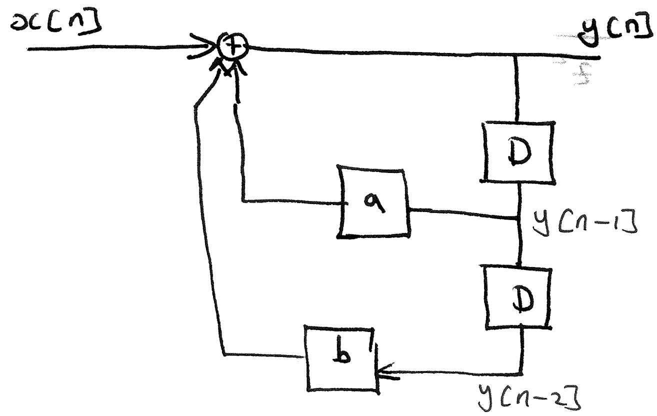
Signals and Systems ELC 321-01
Sample Solution Final Exam

Problem 1



(a)

$$y[n] = a\,y[n-1] + b\,y[n-2] + x[n] \quad \text{— (1)}$$

(b) Taking the z-transform of (1) gives

$$Y(z) = a z^{-1} Y(z) + b z^{-2} Y(z) + X(z)$$

$$Y(z) - a z^{-1} Y(z) - b z^{-2} Y(z) = X(z)$$

$$(1 - a z^{-1} - b z^{-2}) Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1} - b z^{-2}}$$ — The transfer function.

(c) For $a = 0.5$ and $b = 0.25$

$$H(z) = \frac{1}{1 - a z^{-1} - b z^{-2}} = \frac{1}{1 - 0.5 z^{-1} - 0.25 z^{-2}} = \frac{z^2}{z^2 - 0.5z - 0.25}$$

$$= \frac{z^2}{(z - P_1)(z - P_2)} \equiv \frac{z^2}{(z - 0.8090)(z + 0.3090)}$$

The system is Stable since $|P_1| < 1$ and $|P_2| < 1$.

The system is Causal since the order of the numerator of $H(z)$ is the same as the order of the denominator of $H(z)$.

(d)

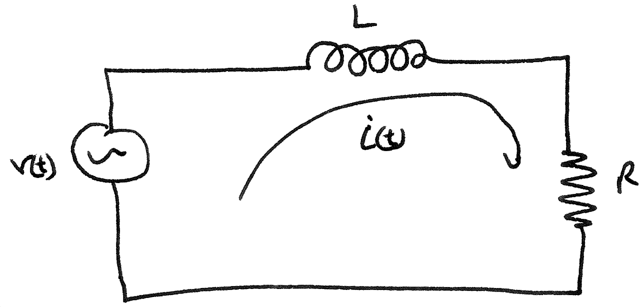
$$H(z) = \frac{z^2}{z^2 - 0.5z - 0.25} = \frac{1}{1 - 0.5z^{-1} - 0.25z^{-2}}$$

We require the inverse system $W(z)$ such that $H(z)W(z) = I$

$$W(z) = \frac{1}{H(z)} = \frac{z^2 - 0.5z - 0.25}{z^2} = \frac{1 - 0.5z^{-1} - 0.25z^{-2}}{1}$$

$$W(z) = 1 - 0.5z^{-1} - 0.25z^{-2}$$

Problem 2



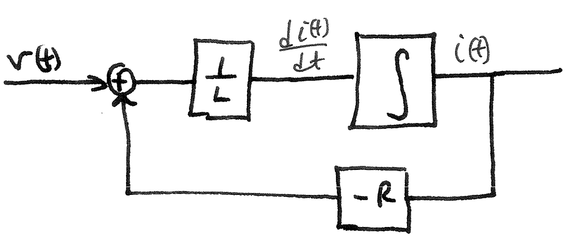
(a)

Applying KVL gives

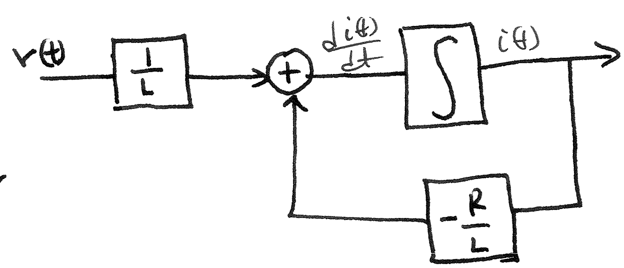
$$v(t) = L\frac{di(t)}{dt} + i(t)R$$

(b) To draw the simulation diagram, we rewrite the above equation as

$$L\frac{di(t)}{dt} = v(t) - i(t)R \qquad \text{or} \qquad \frac{di(t)}{dt} = \frac{1}{L}v(t) - \frac{R}{L}i(t)$$



or



(c)

$$L \frac{d\, i(t)}{dt} + i(t) R = v(t)$$

when $L = 1H$ and $R = 2\Omega$, we have

$$\frac{d\, i(t)}{dt} + 2\, i(t) = v(t)$$

Taking the Laplace transform gives

$$s I(s) + 2 I(s) = V(s)$$

$$(s+2) I(s) = V(s)$$

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{s+2}$$

(d) <u>using the Laplace transform method</u>

$I(s) = H(s) V(s)$ where $H(s) = \frac{1}{s+2}$ and $V(s) = \mathcal{L}[u(t)] = \frac{1}{s}$.

Hence $I(s) = \frac{1}{s+2} \cdot \frac{1}{s} = \frac{1}{s(s+2)}$

Taking partial fractions gives

$$I(s) = \frac{1}{s(s+2)} \equiv \frac{A}{s} + \frac{B}{s+2}$$ with $A = \left.\frac{1}{s+2}\right|_{s=0} = \frac{1}{2}$

$B = \left.\frac{1}{s}\right|_{s=-2} = -\frac{1}{2}$

$$\therefore I(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

$i(t)$ is obtained by taking the inverse Laplace transform

$$i(t) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right) u(t) = \frac{1}{2}\left(1 - e^{-2t}\right) u(t)$$

$$\text{or } \frac{1}{2}\left(1 - e^{-2t}\right), \; t > 0 .$$

<u>Using convolution method.</u>

$$h(t) = \mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}u(t).$$

The current $i(t)$ is given via convolution, as

$$i(t) = h(t) * v(t) = v(t) * h(t)$$

$$= \int_{\tau=-\infty}^{\infty} h(\tau)\,v(t-\tau)\,d\tau = \int_{\tau=-\infty}^{\infty} v(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{\tau=-\infty}^{\infty} e^{-2\tau}u(\tau)\,u(t-\tau)\,d\tau = \int_{\tau=0}^{\infty} e^{-2\tau}u(t-\tau)\,d\tau = \int_{\tau=0}^{t} e^{-2\tau}\,d\tau$$

$$= -\frac{1}{2}e^{-2\tau}\Big|_{\tau=0}^{t} = -\frac{1}{2}e^{-2t} + \frac{1}{2}$$

$$i(t) = \frac{1}{2}\left(1 - e^{-2t}\right), \quad t > 0$$

<u>Using undetermined coefficient method.</u>

<u>Complementary solution</u> $i_c(t) = Ae^{-st}$

Substituting into the homogenous equation $\frac{di(t)}{dt} + 2i(t) = 0$ gives

$$-Ase^{-st} + 2Ae^{-st} = 0 \Rightarrow (-s+2)Ae^{-st} = 0$$

This implies $s = 2$

Hence $i_c(t) = Ae^{-2t}$.

<u>particular Solution</u> $i_p(t) = B$

Substituting into the differential equation $\frac{di(t)}{dt} + 2i(t) = u(t)$

gives $2B = 1 \Rightarrow B = 1/2$.

General solution $i(t) = i_c(t) + i_p(t) = \frac{1}{2} + Ae^{-2t}$

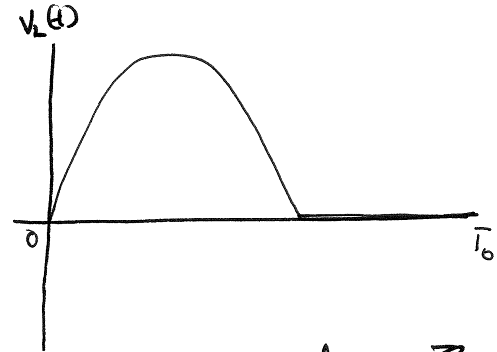
<u>initial condition</u> using $i(0) = 0$

$$\frac{1}{2} + A = 0 \Rightarrow A = -1/2$$

Hence

$$i(t) = \frac{1}{2}\left(1 - e^{-2t}\right); \quad t > 0$$

Problem 3



$$V_L(t) = \begin{cases} A\sin w_0 t, & 0 \le t \le T_0/2 \\ 0, & T_0/2 < t \le T_0 \end{cases}$$

$V_L(t)$ can be expressed ~~as~~ as Fourier series

$$V_L(t) = \sum_{k=-\infty}^{\infty} C_k e^{jkw_0 t} \quad \text{with} \quad C_k = \frac{1}{T_0}\int_{T_0} V_L(t) e^{-jkw_0 t}\,dt.$$

We have

$$C_k = \frac{1}{T_0}\int_0^{T_0/2} A\sin w_0 t\, e^{-jkw_0 t}\,dt = \frac{1}{T_0}\int_0^{T_0/2} \frac{A}{2j}\left(e^{jw_0 t} - e^{-jw_0 t}\right)e^{-jkw_0 t}\,dt$$

$$= \frac{A}{2jT_0}\int_0^{T_0/2}\left(e^{j(1-k)w_0 t} - e^{-j(1+k)w_0 t}\right)dt$$

$$= \left.\frac{A e^{j(1-k)w_0 t}}{2jT_0(1-k)jw_0}\right|_0^{T_0/2} + \left.\frac{A e^{-j(1+k)w_0 t}}{2jT_0(1+k)jw_0}\right|_0^{T_0/2}$$

$$= \frac{A e^{j(1-k)w_0 T_0/2}}{-2w_0T_0(1-k)} + \frac{A}{2w_0T_0(1-k)} + \frac{A e^{-j(1+k)w_0T_0/2}}{-2w_0T_0(1+k)} + \frac{A}{2w_0T_0(1+k)}$$

Recall that $w_0T_0 = 2\pi$ or $T_0 = \frac{2\pi}{w_0}$

Hence

$$C_k = \frac{A e^{-j(k-1)\pi}}{4\pi(k-1)} - \frac{A}{4\pi(k-1)} - \frac{A e^{-j(1+k)\pi}}{4\pi(1+k)} + \frac{A}{4\pi(1+k)}$$

For k even

$$C_k = \frac{-A}{4\pi(k-1)} - \frac{A}{4\pi(k-1)} + \frac{A}{4\pi(1+k)} + \frac{A}{4\pi(1+k)}$$

(Since $e^{-j(k-1)\pi} = -1$, $e^{-j(k+1)\pi} = -1$ for k even)

$$= \frac{-A}{\pi(k^2-1)}$$

For k-odd

$$C_k = \frac{-A}{4\pi(k-1)} + \frac{A}{4\pi(k-1)} - \frac{A}{4\pi(k+1)} + \frac{A}{4\pi(k+1)} = 0.$$

For $k = 1$

$$C_1 = \frac{1}{T_0}\int_0^{T_0/2} A\sin(\omega_0 t)\, e^{-j\omega_0 t}\, dt = \frac{1}{T_0}\int_0^{T_0/2} \frac{A}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) e^{-j\omega_0 t}\, dt$$

$$= \frac{A}{2jT_0}\int_0^{T_0/2}\left(1 - e^{-2j\omega_0 t}\right) dt = \frac{A}{2jT_0}\left[t + \frac{e^{-2j\omega_0 t}}{2j\omega_0}\right]\Bigg|_{t=0}^{t=T_0/2}$$

$$= \frac{A}{2jT_0}\left[\frac{T_0}{2} + \frac{e^{-2j\omega_0 T_0/2}}{2j\omega_0} - \frac{1}{2j\omega_0}\right]$$

$$= \frac{A}{4j} - \frac{Ae^{-j2\pi}}{8\pi} + \frac{A}{8\pi} = \frac{A}{4j} = -j\frac{A}{4}$$

Since $C_{-k} = C_k^*$, it follows that $C_{-1} = C_1^* = j\frac{A}{4}$

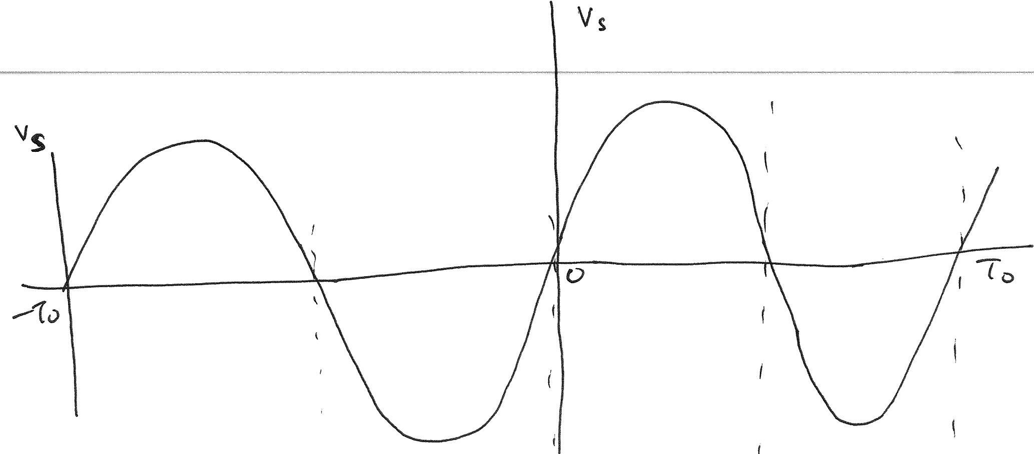
Hence

$$C_k = \begin{cases} 0 \;; & k \text{ odd except at } k = \pm 1 \quad \begin{pmatrix} C_1 = -jA/4 \\ C_{-1} = jA/4 \end{pmatrix}. \\[2ex] -\dfrac{A}{\pi(k^2-1)} & k, \text{ even} \end{cases}$$

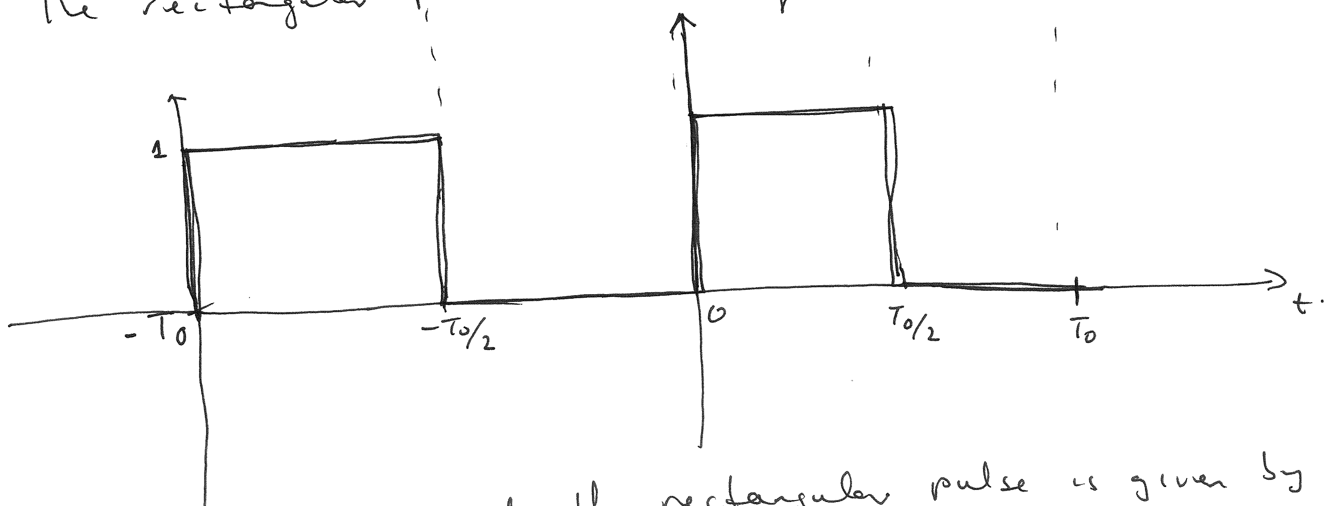
(b) from $C_k = -\dfrac{A}{\pi(k^2-1)}$

when $k = 0$, $C_0 = \dfrac{-A}{-\pi} = \dfrac{A}{\pi}$.

(c)



The rectangular pulse train required is



For a period, the rectangular pulse is given by

$$f(t) = rect(t/T_0) = \begin{cases} 1 & 0 \le t \le T_0/2 \\ 0 & T_0/2 < t \le T_0 \end{cases}$$

Since this is repeated every period, the rectangular pulse train can be expressed as

$$\sum_{K=-\infty}^{\infty} f(t + KT_0) = \sum_{K=-\infty}^{\infty} rect(t+KT_0/T_0)$$

Problem 4

$$y[n] = (1-\beta)\, y[n-1] + \beta\, x[n]$$

a)

Causal: The system is causal because it does not depend on future values of input or output.

Linear: The system is linear because its satisfying the superposition principle.

let $y[n] = a y_1[n] + b y_2[n]$ and

$x[n] = a x_1[n] + b x_2[n]$

Then $y[n] = (1-\beta)\, y[n-1] + \beta\, x[n]$ becomes

$$a y_1[n] + b y_2[n] = (1-\beta) a y_1[n-1] + (1-\beta) b y_2[n] + a\beta x_1[n] + b\beta x_2[n]$$

$$a y_1[n] + b y_2[n] = a\left((1-\beta) y_1[n-1] + \beta x_1[n]\right) + b\left((1-\beta) y_2[n-1] + x_2[n]\right)$$

$$= a y_1[n] \qquad + b y_2[n] \quad \checkmark$$

Time-invariant: The system is time-invariant because the output is independent of the time at which the input is applied.

$$y[n-n_0] = (1-\beta)\, y[n-n_0-1] + \beta\, x[n-n_0]$$

This is is equivalent to delaying the input and the output by n_0 respectively.

b)

$$y[n] = (1-\beta)\, y[n-1] + \beta\, x[n]$$

hence

$$h[n] = (1-\beta)\, h[n-1] + \beta\, \delta[n]$$

$n=0 \Rightarrow$ $h[0] = (1-\beta)\, h[-1] + \beta\, \delta[0] = \beta$ ($h[-1] = 0$ since the system is causal)

$n=1 \Rightarrow$ $h[1] = (1-\beta)\, h[0] + \beta\, \delta[1] = (1-\beta)\beta$

$n=2 \Rightarrow$ $h[2] = (1-\beta)\, h[1] + \beta\, \delta[2] = (1-\beta)^2\beta$

$n=3 \Rightarrow$ $h[3] = (1-\beta)\, h[2] + \beta\, \delta[3] = (1-\beta)^3\beta$

$\vdots$

$h[n] = (1-\beta)^n \beta\, u[n]$

(c)

<u>using convolution</u>

$$h[n] = (1-\beta)^n \beta\, u[n]$$

$$= (0.6^n) \times 0.4\, u[n] = 0.4\,(0.6)^n u[n]$$

$$x[n] = u[n]$$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} 0.4(0.6)^k u[k]\, u[n-k]$$

$$= 0.4 \sum_{k=0}^{\infty} (0.6)^k u[n-k] = 0.4 \sum_{k=0}^{n} (0.6)^k = \frac{0.4\,(1-0.6^{n+1})}{1-0.6}$$

$$= (1-0.6^{n+1})\, u[n]$$

<u>using z-transform method</u>

$$y[n] = (1-\beta)\, y[n-1] + \beta\, x[n]$$

z-transforms gives

$$Y(z) = 0.6 z^{-1} Y(z) + 0.4 X(z)$$

$$Y(z) = \frac{0.4}{1-0.6z^{-1}} X(z) = \frac{0.4z}{z-0.6} X(z)$$

using $X(z) = U(z) = \frac{z}{z-1}$

$$Y(z) = \frac{0.4z^2}{(z-0.6)(z-1)}$$

expressing $Y(z)/z$ in partial fractions gives

$$\frac{Y(z)}{z} = \frac{0.4z}{(z-0.6)(z-1)} = \frac{A}{z-0.6} + \frac{B}{z-1}$$

$$A = \left.\frac{0.4z}{z-1}\right|_{z=0.6} = -0.6$$

$$B = \left.\frac{0.4z}{z-0.6}\right|_{z=1} = 1$$

$$\frac{Y(z)}{z} = \frac{-0.6}{z-0.6} + \frac{1}{z-1}$$

$$Y(z) = \frac{-0.6z}{z-0.6} + \frac{z}{z-1}$$

$y[n]$ is obtained by taking the inverse z-transform of $Y(z)$

$$y[n] = -0.6\, \mathcal{Z}^{-1}\left(\frac{z}{z-0.6}\right) + \mathcal{Z}^{-1}\left(\frac{z}{z-1}\right)$$

$$= -0.6\,(0.6)^n + 1$$

$$= 1 - 0.6^{n+1}, \quad n \geq 0 \quad \text{or} \quad (1-0.6^{n+1})\, u[n].$$

(d)

From (b)

$$h[n] = 0.4\,(0.6)^n\, u[n]$$

Using

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} 0.4\,(0.6)^n\, u[n]\, e^{-j\Omega n} = 0.4 \sum_{n=0}^{\infty} \left(0.6\, e^{-j\Omega}\right)^n$$

$$= \frac{0.4}{1 - 0.6\, e^{-j\Omega}}$$