Mid Term Examination II *

Signals and Systems (ELC 321)
Department of Electrical and Computer Engineering
The College of New Jersey.

Instructions:

1. This is a closed-book examination
2. Attempt all questions (10 Marks). Total score obtainable is 100%.

   Information
   The block diagram of Figure 1 is an electronic oscillator for generating pure sinusoidal signal of a particular frequency, say ω_o. The block comprises of a square wave generator and a filter.

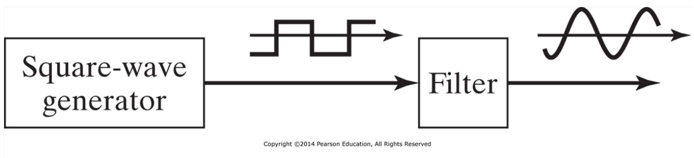

Figure 1: Square Wave Generator

Problem 1 (30 Marks). *Consider the electronic oscillator of Fig. 1 and let the output $V_i(t)$ of the square wave generator be as shown in Fig. 2 and the final sinusoidal output be $V_o(t) = Acos(\omega_o t)$.*

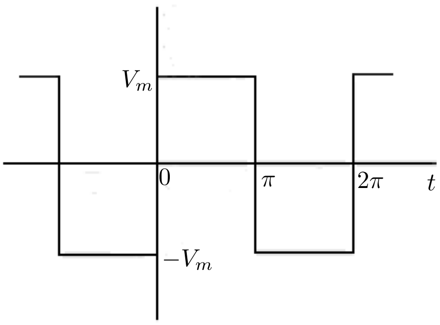

Figure 2: Half-wave Rectifier Circuit

*Dr. Ambrose A. Adegbege

a) Write the expression for the square wave shown in Figs 2.

b) Calculate the average value of the square wave signal.

c) Express the output sinusoidal signal as an exponential Fourier series.

Solution 1. *a) The square-wave can be expressed as:*

$$V_i(t) = \begin{cases} V_m; & 0 \le t \le \pi \\ -V_m; & \pi \le t \le \pi \end{cases} \tag{1}$$

b) The average value of $V_i(t)$ is given by:

$$C_o = \frac{1}{T_o} \int_{T_o} x(t)dt \tag{2}$$

where $T_o = 2\pi$. Thus,

$$\begin{aligned} C_o =& \frac{1}{2\pi} \int_{t=0}^{t=2\pi} V_i(t)dt = \frac{1}{2\pi} \left[\int_{t=0}^{t=\pi} V_m dt - \int_{t=\pi}^{t=2\pi} V_m dt \right] & (3) \\ =& \frac{1}{2\pi} \left[V_m t \Big|_{t=0}^{t=\pi} - V_m t \left|_{t=\pi}^{t=2\pi} \right] = \frac{1}{2\pi} \left[2V_m\pi - 2V_m\pi \right] \right. & (4) \\ =& 0 \ volts. & (5) \end{aligned}$$

c) Using Euler's relation $Cos(\theta) = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$, the output signal $V_o(t)$ as

$$\begin{aligned} V_o(t) =& Acos(\omega_o t) = \frac{A}{2}\left[e^{j\omega_o t} + e^{-j\omega_o t}\right] & (6) \\ =& \sum_{k=-1,1} C_k e^{j\omega_o k t} & (7) \end{aligned}$$

where $C_{-1} = C_1 = \frac{A}{2}$.

Problem 2 (30 Marks). *An infinite impulse response filter is described by the following difference equation:*

$$y[n] = (1-k)y[n-1] + kx[n] \tag{8}$$

where $x[n]$ and $y[n]$ represent the input and output sequences respectively. The coefficient k is an attenuation factor.

a) Determine if the system is a causal linear time-invariant.

b) Determine the impulse response $h[n]$ for the filter.

c) Suppose that $\beta = 0.4$, determine the system response to a unit step input i.e $x[n] = \mu[u]$ assuming zero initial condition, i.e. $y[0] = 0$.

□

Solution 2. *a) The proof is as follows:*

Causality:

The system is causal because it does not depend on future values of the input and output sequence.

Linearity:

Let

$$y[n] = \alpha y_1[n] + \beta y_2[n] \text{ and} \tag{9}$$
$$x[n] = \alpha x_1[n] + \beta x_2[n]. \tag{10}$$

Then

$$\begin{aligned} y[n] =& (1-k)y[n-1] + kx[n] \text{ becomes} && (11)\\ =&(1-k)(\alpha y_1[n] + \beta y_2[n]) + k(\alpha x_1[n] + \beta x_2[n]) && (12)\\ =&\alpha((1-k)y_1[n] + kx_1[n]) + \beta((1-k)y_2[n] + kx_2[n]) && (13)\\ =&\alpha y_1[n] + \beta y_2[n]. && (14) \end{aligned}$$

The system is linear because it satisfies the superposition principle.

Time-invariance:

Delaying the left-hand side of $y[n] = (1-k)y[n-1] + kx[n]$ by n_o gives: $y[n-n_o]$. Also, Delaying the right-hand side of $y[n] = (1-k)y[n-1] + kx[n]$ give: (1-k) y[n-$n_o-1] + kx[n-n_o]$.Sincey[n-$n_o] = (1-k)y[n-n_o-1] + kx[n-n_o]$, the system is time invariant. This implies that the output of the system at any instant of time is independent on the time at which the input is applied.

b) The impulse response of $y[n] = (1-k)y[n-1] + kx[n]$ is its output $h[n] = y[n]$ when the input is the unit sample signal $\delta[n]$. Hence,

$$h[n] = (1-k)h[n-1] + k\delta[n] \tag{15}$$

Since the system is causal, $h[n] = 0$ for $n < 0$.

$$\begin{array}{ll} n=0 & h[0] = (1-k)h[-1] + k\delta[0] = k \\ n=1 & h[1] = (1-k)h[0] + k\delta[1] = (1-k)k \\ n=2 & h[2] = (1-k)h[1] + k\delta[2] = (1-k)^2k \\ n=3 & h[3] = (1-k)h[2] + k\delta[3] = (1-k)^3k \\ \vdots & \vdots \\ n & h[n] = (1-k)^n k\mu[n]. \end{array} \tag{16}$$

c) For $k = 0$, we have $y[n] = 0.6y[n-1] + 0.4x[n]$

Complementary Solution:

Let $y_c[n] = Cz^n$ and substituting in $y[n] - 0.6y[n-1] = 0$. We have

$$Cz^n - 0.6Cz^{n-1} = 0 \tag{17}$$

$$Cz^n(1 - 0.6z^{-1}) = 0. \tag{18}$$

This implies $z = 0.6$. Hence $y_c[n] = C(0.6)^n$.

Particular Solution:

Since $x[n] = \mu[n]$, we set

$$y_p[n] = \begin{cases} P & n \geq 0 \\ 0 & n < 0 \end{cases}. \tag{19}$$

Substituting in $y[n] - 0.6y[n-1] = 0.4x[n]$. We have

$$\begin{array}{lll} P - 0.6P = 0.4 & P = 1 & n \geq 0 \\ P - 0.6P = 0 & P = 0 & n < 0 \end{array}. \tag{20}$$

Hence,

$$y_p[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}. \tag{21}$$

General Solution:

$$y[n] = y_p[n] + y_c[n] \begin{cases} 1 + C(0.6)^n & n \geq 0 \\ C(0.6)^n & n < 0 \end{cases} \tag{22}$$

Initial Condition:

Using $y[0] = 0$, we have

$$y[n] = y_p[n] + y_c[n] \tag{23}$$

$$= \begin{cases} 1 - (0.6)^n & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{24}$$

Information

Figure 3 shows a half-wave rectifier circuit with sinusoidal signal input $V_S(t) = Asin(\omega_o t)$ as shown in Fig 4. The voltage measured across the load resistor R_L is shown in Fig 5 assuming ideal diode behavior and a unity R_L.

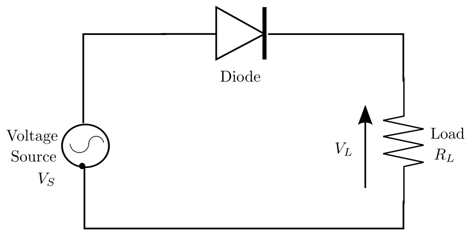

Figure 3: Half-wave Rectifier Circuit

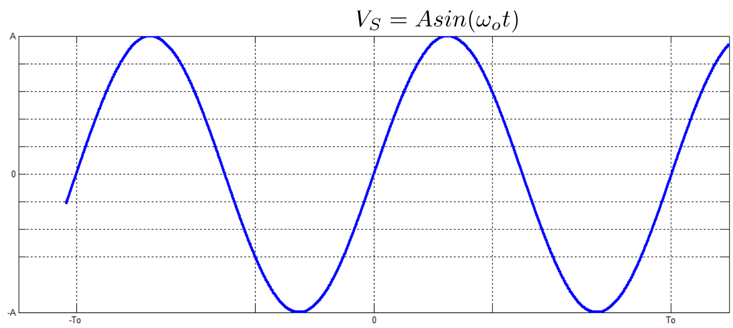

Figure 4: Sinusoidal Input

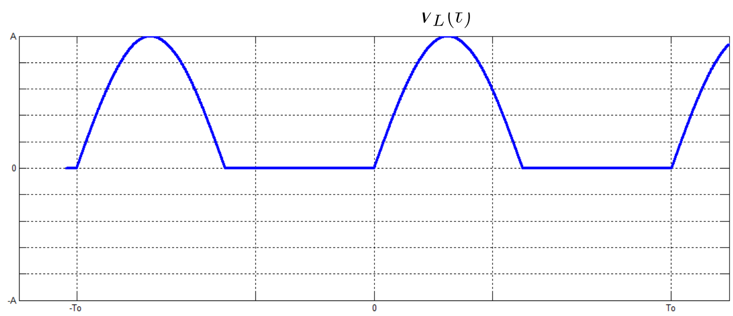

Figure 5: Half-wave rectified Signal

Problem 3 (30 Marks). *a) Express the input sinusoidal signal of Fig. 4 as an exponential Fourier series.*

b) Determine the period of the output signal shown in Figs 5.

c) Calculate the average value of the half-wave rectified signal of Fig 5.

Solution 3. *a) Using Euler's relation* $sin(\theta) = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$*, the input signal* $V_S(t)$ *as*

$$V_S(t) = Asin(\omega_o t) = \frac{A}{2j}\left[e^{j\omega_o t} - e^{-j\omega_o t}\right] \tag{25}$$

$$= \sum_{k=-1,1} C_k e^{j\omega_o k t} \tag{26}$$

where $C_1 = C_{-1}^* = \frac{A}{2j}$.

b) The output signal has same period as the input signal: From $Asin(\omega_o t)$*, we have* $\omega_o = 2\pi f_o$ *where* $f_o = \frac{1}{T_o}$*. Hence, the period* T_o *is given by*

$$T_o = \frac{2\pi}{\omega_o}. \tag{27}$$

c) The average value of $V_L(t)$ *is given by:*

$$C_o = \frac{1}{T_o}\int_{T_o} x(t)dt \tag{28}$$

Thus,

$$C_o = \frac{\omega_o}{2\pi}\int_{t=0}^{t=2\pi/\omega_o} V_L(t)dt = \frac{\omega_o}{2\pi}\int_{t=0}^{t=\pi/\omega_o} Asin(\omega_o t)dt \tag{29}$$

$$= -\frac{A\omega_o}{2\pi\omega_o}\left[cos(\omega_o t)\Big|_{t=0}^{t=\pi/\omega_o}\right] = -\frac{A}{2\pi}\left[cos(\pi) - cos(0)\right] \tag{30}$$

$$= \frac{A}{\pi}. \tag{31}$$

Problem 4 (30 Marks)**.** *The half-wave rectified signal of Fig 5 can also be obtained by multiplying the sinusoidal input of Fig. 4 by a rectangular pulse train.*

a) Sketch the rectangular pulse train that can be used for this purpose

b) Write a mathematical function describing the pulse train.

c) Find the Fourier transform of the sinusoidal input waveform $Vs = Asin(\omega_o t)$.

□

Solution 4. *a) The Rectangular Pulse Train that can be used is:*

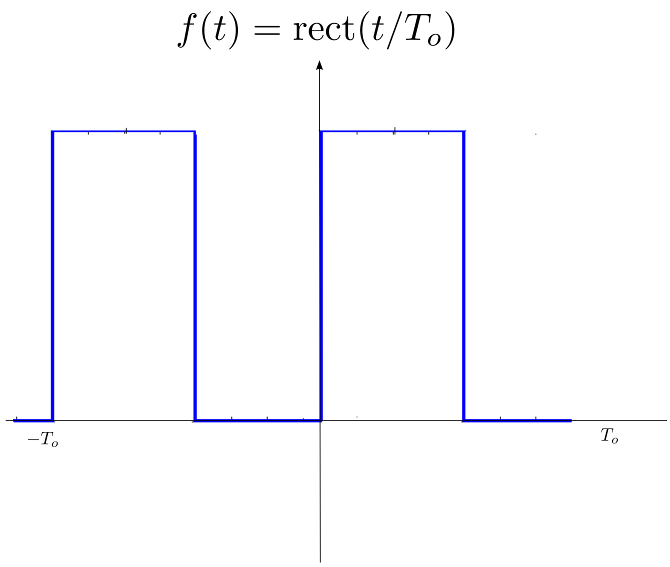

Figure 6: Rectangular Pulse Train

b) For a period, the rectangular pulse can be described as

$$f(t) = rect(t/T_o) = \begin{cases} 1 & 0 \leq t \leq T_o/2 \\ 0 & T_o/2 < t \leq T_o \end{cases} \tag{32}$$

Since this is repeated every period, the rectangular pulse train can be expressed as

$$\sum_{k=-\infty}^{\infty} f(t + kT_o) = \sum_{k=-\infty}^{\infty} rect(\frac{t + kT_o}{T_o}) \tag{33}$$

c) The Fourier transform is computed as

$$V_S(\omega) = \int_{-\infty}^{\infty} V_S(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} Asin(\omega_o t)e^{-j\omega t}dt \tag{34}$$

using the Euler's identity, the transform equation can be re-written as

$$V_S(\omega) = \int_{-\infty}^{\infty} Asin(\omega_o t)e^{-j\omega t}dt = \frac{A}{2j}\int_{-\infty}^{\infty} \left[e^{j\omega_o t}e^{-j\omega t} - e^{-j\omega_o t}e^{-j\omega t}\right]dt \tag{35}$$

This can be expressed as

$$V_S(\omega) = \frac{A}{2j}\left[\mathcal{F}(e^{j\omega_o t}) - \mathcal{F}(e^{-j\omega_o t})\right] \tag{36}$$

where $\mathcal{F}(x(t))$ *is the Fourier transform of* $x(t)$. *Recall that* $\mathcal{F}(e^{j\omega_o t}) = 2\pi\delta(\omega - \omega_o)$. *Hence,*

$$V_S(\omega) = \frac{A}{2j}\left[2\pi\delta(\omega - \omega_o) - 2\pi\delta(\omega + \omega_o)\right] = \quad jA\pi\left[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right]. \tag{37}$$

1 Reference

The Fourier series of a continuous-time signal $x(t)$ is defined as:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega} \tag{38}$$

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega}dt \tag{39}$$

The continuous-time Fourier transform (inverse Fourier transform)of $x(t)$ is defined as

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{40}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega. \tag{41}$$

The magnitude and phase spectra of $X(\omega)$ are given by $|X(\omega)|$ and $\angle X(\omega)$ respectively.
Given that $x(t) = e^{j\omega_o t}$, the Fourier transform of $x(t)$ is given as:

$$X(\omega) = \int_{t=-\infty}^{\infty} e^{j\omega_o t}e^{-j\omega t}dt = 2\pi\delta(\omega - \omega_o). \tag{42}$$

Given a discrete-time signal $x[n]$, its discrete-time Fourier transform (DTFT) (inverse DTFT)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \tag{43}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega \tag{44}$$

$X(\Omega)$ is said to be periodic with respect Ω if $X(\Omega + kT) = X(\Omega)$ where T is the period and k is any integer.