Chapter 1: Introduction to Differential Equations

What is a differential equation?

→ An equation containing derivatives of one or more dependent variables

Where do they come from?

→ For instance: Mathematical Models of physical systems. OR.. Derivatives of functions!

Consider Newtons Second Law:

$$F = ma$$

this can be rewritten as either a first order or a second order differential Equation.

Free Fall with Air Resistance (or viscous damping)

$$\sum F = ma = mg - kv = ma$$

In terms of velocity $a = \frac{dv}{dt}$

this makes Newtons Second Law a first order differential equation!

m = mass, k = drag coefficient

What does the curve look like? (2)
∴ The force is proportional to velocity and the equation can be written as:

$$m\frac{dv}{dt} = mg - kv \quad OR$$

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

This equation can be solved by separating and integrating:

$$\int_0^{v} \frac{dv}{\left(g - \frac{k}{m}v\right)} = \int_0^{v} \frac{dv}{\frac{k}{m}\left(\frac{mg}{k} - v\right)} = \int_0^t dt$$

Let $\frac{mg}{k} = V_{term}$, So...

$$\int_0^{v} \frac{dv}{(V_t - v)} = \int_0^t \frac{k}{m}\,dt = \frac{k}{m}t$$

$u = V_t - v$, $du = -dv$

$$-\left[\ln(V_t - v) - \ln V_t\right] = \frac{k}{m}t$$

$$\ln\left[\frac{V_t - v}{V_t}\right] = -\frac{k}{m}t$$

$$\frac{V_t - V}{V_t} = e^{-\left(\frac{kt}{m}\right)}$$

$$V_t - V = V_t e^{-\left(\frac{kt}{m}\right)} \quad \text{OR} \ldots$$

The curve is exponential!

$$V = V_t\left(1 - e^{-\left(\frac{kt}{m}\right)}\right), \quad V_t = \frac{mg}{k}$$

In most cases the force is proportional to the square of the velocity where:

$$\Sigma F = Ma = mg - kv^2$$

This can be solved the same way

The solution is:

$$\boxed{V = V_t \left(\frac{1 - e^{-\left(\frac{2V_t k}{m}\right)t}}{1 + e^{-\left(\frac{2V_t k}{m}\right)t}}\right)}$$

Hint: $\frac{dv}{dt} = g - \frac{k}{m}v^2$

let $V_t^2 = \frac{mg}{k}$, $\frac{dv}{dt} = \frac{k}{m}\left(V_t^2 - V^2\right)$

OR $\frac{dv}{dt} = \frac{k}{m}\left(V_t + V\right)\left(V_t - V\right)$ Solve! ...

Now lets take a look at Newtons Second law as a Second Order ODE:

$V = \frac{ds}{dt}$ so $\frac{d^2s}{dt^2} + \frac{k}{m}\frac{ds}{dt} = g$

Also $F = Ma = -kx = M\frac{d^2x}{dt^2}$ } Spring!

$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$ OR

$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ for $x(t)$

We know that a spring oscillates.

Is $x(t) = A\cos ct$ a solution?

$\frac{dx}{dt} = -Ac\sin ct$, $\frac{d^2x}{dt^2} = -Ac^2\cos ct$

lets see if they satisfy the above second order ODE:

$-AC^2\cos ct + A\frac{k}{m}\cos ct = 0$

It does if $C^2 = \frac{k}{m}$!

Remember that the frequency w is $w = \sqrt{\frac{k}{m}}$!

But what about the A?

The Solution is $x(t) = A\cos\sqrt{\frac{k}{m}}\,t$

or $x(t) = A\cos\omega t$

This solution is actually an infinite family of curves which satisfies the ODE $x'' + \frac{k}{m}x = 0$ for different values of the amplitude A.

Now lets impose an initial condition that $x(0) = \pi = A\cos(0) = A$

The solution becomes $\boxed{x(t) = \pi\cos\omega t}$

Thus we have a single solution to this ODE because an initial condition was imposed.

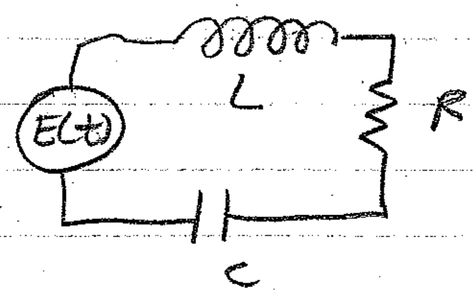
What if we include damping of the spring which is proportional to velocity?

$$F = ma = -kx - bv = -kx - b\frac{dx}{dt}$$

$$\therefore m\frac{d^2x}{dt} + b\frac{dx}{dt} + kx = 0, \quad x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$$

⑥ <u>Series</u> <u>Circuit</u>:

RCL circuit



Resistor - Capacitor - Inductor

<u>Kirchhoff's Second Law</u>:

Sum of voltage drops from passive sources must equal the impressed voltage $E(t)$ → conservation of Energy!

$$V_R + V_c + V_L = E(t)$$

$$V_R = iR = \frac{dq}{dt}R, \; V_c = \frac{1}{c}q, \; V_L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

In terms of q the equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E(t)$$

OR $$q'' + \frac{R}{L}q' + \frac{1}{LC}q = E(t)$$

(Equivalent to a forced, damped harmonic oscillator)

⑦ <u>Falling Bodies</u>

$F = ma$ } Newtons Second Law where $F = -mg$, $a = \frac{d^2s}{dt^2}$

$$\therefore -mg = m\frac{d^2s}{dt^2}$$

$$\frac{d^2s}{dt^2} = -g, \; s(0) = s_0, \; s'(0) = v_0$$

(Remember $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$)

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = -g$$ Separate!

$$\int d\left(\frac{ds}{dt}\right) = -\int g\,dt = -gt + C_1$$

$$\frac{ds}{dt} = V(t)\ ;\ V(0) = s'(0) = V_0 = C_1$$

$$\frac{ds}{dt} = -gt + V_0\ ;$$ Separate again!

$$\int ds = \int [-gt + V_0]\,dt$$

$$S(t) = -\frac{1}{2}gt^2 + V_0 t + C_2$$

Now $S(0) = S_0$ So...

$$S(0) = -\frac{1}{2}g(0) + V_0(0) + C_2 = S_0$$

$$\therefore S(t) = -\frac{1}{2}gt^2 + V_0(t) + S_0$$

Define an ODE from a function

Consider the function: Make it an ODE

$$y = e^{-0.5x^3}$$

Take the derivative dy/dx

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$ Use the chain rule where $u = -0.5x^3$

$$\frac{du}{dx} = -3/2\,x^2$$

$$\frac{dy}{du} = e^u \text{, so}$$

$$\frac{dy}{dx} = -\frac{3}{2}x^2 e^{-0.5x^3} \quad \text{where } e^{-0.5x^3} = y$$

$$\therefore \frac{dy}{dx} = -\frac{3}{2}x^2 y$$ First-order separable differential equation

Differential equations are classified by type, order, and linearity

$\boxed{\text{Type}}$

- ODE : $x'' + \frac{K}{M}x = 0$
- PDE : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\boxed{\text{Order}}$ → order of highest derivative:

$x'' + \frac{k}{m}x = 0$ → Second order Liebnitz notation

$\frac{d^3x}{dt^3} - xt = 0$ → Third order prime notation

General Form of nth order D.E.

$$F(x, y, y' \ldots y^n) = 0 \quad \text{where}$$

F is a real-valued function of n+2 variables.

Normal Form is written as:

$$\frac{d^n y}{dx^n} = f(x, y, y' \ldots y^{(n-1)})$$

Example: Change to normal form:

Normal Form of $4xy' + y = x$ is

$$\frac{dy}{dx} = \frac{(x-y)}{4x} \quad \text{or} \quad y' = \frac{(x-y)}{4x}$$

$\boxed{\text{Classification}}$ — Linear / Non-Linear

Linear: F is linear in $y, y' \ldots y^n$

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(y)\frac{dy}{dx}$$

$$+ a_0(x)y = g(x)$$

e.g. $(y-x)dx + 4x\,dy = 0$

$y'' - 2y' + y = 0$

etc

See pp. 6, 7

Nonlinear: F is non-linear in y or y' or y'' etc.

Examples:

1. $(1-y)y' + 2y = e^x = y' - \underbrace{yy'}_{\text{non-linear term}} + 2y = e^x$

Defines pendulum motion for large angles

2. $y'' + \sin y = 0$ } $\sin y$ is a non-linear function of y

3. $\frac{d^4y}{dx^4} + y^2 = 0$ } y^2 is non-linear i.e. power is not 1

(Non-linear ODES are introduced in Chapter 3)
Goal is to linearize them!

p.5 Example:
Verify that $y = \frac{x^4}{16}$ is a solution of $\frac{dy}{dx} = x y^{1/2}$ on interval $(-\infty, \infty)$

$y' = \frac{1}{4}x^3$, $y^{1/2} = \sqrt{\frac{x^4}{16}} = \frac{x^2}{4}$

$\therefore$ Plug in: $\frac{1}{4}x^3 = x\left(\frac{x^2}{4}\right) = \frac{1}{4}x^3$ ✓

(Try b on p.7)

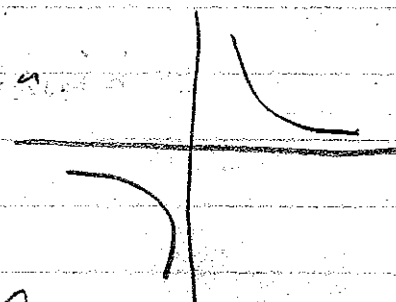
Solution $y=0$ is a trivial solution

p. 6 Function vs Solution

Function is defined from $(-\infty, \infty)$ where the function exists

Solution is defined over an interval I

The function $y = 1/x$ defined on $(-\infty, \infty)$ except for $x = 0$



Solution to differential equation

$xy' + y = 0$ is $y = 1/x$ over the interval $(-\infty, 0)$ or $(0, \infty)$

Solve: $y' = -\frac{y}{x} = -\frac{1}{x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$x(-x^{-2}) + x^{-1} = 0, \quad -x^{-1} + x^{-1} = 0$ ✓

p. 7 Implicit Solution:

Consider the motion of a particle as a function of time t

$x = 5\cos t, \quad y = 5\sin t$

These are parametric equations of the particle which define the circle $x^2 + y^2 = 25$

$\cos^2 t = \frac{x^2}{25}, \quad \sin^2 t = \frac{y^2}{25}, \quad \cos^2 t + \sin^2 t = 1$

where:

$$x^2 = 25\cos^2 t, \quad y^2 = 25\sin^2 t$$

adding x^2 and y^2 we get:

$$x^2+y^2 = 25\cos^2 t + 25\sin^2 t$$
$$= 25(\cos^2 t + \sin^2 t)$$
$$x^2+y^2 = 25 \checkmark \Big\}$$ Particle moves in circular path.

This equation is said to be an <u>implicit</u> solution of the equation

$$\frac{dy}{dx} = -\frac{x}{y}$$; why?

Take the derivative of $x^2+y^2=25$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) = 0$$

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y} \Big\}$$ Separable DE

Now, solve this D.E. on the interval $-5 < x < 5$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$$

$$x^2 + y^2 = c^2, \quad -5 < x < 5$$

Solve for constant c from the interval I:

let $x = 0$, $y^2 = c^2$

$x = 5$, $y^2 = c^2 - 25$

* $y = 0$, $x^2 = c^2$, let $x = 5$, $c^2 = 25$

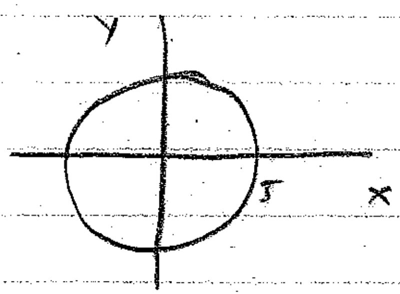
$\therefore$ $x^2 + y^2 = 25$

Now solve $x^2 + y^2 = 25$ for y:

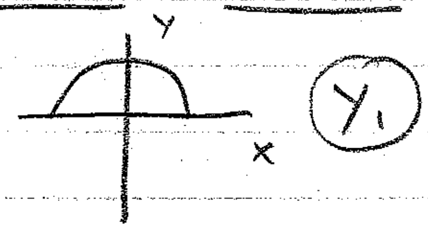
$$y^2 = 25 - x^2, \quad y = \pm\sqrt{25 - x^2}$$

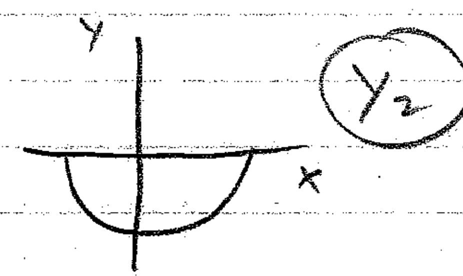
$$y_1 = \sqrt{25 - x^2}, \quad y_2 = -\sqrt{25 - x^2}$$

$\therefore$ Implicit Solution: Explicit Solution



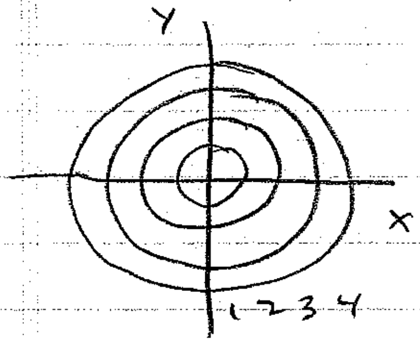
$x^2 + y^2 = 25$





Now, the <u>general</u> <u>solution</u>

$x^2+y^2=c$ satisfys the differential equation $y' = -\frac{x}{y}$ for an infinite number of values of c. Some solutions are as follows:



<u>Family of curves</u> or <u>particular solutions</u> for $x^2+y^2=c^2$

The solution $x^2+y^2=25$ is a <u>particular solution</u> defined by the interval $-5 < x < 5$

p.8 Verify that $y = cx - x\cos x$ is an explicit solution of the first order equation $xy' - y = x^2 \sin x$ on $(-\infty, \infty)$

$$y' = c + x\sin x - \cos x$$

$$x\cancel{c} + x^2\sin x - \cancel{x\cos x} - \cancel{cx} + \cancel{x\cos x} = x^2\sin x$$

$$\therefore x^2 \sin x = x^2 \sin x \checkmark$$

p.10 <u>#12: Verify that</u>

$y = \frac{6}{5} - \frac{6}{5} e^{-20t}$ is an explicit solution of $\frac{dy}{dt} + 20y = 24$

$y' = 24e^{-20t}$, $\therefore 24e^{-20t} + 24 - 24e^{-20t} = 24$

$\therefore 24 = 24$ ✓ over $(0 < t, < \infty)$

p. 8 <u>Piecewise - Defined Solution</u>

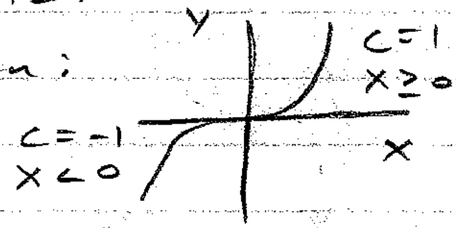
→ Solution is defined for multiple values of c:

Verify that $y = cx^4$ is a solution of $xy' - 4y = 0$ on $(-\infty, \infty)$

$y' = 4cx^3$, $\therefore 4cx^4 - 4cx^4 = 0$

The particular solution:

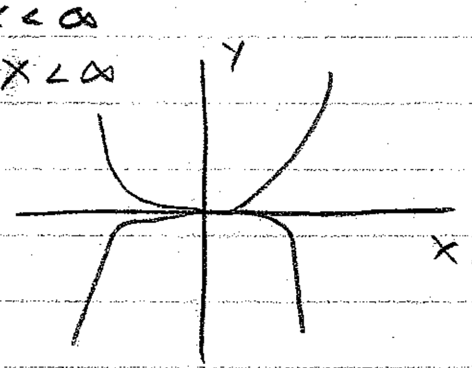
$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases}$$



is obtained by choosing $c = 1$ for $x \geq 0$ and $c = -1$ for $x < 0$. This piecewise solution is not obtained from a <u>single</u> choice for c.

Single choice: $c = 1$ $-\infty < x < \infty$
$c = -1$ $-\infty < x < \infty$

for $y = cx^4$ where



p. 11 # 47: Verify that the family of curves called folia of Descartes

$$x^3 + y^3 = 3cxy$$

is an implicit solution of

$$\frac{dy}{dx} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}$$

Implicitly differentiate:

$$\frac{d}{dx}\left(\frac{x^3 + y^3}{xy}\right) = \frac{d}{dx}(3c)$$

$$\frac{xy(3x^2 + 3y^2y') - (x^3 + y^3)(y + xy')}{(xy)^2} = 0$$

$$3x^3y + 3xy^3y' - x^3y - x^4y' - y^4 - xy^3y' = 0$$

Solve for y':

$$(3xy^3 - x^4 - xy^3)y' = -3x^3y + x^3y + y^4$$

$$y' = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)} \checkmark$$

p.12 Initial-Value Problems

- Solution to a differential equation which includes initial conditions:

$y(x_0)=y_0,\ y'(x_0)=y_1 \ldots y^{n-1}(x_0)=y_{n-1}$

at a single point x_0

p.14

Example 3:

Find a solution to the IVP

$$x''+16x=0,\ x\left(\tfrac{\pi}{2}\right)=-2,\ x'\left(\tfrac{\pi}{2}\right)=1$$

where $x=c_1\cos 4t + c_2\sin 4t$ is a two-parameter family of solutions.

$x'=-4c_1\sin 4t+4c_2\cos 4t$

$x'\left(\tfrac{\pi}{2}\right)=1=-4c_1\sin\left[4\left(\tfrac{\pi}{2}\right)\right]+4c_2\cos\left[4\left(\tfrac{\pi}{2}\right)\right]$

$1=-4c_1\underbrace{\sin(2\pi)}_{0}+4c_2\cos(2\pi)$

$4c_2=1\ ,\ \boxed{c_2=+1/4}$

$x\left(\tfrac{\pi}{2}\right)=-2=c_1\cos\left(4\left(\tfrac{\pi}{2}\right)\right)+\tfrac{1}{4}\sin\left(4\left(\tfrac{\pi}{2}\right)\right)$

$\therefore \boxed{c_1=-2}$

$\therefore x(t)=-2\cos 4t+\tfrac{1}{4}\sin 4t$

In general;

Solve: $\frac{dy}{dx} = f(x,y)$

Subject to: $y(x_0) = y_0$

and

Solve: $\frac{d^2y}{dx^2} = f(x,y,y')$

Subject to: $y(x_0) = y_0,\ y'(x_0) = y_1$

Example: (p.16, #10)

$x = c_1 \cos t + c_2 \sin t$ is a solution to $x'' + x = 0$

Find a particular solution given the initial conditions:

$x\left(\frac{\pi}{4}\right) = \sqrt{2}$, $x'\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$$x\left(\frac{\pi}{4}\right) = \sqrt{2} = c_1 \cos\frac{\pi}{4} + c_2 \sin\frac{\pi}{4}$$

$$\sqrt{2} = c_1 \frac{\sqrt{2}}{2} + c_2 \frac{\sqrt{2}}{2}$$

$$\therefore\ c_1 + c_2 = 2$$

$$x'(t) = -c_1 \sin t + c_2 \cos t = 2\sqrt{2}$$

$$x'\left(\frac{\pi}{4}\right) = -c_1 \sin\frac{\pi}{4} + c_2 \cos\frac{\pi}{4} = 2\sqrt{2}$$

$$-c_1 \frac{\sqrt{2}}{2} + c_2 \frac{\sqrt{2}}{2} = 2\sqrt{2}$$

$$c_1 - c_2 = 4$$
$$c_1 + c_2 = 2$$
$$2c_1 = 6, \quad \boxed{c_1 = 3}$$

$$3 + c_2 = 2, \quad \boxed{c_2 = -1}$$

Check:

$$x\left(\frac{\pi}{4}\right) = \sqrt{2} = 3\cos\frac{\pi}{4} - \sin\frac{\pi}{4}$$
$$\sqrt{2} = 3\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = \sqrt{2} \checkmark$$

Example (p.16, # 12)

$y = c_1 e^x + c_2 e^{-x}$ is a two-parameter family of solutions of $y'' - y = 0$

Find a solution of the IVP

$y(1) = 0, \; y'(1) = e$

$$y(1) = c_1 e + \frac{c_2}{e} = 0$$
$$c_2 = -c_1 e^2$$

$$y' = c_1 e^x - c_2 e^{-x}$$
$$y'(1) = c_1 e - \frac{c_2}{e} = e$$
$$c_1 e + \frac{c_1 e^2}{e} = e$$
$$c_1 = \frac{1}{2}, \; c_2 = -\frac{e^2}{2}, \quad \boxed{y = \frac{1}{2}e^x - \frac{e^2}{2}e^{-x}}$$

Does this Satisfy $y'' - y = 0$?

$y = \frac{1}{2}e^{x} - \frac{e^2}{2}e^{-x}$

$y' = \frac{1}{2}e^{x} + \frac{e^2}{2}e^{-x}$

$y'' = \frac{1}{2}e^{x} - \frac{e^2}{2}e^{-x}$

Plug into $y'' - y = 0$;

$\frac{1}{2}e^{x} - \frac{e^2}{2}e^{-x} - \frac{1}{2}e^{x} + \frac{e^2}{2}e^{-x} = 0$ ✓

p.18 Section 1.3 : Differential Equations as Mathematical Models

Mathematical Model of physical system usually involves time t.

→ Model gives the state of the system, or a description of the system in terms of past, present, and future.

Examples:

① Population dynamics (growth)

$$\frac{dP}{dt} \propto P \quad \text{or} \quad \frac{dP}{dt} = kP \qquad \text{growth } k>0$$

(Population growth is proportional to the total population)

② Radioactive Decay

$$\frac{dA}{dt} \propto A \quad \text{or} \quad \frac{dA}{dt} = kA \qquad \text{decay } k<0$$

(Rate of decay of the nuclei of a substance is proportional to the # of nuclei $A(t)$ remaining at time t

OR Air resistance! $\frac{dv}{dt} \propto kv$!

③ Newtowns Law of Cooling Decay Problem

$$\frac{dT}{dt} \propto T - T_m \quad \text{or} \quad \frac{dT}{dt} = k(T - T_m)$$

$T(t)$ is the temperature of the body

T_m is the temperature of the medium

④ Spread of a disease.

$$\frac{dx}{dt} = kxy$$

$x(t)$ = # of people who have contracted the disease

$y(t)$ = # of people who have not been exposed.

x and y are related by

$x + y = n + 1$ (a population of n people introduces one infected one)

$$\therefore \frac{dx}{dt} = kx(n+1-x) \Bigg\} \begin{array}{l}\text{Initial condition}\\ x(0) = 1\end{array}$$

$$\frac{dx}{dt} = knx + kx - kx^2$$

$$= -kx^2 + k(n+1)x = -k\left(x^2 + (n+1)\right)$$

(Solve by completing the square!)

$$\left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2$$

$$-\frac{1}{k}\frac{dx}{dt} = (x+(n+1))^2 - (n+1)^2$$

$$\frac{dx}{\left[x+(n+1)\right]^2-(n+1)^2} = -k\,dt$$

Integrate ... what do you get?

let $u = x+(n+1)$, $du = dx$

$$\int \frac{du}{u^2-(n+1)^2} = -k\int dt$$

(5) Second Order Chemical Reaction

First order chemical reaction is:

$$\frac{dX}{dt} = kX$$

Example: Conversion of t-butyl chloride into t-butyl alcohol:

$$(CH_3)_3CCl + NaOH \rightarrow (CH_3)_3COH + NaCl$$

For the reaction

$$CH_3Cl + NaOH \rightarrow CH_3OH + NaCl$$

CH_3Cl	$NaOH$	CH_3OH	$NaCl$
Methyl Chloride	sodium hydroxide	methyl Alcohol	sodium chloride

The rate at which the reaction proceeds is proportional to the product of the remaining concentrations of CH_3Cl and $NaOH$.

let X = amount of CH_3OH

α and β are given amounts of chemicals A and B. Instantaneous amounts not converted to Chemical C are $\alpha - X$, $\beta - X$

$$\therefore \quad \frac{dX}{dt} = k(\alpha - X)(\beta - X) \quad \Big\} \text{ Second Order chemical Reaction}$$

Mixtures:

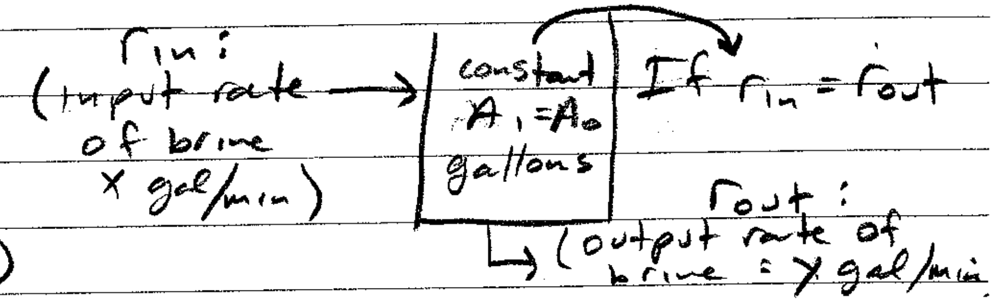
A_0 = Initial amount of Brine in tank

A_1 = Amount of brine in tank at time t

$A(t)$ = amount of salt at time t

$\frac{dA}{dt}$ = Rate at which $A(t)$ changes

Mixing tank : (well stirred)



- The solution is pumped out at the same rate as entering the solution

$$\frac{dA}{dt} = \begin{pmatrix}\text{input rate} \\ \text{of salt}\end{pmatrix} - \begin{pmatrix}\text{output rate} \\ \text{of salt}\end{pmatrix}$$

$$= R_{in} - R_{out}$$

$R_{in} \rightarrow$ input rate at which salt enters the tank or input rate of salt

$$R_{in} = \begin{pmatrix}\text{inflow concentration} \\ \text{of salt}\end{pmatrix}\begin{pmatrix}\text{input rate} \\ \text{of brine}\end{pmatrix}$$

$$R_{out} = \begin{pmatrix}\text{outflow concentration} \\ \text{of salt } c(t)\end{pmatrix}\begin{pmatrix}\text{output rate} \\ \text{of brine}\end{pmatrix}$$

$$c(t) = \frac{A(t)}{A_1} \text{ lb/gal}$$

$$\therefore \frac{dA}{dt} = \begin{pmatrix} \text{inflow concentration} \\ \text{of salt} \end{pmatrix} \begin{pmatrix} \text{input rate} \\ \text{of brine} \end{pmatrix} - \begin{pmatrix} \text{outflow concentration} \\ \text{of salt } c(t) \end{pmatrix} \begin{pmatrix} \text{output rate} \\ \text{of brine} \end{pmatrix}$$

If r_{in} and r_{out} denote the general input and output rates of the brine solutions, then there are three possibilities:

	Gallons in tank
① $r_{in} = r_{out}$	constant A_1
② $r_{in} > r_{out}$	increasing
③ $r_{in} < r_{out}$	decreasing
	(at the net rate $r_{in} - r_{out}$)

Case ②: $r_{in} > r_{out}$ and # gallons of brine in tank is increasing then it is accumulating liquid at a rate of $(r_{in} - r_{out})$ gal/min. After t minutes there are $A_1 = A_0 + (r_{in} - r_{out})t$ gallons

Case ③: $r_{in} < r_{out}$ and # gallons of brine in tank is decreasing then it is losing liquid at a rate $(r_{in} - r_{out})$ gal/min. After t minutes there are $A_1 = A_0 + (r_{in} - r_{out})t$ gallons where $r_{in} - r_{out}$ is negative.