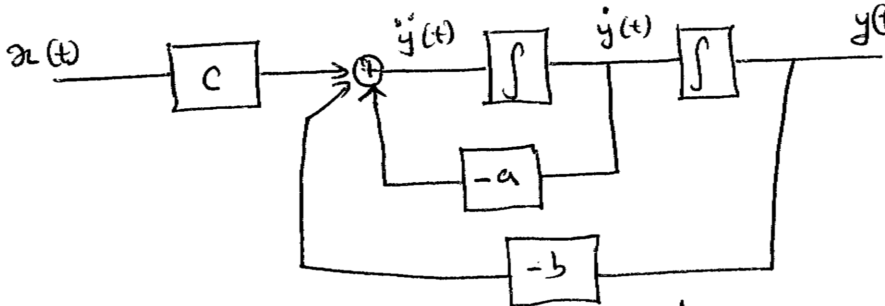
Assignment 5: Solution Guide

(1)



(a) Writing the equation around the summing point gives

$$\frac{d^2y(t)}{dt^2} = -a\frac{dy(t)}{dt} - by(t) + c\,x(t)$$

or

$$\frac{d^2y(t)}{dt^2} + a\frac{dy(t)}{dt} + by(t) = c\,x(t).$$

(b) Taking the Laplace transform of the above equation and assuming zero-initial conditions gives:

$$s^2Y(s) + asY(s) + bY(s) = cX(s)$$

$$(s^2 + as + b)Y(s) = cX(s)$$

or

$$\frac{Y(s)}{X(s)} = \frac{c}{s^2+as+b} = H(s) \text{ [The transfer function]}.$$

(c) The impulse response $h(t)$ can be obtained from $H(s)$ as

$$h(t) = \mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}\left(\frac{c}{s^2+as+b}\right) = \mathcal{L}^{-1}\left(\frac{2}{s^2+5s+6}\right)$$

Expressing $\frac{2}{s^2+5s+6}$ in partial fractions gives

$$\frac{2}{s^2+5s+6} \equiv \frac{2}{(s+3)(s+2)} \equiv \frac{A}{s+3} + \frac{B}{s+2}$$

$$A = \left.\frac{2}{s+2}\right|_{s=-3} = \frac{2}{-1} = -2$$

$$B = \left.\frac{2}{s+3}\right|_{s=-2} = \frac{2}{1} = 2$$

Hence $H(s) = \frac{2}{(s+3)(s+2)} \equiv \frac{-2}{s+3} + \frac{2}{s+2}$

It follows that

$$h(t) = \mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}\left(\frac{-2}{s+3} + \frac{2}{s+2}\right) = \mathcal{L}^{-1}\left(\frac{-2}{s+3}\right) + \mathcal{L}^{-1}\left(\frac{2}{s+2}\right)$$

$$= -2e^{-3t}u(t) + 2e^{-2t}u(t)$$

$$= \left(-2e^{-3t} + 2e^{-2t}\right)u(t).$$

$$= 2\left[e^{-2t} - e^{-3t}\right]u(t).$$

(d) Recall

$$H(s) = \frac{2}{s^2+5s+6} = \frac{2}{(s+3)(s+2)}$$

The poles of the system are at $s=-3$ and $s=-2$. Since the poles are on the left half s-plane, the system is '<u>stable</u>'.

Also, from the impulse response $h(t) = 2\left[e^{-2t} - e^{-3t}\right]u(t)$. We have that $h(t) = 0$ for $t < 0$ and hence the system is '<u>Causal</u>'

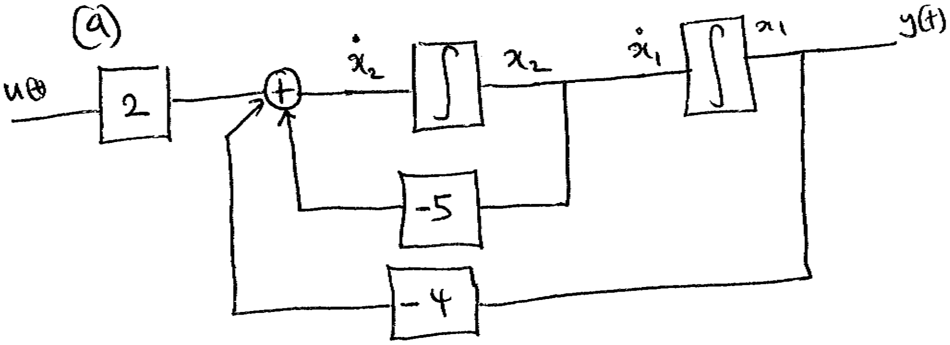
(2)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -4x_1 - 5x_2 + 2u$$

$$y = x_1$$

(a)



(b) Taking the Laplace transform of the differential equations (assuming zero initial conditions

$$sX_1(s) = X_2(s) \quad \text{—— (1)}$$

$$sX_2(s) = -4X_1(s) - 5X_2(s) + 2U(s) \quad \text{—— (2)}$$

$$Y(s) = X_1(s) \quad \text{—— (3)}$$

Eliminating $X_2(s)$ from (2) using (1) gives

$$s^2 X_1(s) = -4X_1(s) - 5sX_1(s) + 2U(s) \quad \text{—— (4)}$$

Now substituting (3) into (4) gives

$$s^2 Y(s) = -4Y(s) - 5sY(s) + 2U(s)$$

Re-arranging gives

$$(s^2 + 5s + 4)Y(s) = 2U(s)$$

or

$$\frac{Y(s)}{U(s)} = \frac{2}{s^2 + 5s + 4} = H(s) \text{ [Transfer function].}$$

(c) $h(t) = \mathcal{L}^{-1}\left(H(s)\right)$

$$= \mathcal{L}^{-1}\left(\frac{2}{s^2+5s+4}\right) = \mathcal{L}^{-1}\left(\frac{2}{(s+1)(s+4)}\right)$$

$$\frac{2}{(s+1)(s+4)} \equiv \frac{A}{s+1} + \frac{B}{s+4}$$

$$A = \left.\frac{2}{s+4}\right|_{s=-1} = \frac{2}{3}$$

$$B = \left.\frac{2}{s+1}\right|_{s=-4} = \frac{2}{-3} = -\frac{2}{3}$$

$$\therefore \frac{2}{(s+1)(s+4)} = \frac{2/3}{s+1} - \frac{2/3}{s+4}$$

It follows that

$$h(t) = \mathcal{L}^{-1}\left(\frac{2/3}{s+1} - \frac{2/3}{s+4}\right)$$

$$= \frac{2}{3}e^{-t}u(t) - \frac{2}{3}e^{-4t}u(t)$$

$$= \frac{2}{3}\left(e^{-t} - e^{-4t}\right)u(t).$$

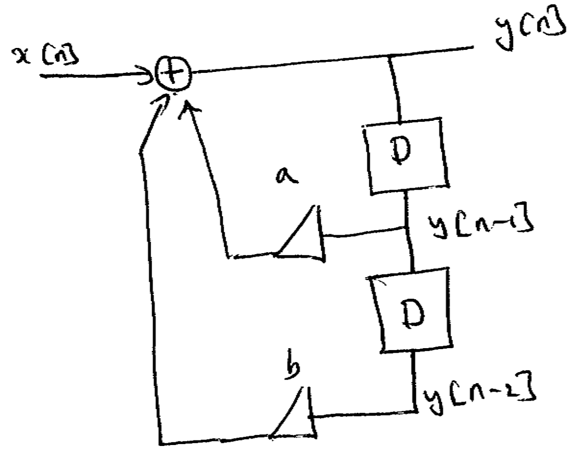
(d)

$$H(s) = \frac{2}{(s+1)(s+4)}$$

The poles of the system are at $s=-1$ and $s=-4$. Since the poles are located on the left-half s-plane, the system is "stable"

Also $h(t) = \frac{2}{3}\left(e^{-t} - e^{-4t}\right)u(t)$ which implies $h(t) = 0$ for $t < 0$ and hence the system is "causal".

(3)



(a) Writing the equation around the summing point gives

$$y[n] = a\,y[n-1] + b\,y[n-2] + x[n].$$

or

$$y[n] - a\,y[n-1] - b\,y[n-2] = x[n].$$

(b) Taking the z-transform of the above difference equation and assumming zero-initial conditions gives

$$Y(z) - a z^{-1} Y(z) - b z^{-2} Y(z) = X(z)$$

or

$$(1 - a z^{-1} - b z^{-2}) Y(z) = X(z)$$

Hence $$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1} - b z^{-2}} \quad \text{or} \quad \frac{z^2}{z^2 - a z - b}$$

(c) Setting $a = 0.5$ and $b = 0.25$, we have

$$H(z) = \frac{1}{1 - 0.5 z^{-1} - 0.25 z^{-2}} = \frac{z^2}{z^2 - 0.5 z - 0.25}$$

Solving for the poles gives

$z^2 - 0.5z - 0.25 = 0 \Rightarrow z_1 = 0.8090$ and $z_2 = 0.3090$.

Since $|z_1| = 0.8090 < 1$ and $|z_2| = 0.3090 < 1$

The system is "stable".

Also from

$$y[n] = a\,y[n-1] + b\,y[n-2] + x[n] \quad \text{or}$$

$$\underset{\substack{\uparrow \\ \text{Current output}}}{y[n]} = 0.5\,\underbrace{y[n-1] + 0.25\,y[n-2]}_{\text{past output}} + \underset{\substack{\uparrow \\ \text{Current input}}}{x[n]}$$

It follows that the current output "y[n]" depends only on "past outputs" and on current input 'x[n]'. Hence the system is Causal.

(d) $x[n] = u[n]$

using $Y(z) = H(z)X(z)$

where $H(z) = \dfrac{z^2}{z^2 - 0.5z - 0.25}$ and $X(z) = \dfrac{z}{z-1}$

Thus

$$Y(z) = \frac{z^3}{(z-1)(z^2-0.5z-0.25)} = \frac{z^3}{(z-1)(z-0.809)(z+0.309)}$$

Expressing this in partial fractions gives

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)(z-0.809)(z+0.309)} \equiv \frac{A}{z-1} + \frac{B}{z-0.809} + \frac{C}{z+0.309}$$

with

$$A = \left.\frac{z^2}{(z-0.809)(z+0.309)}\right|_{z=1} = \frac{1}{0.191 * 1.309} = 3.9997$$

$$B = \left.\frac{z^2}{(z-1)(z+0.309)}\right|_{z=0.809} = -3.0649$$

$$C = \left.\frac{z^2}{(z-1)(z-0.809)}\right|_{z=-0.309} = 0.0652$$

So

$$Y(z) = \frac{3.9997\,z}{z-1} - \frac{3.0649\,z}{z-0.809} + \frac{0.0652\,z}{z+0.309}$$

Taking the inverse z-transform gives

$$y[n] = Z^{-1}(Y(z)) = Z^{-1}\left(\frac{3.9997z}{z-1} - \frac{3.0649}{z-0.809}z + \frac{0.0652}{z+0.309}z\right)$$

$$= 3.9997\,u[n] - 3.0649(0.809)^n u[n] + 0.0652(-0.309)^n u[n]$$

or

$$\left[3.9997 - 3.0649(0.809)^n + 0.0652(-0.309)^n\right] u[n].$$

Method II: using undetermined coefficients.

$$y[n] - 0.5y[n-1] - 0.25y[n-2] = x[n]$$

Complementary solution: Set $y_c[n] = Az^n$

substituting into $y_c[n] - 0.5y_c[n-1] - 0.25y_c[n-2] = 0$ gives

$$Az^n - 0.5Az^{n-1} - 0.25Az^{n-2} = 0$$

$$Az^n\left[1 - 0.5z^{-1} - 0.25z^{-2}\right] = 0$$

The non-trivial solution is given by

$$1 - 0.5z^{-1} - 0.25z^{-2} = 0 \quad \text{or} \quad z^2 - 0.5z - 0.25 = 0$$

This gives $z = 0.809$ or -0.309

Hence the solution of the homogenous equation may be expressed as:

$$y_c[n] = A_1 z_1^n + A_2 z_2^n = \underline{A_1(0.809)^n + A_2(-0.309)^n}.$$

particular solution set $y_p[n] = P$.

substituting into $y_p[n] - 0.5y_p[n-1] - 0.25y_p[n-2] = u[n]$ gives

$$P - 0.5P - 0.25P = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases} \Rightarrow P = \begin{cases} 4; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

The general solution: set $y[n] = y_c[n] + y_p[n]$

$$y[n] = \begin{cases} A_1(0.809)^n + A_2(-0.309)^n + 4 & ; n \geq 0 \\ A_1(0.809)^n + A_2(-0.309)^n & ; n < 0 \end{cases}$$

Applying initial conditions (Assuming $y[-2]=0$ and $y[-1]=0$)

For $n \geq 0$, we have

$$0 = A_1(0.809)^{-1} + A_2(-0.309)^{-1} + 4 \text{ and}$$

$$0 = A_1(0.809)^{-2} + A_2(-0.309)^{-2} + 4$$

or

$$\begin{bmatrix} 0.809^{-1} & (-0.309)^{-1} \\ 0.809^{-2} & (-0.309)^{-2} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

Solving for A_1 and A_2 gives

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -3.0652 \\ 0.0652 \end{bmatrix}$$

for $n < 0$ we have

$$0 = A_1(0.809)^{-1} + A_2(-0.309)^{-1}$$

$$0 = A_1(0.809)^{-2} + A_2(-0.309)^{-2}$$

or

$$\begin{bmatrix} 0.809^{-1} & (-0.309)^{-1} \\ 0.809^{-2} & (-0.309)^{-2} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence the final solution is

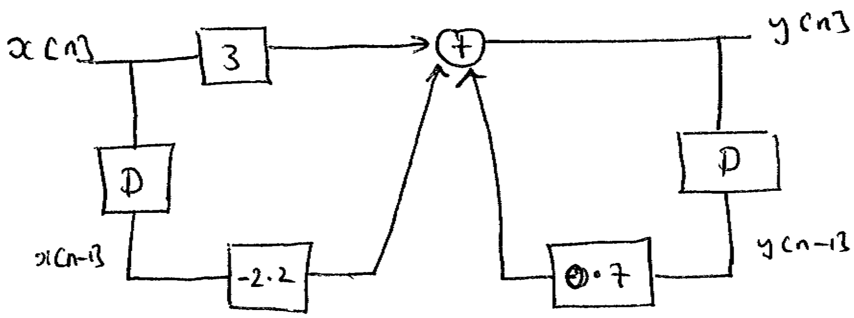
$$y[n] = -3.0652(0.809)^n + 0.0652(-0.309)^n + 4; \; n \geq 0$$

or

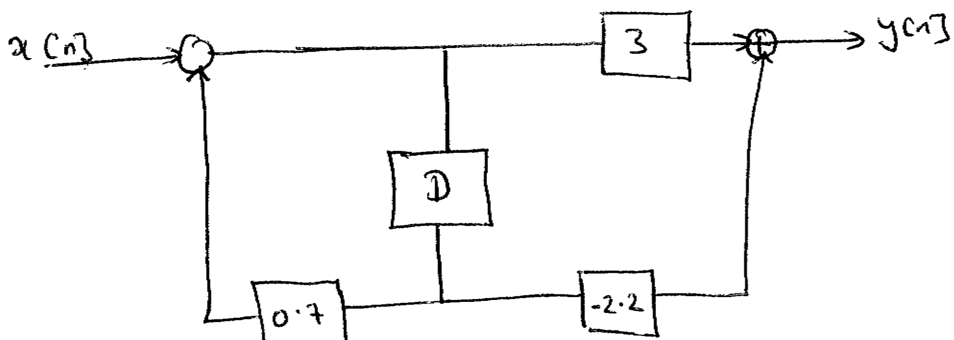
$$y[n] = \left[4 - 3.0652(0.809)^n + 0.0652(-0.309)^n\right] u[n].$$

(4) $y[n] - 0.7y[n-1] = 3x[n] - 2.2x[n-1]$

(a)



Direct form I



Direct form II.

(b)

$$y[n] = 0.7y[n-1] + 3x[n] - 2.2x[n-1]$$

setting $x[n] = \delta[n]$ gives

$$h[n] = 0.7h[n-1] + 3\delta[n] - 2.2\delta[n-1]$$

Note that since the system is causal, $h[-1] = 0$. So

$n=0$: $h[0] = 0.7\cancel{h[-1]}^{0} + 3\delta[0] - 2.2\cancel{\delta[-1]}^{0} = 3$

$n=1$: $h[1] = 0.7h[0] + 3\cancel{\delta[1]}^{0} - 2.2\delta[0] = 0.7(3) - 2.2 = -0.1$

$n=2$: $h[2] = 0.7h[1] + 3\cancel{\delta[2]}^{0} - 2.2\cancel{\delta[1]}^{0} = 0.7(-0.1) = -0.07$

$n=3$: $h[3] = 0.7h[2] + 3\cancel{\delta[3]}^{0} - 2.2\cancel{\delta[2]}^{0} = 0.7(-0.07) = -0.049$

$n=4$: $h[4] = 0.7h[3] + 3\cancel{\delta[4]}^{0} - 2.2\cancel{\delta[3]}^{0} = 0.7(-0.049) = -0.0343$

Hence $h[n] = \{3, -0.1, -0.07, -0.049, -0.0343\}, \quad 0 \le n \le 4.$

Alternatively

Taking the z-transform ~~of~~ of

$y[n] - 0.7\,y[n-1] = 3\,x[n] - 2.2\,x[n-1]$ gives

$$Y(z) - 0.7 z^{-1} Y(z) = 3X(z) - 2.2 z^{-1} X(z)$$

or

$$(1 - 0.7z^{-1})\,Y(z) = (3 - 2.2z^{-1})\,X(z)$$

or

$$\frac{Y(z)}{X(z)} = H(z) = \frac{3 - 2.2z^{-1}}{1 - 0.7z^{-1}} = \frac{3z - 2.2}{z - 0.7}$$

Carrying out Long division gives

$$\begin{array}{r|l}
 & 3 - 0.1z^{-1} - 0.07z^{-2} - 0.049z^{-3} - 0.0343z^{-3} \\
\hline
z - 0.7 & 3z - 2.2 \quad \ominus \\
 & \underline{3z - 2.1} \\
 & -0.1 \quad \ominus \\
 & \underline{-0.1 + 0.07z^{-1}} \\
 & -0.07z^{-1} \quad \ominus \\
 & \underline{-0.07z^{-1} + 0.049z^{-2}} \\
 & -0.049z^{-2} \\
 & \underline{-0.049z^{-2} + 0.0343z^{-3}} \\
 & -0.0343z^{-3} \\
 & \vdots
\end{array}$$

$$\therefore\ H(z) = 3 - 0.1z^{-1} - 0.07z^{-2} - 0.049z^{-3} - 0.0343z^{-3} + \cdots$$

$$\equiv \sum_{n=0}^{\infty} h[n]\,z^{-n}$$

Hence for $0 \le n \le 4$,

$$h[n] = \{3,\ -0.1,\ -0.07,\ -0.049,\ -0.0343\},\quad 0 \le n \le 4$$

(c) using $Y(z) = H(z)X(z)$

where $H(z) = \dfrac{3z - 2.2}{z - 0.7}$ and $X(z) = \dfrac{z}{z-1}$

$$\therefore Y(z) = \frac{3z^2 - 2.2z}{(z-0.7)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{3z - 2.2}{(z-0.7)(z-1)} \equiv \frac{A}{z-0.7} + \frac{B}{z-1}$$

$$A = \left.\frac{3z-2.2}{z-1}\right|_{z=0.7} = \frac{-0.1}{-0.3} = \frac{1}{3}$$

$$B = \left.\frac{3z-2.2}{z-0.7}\right|_{z=1} = \frac{0.8}{0.3} = \frac{8}{3}$$

Hence
$$Y(z) = \frac{\frac{1}{3}z}{z-0.7} + \frac{\frac{8}{3}z}{z-1}$$

Taking the inverse z-transform gives

$$y[n] = \frac{1}{3}(0.7)^n u[n] + \frac{8}{3}u[n]$$

$$= \frac{1}{3}\left(8 + (0.7)^n\right)u[n].$$

(d) Again using z-transform with $H(z) = \dfrac{3z-2.2}{z-0.7}$ and

$$X(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8}$$

$$Y(z) = \frac{3z^2 - 2.2z}{(z-0.7)(z-0.8)}$$

$$\frac{Y(z)}{z} = \frac{3z-2.2}{(z-0.7)(z-0.8)} \equiv \frac{A}{z-0.7} + \frac{B}{z-0.8}$$

$$A = \left.\frac{3z-2.2}{z-0.8}\right|_{z=0.7} = \frac{-0.1}{-0.1} = 1$$

$$B = \left.\frac{3z-2.2}{z-0.7}\right|_{z=0.8} = \frac{0.2}{0.1} = 2$$

Therefore

$$Y(z) = \frac{z}{z-0.7} + \frac{2z}{z-0.8}$$

Taking inverse z-transform gives

$$y[n] = 0.7^n u[n] + 2(0.8)^n u[n] \quad \text{or}$$

$$\left[0.7^n + 2(0.8)^n\right] u[n]$$