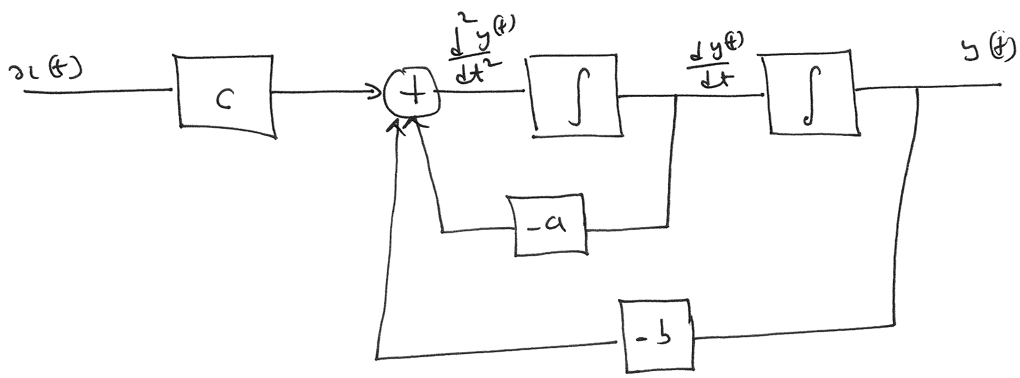
Signals and Systems ELC 321-02

Sample solution Final Exam.

Problem 1:



(a)

$$\frac{d^2y(t)}{dt^2} = -a\frac{dy(t)}{dt} - b\,y(t) + c\,x(t)$$

$$\frac{d^2y(t)}{dt^2} + a\frac{dy(t)}{dt} + b\,y(t) = c\,x(t).$$

(b) Taking Laplace transform, assuming zero initial conditions gives

$$s^2 Y(s) + a s Y(s) + b Y(s) = c X(s)$$

$$(s^2 + as + b) Y(s) = c X(s)$$

The transfer function is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{c}{s^2 + as + b}.$$

(c) If $a = 5$, $b = 6$ and $c = 10$

we have $$H(s) = \frac{10}{s^2 + 6s + 5} = \frac{10}{(s+3)(s+2)}$$

$h(t)$ is the inverse Laplace transform of $H(s)$.

$$H(s) = \frac{A}{s+3} + \frac{B}{s+2} \quad \text{where } A = \left.\frac{10}{s+2}\right|_{s=-3} = -10, \quad B = \left.\frac{10}{s+3}\right|_{s=-2} = 10$$

$$H(s) = \frac{-10}{s+3} + \frac{10}{s+2}$$

Hence

$$h(t) = \mathcal{L}^{-1}(H(s)) = -10\,\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + 10\,\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= -10e^{-3t} + 10e^{-2t}.$$

(d)

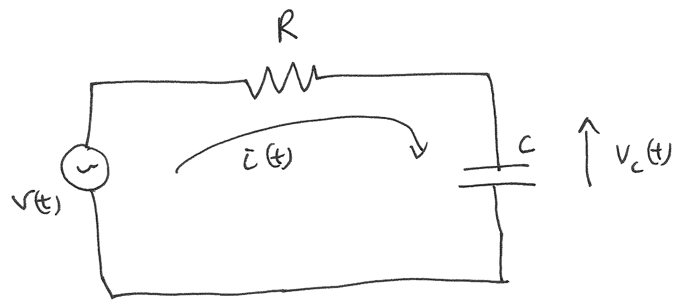
from (b)

$$H(s) = \frac{10}{s^2+5s+5}$$

Since the poles ~~are~~ $s=-2$ and -3 are on the left half s-plane, the system is stable.

The order of the denominator polynomial is higher than that of the numerator, the system is causal.

Problem 2:



(a) Writing KVL around the loop gives

$$v(t) = i(t)R + \frac{1}{C}\int i(t)\,dt$$

where $V_c(t) = \frac{1}{C}\int i(t)\,dt$ or $i(t) = C\frac{dV_c(t)}{dt}$

Rewriting the above equation gives

$$v(t) = RC\frac{dV_c(t)}{dt} + V_c(t)$$

(b)

Let $v(t) = \alpha V_1(t) + \beta V_2(t)$

and $V_i(t) = \alpha V_{c_1}(t) + \beta V_{c_2}(t)$

Hence $v(t) = RC \dfrac{dV_c(t)}{dt} + V_c(t)$ becomes

$$\alpha V_i(t) + \beta V_2(t) = \alpha RC \frac{dV_{c_1}(t)}{dt} + \beta RC \frac{dV_{c_2}(t)}{dt} + \alpha V_{c_1}(t) + \beta V_{c_2}(t)$$

$$= \alpha\left(RC \frac{dV_{c_1}(t)}{dt} + V_{c_1}(t)\right) + \beta\left(RC \frac{dV_{c_2}(t)}{dt} + V_{c_2}(t)\right)$$

$$= \alpha V_1(t) \quad + \quad \beta V_2(t)$$

Linear

Let $v(t-t_0) = RC \dfrac{dV_c(t-t_0)}{dt} + V_c(t-t_0)$

which is equivalent to delaying the input $v(t)$ and the output $v_c(t)$ by t_0. Hence the system is time-invariant.

The circuits current at time t does not depend on future values of the input voltage $v(t)$ but on the current and past input voltage. — Causal

(c)

Recall

$$RC \frac{dV_c(t)}{dt} + V_c(t) = v(t)$$

with $R = 0.5\Omega$ and $C = 0.25F$, we have

$$0.125 \frac{dV_c(t)}{dt} + V_c(t) = v(t)$$

or

$$\frac{dV_c(t)}{dt} + 8V_c(t) = 8v(t)$$

Using Laplace method.

Taking the Laplace transform of the above differential equation gives

$$sV_c(s) + 8V_c(s) = 8V(s)$$

$$(s+8)V_c(s) = 8V(s)$$

We have

$$V_c(s) = \frac{8}{s+8} V(s)$$

with $v(t) = u(t)$, $V(s) = \frac{1}{s}$

Hence $$V_c(s) = \frac{8}{s(s+8)} = \frac{A}{s} + \frac{B}{s+8}$$

with $$A = \left.\frac{8}{s+8}\right|_{s=0} = 1$$

$$B = \left.\frac{8}{s}\right|_{s=-8} = -1$$

We have
$$V_c(s) = \frac{8}{s(s+8)} = \frac{1}{s} - \frac{1}{s+8}$$

Taking the inverse Laplace transform of $V_c(s)$ gives

$$v_c(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{1+8}\right)$$
$$= \left(1 - e^{-8t}\right)u(t) \quad \text{or} \quad 1 - e^{-8t},\ t > 0.$$

<u>using undetermined coefficient method</u>

<u>Complementary solution:</u>

Let $v_c^c(t) = Ae^{st}$

substituting into the homogenous equation $\frac{dv_c(t)}{dt} + 8v_c(t) = 0$ gives

$$Ase^{st} + 8Ae^{st} = 0 \Rightarrow Ae^{st}(s+8) = 0$$

This implies $s = -8$.

The complementary solution is thus $v_c^c(t) = Ae^{-8t}$

<u>Particular solution</u>

let $v_c^p(t) = B$ (since $v(t) = u(t)$)

substituting into the differential equation gives

$$8B = 8 \Rightarrow B = 1$$

<u>General solution</u>

$$v_c(t) = v_c^c(t) + v_c^p(t) = 1 + Ae^{-8t}.$$

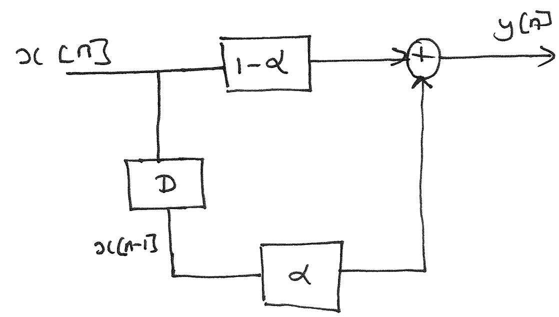
<u>Initial Condition:</u> $0 = 1 + Ae^{0} \Rightarrow A = -1$

Hence $v_c(t) = 1 - e^{-8t}$, $t > 0$

Problem 3

$$y[n] = (1-\alpha)x[n] + \alpha x[n-1]$$

(a)



(b) $$y[n] = (1-\alpha)x[n] + \alpha x[n-1]$$

$$h[n] = (1-\alpha)\delta[n] + \alpha\delta[n-1]$$

Since the system is causal, $h[n] = 0$ for $n < 0$.

Hence
$$h[0] = (1-\alpha)\delta[0] + \alpha\delta[-1] = 1-\alpha$$
$$h[1] = (1-\alpha)\delta[1] + \alpha\delta[0] = \alpha$$
$$h[2] = (1-\alpha)\delta[2] + \alpha\delta[1] = 0$$
$$\vdots \qquad \vdots$$

Hence
$$h[n] = \begin{cases} 1-\alpha & ; \quad n=0 \\ \alpha & ; \quad n=1 \\ 0 & ; \quad n \geq 2 \end{cases}$$

(c) using $h[n] = (1-\alpha)\delta[n] + \alpha\delta[n-1]$

and $$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty}\left[(1-\alpha)\delta[n] + \alpha\delta[n-1]\right]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty}(1-\alpha)\delta[n]e^{-j\Omega n} + \sum_{n=-\infty}^{\infty}\alpha\delta[n-1]e^{-j\Omega n}$$

$$= (1-\alpha)e^{0} + \alpha e^{-j\Omega} = 1-\alpha+\alpha e^{-j\Omega}.$$

d

$y[n] = (1-\alpha)x[n] + \alpha x[n-1]$

Taking the z-transform of the difference equation

$$Y(z) = (1-\alpha)X(z) + \alpha z^{-1} X(z)$$

$$Y(z) = \left((1-\alpha) + \alpha z^{-1}\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1-\alpha + \alpha z^{-1}$$

or

From (b)

$$h[n] = (1-\alpha)\,\delta[n] + \alpha\,\delta[n-1]$$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty}\left((1-\alpha)\delta[n] + \alpha\,\delta[n-1]\right) z^{-n}$$

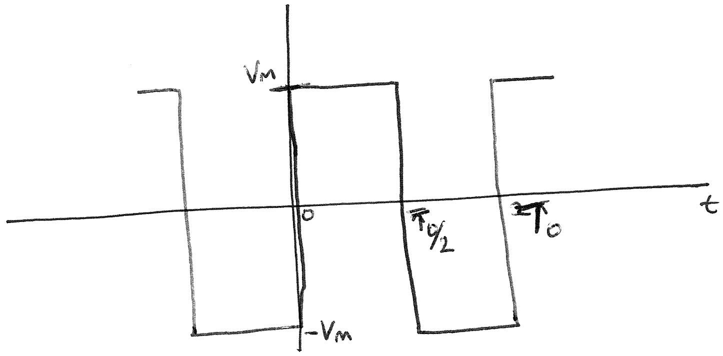
$$= \sum_{n=0}^{\infty} (1-\alpha)\,\delta[n] z^{-n} + \sum_{n=0}^{\infty} \alpha\,\delta[n-1] z^{-n}$$

$$= (1-\alpha) z^{0} + \alpha z^{-1}$$

$$= 1-\alpha + \alpha z^{-1}$$

No, the filter cannot go unstable because it has no poles. It is a finite impulse response (FIR) filter.

Problem 4



$T_0 = \frac{2\pi}{w_0}$

$\Rightarrow T_0 w_0 = 2\pi$

(a)

The square wave can be described for a period as

$$x(t) = \begin{cases} V_m \ ; & 0 \le t \le T_0/2 \\ -V_m \ ; & \frac{T_0}{2} < t \le T_0 \end{cases}$$

To express this in Fourier Series, we have

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jw_0 k t} \quad \text{with} \quad c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jkw_0 t}\, dt$$

We have

$$c_k = \frac{1}{T_0}\left[\int_0^{T_0/2} V_m e^{-jkw_0 t}\, dt + \int_{T_0/2}^{T_0} -V_m e^{-jkw_0 t}\, dt\right]$$

$$= \frac{V_m}{-jkw_0 T_0}\left[e^{-jkw_0 t}\Big|_0^{T_0/2} - e^{-jkw_0 t}\Big|_{T_0/2}^{T_0}\right]$$

$$= \frac{jV_m}{kw_0 T_0}\left[e^{-jkw_0 T_0/2} - 1 - e^{-jkw_0 T_0} + e^{-jkw_0 T_0/2}\right]$$

using $T_0 w_0 = 2\pi$ gives

$$c_k = \frac{jV_m}{2\pi k}\left[2e^{-jk\pi} - 1 - e^{-j2\pi k}\right]$$

for K even

$$C_K = \frac{jV_m}{2\pi K}\left[2-1-1\right] = 0$$

For K odd

$$C_K = \frac{jV_m}{2\pi K}\left[-2-1-1\right] = \frac{-j4V_m}{2\pi K} = \frac{-j2V_m}{\pi K}.$$

Hence

$$C_K = \begin{cases} 0 \,; & K \text{ even} \\ \dfrac{-j2V_m}{\pi K} \,; & K \text{ odd} \end{cases}$$

Hence

$$x(t) = \sum_{K=-\infty}^{\infty} \frac{-j2V_m}{\pi K} e^{jKw_0 t} \;;\; K \text{ odd}.$$

or

$$x(t) = \sum_{K=-\infty}^{\infty} \frac{2V_m}{\pi K} e^{j(Kw_0 t - \pi/2)}.$$

(b)

Average value ⇒

$$C_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\left[\int_0^{T_0/2} V_m\,dt + \int_{T_0/2}^{T_0} -V_m\,dt\right]$$

$$= \frac{1}{T_0}\left[V_m t\Big|_0^{T_0/2} - V_m t\Big|_{T_0/2}^{T_0}\right] = \frac{1}{T_0}\left[\frac{V_m T_0}{2} - V_m T_0 + \frac{V_m T_0}{2}\right] = 0$$

(c)

From (a) the harmonic contents of the Square-wave includes w_0, $3w_0$, $5w_0$, $7w_0$, $9w_0$, ---- while the output sinusoidal signal has only w_0.

It follows that $3w_0$, $5w_0$, $7w_0$, $9w_0$ + ---- must be filtered out.

A Low-pass filter will be needed.