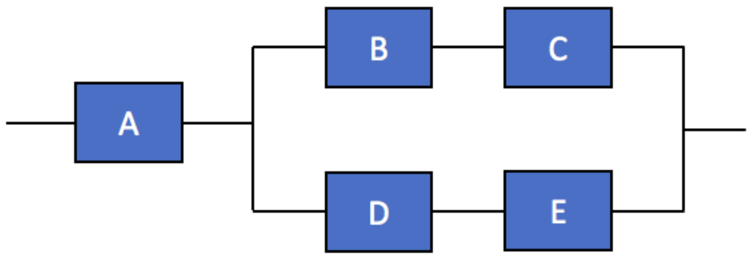
ENG 342: Advanced Engineering Math II

Quiz #5

November 22, 2016

Problem 1 [3 pts]

Consider the system comprised of components A – E below.



(i) Let Z be the event that the system functions, and let $A, ..., E$ be the events that each component functions. Express Z in terms of $A, ..., E$ (*i.e.*, as a combination of union, intersection, and/or complement operations).

B and C are in series, as are D and E, so the events on these branches are $B \cap C$, $D \cap E$. These subsystems are in parallel, so the combination of the four is $(B \cap C) \cup (D \cap E)$. Since this is in series with A, the full event Z is

$$Z = A \cap ((B \cap C) \cup (D \cap E))$$

(ii) What is $P(Z)$ in terms of $P(A), ..., P(E)$? Assume each component functions independently.

For independent events A_1 and A_2, $P(A_1 \cap A_2) = P(A_1)P(A_2)$ and $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1)P(A_2)$. Therefore,

$$\begin{aligned}
P(Z) &= P(A)P\left((B \cap C) \cup (D \cap E)\right) \\
&= P(A)\left(P(B \cap C) + P(D \cap E) - P\left((B \cap C) \cap (D \cap E)\right)\right) \\
&= P(A)\left(P(B)P(C) + P(D)P(E) - P(B \cap C)P(D \cap E)\right) \\
&= P(A)\left(P(B)P(C) + P(D)P(E) - P(B)P(C)P(D)(E)\right)
\end{aligned}$$

(iii) Suppose A has a 90% chance of functioning properly, B and D also have 90%, and C and E have 80%. What is the probability that the system functions?

Using these values,

$$P(Z) = 0.9(0.9 \cdot 0.8 + 0.9 \cdot 0.8 - 0.9 \cdot 0.8 \cdot 0.9 \cdot 0.8) = 0.8294$$

or about 83%.

Problem 2 [3 pts]

A box of 20 resistors contains 5 that are out of spec. We select two of them at random to complete our circuit.

(i) How many choices of resistors are possible?

This is a combination operation because order does not matter: picking A followed by B is the same as B followed by A.

$$\binom{20}{2} = 190$$

(ii) What is the probability that the first resistor is out of spec?

Let A be the event that the first resistor is out of spec. Out of the 20 possibilities, 5 are out of spec. So,

$$P(A) = 5/20 = 0.25$$

(iii) What is the probability that both resistors are out of spec?

Let B be the event the second resistor is out of spec.

$$P(A \cap B) = P(A)P(B|A) = (5/20)(4/19) = 0.0526$$

(iv) What is the probability that the second resistor is out of spec?

To find $P(B)$, we use the law of total probability:

$$P(B) = P(A \cap B) + P(A^c \cap B) = (5/20)(4/19) + (15/20)(5/19) = 0.25$$

(v) If the first resistor was out of spec, what is the probability that the second resistor is out of spec?

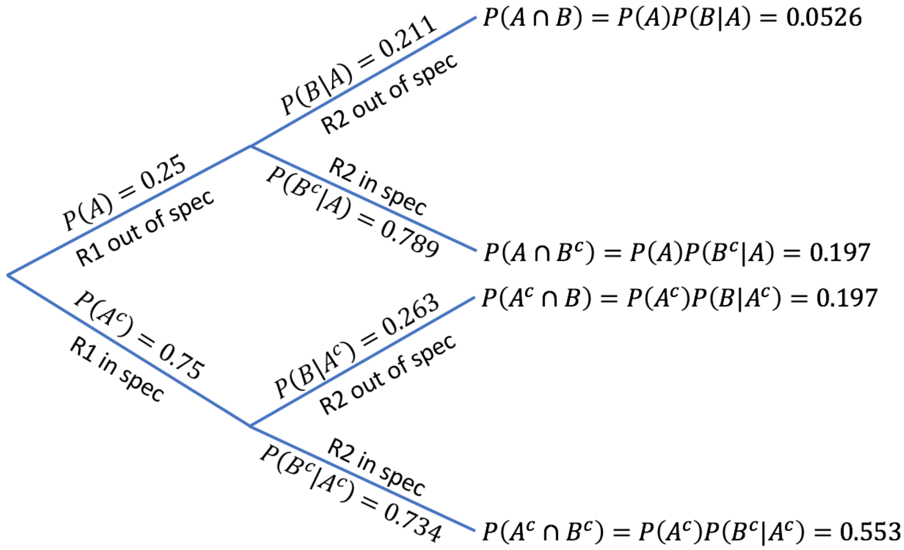
This is $P(B|A)$. That is calculated by

$$P(B|A) = P(B \cap A)/P(A) = 0.0526/0.25 = 0.2104$$

(vi) Are the events "first resistor out of spec" and "second resistor out of spec" independent? Are they mutually exclusive? Explain.

They are not independent because $P(B|A) = 0.2104 \neq P(B) = 0.25$, *i.e.*, A occurring changes the chance that B will occur. They are not mutually exclusive because $P(A \cap B) = 0.0526 \neq 0$, *i.e.*, A and B can occur together.

Below is the tree diagram for this problem, illustrating the different probabilities.



Problem 3 [3 pts]

Through repeated measurements, we have determined that a random variable X has a mean $\mu_X = 0.8$ and standard deviation $\sigma_X = 0.16$.

(i) Determine an upper bound on the probability that X falls outside the range $(0.6, 1.0)$.

This range is ± 0.2 around the mean, which is $k = 0.20/\sigma_X = 1.25$ standard deviations. Since we don't know the distribution of X, we use the Chebyschev inequality:

$$P(|X - \mu_X| \geq 1.25\sigma_X) \leq 1/1.25^2 = 0.64$$

So, the probability of X falling outside this range is at most 64%.

(ii) We use our data to fit X to the cumulative distribution function (CDF)

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ c \cdot x^4 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

for some constant c. What value of c will make this a valid CDF? Why?

We know immediately that $c \in [0, 1]$, since the density cannot be negative and $F_X(x) \leq 1$ for all values of x. But if $c \neq 1$, then there will be a jump discontinuity in $F_X(x)$ at $x = 1$, which would require $P(X = 1) > 0$; if X is a continuous random variable (which it obviously is), then this cannot happen, because the probability of any particularly value occurring is 0. So, $c = 1$.

(iii) Find the probability density function (PDF) $f_X(x)$.

Since $f_X(x) = \frac{d}{dx} F_X(x)$,

$$f_X(x) = \begin{cases} 0 & x \leq 0 \\ 4x^3 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

(iv) Show that $\mu_X = 0.8$ and $\sigma_X \approx 0.16$ with this PDF.

The mean and variance are calculated as:

$$
\begin{aligned}
\mu_X &= \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 4x^4\, dx = \frac{4}{5}x^5\Big|_0^1 = 0.8 \\
\sigma_X^2 &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - \mu_X^2 = \int_0^1 4x^5\, dx - (0.8)^2 = \frac{2}{3} - 0.64 = 0.0267
\end{aligned}
$$

So, $\sigma_X = \sqrt{0.0267} = 0.1633 \approx 0.16$.

(v) What is the actual probability that X falls outside $(0.6, 1.0)$?

We could integrate $f_X(x)$, or just use the CDF directly: Since $P(X > 1) = 0$,

$$P(X < 0.6 \text{ or } X > 1.0) = F_X(0.6) = 0.1296$$

So, the actual probability is about 13%, which is substantially lower than the 64% from (i).