ENG 342: Advanced Engineering Math II

Quiz #2

September 27, 2016

Problem 1 [5 pts]

Let $f(x) = \begin{cases} 0 & 0 < x < 1/2 \\ 1 & 1/2 \leq x < 1 \end{cases}$

(a) Expand $f(x)$ as a complex Fourier series. Write it as a summation. [3 pts]

To calculate c_n in the series, we identify the period $T = 1$, and therefore $p = T/2 = 1/2$ in the formula. Then:

$$
\begin{aligned}
c_n &= \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}\,dt = \frac{1}{2\times\frac{1}{2}}\int_0^1 f(t)e^{-2in\pi x}\,dx \\
&= \frac{1}{1}\int_0^{1/2} 0e^{-2in\pi x}\,dx + \frac{1}{1}\int_{1/2}^{1} 1e^{-2in\pi x}\,dx \\
&= -\frac{1}{2in\pi}e^{-2in\pi x}\Big|_{1/2}^{1} = \frac{i}{2n\pi}\left(e^{-2in\pi} - e^{-in\pi}\right) \\
&= \frac{i}{2n\pi}\left(1 - (-1)^n\right)
\end{aligned}
$$

This term is not defined for $n = 0$. So we must calculate that one separately:

$$c_0 = \frac{1}{1}\int_0^1 f(x)\,dx = \int_{1/2}^{1} 1\,dx = \frac{1}{2}$$

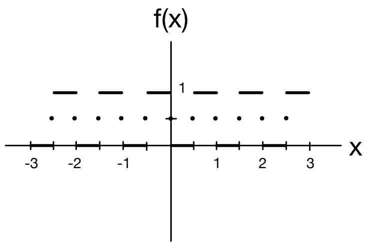
Therefore, the complex Fourier series is:

$$f(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{2n\pi}\left(1 - (-1)^n\right)e^{2in\pi x}$$

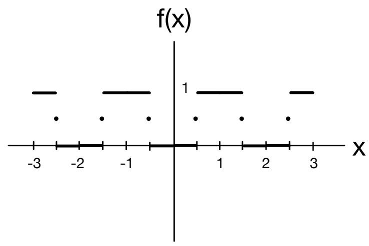
(b) Suppose $f(x)$ is expanded in a cosine series, a sine series, and a Fourier series. Sketch what these three series will converge to over $(-3, 3)$. [2 pts]

In all cases, the Fourier series will be periodic; the question is what the periodic function will look like.

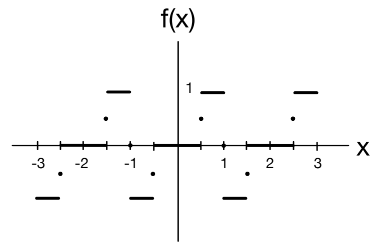
- Expanding $f(x)$ as a Fourier series (as above) will lead to a repeated version of the function outside of $(0, 1)$, shown in (a) below.
- Expanding it as a cosine series will lead to the even function in (b), or repeated versions of the function reflected over the y-axis on $(-1, 1)$.



(a) Fourier series



(b) Cosine series



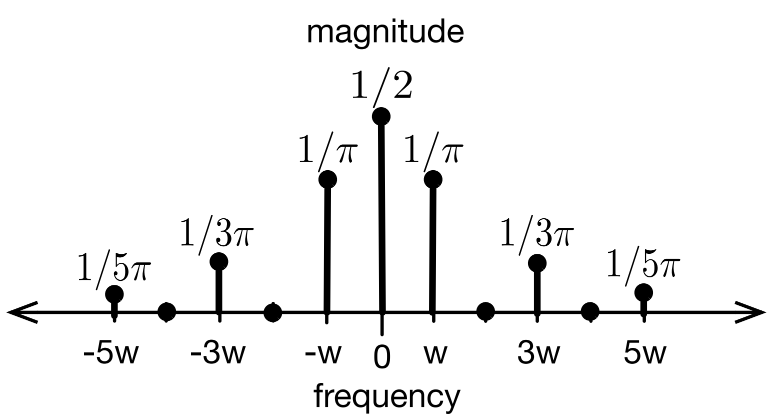
(c) Sine series

- Expanding it as a sine series will lead to the odd function in (c), or repeated versions of the function reflected over the origin on $(-1, 1)$.

(c) If x is time, then the fundamental period is $T = 1$, so the fundamental angular frequency is $w = 2\pi/T = 2\pi$. The frequency components of the signal are at multiples of w, *i.e.*, nw for n in the Fourier series expansion. The magnitudes at these points are:

$$|c_n| = \begin{cases} 1/2 & n = 0 \\ 1/n\pi & n = \pm 1, \pm 3, \pm 5, ... \\ 0 & n = \pm 2, \pm 4, \pm 6, ... \end{cases}$$

Which gives the following spectrum plot:



Problem 2 [5 pts]

Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} = 0$$

using the separation of variables method. (There are three cases to consider.)

Assuming $u(x, y) = X(x)Y(y)$, with the shorthand $u = XY$, the PDE becomes $X''Y + 4XY'' = 0$, or $X''Y = -4XY''$. Dividing both sides by X and Y, we have:

$$\frac{X''}{X} = -4\frac{Y''}{Y} = -\lambda$$

for the separation constant $\lambda \in (-\infty, +\infty)$. We then have two differential equations

$$X'' + \lambda X = 0 \qquad Y'' - \frac{\lambda}{4}Y = 0$$

For which the auxiliary equations are

$$m^2 + \lambda = 0 \qquad m^2 - \frac{\lambda}{4} = 0$$

There are three cases of λ that will change the nature of the solution: $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. We must consider each of them separately.

Case I: $\lambda = -\alpha^2 < 0, \alpha > 0$
The roots of the auxiliary equations are

$$m = \pm\alpha \qquad m = \pm\frac{\alpha}{2}i$$

Which give the solutions

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x} \qquad Y(y) = c_3 \cos\frac{\alpha}{2}y + c_4 \sin\frac{\alpha}{2}y$$

And the product solution

$$\begin{aligned} u(x,y) &= \left(c_1 e^{\alpha x} + c_2 e^{-\alpha x}\right)\left(c_3 \cos\frac{\alpha}{2}y + c_4 \sin\frac{\alpha}{2}y\right) \\ &= Ae^{\alpha x}\cos\frac{\alpha}{2}y + Be^{-\alpha x}\cos\frac{\alpha}{2}y + Ce^{\alpha x}\sin\frac{\alpha}{2}y + Dc_2 e^{-\alpha x}\sin\frac{\alpha}{2}y \end{aligned}$$

Case II: $\lambda = 0$

The roots of the auxiliary equations are

$$m = \pm 0 \qquad m = \pm 0$$

Which give the solutions

$$X(x) = c_1 + c_2 x \qquad Y(y) = c_3 + c_4 y$$

And the product solution

$$\begin{aligned} u(x,y) &= (c_1 + c_2 x)(c_3 + c_4 y) \\ &= A + Bx + Cy + Dxy \end{aligned}$$

Case III: $\lambda = +\alpha^2 > 0$, $\alpha > 0$
The roots of the auxiliary equations are

$$m = \pm \alpha i \qquad m = \pm \frac{\alpha}{2}$$

Which give the solutions

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad Y(y) = c_3 e^{\alpha y/2} + c_4 e^{-\alpha y/2}$$

And the product solution

$$\begin{aligned} u(x,y) &= (c_1 \cos \alpha x + c_2 \sin \alpha x)\left(c_3 e^{\alpha y/2} + c_4 e^{-\alpha y/2}\right) \\ &= A e^{\alpha y/2} \cos \alpha x + B e^{\alpha y/2} \sin \alpha x + C e^{-\alpha y/2} \cos \alpha x + D e^{-\alpha y/2} \sin \alpha x \end{aligned}$$