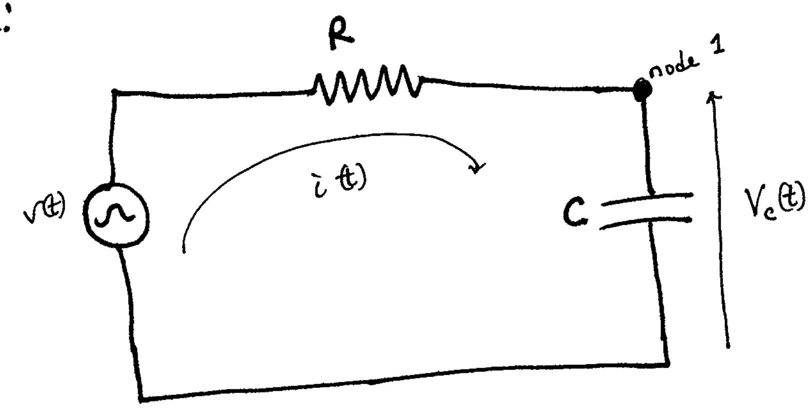
Signals and Systems ELC 321

Final Examinations

Spring 2015

Sample solutions

Problem 1:



(a) Applying KCL at node 1 yields

$$\underbrace{\frac{v(t) - V_c(t)}{R}}_{\text{Current flowing towards the node}} = \underbrace{C\frac{d\,V_c(t)}{dt}}_{\substack{\to \text{ Current flowing out the node} \\ \to \text{ current through the capacitor.}}}$$

Rearranging gives

$$\boxed{RC\frac{d\,V_c(t)}{dt} + V_c(t) = V(t)}$$

The differential equation model relating $V_c(t)$ to $V(t)$.

5 MARKS

(b) impulse response

Recall that

$$h(t) = \mathcal{L}^{-1}(H(s))$$

Taking the Laplace transform of

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v(t) \qquad \text{gives}$$

$$RCsV_c(s) + V_c(s) = V(s)$$

The transfer function $H(s)$ is obtained as

$$H(s) = \frac{V_c(s)}{V(s)} = \frac{1}{RCs+1} = \frac{1/RC}{s+1/RC}$$

Hence $h(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{1/RC}{s+1/RC}\right]$

$$\boxed{h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)}$$

5 MARKS

(c) <u>step response</u>

<u>Method 1</u>: from convolution

$$v_c(t) = h(t) * v(t) = \int_{\tau=-\infty}^{\infty} h(t-\tau)v(\tau)\,d\tau = \int_{\tau=-\infty}^{\infty} v(t-\tau)h(\tau)\,d\tau$$

Since $h(t) = \frac{1}{RC}e^{-1/RC\,t}u(t)$ and $v(t) = u(t)$

we have

$$v_c(t) = \frac{1}{RC}\int_{\tau=-\infty}^{\infty} u(t-\tau)e^{-\frac{1}{RC}\tau}u(t)\,d\tau = \frac{1}{RC}\int_{\tau=0}^{\infty} u(t-\tau)e^{-\frac{1}{RC}\tau}\,d\tau$$

$$V_c(t) = \frac{1}{RC}\int_{\tau=0}^{\infty} u(t-\tau)\, e^{-\frac{1}{RC}\tau}\, d\tau = \frac{1}{RC}\int_{\tau=0}^{t} e^{-\frac{1}{RC}\tau}\, d\tau$$

$$= -\left[e^{-\frac{1}{RC}\tau} \right]_{\tau=0}^{\tau=t} = -\left[e^{-t/RC} - e^{0} \right]$$

$$= 1 - e^{-t/RC}$$

Hence the step response is given by

$$\boxed{V_c(t) = 1 - e^{-t/RC}\, u(t)}$$

5 marks

<u>Method II</u>: Laplace transform method.

$$V_c(s) = H(s)\, V(s)$$

Since $v(t) = u(t)$, we have $V(s) = \frac{1}{s}$

Also, $H(s) = \dfrac{1/RC}{s + 1/RC}$

Hence $V_c(s) = H(s)\, V(s) = \dfrac{\frac{1}{RC}}{s\left(s + 1/RC\right)} \equiv \dfrac{A}{s} + \dfrac{B}{s + 1/RC}$

$$A = \left.\frac{1/RC}{s + 1/RC}\right|_{s=0} = 1$$

$$B = \left.\frac{1/RC}{s}\right|_{s=-1/RC} = -1$$

$$\therefore \quad V_c(s) = \frac{1}{s} - \frac{1}{s + 1/RC}$$

Taking inverse Laplace transform gives

$$v_c(t) = u(t) - e^{-\frac{1}{RC}t}u(t)$$

$$= \left[1 - e^{-\frac{1}{RC}t}\right]u(t).$$

Method III : undetermined coefficients.

<u>step I</u> : Complementary solution:

set $v_c^c(t) = Ae^{st}$

substituting into $RC\frac{dv_c(t)}{dt} + v_c(t) = 0$ yields

$$RCAse^{st} + Ae^{st} = 0$$

$$(RCs+1)Ae^{st} = 0$$

This gives $RCs+1 = 0 \Rightarrow s = -\frac{1}{RC}$

Hence $v_c^c(t) = Ae^{-\frac{1}{RC}t}$.

<u>step II</u> : particular solution:

since $v(t) = u(t)$, set $v_c^p(t) = P \; (t \geq 0)$

substituting into $RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$ yields

$$P = 1 \; (t \geq 0)$$

<u>step III</u> : General solution

$$v_c(t) = v_c^c(t) + v_c^p(t)$$

$$= \left(Ae^{-\frac{1}{RC}t} + 1\right) \quad t \geq 0.$$

<u>step IV</u> : Initial conditions: $v_c(0) = 0$

$$0 = Ae^0 + 1 \Rightarrow A = -1$$

Hence $v_c(t) = 1 - e^{-\frac{1}{RC}t}, \; t \geq 0$ or

$$\boxed{v_c(t) = \left[1 - e^{-t/RC}\right]u(t)}$$

Note that the impulse response of the system can be obtained from the step response as follows:

$$h(t) = \frac{d}{dt}\left(\text{step response}\right).$$

It follows that

$$h(t) = \frac{d}{dt}\left(1 - e^{-t/RC}\right) \quad t \geq 0$$

$$= \frac{1}{RC} e^{-t/RC} \quad \text{As before.}$$

(d) Recall

$$RC \frac{dV_c(t)}{dt} + V_c(t) = V(t)$$

Substituting for $\frac{dV_c(t)}{dt} = \frac{V_c(t) - V_c(t-T_s)}{T_s}$ gives

$$RC\left[\frac{V_c(t) - V_c(t-T_s)}{T_s}\right] + V_c(t) = V(t)$$

$$\frac{RC}{T_s}\left[V_c(t) - V_c(t-T_s)\right] + V_c(t) = V(t)$$

or

$$\left(1 + \frac{RC}{T_s}\right)V_c(t) - \frac{RC}{T_s} V_c(t-T_s) = V(t)$$

At times $t = nT_s$, we have

$$\left(1 + \frac{RC}{T_s}\right) V_c(nT_s) - \frac{RC}{T_s} V_c(nT_s - T_s) = V(nT_s)$$

Re-arranging gives

$$\left(\frac{T_s + RC}{T_s}\right) V_c(nT_s) - \frac{RC}{T_s} V_c([n-1]T_s) = V(nT_s)$$

or

$$V_c(nT_s) = \left(\frac{T_s}{T_s + RC}\right) \times \left(\frac{RC}{T_s}\right) V_c([n-1]T_s) + \left(\frac{T_s}{T_s + RC}\right) V(nT_s)$$

Simplifying and using $V_c(nT_s) = V_c[n]$ gives

$$V_c[n] = \frac{RC}{T_s + RC} V_c[n-1] + \frac{T_s}{T_s + RC} V[n]$$

Comparing with

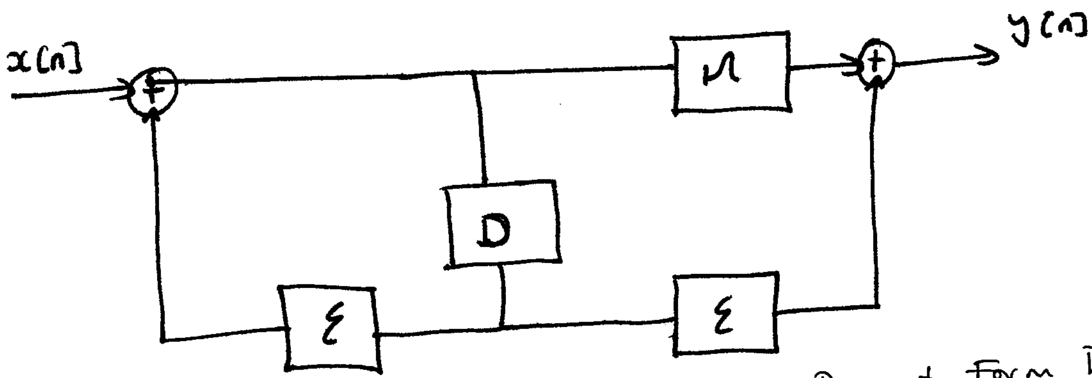
$$V_c[n] = \alpha V_c[n-1] + \beta V[n]$$

we have $\boxed{\alpha = \frac{RC}{T_s + RC}}$ and $\boxed{\beta = \frac{T_s}{T_s + RC}}$ 5 MARKS

Problem 2

$$y[n] = \xi\, y[n-1] + \eta\, x[n] + \xi\, x[n-1]$$

(a)



5 MARKS

Direct Form II

(b) Taking the z-transform of

$y[n] = \xi\, y[n-1] + \eta\, x[n] + \xi\, x[n-1]$ gives

$$Y(z) = \xi z^{-1} Y(z) + \eta X(z) + \xi z^{-1} X(z)$$

Re-arranging

$$(1 - \xi z^{-1}) Y(z) = (\eta + \xi z^{-1}) X(z)$$

Hence, the system function is given by

$$\boxed{\frac{Y(z)}{X(z)} = \frac{\eta + \xi z^{-1}}{1 - \xi z^{-1}}}$$

5 MARKS

(c) Yes, the filter can be unstable. For stability, the pole of the system must be inside the unit circle.

Solving for the pole gives

$$1 - \xi z^{-1} = 0$$

$$\frac{\xi}{z} = 1 \Rightarrow z = \xi$$

For stability $|z| < 1$ or $|\xi| < 1$.

Hence ξ must take values from the range

$$\boxed{-1 < \xi < 1}$$

5 MARKS

(d) Method I: Z-transform.

$$H(z) = \frac{\eta + \xi z^{-1}}{1 - \xi z^{-1}} = \frac{1 + 0.6 z^{-1}}{1 - 0.6 z^{-1}} = \frac{z + 0.6}{z - 0.6}$$

$$X(z) = \frac{1}{1 - 0.8 z^{-1}} = \frac{z}{z - 0.8}$$

$$Y(z) = H(z)X(z) = \left(\frac{z+0.6}{z-0.6}\right)\left(\frac{z}{z-0.8}\right)$$

$$\frac{Y(z)}{z} = \frac{z+0.6}{(z-0.6)(z-0.8)} \equiv \frac{A}{z-0.6} + \frac{B}{z-0.8}$$

$$A = \left.\frac{z+0.6}{z-0.8}\right|_{z=0.6} = \frac{1.2}{-0.2} = -6$$

$$B = \left.\frac{z+0.6}{z-0.6}\right|_{z=0.8} = \frac{1.4}{0.2} = 7$$

Hence

$$\frac{Y(z)}{z} = \frac{-6}{z-0.6} + \frac{7}{z-0.8} \quad \text{or} \quad Y(z) = \frac{-6z}{z-0.6} + \frac{7z}{z-0.8}$$

Taking the inverse z-transform gives

$$y[n] = -6(0.6)^n u[n] + 7(0.8)^n u[n]$$

$$\boxed{y[n] = \left[-6(0.6)^n + 7(0.8)^n\right] u[n]}$$ 5 MARKS

<u>Method II</u>: undetermined coefficients

<u>Step I</u> complementary solution

set $y_c[n] = Az^n$ in

$$y[n] - 0.6\,y[n-1] = 0$$

$$\Rightarrow \quad Az^n - 0.6\,Az^{n-1} = 0$$

$$Az^n\left[1 - 0.6z^{-1}\right] = 0 \quad \Rightarrow \quad z = 0.6$$

Hence $y_c[n] = A(0.6)^n$

<u>Step II</u>: particular solution

set $y_p[n] = B(0.8)^n \ (n \geq 0)$ since $x[n] = 0.8^n u[n]$

substituting into $y[n] - 0.6\,y[n-1] = x[n] + 0.6\,x[n-1]$ yields

$$B(0.8)^n - 0.6\,B(0.8)^{n-1} = 0.8^n + 0.6(0.8)^{n-1}$$

$$\Rightarrow \left[B - \frac{0.6}{0.8}B\right]0.8^n = \left[1 + \frac{0.6}{0.8}\right]0.8^n$$

$$\text{Hence} \quad B = \frac{1 + 0.6/0.8}{1 - 0.6/0.8} = \frac{1.4}{0.8} \times \frac{0.8}{0.2} = 7$$

<u>Step III</u>: General solution

$$y[n] = y_c[n] + y_p[n] = \left[A(0.6)^n + 7(0.8)^n\right]u[n]$$

Step IV : initial conditions:

from

$$y[n] = 0.6\,y[n-1] + x[n] + 0.6\,x[n-1]$$

we have

$$y[0] = 0.6\,y[-1] + x[0] + 0.6\,x[-1]$$

Applying zero initial condition and the fact that $x[n] = 0.8^n u[n]$

$$y[0] = 0.6\,\cancel{y[-1]}^{\,0} + 0.8^0 u[0] + \cancel{0.6(0.8)^{-1} u[-1]}^{\,0}$$

$$y[0] = 1.$$

using this value in the general solution

$$y[n] = \left[A(0.6)^n + 7(0.8)^n\right] u[n]$$ gives

$$y[0] = A(0.6)^0 + 7(0.8)^0 = 1$$

$$\Rightarrow \quad A + 7 = 1$$

$$\Rightarrow \quad A = -6$$

Hence $$\boxed{y[n] = \left[-6(0.6)^n + 7(0.8)^n\right] u[n]}$$

Method III : Convolution Sum:

$$y[n] = h[n] \circledast x[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

From $$H(z) = \frac{1+0.6z^{-1}}{1-0.6z^{-1}} = \frac{z+0.6}{z-0.6} = \frac{z}{z-0.6} + \frac{0.6}{z-0.6}$$

we have $$h[n] = (0.6)^n u[n] + 0.6\,(0.6)^{n-1} u[n-1]$$

$$= 0.6^n\left[u[n] + u[n-1]\right]$$

Also $$x[n] = 0.8^n u[n]$$

It follows that

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} 0.8^{k} u[k] \cdot 0.6^{n-k} \left[u[n-k] + u[n-k-1]\right]$$

$$= \sum_{k=0}^{\infty} 0.8^{k}\, 0.6^{n-k} \left[u[n-k] + u[n-k-1]\right]$$

$$= \sum_{k=0}^{\infty} 0.8^{k}\, 0.6^{n-k} u[n-k] + \sum_{k=0}^{\infty} 0.8^{k}\, 0.6^{n-k} u[n-k-1]$$

$$= \sum_{k=0}^{n} 0.8^{k}\, 0.6^{n-k} + \sum_{k=0}^{n-1} 0.8^{k}\, 0.6^{n-k}$$

$$= 0.6^{n} \sum_{k=0}^{n} \left(\frac{0.8}{0.6}\right)^{k} + 0.6^{n} \sum_{k=0}^{n-1} \left(\frac{0.8}{0.6}\right)^{k}$$

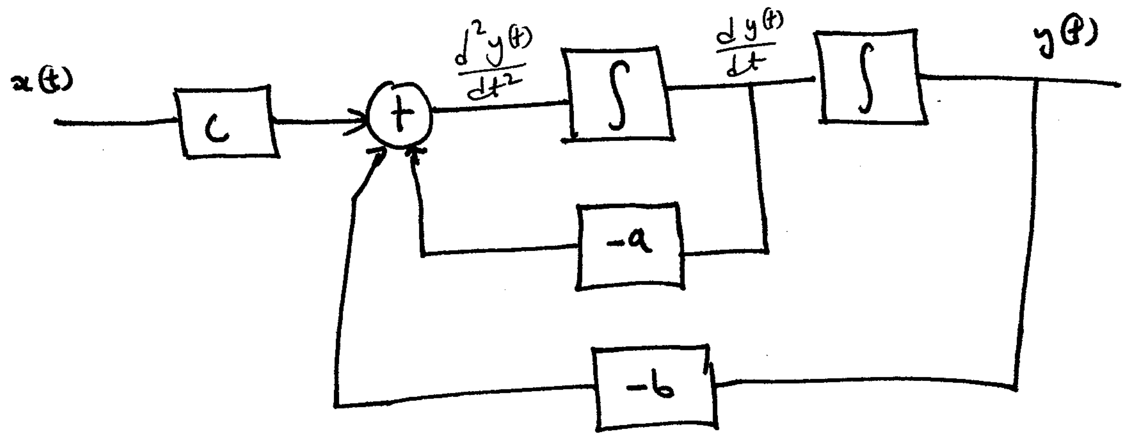
$$= 0.6^{n} \left[\frac{1 - \left(\frac{0.8}{0.6}\right)^{n+1}}{1 - 0.8/0.6}\right] + 0.6^{n} \left[\frac{1 - \left(0.8/0.6\right)^{n}}{1 - 0.8/0.6}\right].$$

$$= -3\left[0.6^{n} - \frac{0.8}{0.6}(0.8^{n})\right] - 3\left[0.6^{n} - 0.8^{n}\right]$$

$$= -6(0.6)^{n} + 7(0.8)^{n}$$

$$\therefore \boxed{y[n] = \left[-6(0.6)^{n} + 7(0.8)^{n}\right] u[n]}.$$

Problem 3:



(a) Writing the equation around the summing point gives

$$\frac{d^2y(t)}{dt^2} = -a\frac{dy(t)}{dt} - b\,y(t) + c\,x(t)$$

or

$$\boxed{\frac{d^2y(t)}{dt^2} + a\frac{dy(t)}{dt} + b\,y(t) = c\,x(t)}$$

5 MARKS

(b) Taking the Laplace transform (and assuming zero initial conditions) gives

$$s^2Y(s) + as\,Y(s) + b\,Y(s) = c\,X(s)$$

$$(s^2 + as + b)\,Y(s) = c\,X(s)$$

Hence $$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{c}{s^2 + as + b}}$$

5 MARKS

(c)

$$h(t) = 2\left[e^{-2t} - e^{-3t}\right]u(t).$$

Since $H(s) = \mathcal{L}(h(t))$

Taking the Laplace transform of $h(t)$ gives

$$H(s) = \frac{2}{s+2} - \frac{2}{s+3} = \frac{2(s+3)-2(s+2)}{(s+2)(s+3)}$$

$$= \frac{2s+6-2s-4}{s^2+5s+6} = \frac{2}{s^2+5s+6}$$

Comparing with $\dfrac{c}{s^2+as+b}$

we have $\boxed{a=5,\ b=6 \text{ and } c=2}$. 5 MARKS

(d) step response

Method I. Laplace transform.

$x(t) = u(t) \Rightarrow X(s) = \frac{1}{s}$

Hence

$$Y(s) = H(s)X(s) = \frac{2}{s^2+5s+6} \times \frac{1}{s} = \frac{2}{s(s^2+5s+6)}$$

$$Y(s) = \frac{2}{s(s+2)(s+3)} \equiv \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$A = \left.\frac{2}{(s+2)(s+3)}\right|_{s=0} = \frac{2}{6} = \frac{1}{3}.$$

$$B = \left. \frac{2}{s(s+3)} \right|_{s=-2} = \frac{2}{-2} = -1$$

$$c = \left. \frac{2}{s(s+2)} \right|_{s=-3} = \frac{2}{3}$$

hence $Y(s) = \frac{1/3}{s} - \frac{1}{s+2} + \frac{2/3}{s+3}$

Taking the inverse Laplace transform gives

$$y(t) = \frac{1}{3}u(t) - e^{-2t}u(t) + \frac{2}{3}e^{-3t}u(t)$$

$$\boxed{y(t) = \left[\frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}\right]u(t)}$$

5 MARKS

<u>Method II</u>: Convolution

$$h(t) = 2\left[e^{-2t} - e^{-3t}\right]u(t)$$

$$x(t) = u(t)$$

Hence $y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

$$y(t) = 2\int_{\tau=-\infty}^{\infty} \left[e^{-2\tau} - e^{-3\tau}\right]u(\tau)\,u(t-\tau)\,d\tau$$

$$= 2\int_{\tau=0}^{\infty} \left(e^{-2\tau} - e^{-3\tau}\right)u(t-\tau)\,d\tau = 2\int_{\tau=0}^{\infty} e^{-2\tau}u(t-\tau)\,d\tau - 2\int_{\tau=0}^{\infty} e^{-3\tau}u(t-\tau)\,d\tau$$

$$y(t) = 2\int_{\tau=0}^{t} e^{-2\tau}\,d\tau - 2\int_{\tau=0}^{t} e^{-3\tau}\,d\tau$$

$$= -e^{-2\tau}\Big|_{\tau=0}^{t} + \frac{2}{3}e^{-3\tau}\Big|_{\tau=0}^{t}$$

$$= -e^{-2t} + 1 + \frac{2}{3}e^{-3t} - \frac{2}{3}$$

$$= \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}$$

Hence $\boxed{y(t) = \left[\frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}\right]u(t)}$

Method III undetermined coefficients:

Step I: Complementary soln

set $y_c(t) = Ae^{st}$

and substitute into $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 0$

$$s^2Ae^{st} + 5sAe^{st} + 6Ae^{st} = 0$$

$$(s^2 + 5s + 6)Ae^{st} = 0$$

$$\Rightarrow (s+2)(s+3) = 0 \Rightarrow s = -2 \text{ or } -3$$

Hence $y_c(t) = A_1e^{-2t} + A_2e^{-3t}$.

Step II: particular soln.

set $y_p(t) = P$ (since $x(t) = u(t)$) and substitute

into $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 2x(t)$

We have

$$6p = 2 \quad (t \geq 0)$$

$$\Rightarrow p = 1/3 .$$

Step III: General solution $y(t) = y_c(t) + y_p(t)$

$$y(t) = A_1 e^{-2t} + A_2 e^{-3t} + 1/3$$

Step IV: Initial conditions: i.e $y(0) = 0$, $\dfrac{dy(0)}{dt} = 0$

from $\quad y(t) = A_1 e^{-2t} + A_2 e^{-3t} + 1/3$

$$\frac{dy(t)}{dt} = -2A_1 e^{-2t} - 3A_2 e^{-3t}$$

Applying initial conditions give

$$A_1 + A_2 + 1/3 = 0 \quad \text{and}$$

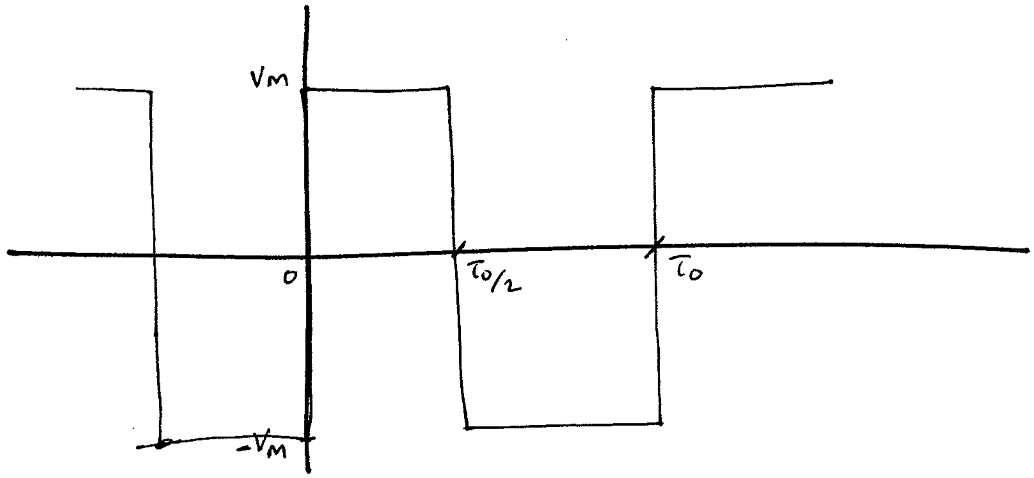
$$-2A_1 - 3A_2 = 0$$

manipulating gives

$$A_1 = -1 \quad \text{and} \quad A_2 = 2/3 .$$

Hence $\boxed{y(t) = \left[\frac{1}{3} - e^{-2t} + \frac{2}{3} e^{-3t}\right] u(t)}$.

Problem 4



(a)

$$x(t) = \begin{cases} V_m \; ; & 0 \le t \le T_0/2 \\ -V_m \; ; & T_0/2 \le t \le T_0 \end{cases}$$

By observation, the average value = 0.

Alternatively,

using $$c_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} V_m\,dt - \int_{T_0/2}^{T_0} V_m\,dt\right]$$

$$= \frac{1}{T_0}\left[V_m t\Big|_{t=0}^{T_0/2} - V_m t\Big|_{t=T_0/2}^{T_0}\right]$$

$$= \frac{1}{T_0}\left[\frac{V_m T_0}{2} - V_m T_0 + \frac{V_m T_0}{2}\right] = 0.$$

5 MARKS

(b)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jw_0 k t} \quad \text{with} \quad c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jkw_0 t} dt.$$

we have

$$c_k = \frac{1}{T_0}\left[\int_0^{T_0/2} V_m e^{-jw_0 k t} dt + \int_{T_0/2}^{T_0} -V_m e^{-jw_0 k t} dt\right]$$

$$= \frac{V_m}{-jw_0 k T_0}\left[e^{-jw_0 k t}\Big|_{t=0}^{T_0/2} - e^{-jw_0 k t}\Big|_{t=T_0/2}^{T_0}\right]$$

$$= \frac{jV_m}{k w_0 T_0}\left[e^{-jkw_0 T_0/2} - 1 - e^{-jkw_0 T_0} + e^{-jkw_0 T_0/2}\right]$$

using $T_0 w_0 = 2\pi$ give

$$c_k = \frac{jV_m}{2\pi k}\left[2e^{-jk\pi} - 1 - e^{-j2\pi k}\right]$$

For even k

$$c_k = \frac{jV_m}{2\pi k}[2-1-1] = 0$$

For odd k

$$c_k = \frac{jV_m}{2\pi k}[-2-1-1] = \frac{-j4V_m}{2\pi k} = \frac{-j2V_m}{\pi k}$$

Hence $$c_k = \begin{cases} 0 & ; \ k \text{ even} \\ -j\dfrac{2V_m}{\pi k} & ; \ k \text{ odd} \end{cases}$$

5 MARKS

It follow that $\boxed{x(t) = \sum_{k=-\infty}^{\infty} \frac{-j2V_m}{\pi k} e^{jkw_0 t} : k \text{ odd}}$ or $\boxed{\sum_{k=-\infty}^{\infty} \frac{2V_m}{\pi k} e^{j(kw_0 t - \pi/2)} \quad k \text{ odd}.}$

(c)

From (b) the harmonics contents of the square-wave include ω_0, $3\omega_0$, $5\omega_0$, $7\omega_0$, $9\omega_0$, ...
while the output sinusoidal signal has only ω_0.
It follows that $3\omega_0$, $5\omega_0$, $7\omega_0$, $9\omega_0$, ... must be filtered out.

5 MARKS

A low-pass filter will be required.

(d)

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 \, dt$$

Set $t_0 = 0$, we have

$$P = \frac{1}{T_0}\left[\int_0^{T_0/2} |V_m|^2 dt + \int_{T_0/2}^{T_0} |-V_m|^2 dt\right]$$

$$= \frac{1}{T_0}\left[V_m^2 t \Big|_{t=0}^{T_0/2} + V_m^2 t \Big|_{t=T_0/2}^{T_0}\right]$$

$$= \frac{1}{T_0}\left[\frac{V_m^2 T_0}{2} + V_m^2 T_0 - \frac{V_m^2 T_0}{2}\right] = V_m^2$$

5 MARKS