) Solve for $u(x, y)$ using separation of variables. Be sure to consider all possible cases, showing which are trivial and which are not. Your final answer should be in terms of $f(x)$. [2 pts]

Assuming $u = XY$, we have

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

This gives two differential equations:

$$X'' + \lambda X = 0 \qquad Y'' - \lambda Y = 0$$

Starting with $X'' + \lambda X = 0$, there are three cases. We consider those together with the boundary conditions $X'(0) = X'(\pi) = 0$:

Case I: $\lambda = -\alpha^2 < 0, \alpha > 0$
The roots are real: $m = \pm\alpha$, so $X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. Since $X'(x) = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x$, $X'(0) = 0$ implies $c_2 = 0$ and $X'(\pi) = 0$ implies $c_2 \alpha \sinh \alpha\pi = 0$. Since $\sinh \alpha\pi$ is always positive for $\alpha\pi > 0$, $c_1 = 0$ also. Therefore this case is trivial.

Case II: $\lambda = 0$
This is a repeated root $m = 0, 0$. So, $X(x) = c_1 + c_2 x$, and $X'(x) = c_2$. $X'(0) = 0$ means $c_2 = 0$, and $X'(1) = 0$ doesn't give any more information. So, $X(x) = c_1$ is a constant solution.

Case III: $\lambda = +\alpha^2, \alpha > 0$
The roots are imaginary: $m = \pm\alpha i$. So, $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$, and $X'(x) = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$. $X'(0) = 0$ implies $c_2 = 0$. $X'(\pi) = 0$ then implies $-\alpha c_1 \sin \alpha\pi = 0$, which means $\alpha\pi = n\pi,\ n = 1, 2, 3, ...$
Putting Cases II and III together, we have

$$\alpha_n = \begin{cases} 0 & n = 0 \\ n & n = 1, 2, ... \end{cases}$$

And corresponding eigenfunctions

$$X_n(x) = \begin{cases} c_0 & n = 0 \\ c_n \cos nx & n = 1, 2, 3, ... \end{cases}$$

As for the Y component, we consider $\lambda = 0$ and $\lambda_n = n^2$ with the condition $Y(0) = 0$.

For $\lambda = 0$, we have repeated roots: $m = 0, 0$. So, $Y(y) = d_1 + d_2 y$. $Y(0) = 0$ means $d_1 = 0$, so $Y(y) = d_2 y$.

For $\lambda = n^2$, we have distinct roots: $m = n, -n$. Then, $Y(y) = d_3 \cosh ny + d_4 \sinh ny$. $Y(0) = 0$ means that $d_3 = 0$, so $Y(y) = d_4 \sinh ny$.

The product of X and Y is

$$u_n(x,y) = \begin{cases} C_0 y & n = 0 \\ C_n \sinh ny \cos nx & n = 1, 2, ... \end{cases}$$

By superposition of the product solutions $u_n(x,y)$ $n = 1, 2, ...$, we have

$$u(x,y) = C_0 y + \sum_{n=1}^{\infty} C_n \sinh ny \cos nx$$

Then applying the initial condition,

$$u(x, 2\pi) = 2\pi C_0 + \sum_{n=1}^{\infty} C_n \sinh 2\pi n \cos nx = f(x)$$

This is a Fourier cosine series. Using the formula for $2\pi C_0$,

$$2\pi C_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx \quad \rightarrow \quad C_0 = \frac{1}{2\pi^2}\int_0^{\pi} f(x)dx$$

Using the one for $C_n \sinh 2\pi n$,

$$C_n \sinh 2\pi n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx \quad \rightarrow \quad C_n = \frac{2}{\pi \sinh 2\pi n}\int_0^{\pi} f(x)\cos nx \, dx$$

So,

$$u(x,y) = \left[\frac{1}{2\pi^2}\int_0^{\pi} f(x)dx\right] y + \sum_{n=1}^{\infty}\left[\frac{2}{\pi \sinh 2\pi n}\int_0^{\pi} f(x)\cos nx \, dx\right] \sinh ny \cos nx$$

(c) Find $u(x,y)$ when

$$f(x) = \begin{cases} 0 & 0 < x < \frac{\pi}{2} \\ 10(x - \pi/2)/(\pi/2) & \frac{\pi}{2} \le x < \pi \end{cases}$$

[1.5 pts]

$$\frac{1}{2\pi^2}\int_0^{\pi} f(x)dx = \frac{1}{2\pi^2}\int_{\pi/2}^{\pi} 10(x - \pi/2)/(\pi/2) \, dx = \frac{1}{2\pi^2} \cdot 2.5\pi$$

$$
\begin{aligned}
\frac{2}{\pi \sinh 2\pi n} \int_0^{\pi} f(x) \cos nx \; dx &= \frac{2}{\pi \sinh 2\pi n} \int_{\pi/2}^{\pi} 10(x - \pi/2)/(\pi/2) \; dx \\
&= \frac{2}{\pi \sinh 2\pi n} \cdot \frac{20\left((-1)^n - \cos(n\pi/2)\right)}{\pi n^2} \\
&= 40 \frac{(-1)^n - \cos(n\pi/2)}{(\pi n)^2 \sinh 2\pi n}
\end{aligned}
$$

So,

$$
u(x, y) = \frac{1.25}{\pi} y \; + \; \sum_{n=1}^{\infty} 40 \frac{(-1)^n - \cos(n\pi/2)}{(\pi n)^2 \sinh 2\pi n} \sinh ny \cos nx
$$

Problem 2 [5 pts]

Consider a quarter-circular plate of radius c with radial boundaries $\theta = 0$ and $\theta = \pi/2$ held at temperature 0, and angular boundary $r = c$ maintained at $f(\theta)$.

(a) Set up the boundary-value problem for the steady-state temperature $u(r, \theta)$ at any given point (r, θ) on the plate. Draw a diagram to represent the situation. [1.5 pts]

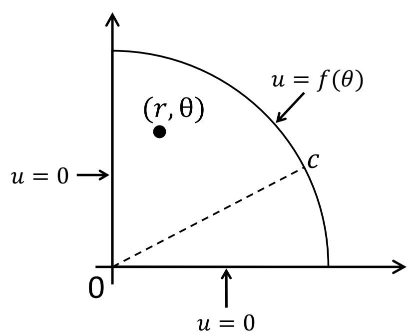
From Laplace's Equation in polar coordinates, the boundary-value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad 0 < r < c,\ 0 < \theta < \pi/2$$

$$u(r, 0) = 0,\ u(r, \pi/2) = 0 \qquad 0 < r < c$$

$$u(c, \theta) = f(\theta) \qquad 0 < \theta < \pi/2$$

Here is a diagram describing this BVP:



(b) Solve for $u(r, \theta)$ using separation of variables. Be sure to consider all possible cases, showing which are trivial and which are not, accounting for continuity and periodicity as needed. Your final answer should be in terms of $f(\theta)$. [2 pts]

Assuming $u = R\Theta$, we have

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Dividing by $R\Theta$ and multiplying by r^2, this becomes

$$\frac{\Theta''}{\Theta} = -r^2\frac{R''}{R} - r\frac{R'}{R} = -\lambda$$

This gives two differential equations:

$$\Theta'' + \lambda\Theta = 0 \qquad r^2R'' + rR' - \lambda R = 0$$

Out of the two equations, it's better to start with $\Theta'' + \lambda\Theta = 0$ since we have the two boundary conditions $\Theta(0) = 0$ and $\Theta(\pi/2) = 0$. As with the previous problem, there are three cases:

Case I: $\lambda = -\alpha^2 < 0, \alpha > 0$ The roots are real: $m = \pm\alpha$, so $\Theta(\theta) = c_1 \cosh\alpha\theta + c_2 \sinh\alpha\theta$. $\Theta(0) = 0$ implies $c_1 = 0$. $\Theta(\pi/2) = c_2 \sinh\alpha\pi/2 = 0$ implies $c_2 = 0$ because sinh is always positive for a positive argument. Therefore this case is trivial.

Case II: $\lambda = 0$
This is a repeated root $m = 0, 0$. So, $\Theta(\theta) = c_1 + c_2\theta$. $\Theta(0) = 0$ means $c_1 = 0$, and $\Theta(\pi/2) = 0$ means $c_2 = 0$. Therefore this case is trivial.

Case III: $\lambda = +\alpha^2, \alpha > 0$
The roots are imaginary conjugates: $m = \pm\alpha i$. So, $\Theta(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta$. $\Theta(0) = 0$ means that $c_1 = 0$. Then, $\Theta(\pi/2) = c_2 \sin\alpha\pi/2 = 0$ requires that $\alpha\pi/2 = n\pi$, or $\alpha_n = 2n, n = 1, 2, ...$ The eigenfunctions are

$$\Theta_n(\theta) = c_n \sin 2n\theta$$

As for the R component, we have the eigenvalues $\lambda_n = (2n)^2$. The differential equation becomes

$$r^2R'' + rR' - (2n)^2R = 0$$

To solve this Cauchy-Euler Equation, we find the roots of the auxiliary equation

$$1m^2 + (1-1)m - (2n)^2 = 0$$

for $n > 0$. So, $m = \pm 2n$. This gives the solution

$$R(r) = c_3r^{2n} + c_4r^{-2n}$$

Since $R(r)$ must be defined for $0 \le r \le c$, $c_4 = 0$. So, $R(r) = c_3r^{2n}$. The product solution is then

$$u_n(r, \theta) = C_nr^{2n} \sin 2n\theta$$

By superposition,

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n r^{2n} \sin 2n\theta$$

We then apply the final condition $u(c,\theta) = f(\theta)$:

$$u(c,\theta) = \sum_{n=1}^{\infty} C_n c^{2n} \sin 2n\theta = f(\theta)$$

The left hand side is the sine-series expansion of $f(\theta)$ over $(0, \pi/2)$. So,

$$C_n c^{2n} = \frac{2}{\pi/2} \int_0^{\pi/2} f(\theta) \sin 2n\theta \; d\theta \to C_n = \frac{4}{\pi c^{2n}} \int_0^{\pi/2} f(\theta) \sin 2n\theta \; d\theta$$

Therefore,

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{4}{\pi} \left[\int_0^{\pi/2} f(\theta) \sin 2n\theta \; d\theta \right] \left(\frac{r}{c} \right)^{2n} \sin 2n\theta$$

(c) Find $u(r,\theta)$ when $f(\theta) = \sin(4\theta)$, $0 < \theta < \pi/2$. [1.5 pts]

This means that

$$u(c,\theta) = \sum_{n=1}^{\infty} C_n c^{2n} \sin 2n\theta = \sin 4\theta$$

Since the left side turns to a sinusoid with argument 4θ at $n = 2$, we can set $C_n = 0$ for $n \neq 2$, and then for $n = 2$,

$$C_2 c^4 \sin 4\theta = \sin 4\theta \to C_2 = 1/c^4$$

So, the solution is

$$u(r,\theta) = \left(\frac{r}{c} \right)^4 \sin 4\theta$$

You could also integrate $\int_0^{\pi/2} f(\theta) \sin 2n\theta \; d\theta$ directly, from which you will get 0 when $n \neq 2$ and $\pi/4$ when $n = 2$.