p.103 Chapter 3:

Higher-Order Differential Equations

n^{th}-order Initial Value Problems:

Solve: $a_n \dfrac{d^n y}{dx^n} + a_{n-1}(x) \dfrac{d^{n-1} y}{dx^{n-1}} + \ldots$

$$+ a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

Subject to: $y(x_0) = y_0$,

$y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$

Theorem 3.1.1 Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x) \ldots a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem above exists on the interval and is unique

$\Rightarrow$ Particular Solution!

p.105 <u>Boundary Value Problem</u> (BVP)

Solve: $a_2(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x)$

Subject to: $y(a) = y_0$, $y(b) = y_1$

where $y(a)$ and $y(b)$ are <u>boundary</u> <u>conditions</u>

p.106 An equation is <u>homogeneous</u> if $g(x) = 0$ and <u>non</u>-<u>homogeneous</u> if $g(x) \neq 0$

<u>differential</u> <u>operator</u>:

$$\frac{dy}{dx} = Dy \;\ldots\; \frac{d^ny}{dx^n} = D^n y$$

<u>Superposition Principle for homogeneous equations</u>:

If $y_1, y_2 \ldots y_k$ are solutions to an n^{th} order D.E on interval I, then $y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_k y_k(x)$, a linear combination of the solutions is also a solution.

p.107 Definition 3.1.1: Linear Dependence / Independence:

A set of functions $f_1(x), f_2(x) \ldots f_n(x)$ is said to be linearly dependent on interval I, if there exists constants $c_1, c_2 \ldots c_n$, not all zero such that:

$$c_1 f_1(x) + c_2 f_2(x) + \ldots c_n f_n(x) = 0$$

for every x in the interval.

If the set of functions is not linearly dependent, then they are linearly independent.

Example: (p.114 #18)

Are the set of functions linearly dependent?

$f_1(x) = \cos 2x$, $f_2(x) = 1$, $f_3(x) = \cos^2 x$

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \,?$$

$$c_1(2\cos^2 x - 1) + c_2(1) + c_3(\cos^2 x) = 0$$

let $c_1 = 1$, $c_2 = 1$, $c_3 = -2$

$\therefore 2\cos^2 x - 1 + 1 - 2\cos^2 x = 0$

$\therefore$ linearly dependant ✓

Definition:

p.109 <u>Wronskian:</u>

What is it?

Suppose each of the functions:

$f_1(x), f_2(x) \dots f_n(x)$ possesses

at least $n-1$ derivatives,

The determinant:

$$W(f_1 \dots f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

where the primes denote derivatives is called the <u>Wronskian</u> of the function

A set of solutions $y_1 \dots y_n$ of the homogeneous linear n^{th} order differential equation on an interval I is <u>linearly independent</u> if and only if $\boxed{W(y_1, y_2 \dots y_n) \neq 0}$ for every x in the interval

$\therefore$ $W \neq 0$ for linear independence and solutions form a <u>fundamental set of solutions</u> on the Interval

p. 109 Theorem 3.1.5 General Solution - Homogeneous Equations:

Let $y_1, y_2 \ldots y_n$ be a fundamental set of solutions of an nth order homogeneous linear D.E. on interval I, the general solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

where c_i, $i = 1, 2 \ldots n$ are arbitrary constants

Example (p. 114 #26)

Verify that $e^{x/2}$, $xe^{x/2}$ are solutions to $4y'' - 4y' + y = 0$ on $(-\infty, \infty)$ and form a general solution

$$W = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & e^{x/2} + \frac{1}{2}xe^{x/2} \end{vmatrix}$$

$$= e^x\left(1 + \tfrac{1}{2}x\right) - e^x\left(\tfrac{1}{2}x\right)$$

$$= e^x \neq 0 \text{ for any } x$$

$\therefore$ Solutions are independent

$$y_g = c_1 e^{x/2} + c_2 x e^{x/2}$$

5 a

Now lets revisit the set of linearly dependent equations:

$f_1(x) = \cos 2x$, $f_2(x) = 1$, $f_3(x) = \cos^2 x$

Prove dependence using the Wronskian

We already showed dependence using $C_1 f_1(x) + C_2 f_2(x) + C_3 f_3(x) = 0$

$C_1(2\cos^2 x - 1) + C_2 + C_3(\cos^2 x) = 0$

$2C_1 \cos^2 x + C_3 \cos^2 x - C_1 + C_2 = 0$

$(2C_1 + C_3)(\cos^2 x) + (C_2 - C_1) = 0$

$2C_1 = -C_3$, $C_2 = C_1$

let $C_1 = 1 = C_2$, $C_3 = -2$

Since $2\cos^2 x - 1 + 1 - 2\cos^2 x = 0$ ✓

Now use the Wronskian ...

$f_1(x) = \cos 2x = 2\cos^2 x - 1$

$f_1'(x) = 2[-2\cos x \sin x]$

$f_2'(x) = 0$, $f_3'(x) = -2\cos x \sin x$

5 (b)

$$W = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$f_1''(x) = +4\sin x \sin x - 4\cos x \cos x$$
$$= 4(\sin^2 x - \cos^2 x)$$

$$f_3''(x) = 2(\sin^2 x - \cos^2 x)$$

$$W = \begin{vmatrix} (2\cos^2 x - 1) & 1 & \cos^2 x \\ -4\cos x \sin x & 0 & -2\cos x \sin x \\ 4(\sin^2 x - \cos^2 x) & 0 & 2(\sin^2 x - \cos^2 x) \end{vmatrix}$$

$$W = \Big(-8(\cos x \sin x)(\sin^2 x - \cos^2 x) +$$
$$8(\cos x \sin x)(\sin^2 x - \cos^2 x) = 0 \checkmark$$

5 C

p.114 #22 $f_1(x) = e^x$, $f_2(x) = e^{-x}$, $f_3(x) = \sinh x$

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$?

$c_1 e^x + c_2 e^{-x} + c_3 \sinh x = 0$

let $c_1 = 1/2$, $c_2 = -1/2$

$\frac{1}{2}(e^x - e^{-x}) = \sinh x$

now let $c_3 = -1$

$\therefore \frac{1}{2}e^x - \frac{1}{2}e^{-x} - \sinh x = 0$ ✓

Linearly dependent.

Wronskian:

$f_1'(x) = e^x$, $f_2'(x) = -e^{-x}$

$f_3'(x) = \cosh x$, $f_3''(x) = \sinh x$

$$W = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

(d)

$$W = \begin{vmatrix} e^{x} & e^{-x} & \sinh x \\ e^{x} & -e^{-x} & \cosh x \\ e^{x} & e^{-x} & \sinh x \end{vmatrix}$$

$$e^{x}\left(e^{-x}\sinh x - e^{-x}\cosh x\right)$$

$$- e^{x}\left(e^{-x}\sinh x - e^{-x}\sinh x\right)$$

$$+ e^{x}\left(e^{-x}\cosh x - e^{-x}\sinh x\right)$$

$$= 0 \checkmark$$ Linearly dependent!

5 (e)

p.114 Verify that $\cosh 2x$, $\sinh 2x$ form a fundamental set of solutions of $y'' - 4y = 0$

$$W = \begin{vmatrix} \cosh 2x & \sinh 2x \\ 2\sinh 2x & 2\cosh 2x \end{vmatrix}$$

$$= 2\cosh^2 2x - 2\sinh^2 2x \neq 0$$

$$\therefore Y(x) = C_1 \cosh 2x + C_2 \sinh 2x$$

Solve this?:

$$y'' - 4y = 0 = (D^2 - 4)y = 0$$

Solve by roots where

$D^2 - 4 = 0$, or $D = \pm 2$ are the roots of the general form

$$y = C_1 e^{-\alpha x}$$

$$\therefore y(x) = C_1 e^{2x} + C_2 e^{-2x}$$

How do we get to hyperbolic sine and cosine?

First choose $c_1 = c_2 = \frac{1}{2}$

then $y(x) = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$

$= \cosh 2x$

Now choose $c_1 = \frac{1}{2}$ $c_2 = -\frac{1}{2}$

then $y(x) = \frac{1}{2}\left(e^{2x} - e^{-2x}\right) = \sinh 2x$

Since we proved that $\cosh 2x$ and $\sinh 2x$ are linearly independent then an alternative form of the solution is

$$y(x) = c_1 \cosh 2x + c_2 \sinh 2x$$

We get our solution!