p.64 **Section 2.5:**
Solutions by Substitutions

Sometimes ~~It is often~~ necessary to transform a differential equation into another differential equation by means of a substitution:

For instance $\frac{dy}{dx} = f(x,y)$

let $y = g(x,u)$ where u is a function of x, then by the Chain rule:

$$\frac{dy}{dx} = g_x(x,u) + g_u(x,u)\frac{du}{dx}$$

but $\frac{dy}{dx} = f(x,y)$ and $y = g(x,u)$

$$\therefore f(x, g(x,u)) = g_x(x,u)\frac{dx}{dx} + g_u(x,u)\frac{du}{dx}$$

$$\frac{du}{dx} = \frac{f(x,g(x,u)) - g_x(x,u)}{g_u(x,u)} = F(x,u)$$

If $u = \phi(x)$ is a solution, then $y = g(x, \phi(x))$ is a solution to the original equation

We will discuss:

1. Homogeneous Equations
2. Bernoulli Equations
3. General Substitution

p.64 Homogeneous Equations

A function is a homogeneous function of degree α if:

$$f(tx, ty) = t^{\alpha} f(x, y)$$ for

some real number α.

Example: Let $f(x, y) = x^3 + y^3$

$$f(tx, ty) = t^3 x^3 + t^3 y^3 = t^3(x^3 + y^3)$$
$$= t^3 f(x, y)$$

$\therefore$ $f(x, y) = x^3 + y^3$ is homogeneous.

Try $f(x, y) = x^3 + y^3 + 1$

$$f(tx, ty) = t^3 x^3 + t^3 y^3 + 1 \neq t^{\alpha} f(x, y)$$

$f(x, y) = x^3 + y^3 + 1$ is NOT homogeneous

A First-Order DE in differential form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is homogeneous if

$$M(tx, ty) = t^{\alpha} M(x, y)$$ and

$$N(tx, ty) = t^{\alpha} N(x, y)$$

If M and N are homogeneous functions of degree α then,

$$M(x,y) = x^{\alpha} M(1,u) \quad \text{and} \quad N(x,y) = x^{\alpha} N(1,u)$$

where $u = y/x$

leave out

$$\left\{ \begin{array}{l} M(x,y) = y^{\alpha} M(v,1) \text{ and } N(x,y) = y^{\alpha} N(v,1) \\ \quad \text{where } v = x/y \text{ OR } x = vy \end{array} \right.$$

Rewriting the homogeneous equation $M(x,y)dx + N(x,y)dy = 0$, we have:

$$x^{\alpha} M(1,u)\,dx + x^{\alpha} N(1,u)\,dy = 0 \quad \text{OR}$$

$$M(1,u)\,dx + N(1,u)\,dy = 0$$

where $u = y/x$ or $\boxed{y = ux}$

Since $\frac{dy}{dx} = u\frac{dx}{dx} + x\frac{du}{dx}$ we have

the differential $dy = u\,dx + x\,du$

Substituting we have:

$$M(1,u)\,dx + N(1,u)[u\,dx + x\,du] = 0$$

then $[M(1,u) + uN(1,u)]dx + xN(1,u)\,du = 0$

OR $$\frac{dx}{x} + \frac{N(1,u)\,du}{M(1,u) + uN(1,u)} = 0$$

This is a separable ODE!

p.65 Example 1: Solve the Homogeneous DE

$$(x^2+y^2)dx + (x^2-xy)dy = 0$$

Is it homogeneous?

$M(x,y) = x^2+y^2$, $N(x,y) = x^2-xy$

$M(tx,ty) = t^2x^2+t^2y^2 = t^2(x^2+y^2) = t^2M(x,y)$

$N(tx,ty) = t^2x^2 - tx\,ty = t^2(x^2-xy) = t^2N(x,y)$

Yes, it is homogeneous of degree 2

$\therefore$ let $y=ux$, then $dy = u\,dx + x\,du$

Substituting:

$$(x^2+u^2x^2)\,dx + (x^2-x(ux))(u\,dx+x\,du) = 0$$

$$x^2(1+u^2)\,dx + x^2(1-u)(u\,dx+x\,du) = 0$$

$$(x^2+\cancel{u^2x^2}+x^2u-\cancel{x^2u^2})dx + x^3(1-u)\,du = 0$$

$$x^2(1+u)\,dx + x^3(1-u)\,du = 0$$

Separating: $\dfrac{dx}{x} + \dfrac{(1-u)}{(1+u)}\,du = 0$

$$\begin{array}{r|l} & -1 \\ \hline u+1 & -u+1 \\ & \underline{-u-1} \\ & \quad 2 \end{array}$$

; $\dfrac{(1-u)}{(1+u)} = -1 + \dfrac{2}{1+u}$

$$\frac{dx}{x} + \left[-1 + \frac{2}{1+u}\right] du = 0$$

$$\int\left[-1+\frac{2}{1+u}\right]du+\int\frac{dx}{x}=\int 0\,dx$$

$$-u+2\ln|1+u|+\ln|x|=c_1$$

Substituting $u=y/x$:

$$-\frac{y}{x}+2\ln\left|1+\frac{y}{x}\right|+\ln|x|=c_1=\ln|c|$$

$$2\ln\left|1+\frac{y}{x}\right|+\ln|x|-\ln|c|=\frac{y}{x}$$

Since $1+\frac{y}{x}=\frac{x+y}{x}$ and $2\ln\left|\frac{x+y}{x}\right|=\ln\left(\frac{x+y}{x}\right)^2$

$$\ln\left(\frac{x+y}{x}\right)^2+\ln|x|-\ln c=\frac{y}{x}$$

$$\ln\left|\left(\frac{x+y}{x}\right)^2\cdot x\div c\right|=\frac{y}{x}$$

OR $$\ln\left|\frac{(x+y)^2}{cx}\right|=\frac{y}{x}$$

$$\therefore\ \boxed{(x+y)^2=cx\,e^{y/x}}$$

Example (#2, p. 67):

$(x+y)\,dx + x\,dy = 0$ Homogeneous?

$M(tx, ty) = t(x+y)$, $N(tx, ty) = tx$ ✓

Homogeneous of degree 1

$\therefore\ y = ux$, $dy = u\,dx + x\,du$

$(x+ux)\,dx + x(u\,dx + x\,du) = 0$

$(x+2ux)\,dx + x^2\,du = 0$

$\cancel{x}(1+2u)\,dx + x^{\cancel{2}}\,du = 0$

$$\frac{dx}{x} + \frac{du}{(1+2u)} = 0$$

$$\ln|x| + \frac{1}{2}\ln|(1+2u)| = C$$

$$\ln\left[x\left(\sqrt{1+2u}\right)\right] = C$$

$$x\sqrt{1+2u} = C_1$$

$$x^2\left(1+\frac{2y}{x}\right) = C_2$$

$$\boxed{x^2 + 2xy = C_2}$$

Example (#4, p.67):

$y\,dx = 2(x+y)\,dy$ Homogeneous?

$M(tx,ty) = ty$; $N(tx,ty) = 2t(x+y)$ ✓ yes.

$y\,dx - 2(x+y)\,dy = 0$

Try $y = ux$ or $x = vy$?

$ux\,dx - 2(x+ux)(u\,dx + x\,du) = 0$

Would $x = vy$ be easier?

$y(v\,dy + y\,dv) - 2(vy+y)\,dy = 0$

$(yv - 2vy - 2y)\,dy + y^2\,dv = 0$

$-(vy + 2y)\,dy + y^2\,dv = 0$

$-(v+2)\,dy = -y\,dv$

$\dfrac{dy}{y} = \dfrac{dv}{(v+2)}$, $\ln|y| = \ln|v+2| + c$

$\ln\left|\dfrac{y}{v+2}\right| = c_1$

$\ln\left|\dfrac{y}{\frac{x}{y}+2}\right| = c_1$, $\ln\left|\dfrac{y^2}{x+2y}\right| = c_1$

$$\boxed{y^2 = c(x+2y)}$$

Try solving with $y = ux$:

$$ux\,dx - 2\left[xu\,dx + x^2du + u^2x\,dx + ux^2du\right] = 0$$

Collect terms :

$$(ux - 2ux - 2u^2x)\,dx - (x^2 + ux^2)\,du = 0$$

$$-x(2u^2 + 2u - u)\,dx - x^2(1+u)\,du = 0$$

$$\frac{dx}{x} + \frac{(1+u)}{u(2u+1)}\,du = 0$$

$$\frac{dx}{x} + \left[\frac{1}{u(2u+1)} + \frac{1}{(2u+1)}\right] du = 0$$

Partial Fractions: $\frac{1}{u(2u+1)} = \frac{A}{u} + \frac{B}{(2u+1)}$

$1 = A(2u+1) + Bu \Rightarrow u = 0,\ A = 1$

$u = -1/2,\ B = -2$

$$\therefore \int \frac{dx}{x} - \int \frac{1\,du}{(2u+1)} + \int \frac{du}{u} = 0$$

$$\ln|x| - \ln|2u+1| + \ln|u| = C_1$$

$$\ln\left|\frac{x \cdot u}{2u+1}\right| = C_1 \quad ,\ u = \frac{y}{x}$$

$$\frac{xy}{2\left(2\frac{y}{x}+1\right)} = c = \frac{xy}{2\left(\frac{2y+x}{x}\right)} = c$$

$$\boxed{x^2 = c\left[\frac{2y+x}{y}\right]}$$

Example (#12, p. 67)

Solve the initial value problem:

$$(x^2+2y^2)\frac{dx}{dy} = xy \ , \ y(-1)=1$$

let $y = ux$, $dy = u\,dx + x\,du$

$$(x^2+2u^2x^2)\,dx - ux^2(u\,dx + x\,du) = 0$$

$$x^2(1+u^2)\,dx - ux^3\,du = 0$$

$$\frac{dx}{x} - \frac{u\,du}{1+u^2} = 0$$

$$\ln|x| - \frac{1}{2}\ln(1+u^2) = c$$

$$\frac{x^2}{1+u^2} = c_1 \ , \ x^4 = c_1(x^2+y^2)$$

For the point $(-1, 1)$ we have

$$1 = c_1(1+1) \ , \ c_1 = 1/2$$

$\therefore$ The solution is

$$\boxed{x^4 = \frac{1}{2}(x^2+y^2)}$$