

**EE160: Analog and Digital Communications**

**SOLVED PROBLEMS**

# Digital communication systems

1. With reference to Fig. 1.2 of the textbook, illustrating the basic elements of a digital communication system, answer the following questions:
  - (a) What is *source coding*?
  - (b) What is the purpose of the *channel encoder* and *channel decoder*?
  - (c) What is the purpose of the *digital modulator* and *digital demodulator*?
  - (d) Explain how is the performance of a digital communication system measured.

## Solution:

- (a) *Source coding* is the process of efficiently converting the output of either an analog or a digital source, with as little or no redundancy, into a sequence of binary digits.
  - (b) The *channel encoder* introduces, in a controlled (structured) manner, certain amount of redundancy that can be used at the receiver to overcome the effects of noise and interference encountered in the transmission of the signal through the channel. This serves to increase the reliability of the received data and improves the quality of the received signal. The *channel decoder* attempts to reconstruct the original information sequence from knowledge of the code used by the channel encoder, the digital modulation scheme and the redundancy contained in the received sequence.
  - (c) The *digital modulator* serves as the interface to the communications channel. Its primary purpose is to map the information sequence into signal waveforms. The *digital demodulator* processes the corrupted transmitted waveform and reduces each waveform to a single number that represents an estimate of the transmitted data symbol. If this number is quantized into more levels than those used in the modulator, the demodulator is said to produce a *soft output*. In this case, the channel decoder is known as a *soft-decision decoder*. Otherwise, the demodulator produces *hard outputs* that are processed by a *hard-decision decoder*.
  - (d) The performance of a digital communication system is typically measured by the frequency with which errors occur in the reconstructed information sequence. The *probability of a symbol error* is a function of the channel code and modulation characteristics, the waveforms used, the transmitted signal power, the characteristics of the channel — e.g., noise power — and the methods of demodulation and channel decoding.
2. What are the dominant sources of noise limiting performance of communication systems in the VHF and UHF bands?

**Solution:** The dominant noise limiting performance of communication systems in the VHF and UHF bands is thermal noise generated in the front end of the receiver.

3. Explain how storing data on a magnetic or optical disk is equivalent to transmitting a signal over a radio channel.

**Solution:** The process of storing data on a magnetic tape, magnetic disk or optical disk is equivalent to transmitting a signal over a wired or wireless channel. The readback process and the signal processing used to recover the stored information is equivalent to the functions performed by a communications system to recover the transmitted information sequence.

4. Discuss the advantages and disadvantages of digital processing versus analog processing. Do a web search. An interesting, albeit non-technical, discussion was found at [http://www.usatoday.com/tech/bonus/2004-05-16-bonus-analog\\_x.htm](http://www.usatoday.com/tech/bonus/2004-05-16-bonus-analog_x.htm)

**Solution:** A digital communications system does not accumulate errors. Analog signals are prone to interference and noise. There is no equivalent in an analog system to the correction of errors. However, a digital system degrades the quality of the original signal through quantization (analog-to-digital conversion). Also, a digital system requires more bandwidth than an analog system and, in general, relatively complex synchronization circuitry is required at the receiver. Analog systems are very sensitive to temperature and component value variations. It should be noted that no digital technology is used today in the front end of a transmitter and receiver (RF frequency bands of 1GHz and above), where mixers, channel filters, amplifiers and antennas are needed. The world today is still a mix of analog and digital components and will continue to be so for a long time. A key feature of digital technology is *programmability*, which has resulted in new concepts, such as software-defined radios and cognitive radio communications systems.

## Fourier analysis of signals and systems

5. Show that for a real and periodic signal  $x(t)$ , we have

$$\begin{aligned}x_e(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(2\pi \frac{n}{T_0} t\right), \\x_o(t) &= \sum_{n=1}^{\infty} b_n \sin\left(2\pi \frac{n}{T_0} t\right),\end{aligned}$$

where  $x_e(t)$  and  $x_o(t)$  are the *even and odd parts* of  $x(t)$ , defined as

$$\begin{aligned}x_e(t) &= \frac{x(t) + x(-t)}{2}, \\x_o(t) &= \frac{x(t) - x(-t)}{2}.\end{aligned}$$

**Solution:** It follows directly from the uniqueness of the decomposition of a real signal in an even and odd part. Nevertheless for a real periodic signal

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(2\pi \frac{n}{T_0} t) + b_n \sin(2\pi \frac{n}{T_0} t) \right]$$

The even part of  $x(t)$  is

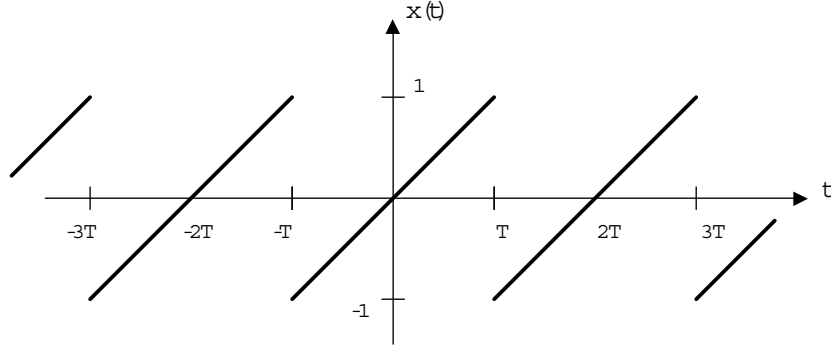
$$\begin{aligned}x_e(t) &= \frac{x(t) + x(-t)}{2} \\&= \frac{1}{2} \left( a_0 + \sum_{n=1}^{\infty} a_n (\cos(2\pi \frac{n}{T_0} t) + \cos(-2\pi \frac{n}{T_0} t)) \right. \\&\quad \left. + b_n (\sin(2\pi \frac{n}{T_0} t) + \sin(-2\pi \frac{n}{T_0} t)) \right) \\&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi \frac{n}{T_0} t)\end{aligned}$$

The last is true since  $\cos(\theta)$  is even so that  $\cos(\theta) + \cos(-\theta) = 2\cos\theta$  whereas the oddness of  $\sin(\theta)$  provides  $\sin(\theta) + \sin(-\theta) = \sin(\theta) - \sin(\theta) = 0$ .

Similarly, the odd part of  $x(t)$  is

$$\begin{aligned} x_o(t) &= \frac{x(t) - x(-t)}{2} \\ &= \sum_{n=1}^{\infty} b_n \sin(2\pi \frac{n}{T_0} t) \end{aligned}$$

6. Determine the Fourier series expansion of the sawtooth waveform, shown below



**Solution:** The signal is periodic with period  $2T$ . Since the signal is odd we obtain  $x_0 = 0$ . For  $n \neq 0$

$$\begin{aligned} x_n &= \frac{1}{2T} \int_{-T}^T x(t) e^{-j2\pi \frac{n}{2T} t} dt = \frac{1}{2T} \int_{-T}^T \frac{t}{T} e^{-j2\pi \frac{n}{2T} t} dt \\ &= \frac{1}{2T^2} \int_{-T}^T t e^{-j\pi \frac{n}{T} t} dt \\ &= \frac{1}{2T^2} \left( \frac{jT}{\pi n} t e^{-j\pi \frac{n}{T} t} + \frac{T^2}{\pi^2 n^2} e^{-j\pi \frac{n}{T} t} \right) \Big|_{-T}^T \\ &= \frac{1}{2T^2} \left[ \frac{jT^2}{\pi n} e^{-j\pi n} + \frac{T^2}{\pi^2 n^2} e^{-j\pi n} + \frac{jT^2}{\pi n} e^{j\pi n} - \frac{T^2}{\pi^2 n^2} e^{j\pi n} \right] \\ &= \frac{j}{\pi n} (-1)^n \end{aligned}$$

7. By computing the Fourier series coefficients for the periodic signal  $\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ , show that

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn \frac{2\pi t}{T_s}}.$$

Using this result, show that for any signal  $x(t)$  and any period  $T_s$ , the following identity holds

$$\sum_{n=-\infty}^{\infty} x(t - nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) e^{jn \frac{2\pi t}{T_s}}.$$

From this, conclude the following relation, known as *Poisson's sum formula*:

$$\sum_{n=-\infty}^{\infty} x(nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right).$$

**Solution:**

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x(t - nT_s) &= x(t) \star \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} x(t) \star \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{n}{T_s} t} \\
&= \frac{1}{T_s} \mathcal{F}^{-1} \left[ X(f) \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right) \right] \\
&= \frac{1}{T_s} \mathcal{F}^{-1} \left[ \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) \delta\left(f - \frac{n}{T_s}\right) \right] \\
&= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) e^{j2\pi \frac{n}{T_s} t}
\end{aligned}$$

If we set  $t = 0$  in the previous relation we obtain Poisson's sum formula

$$\sum_{n=-\infty}^{\infty} x(-nT_s) = \sum_{m=-\infty}^{\infty} x(mT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right)$$

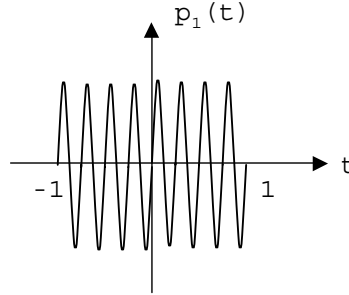
8. Find the Fourier transform  $P_1(f)$  of a pulse given by

$$p_1(t) = \sin(8\pi t) \Pi\left(\frac{t}{2}\right),$$

where

$$\Pi(t) \triangleq \begin{cases} 1, & |t| \leq \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases},$$

and shown in the figure below:



(Hint: Use the convolution theorem.)

**Solution:** Using the Fourier transform pair  $\Pi(t) \iff \text{sinc}(f)$  and the time scaling property (from the table of Fourier transform properties), we have that

$$\Pi\left(\frac{t}{2}\right) \iff 2 \text{sinc}(2f).$$

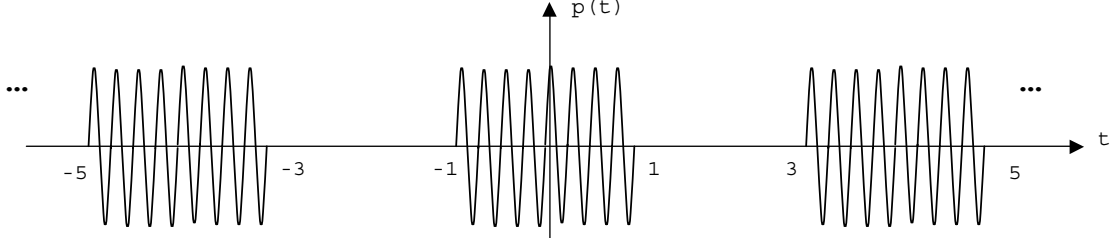
From the pair  $\sin(2\pi f_0 t) \iff \frac{1}{2j} [-\delta(f + f_0) + \delta(f - f_0)]$  and the convolution property, we arrive to the result

$$P_1(f) = j \{ \text{sinc}[2(f + 4)] - \text{sinc}[2(f - 4)] \}.$$

9. Determine the Fourier series expansion of the periodic waveform given by

$$p(t) = \sum_{n=-\infty}^{\infty} p_1(t - 4n),$$

and shown in the figure below:



(Hint: Use the Fourier transform  $P_1(f)$  found in the previous problem, and the following equation to find the Fourier coefficients:  $p_n = \frac{1}{T} F_1\left(\frac{n}{T}\right)$ .)

**Solution:** The signal  $p(t)$  is periodic with period  $T = 4$ . Consequently, the Fourier series expansion of  $p(t)$  is

$$p(t) = \sum_{n=-\infty}^{\infty} p_n \exp\left(j\frac{\pi}{2}tn\right),$$

where

$$p_n = \frac{1}{4} P_1\left(\frac{n}{4}\right) = \frac{1}{4j} \left\{ \text{sinc}\left[2\left(\frac{n}{4} + 4\right)\right] - \text{sinc}\left[2\left(\frac{n}{4} - 4\right)\right] \right\}.$$

10. Classify each of the following signals as an energy signal or a power signal, by calculating the energy  $E$ , or the power  $P$  ( $A$ ,  $\theta$ ,  $\omega$  and  $\tau$  are real positive constants).

- (a)  $x_1(t) = A |\sin(\omega t + \theta)|$ .
- (b)  $x_2(t) = A\tau / \sqrt{\tau + jt}$ ,  $j = \sqrt{-1}$ .
- (c)  $x_3(t) = At^2 e^{-t/\tau} u(t)$ .
- (d)  $x_4(t) = \Pi(t/\tau) + \Pi(t/2\tau)$ .

**Solution:**

- (a) Power. The signal is periodic, with period  $\pi/\omega$ , and

$$P_1 = \frac{\omega}{\pi} \int_0^{\pi/\omega} A^2 |\sin(\omega t + \theta)|^2 dt = \frac{A^2}{2}.$$

- (b) Neither:

$$E_2 = \lim_{T \rightarrow \infty} \int_{-T}^T \frac{(A\tau)^2}{\sqrt{\tau + jt} \sqrt{\tau - jt}} dt \rightarrow \infty,$$

and

$$P_2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \frac{(A\tau)^2}{\sqrt{\tau^2 + t^2}} dt = 0.$$

(c) Energy:

$$E_3 = \int_0^\infty A^2 t^4 \exp(-2t/\tau) dt = \frac{3A^2\tau^5}{4}.$$

(d) Energy:

$$E_4 = 2 \left( \int_0^{\tau/2} (2)^2 dt + \int_{\tau/2}^\tau (1)^2 dt \right) = 5\tau.$$

11. Sketch or plot the following signals:

(a)  $x_1(t) = \Pi(2t + 5)$

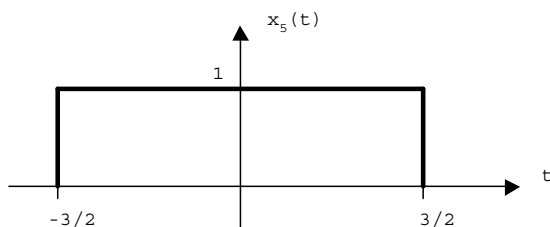
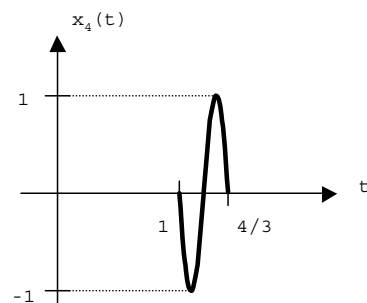
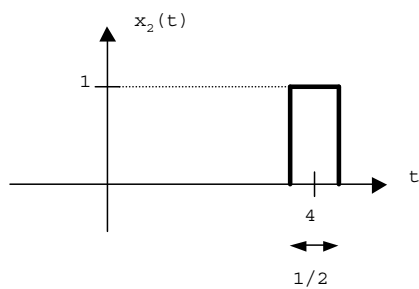
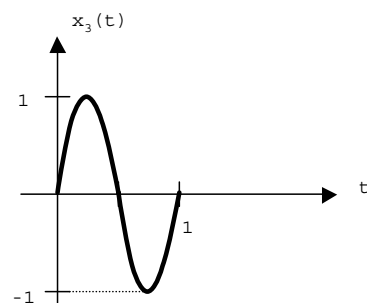
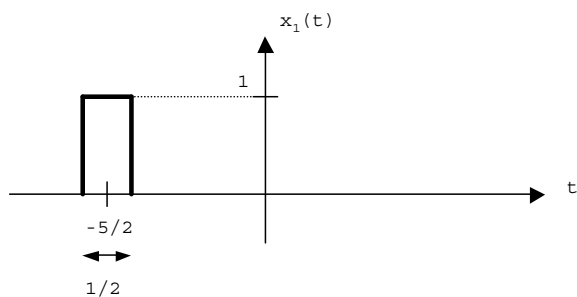
(b)  $x_2(t) = \Pi(-2t + 8)$

(c)  $x_3(t) = \Pi(t - \frac{1}{2}) \sin(2\pi t)$

(d)  $x_4(t) = x_3(-3t + 4)$

(e)  $x_5(t) = \Pi(-\frac{t}{3})$

**Solution:**

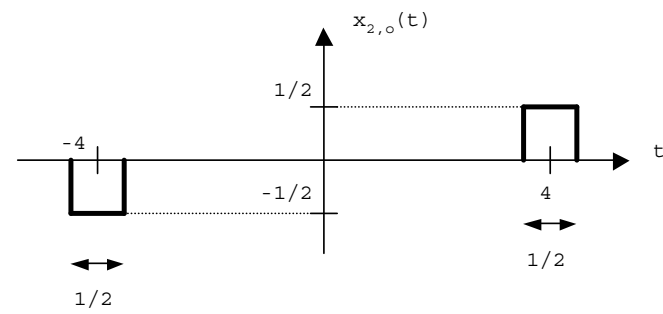
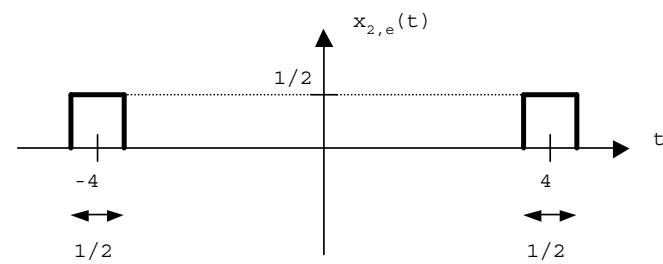
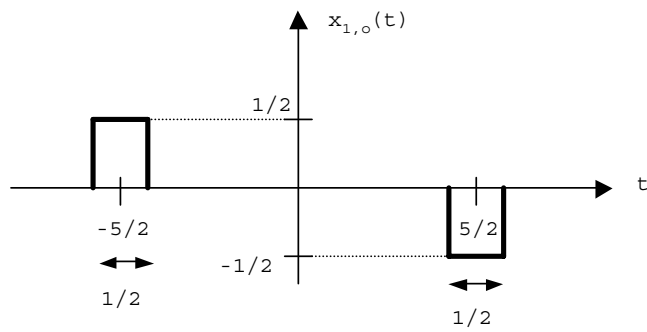
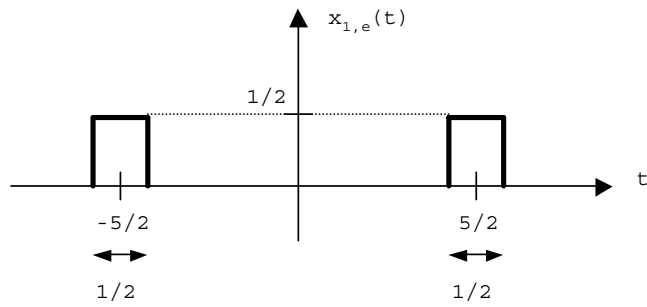


12. Classify each of the signals in the previous problem into even or odd signals, and determine the even and odd parts.

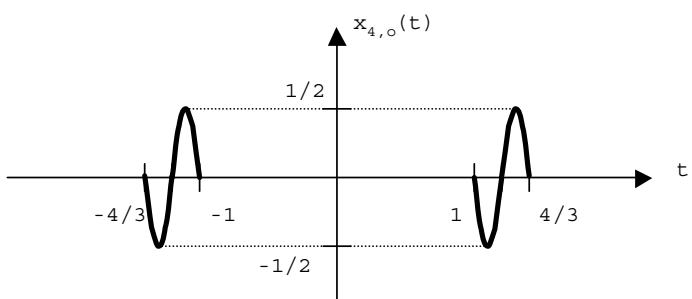
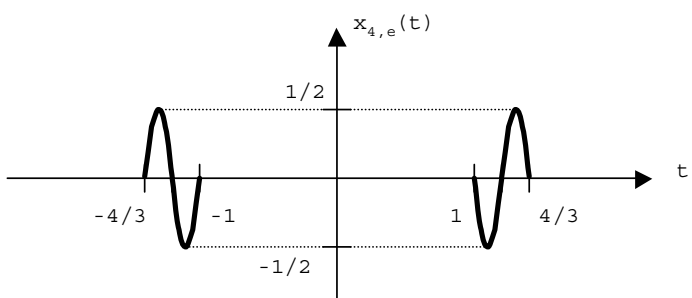
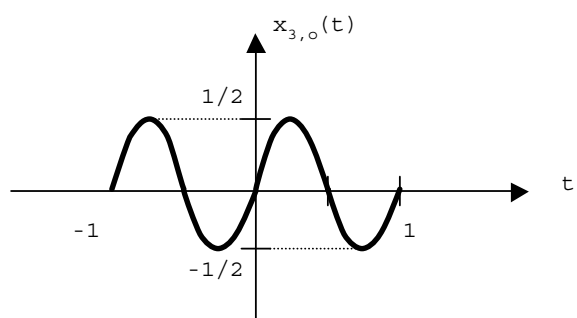
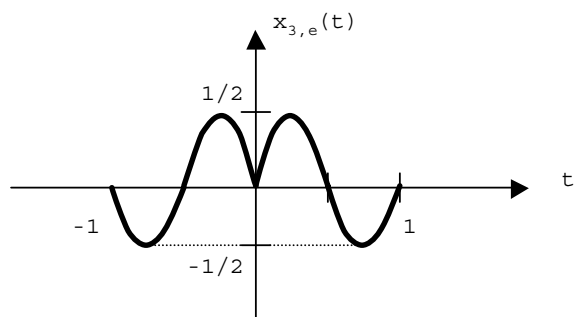
**Solution:**

The signal  $x_i(t)$ , for  $1 \leq i \leq 4$ , is neither even nor odd. The signal  $x_5(t)$  is even symmetric.

For each signal  $x_i(t)$ , with  $1 \leq i \leq 4$ , the figures below are sketches of the even part  $x_{i,e}(t)$  and the odd part  $x_{i,o}(t)$ . Evidently,  $x_{5,e} = x_5(t)$  and  $x_{5,o}(t) = 0$ .







13. Generalized Fourier series

- (a) Given the set of orthogonal functions

$$\phi_n(t) = \Pi\left(\frac{4[t - (2n - 1)T/8]}{T}\right), \quad n = 1, 2, 3, 4,$$

sketch and dimension accurately these functions.

- (b) Approximate the ramp signal

$$x(t) = \frac{t}{T} \Pi\left(\frac{t - T/2}{T}\right)$$

by a generalized Fourier series using these functions.

- (c) Do the same for the set

$$\phi_n(t) = \Pi\left(\frac{2[t - (2n - 1)T/4]}{T}\right), \quad n = 1, 2.$$

- (d) Compare the integral-squared error (ISE)  $\epsilon_N$  for both parts (b) and (c). What can you conclude about the dependency of  $\epsilon_N$  on  $N$ ?

**Solution:**

- (a) These are unit-amplitude rectangular pulses of width  $T/4$ , centered at  $t = T/8, 3T/8, 5T/8$ , and  $7T/8$ . Since they are spaced by  $T/4$ , they are adjacent to each other and fill the interval  $[0, T]$ .
- (b) Using the expression for the generalized Fourier series coefficients,

$$X_n = \frac{1}{c_n} \int_T x(t) \phi_n(t) dt,$$

where

$$c_n = \int_T |\phi_n(t)|^2 dt = \frac{T}{4},$$

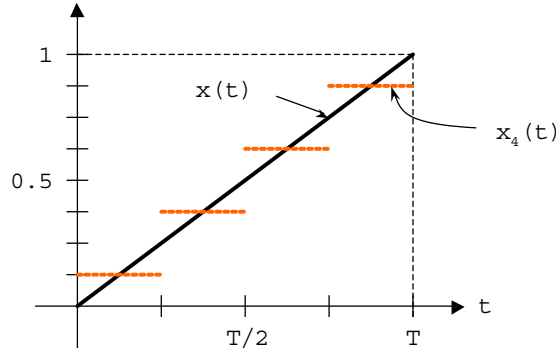
we have that

$$X_1 = \frac{1}{8}, \quad X_2 = \frac{3}{8}, \quad X_3 = \frac{5}{8}, \quad X_4 = \frac{7}{8}.$$

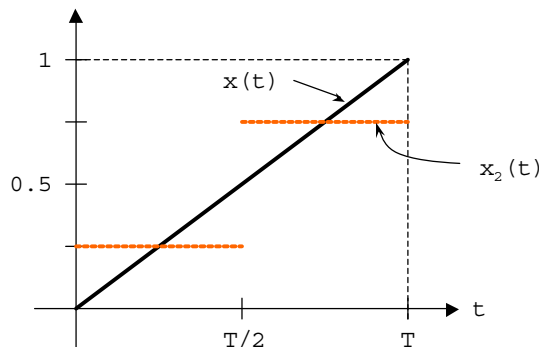
Thus, the ramp signal is approximated by

$$x_4(t) = \sum_{n=1}^4 X_n \phi_n(t) = \frac{1}{8} \phi_1(t) + \frac{3}{8} \phi_2(t) + \frac{5}{8} \phi_3(t) + \frac{7}{8} \phi_4(t), \quad 0 \leq t \leq T.$$

This is shown in the figure below:



- (c) These are unit-amplitude rectangular pulses of width  $T/2$  and centered at  $t = T/4$  and  $3T/4$ . We find that  $X_1 = 1/4$  and  $X_2 = 3/4$ . The approximation is shown in the figure below:



- (d) Use the relation

$$\epsilon_N = \int_T |x(t)|^2 dt - \sum_{n=1}^N c_n |X_n|^2,$$

and note that

$$\int_T |x(t)|^2 dt = \int_0^T \left(\frac{t}{T}\right)^2 dt = \frac{T}{3}.$$

It follows that the ISE for part (b) is given by

$$\epsilon_4 = \frac{T}{3} - \frac{T}{4} \left( \frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64} \right) = 5.208 \times 10^{-3} T,$$

and for part (c)

$$\epsilon_2 = \frac{T}{3} - \frac{T}{2} \left( \frac{1}{16} + \frac{9}{16} \right) = 2.083 \times 10^{-2} T.$$

Evidently, increasing the value of  $N$  decreases the approximation error  $\epsilon_N$ .

14. Show that the time-average signal correlation

$$R_x(\tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau)dt$$

can be written in terms of a convolution as

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} [x(t) \star x(-t)]_{t=\tau}.$$

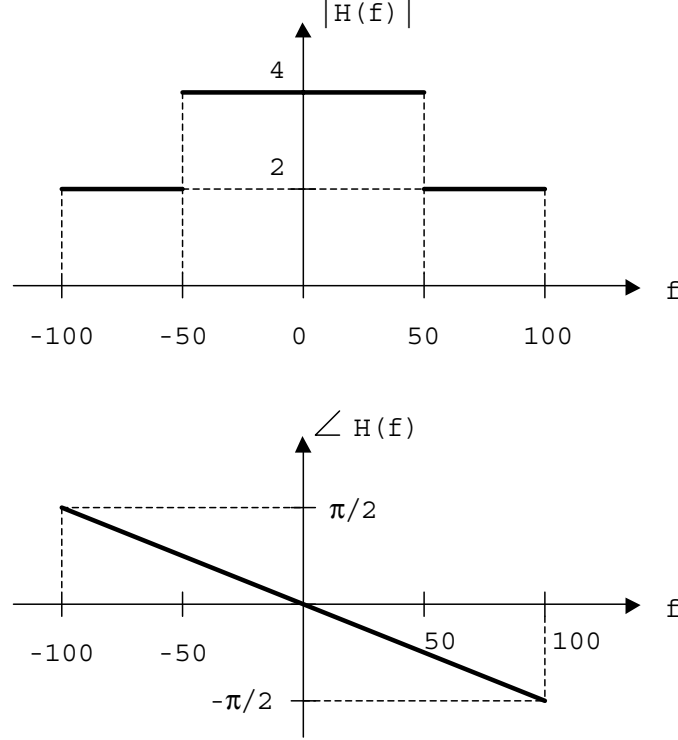
**Solution:** Note that:

$$x(t) \star x(-t) = \int_{-\infty}^{\infty} x(-\tau)x(t-\tau) d\tau = \int_{-\infty}^{\infty} x(u)x(t+u) du,$$

where  $u = -\tau$ . Rename variables to obtain

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(\beta)x(\tau + \beta) d\beta.$$

15. A filter has amplitude and phase responses as shown in the figure below:



Find the output to each of the inputs given below. For which cases is the transmission distortionless? For the other cases, indicate what type of distortion is imposed.

- (a)  $\cos(48\pi t) + 5 \cos(126\pi t)$
- (b)  $\cos(126\pi t) + 0.5 \cos(170\pi t)$
- (c)  $\cos(126\pi t) + 3 \cos(144\pi t)$
- (d)  $\cos(10\pi t) + 4 \cos(50\pi t)$

**Solution:** Note that the four input signals are of the form  $x_i(t) = a \cos(2\pi f_1 t) + b \cos(2\pi f_2 t)$ , for  $i = 1, 2, 3, 4$ . Consequently, their Fourier transforms consist of four impulses:

$$X_i(f) = \frac{a}{2} [\delta(f + f_1) + \delta(f - f_1)] + \frac{b}{2} [\delta(f + f_2) + \delta(f - f_2)], \quad i = 1, 2, 3, 4.$$

With this in mind, we have the following

- (a) Amplitude distortion; no phase distortion.
- (b) No amplitude distortion; phase distortion.
- (c) No amplitude distortion; no phase distortion.
- (d) No amplitude distortion; no phase distortion.

16. Determine the Fourier series expansion of the following signals:

- (a)  $x_4(t) = \cos(t) + \cos(2.5t)$
- (b)  $x_8(t) = |\cos(2\pi f_0 t)|$
- (c)  $x_9(t) = \cos(2\pi f_0 t) + |\cos(2\pi f_0 t)|$

**Solution:**

- (a) The signal  $\cos(t)$  is periodic with period  $T_1 = 2\pi$  whereas  $\cos(2.5t)$  is periodic with period  $T_2 = 0.8\pi$ . The ratio  $T_1/T_2 = 5/2$  and LCM  $(2, 5) = 10$ . It follows then that  $\cos(t) + \cos(2.5t)$  is periodic with period  $T = 2(2\pi) = 5(0.8\pi) = 4\pi$ . The trigonometric Fourier series of the even signal  $\cos(t) + \cos(2.5t)$  is

$$\begin{aligned}\cos(t) + \cos(2.5t) &= \sum_{n=1}^{\infty} \alpha_n \cos(2\pi \frac{n}{T_0} t) \\ &= \sum_{n=1}^{\infty} \alpha_n \cos(\frac{n}{2} t)\end{aligned}$$

By equating the coefficients of  $\cos(\frac{n}{2}t)$  of both sides we observe that  $a_n = 0$  for all  $n$  unless  $n = 2, 5$  in which case  $a_2 = a_5 = 1$ . Hence  $x_{4,\pm 2} = x_{4,\pm 5} = \frac{1}{2}$  and  $x_{4,n} = 0$  for all other values of  $n$ .

- (b) The signal  $x_8(t)$  is real, even symmetric, and periodic with period  $T_0 = \frac{1}{2f_0}$ . Hence,  $x_{8,n} = a_{8,n}/2$  or

$$\begin{aligned}x_{8,n} &= 2f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t) \cos(2\pi n 2f_0 t) dt \\ &= f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t) dt + f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t) dt \\ &= \frac{1}{2\pi(1+2n)} \sin(2\pi f_0(1+2n)t) \Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)} \sin(2\pi f_0(1-2n)t) \Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\ &= \frac{(-1)^n}{\pi} \left[ \frac{1}{(1+2n)} + \frac{1}{(1-2n)} \right]\end{aligned}$$

- (c) The signal  $x_9(t) = \cos(2\pi f_0 t) + |\cos(2\pi f_0 t)|$  is even symmetric and periodic with period  $T_0 = 1/f_0$ . It is equal to  $2\cos(2\pi f_0 t)$  in the interval  $[-\frac{1}{4f_0}, \frac{1}{4f_0}]$  and zero in the interval  $[\frac{1}{4f_0}, \frac{3}{4f_0}]$ . Thus

$$\begin{aligned}x_{9,n} &= 2f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t) \cos(2\pi n f_0 t) dt \\ &= f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+n)t) dt + f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-n)t) dt \\ &= \frac{1}{2\pi(1+n)} \sin(2\pi f_0(1+n)t) \Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-n)} \sin(2\pi f_0(1-n)t) \Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\ &= \frac{1}{\pi(1+n)} \sin(\frac{\pi}{2}(1+n)) + \frac{1}{\pi(1-n)} \sin(\frac{\pi}{2}(1-n))\end{aligned}$$

Thus  $x_{9,n}$  is zero for odd values of  $n$  unless  $n = \pm 1$  in which case  $x_{9,\pm 1} = \frac{1}{2}$ . When  $n$  is even ( $n = 2\ell$ ) then

$$x_{9,2\ell} = \frac{(-1)^\ell}{\pi} \left[ \frac{1}{1+2\ell} + \frac{1}{1-2\ell} \right]$$

17. A triangular pulse can be specified by

$$\Lambda(t) = \begin{cases} t+1, & -1 \leq t \leq 0; \\ -t+1, & 0 \leq t \leq 1. \end{cases}$$

(a) Sketch the signal

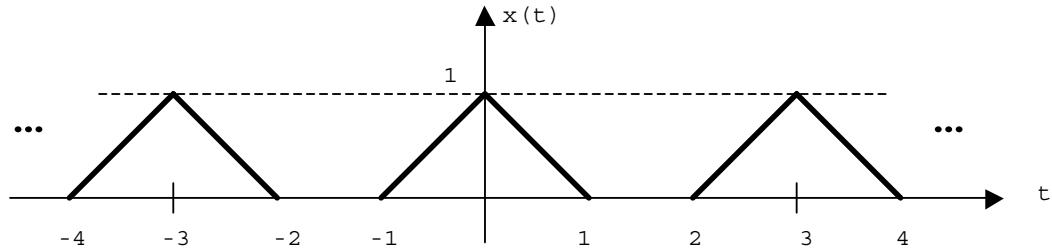
$$x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t+3n).$$

(b) Find the Fourier series coefficients,  $x_n$ , of  $x(t)$ .

(c) Find the Fourier series coefficients,  $y_n$ , of the signal  $y(t) = x(t-t_0)$ , in terms of  $x_n$ .

**Solution:**

(a) Sketch:



(b) The signal  $x(t)$  is periodic with  $T_0 = 3$ . The Fourier series coefficients are obtained from the Fourier transform,  $X_{T_0}(f)$ , of the truncated signal  $x_{T_0}(t)$  as

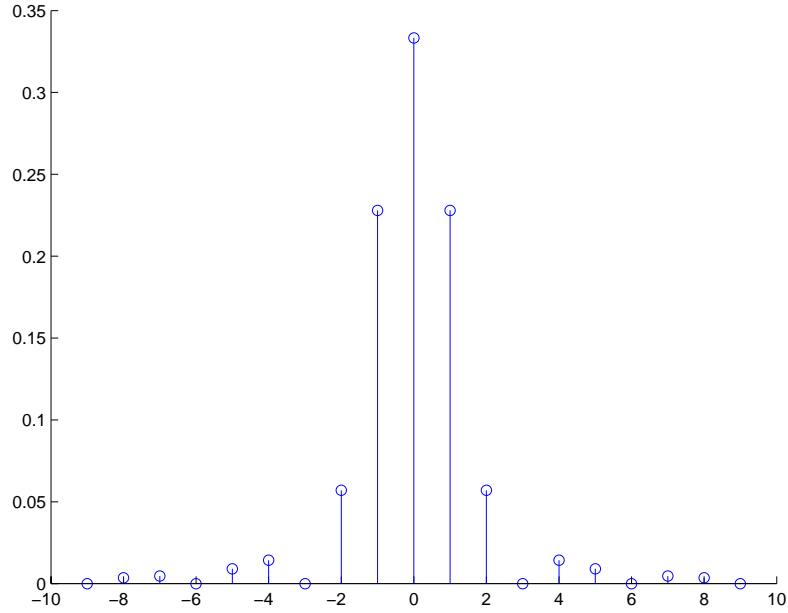
$$x_n = \frac{1}{T_0} X_{T_0}(f)|_{f=\frac{n}{T_0}}.$$

In this case,

$$x_{T_0}(t) = \Lambda(t) \iff X_{T_0}(f) = \text{sinc}^2(f).$$

Consequently,

$$x_n = \frac{1}{3} \text{sinc}^2\left(\frac{n}{3}\right).$$



(c) Using the time-shift property of the Fourier transform, we have

$$y_{T_0}(t) = x_{T_0}(t - t_0) = \Lambda(t - t_0) \quad \Longleftrightarrow \quad Y_{T_0}(f) = X_{T_0}(f) e^{-j2\pi f t_0} = \text{sinc}^2(f) e^{-j2\pi f t_0},$$

and it follows that

$$y_n = x_n e^{-j2\pi(\frac{n}{3})t_0} = \frac{1}{3} \text{sinc}^2\left(\frac{n}{3}\right) e^{-j2\pi(\frac{n}{3})t_0}.$$

18. For each case below, sketch the signal and find its Fourier series coefficients.

(a)  $x(t) = \cos(2\pi t) + \cos(3\pi t)$ . (Hint: Find  $T_0$ . Use symmetry.)

(b)  $y(t) = |\cos(2\pi f_0 t)|$ . (Full-wave rectifier output.)

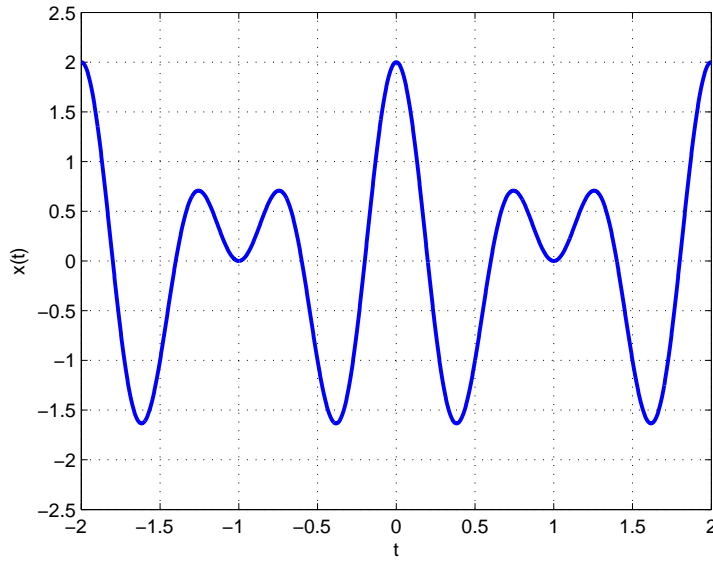
(c)  $z(t) = |\cos(2\pi f_0 t)| + \cos(2\pi f_0 t)$ . (Half-wave rectifier output.)

**Solution:**

(a) The signals  $\cos(2\pi t)$  and  $\cos(3\pi t)$  are periodic with periods  $T_1 = 1$  and  $T_2 = \frac{2}{3}$ , respectively. The period  $T_0$  of  $x(t)$  is the “least common multiple” of  $T_1$  and  $T_2$ :

$$T_0 = \text{“lcm”} \left(1, \frac{2}{3}\right) = \frac{1}{3} \text{lcm}(3, 2) = \frac{6}{3} = 2.$$

Sketch:



Using Euler's formula:

$$x(t) = \frac{1}{2} [e^{j2\pi t} + e^{-j2\pi t} + e^{j3\pi t} + e^{-j3\pi t}]. \quad (1)$$

Comparing (1) with the Fourier series expansion of  $x(t)$ , with  $T_0 = 2$ :

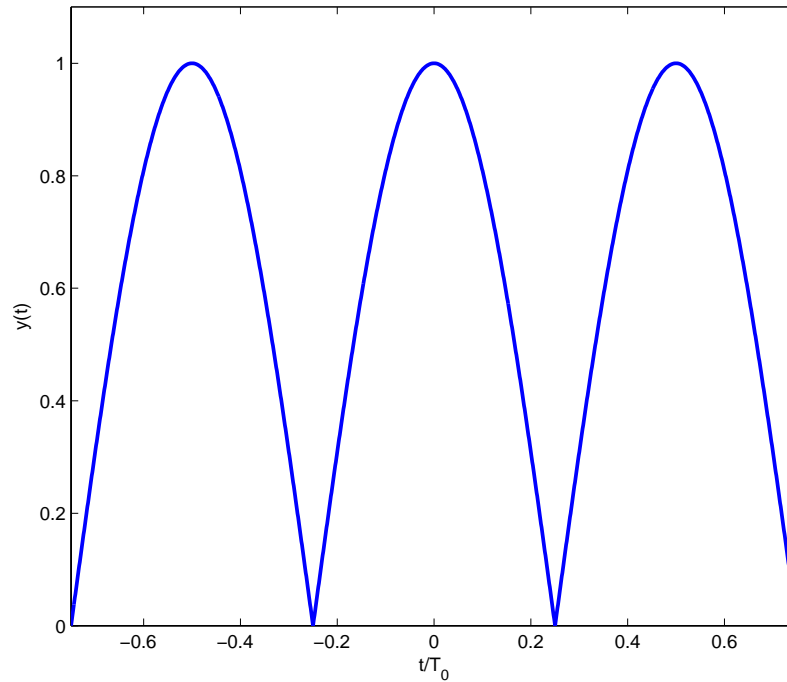
$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j\pi n t},$$

we conclude that

$$x_{\pm 2} = x_{\pm 3} = \frac{1}{2},$$

and  $x_n = 0$  for all other values of  $n$ .

(b) Sketch:





Note that  $y(t)$  is periodic with period  $T_1 = T_0/2$ . A fortunate choice of a truncated signal  $y_{T_1}(t)$ , over an interval of length  $T_1$  seconds, is given by

$$y_{T_1}(t) = \cos(2\pi f_0 t) \Pi\left(\frac{2t}{T_0}\right),$$

with Fourier transform (modulation property)

$$\begin{aligned} Y_{T_1}(f) &= \frac{1}{2} [\delta(f + f_0) + \delta(f - f_0)] \star \frac{T_0}{2} \text{sinc}\left(\frac{T_0}{2}f\right) \\ &= \frac{T_0}{4} \left[ \text{sinc}\left(\frac{T_0}{2}(f + f_0)\right) + \text{sinc}\left(\frac{T_0}{2}(f - f_0)\right) \right]. \end{aligned}$$

It follows that (with  $f_0 = \frac{1}{T_0}$ )

$$\begin{aligned} y_n &= \frac{1}{T_1} Y_{T_1}(f)|_{f=\frac{n}{T_1}=\frac{2n}{T_0}} \\ &= \frac{1}{2} \left[ \text{sinc}\left(\frac{1}{2}(2n + 1)\right) + \text{sinc}\left(\frac{1}{2}(2n - 1)\right) \right]. \end{aligned} \quad (2)$$

The above result can be further simplified by using the definition of the sinc function,  $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ , noticing that

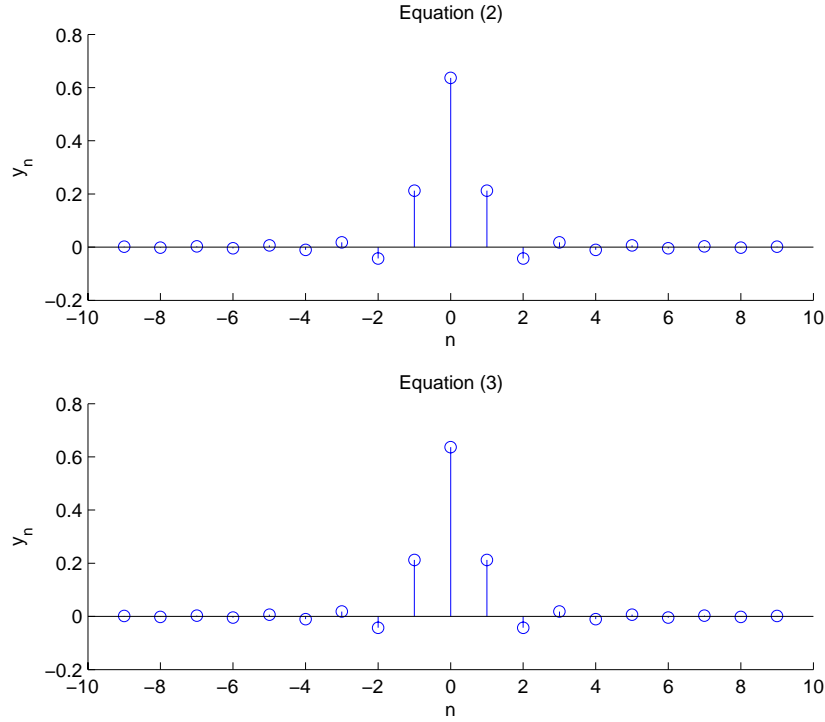
$$\begin{aligned} \sin\left(\frac{\pi}{2}(2n + 1)\right) &= \begin{cases} +1, & n = 0, 2, 4, \dots \\ -1, & n = 1, 3, 5, \dots \end{cases} \\ &= (-1)^n, \end{aligned}$$

and using the odd symmetry of the sine function for negative values of  $n$ . This gives (details omitted):

$$y_n = \frac{(-1)^n}{\pi} \left[ \frac{1}{1 + 2n} + \frac{1}{1 - 2n} \right]. \quad (3)$$

You are invited to verify that both (2) and (3) yield the same result. For example, you can do this using Matlab with the commands:

```
n=-9:1:9;
subplot(2,1,1)
stem(n,0.5*(sinc((2*n+1)/2)+sinc((2*n-1)/2)))
subplot(2,1,2)
stem(n,((-1).^n/pi) .* ( (1./(2*n+1)) + (1./(1-2*n)) ) )
```



(c) The sketch of  $z(t)$  is shown in the following page. Here the period is  $T_0$ . The truncated signal is

$$z_{T_0}(t) = \cos(2\pi f_0 t) \Pi\left(\frac{2t}{T_0}\right),$$

with Fourier transform

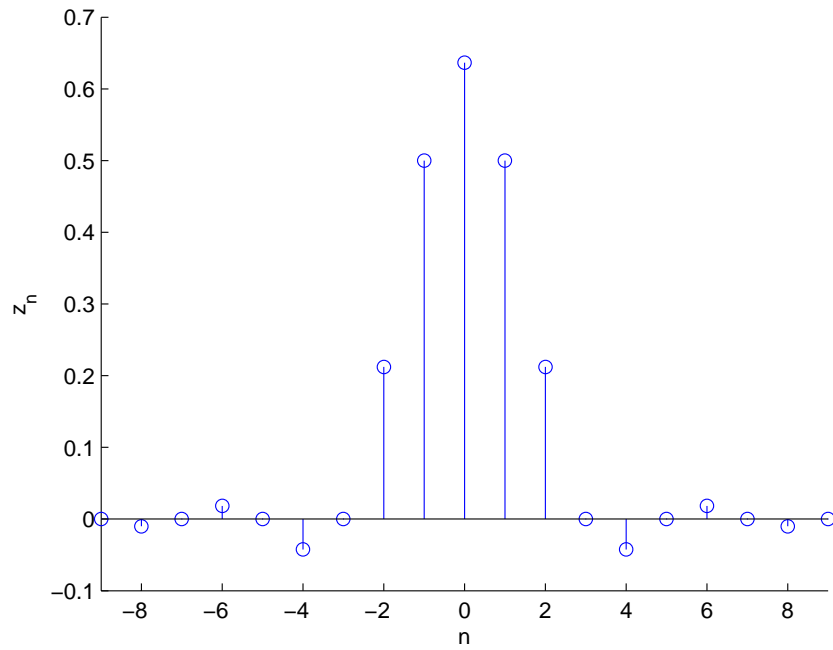
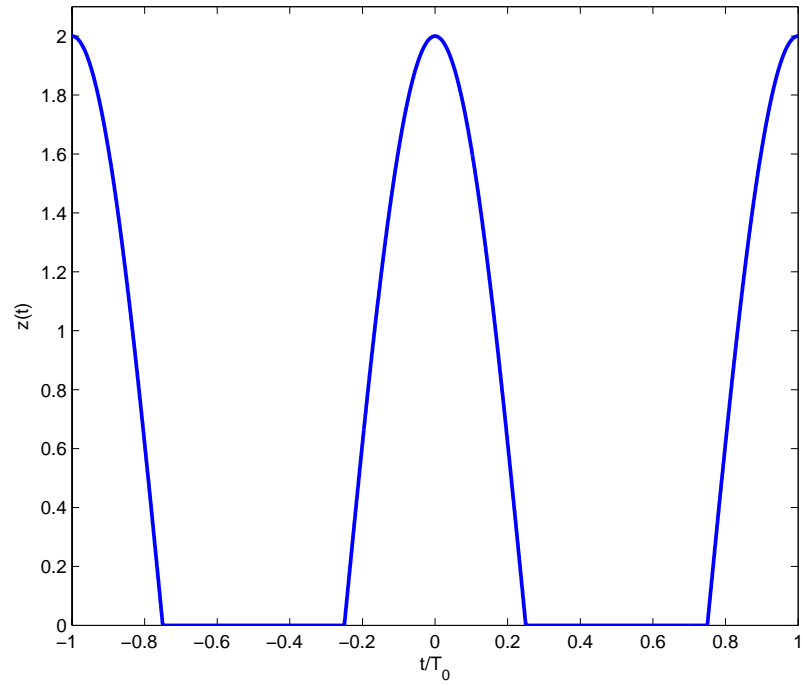
$$Z_{T_0}(f) = \frac{T_0}{4} \left[ \text{sinc}\left(\frac{T_0}{2}(f + f_0)\right) + \text{sinc}\left(\frac{T_0}{2}(f - f_0)\right) \right].$$

(Remarkably,  $Z_{T_0}(f) = Y_{T_1}(f)$ .) Therefore,

$$z_n = \frac{1}{T_0} Z_{T_0}(f)|_{f=\frac{n}{T_0}} = \frac{1}{2} \left[ \text{sinc}\left(\frac{1}{2}(n+1)\right) + \text{sinc}\left(\frac{1}{2}(n-1)\right) \right].$$

As before, there is a simplification possible (but not necessary!) using the definition of the sinc function. This gives,  $z_{\pm 1} = \frac{1}{2}$  and

$$z_n = \frac{(-1)^\ell}{\pi} \left[ \frac{1}{1+2\ell} + \frac{1}{1-2\ell} \right], \quad n = 2\ell, \quad \ell \text{ integer}.$$

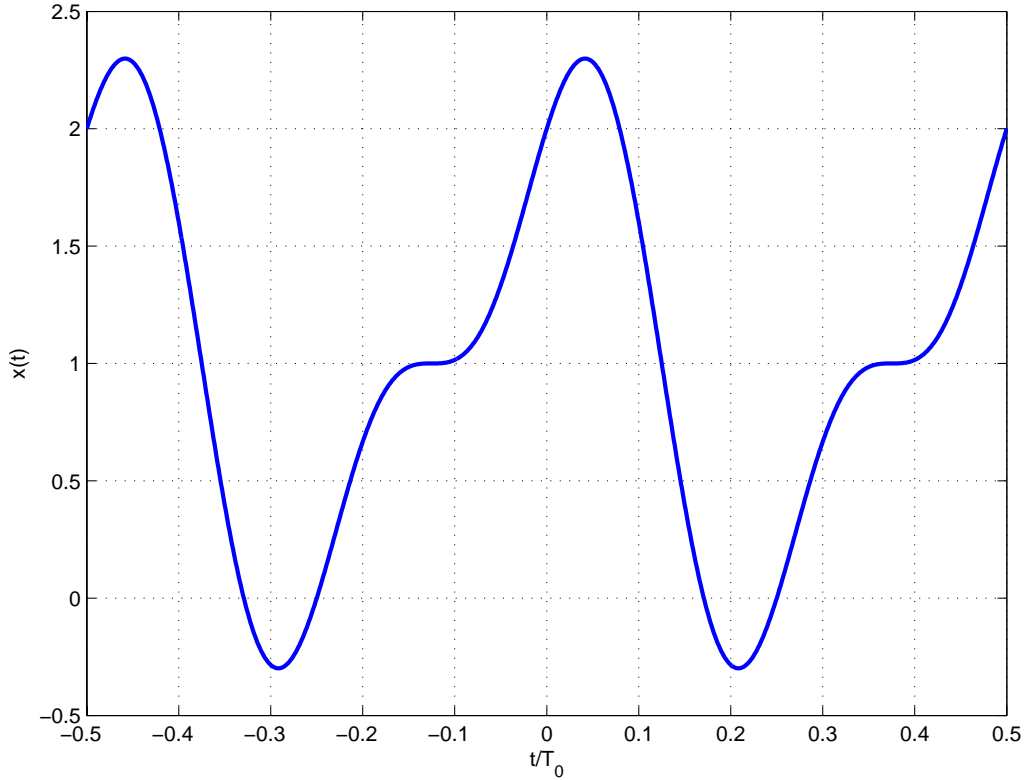


19. Sketch the signal  $x(t)$  whose Fourier series coefficients are given by

$$x_n = \begin{cases} 1, & n = 0; \\ \frac{1}{2}, & n = -2, +2; \\ +\frac{1}{4}j, & n = -4; \\ -\frac{1}{4}j, & n = +4; \\ 0, & \text{elsewhere.} \end{cases}$$

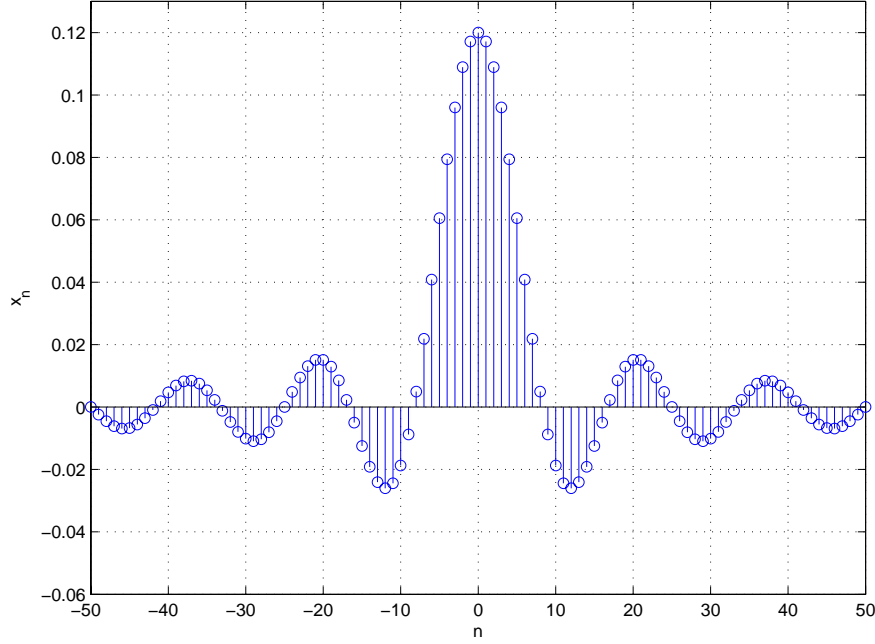
**Solution:** We are given the Fourier series coefficients. Therefore,

$$\begin{aligned}
 x(t) &= \sum_{n=-\infty}^{\infty} x_n e^{j2\pi\left(\frac{n}{T_0}\right)t} \\
 &= 1 + \frac{1}{2} \left[ e^{-j2\pi\left(\frac{2}{T_0}\right)t} + e^{j2\pi\left(\frac{2}{T_0}\right)t} \right] + \frac{1}{4j} \left[ e^{j2\pi\left(\frac{4}{T_0}\right)t} - e^{-j2\pi\left(\frac{4}{T_0}\right)t} \right] \\
 &= 1 + \cos(4\pi f_0 t) + \frac{1}{2} \sin(8\pi f_0 t).
 \end{aligned}$$



20. Modify the Matlab script `example1s05.m` in the web site, to compute the Fourier series coefficients  $x_n$  of an even-symmetric train of rectangular pulses of duty cycle equal to 0.12 over the range  $-50 \leq n \leq 50$ . Attach a printout of the resulting plot.

**Solution:** Using the Matlab script `homework3s05.m` (available in the web site) we obtain:



21. Let  $x_n$  and  $y_n$  denote the Fourier series coefficients of  $x(t)$  and  $y(t)$ , respectively. Assuming the period of  $x(t)$  is  $T_0$ , express  $y_n$  in terms of  $x_n$  in each of the following cases:

- (a)  $y(t) = x(t - t_0)$
- (b)  $y(t) = x(\alpha t)$

**Solution:**

(a) The signal  $y(t) = x(t - t_0)$  is periodic with period  $T = T_0$ .

$$\begin{aligned}
 y_n &= \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t - t_0) e^{-j2\pi \frac{n}{T_0} t} dt \\
 &= \frac{1}{T_0} \int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v) e^{-j2\pi \frac{n}{T_0} (v+t_0)} dv \\
 &= e^{-j2\pi \frac{n}{T_0} t_0} \frac{1}{T_0} \int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v) e^{-j2\pi \frac{n}{T_0} v} dv \\
 &= x_n e^{-j2\pi \frac{n}{T_0} t_0}
 \end{aligned}$$

where we used the change of variables  $v = t - t_0$ .

(b) The signal  $y(t)$  is periodic with period  $T = T_0/\alpha$ .

$$\begin{aligned}
 y_n &= \frac{1}{T} \int_{\beta}^{\beta+T} y(t) e^{-j2\pi \frac{n}{T} t} dt = \frac{\alpha}{T_0} \int_{\beta}^{\beta+\frac{T_0}{\alpha}} x(\alpha t) e^{-j2\pi \frac{n\alpha}{T_0} t} dt \\
 &= \frac{1}{T_0} \int_{\beta\alpha}^{\beta\alpha+T_0} x(v) e^{-j2\pi \frac{n}{T_0} v} dv = x_n
 \end{aligned}$$

where we used the change of variables  $v = \alpha t$ .

22. Determine whether these signals are energy-type or power-type. In each case, find the energy or power spectral density and also the energy or power content of the signal.

(a)  $x(t) = e^{-\alpha t}u(t)$ ,  $\alpha > 0$

(b)  $x(t) = \text{sinc}(t)$

(c)  $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t - 2n)$

(d)  $x(t) = u(t)$

(e)  $x(t) = \frac{1}{t}$

**Solution:**

(a)  $x(t) = e^{-\alpha t}u(t)$ . The spectrum of the signal is  $X(f) = \frac{1}{\alpha + j2\pi f}$  and the energy spectral density

$$\mathcal{G}_X(f) = |X(f)|^2 = \frac{1}{\alpha^2 + 4\pi^2 f^2}$$

Thus,

$$R_X(\tau) = \mathcal{F}^{-1}[\mathcal{G}_X(f)] = \frac{1}{2\alpha} e^{-\alpha|\tau|}$$

The energy content of the signal is

$$E_X = R_X(0) = \frac{1}{2\alpha}$$

(b)  $x(t) = \text{sinc}(t)$ . Clearly  $X(f) = \Pi(f)$  so that  $\mathcal{G}_X(f) = |X(f)|^2 = \Pi^2(f) = \Pi(f)$ . The energy content of the signal is

$$E_X = \int_{-\infty}^{\infty} \mathcal{G}_X(f) df = \int_{-\infty}^{\infty} \Pi(f) df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(f) df = 1$$

(c)  $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t - 2n)$ . The signal is periodic and thus it is not of the energy type. The power content of the signal is

$$\begin{aligned} P_x &= \frac{1}{2} \int_{-1}^1 |x(t)|^2 dt = \frac{1}{2} \int_{-1}^0 (t+1)^2 dt + \int_0^1 (-t+1)^2 dt \\ &= \frac{1}{2} \left( \frac{1}{3} t^3 + t^2 + t \right) \Big|_{-1}^0 + \frac{1}{2} \left( \frac{1}{3} t^3 - t^2 + t \right) \Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

The same result is obtain if we let

$$\mathcal{S}_X(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - \frac{n}{2})$$

with  $x_0 = \frac{1}{2}$ ,  $x_{2l} = 0$  and  $x_{2l+1} = \frac{2}{\pi(2l+1)}$  (see Problem 2.2). Then

$$\begin{aligned} P_X &= \sum_{n=-\infty}^{\infty} |x_n|^2 \\ &= \frac{1}{4} + \frac{8}{\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} = \frac{1}{4} + \frac{8}{\pi^2} \frac{\pi^2}{96} = \frac{1}{3} \end{aligned}$$

(d)

$$E_X = \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T \rightarrow \infty} \int_0^{\frac{T}{2}} dt = \lim_{T \rightarrow \infty} \frac{T}{2} = \infty$$

Thus, the signal is not of the energy type.

$$P_X = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \frac{T}{2} = \frac{1}{2}$$

Hence, the signal is of the power type and its power content is  $\frac{1}{2}$ . To find the power spectral density we find first the autocorrelation  $R_X(\tau)$ .

$$\begin{aligned} R_X(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u_{-1}(t) u_{-1}(t - \tau) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{\tau}^{\frac{T}{2}} dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \left( \frac{T}{2} - \tau \right) = \frac{1}{2} \end{aligned}$$

Thus,  $\mathcal{S}_X(f) = \mathcal{F}[R_X(\tau)] = \frac{1}{2} \delta(f)$ .

(e) Clearly  $|X(f)|^2 = \pi^2 \text{sgn}^2(f) = \pi^2$  and  $E_X = \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \pi^2 dt = \infty$ . The signal is not of the energy type for the energy content is not bounded. Consider now the signal

$$x_T(t) = \frac{1}{t} \Pi\left(\frac{t}{T}\right)$$

Then,

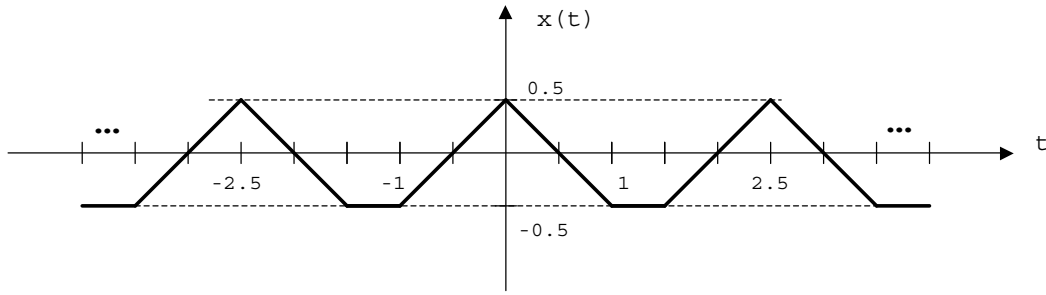
$$X_T(f) = -j\pi \text{sgn}(f) \star T \text{sinc}(fT)$$

and

$$\mathcal{S}_X(f) = \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2}{T} = \lim_{T \rightarrow \infty} \pi^2 T \left| \int_{-\infty}^f \text{sinc}(vT) dv - \int_f^{\infty} \text{sinc}(vT) dv \right|^2$$

However, the squared term on the right side is bounded away from zero so that  $\mathcal{S}_X(f)$  is  $\infty$ . The signal is not of the power type either.

23. Consider the periodic signal depicted in the figure below.



- (a) Find its Fourier transform  $X(f)$  and sketch it carefully.
- (b) The signal  $x(t)$  is passed through an LTI system with impulse response  $h(t) = \text{sinc}(t/2)$ . Find the power of the output  $y(t)$ .

**Solution:**

- (a)  $T_0 = 5/2$  and

$$x_{T_0}(t) = \Lambda\left(\frac{t}{2}\right) - \frac{1}{2}\Pi\left(\frac{2t}{5}\right).$$

As a result

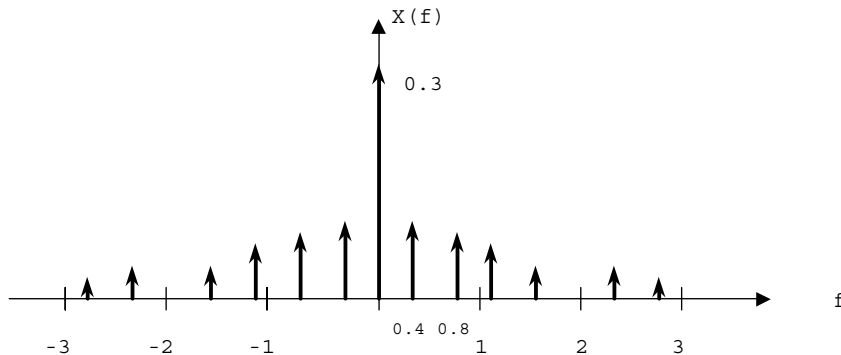
$$X_{T_0}(f) = 2 \text{sinc}^2(2f) - \frac{5}{4} \text{sinc}\left(\frac{5f}{2}\right).$$

Fourier series coefficients:

$$\begin{aligned} x_n &= \frac{1}{T_0} X_{T_0}\left(\frac{n}{T_0}\right) = \frac{2}{5} \cdot 2 \text{sinc}^2\left(\frac{4}{5}n\right) - \frac{2}{5} \cdot \frac{5}{4} \text{sinc}(n) \\ &= \frac{3}{10} \delta(n) + \frac{4}{5} \text{sinc}^2\left(\frac{4n}{5}\right). \end{aligned}$$

Fourier transform:

$$X(f) = \sum_{n=-\infty}^{\infty} \left[ \frac{3}{10} \delta(n) + \frac{4}{5} \text{sinc}^2\left(\frac{4n}{5}\right) \right] \delta\left(f - \frac{2}{5}n\right) = \frac{3}{10} \delta(f) + \frac{8}{5} \sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{4n}{5}\right) \delta\left(f - \frac{2}{5}n\right)$$



- (b)  $H(f) = 2 \Pi(2f)$ . Therefore,  $Y(f) = \frac{6}{10} \delta(f)$  and  $P_y = \left(\frac{6}{10}\right)^2 = 0.36$ .

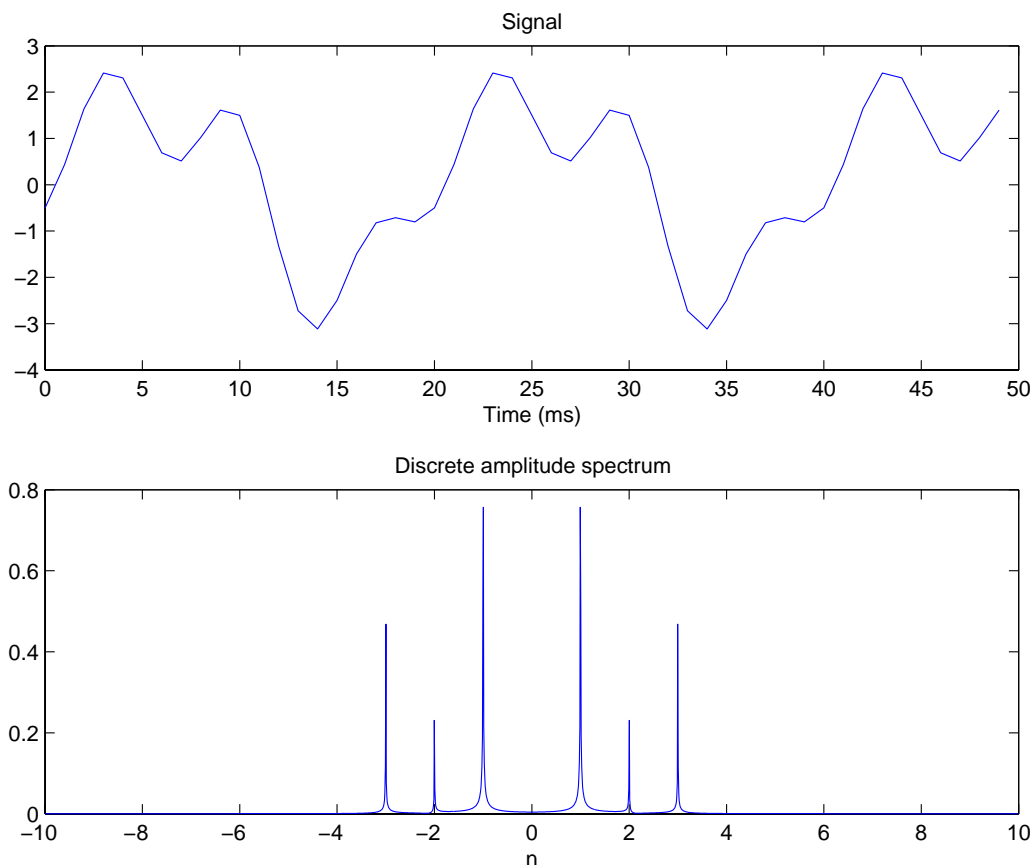
24. Matlab problem. This problem needs the Matlab script `homework1f04.m`, available in the class web site. The script uses the fast Fourier transform (FFT) to compute the discrete amplitude spectrum of the periodic signal  $x(t) = 2\sin(100\pi t) + 0.5\cos(200\pi t) - \cos(300\pi t)$ .



- (a) Run the script `homework1f04.m`. To do this, you must save the file to a local directory, change the working directory in MATLAB to that location, and enter `homework1f04` at the prompt in the command window. You will be requested to enter your student ID number. The script produces a figure that you are required to either print or sketch. Also, record in your solution the value of the magic number that will appear in the command window after execution of the script.
- (b) Verify the results of part (a) by computing the Fourier series coefficients of  $x(t)$ .

**Solution:**

(a)



Magic number: 0.53490560606366733

- (b)  $x(t)$  is a periodic signal. The signal  $\sin(100\pi t)$  has fundamental frequency  $f_0 = 50$ , while the signals  $\cos(200\pi t)$  and  $\cos(300\pi t)$  have fundamental frequencies  $2f_0 = 100$  and  $3f_0 = 150$ , respectively. Consequently,  $f_0$  is the fundamental frequency of  $x(t)$ . Expand  $x(t)$  using Euler's formula:

$$\begin{aligned} x(t) &= 2 \sin(100\pi t) + 0.5 \cos(200\pi t) - \cos(300\pi t) \\ &= -j [e^{j100\pi t} - e^{-j100\pi t}] + \frac{1}{4} [e^{j200\pi t} + e^{-j200\pi t}] - \frac{1}{2} [e^{j300\pi t} + e^{-j300\pi t}]. \end{aligned}$$

It follows that  $|x_{\pm 1}| = 1$ ,  $|x_{\pm 2}| = 0.25$ , and  $|x_{\pm 3}| = 0.5$ . The script gives correctly the three nonzero components of the discrete spectrum of  $x(t)$ . We note that the amplitude values of the Fourier series coefficients are not correct, although their ratios are close to the correct values, that is  $|x_{\pm 1}|/|x_{\pm 3}| = |x_{\pm 3}|/|x_{\pm 2}| = 2$ . This is believed to be an artifact that results from the use of the FFT.

25. Determine the Fourier transform of each of the following signals:

- (a)  $\Pi(t - 3) + \Pi(t + 3)$
- (b)  $\text{sinc}^3(t)$

**Solution:**

(a)) Using the time-shifting property of the Fourier transform,

$$\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t - 3) + \Pi(t + 3)] \\ &= \text{sinc}(f) e^{-j2\pi f(3)} + \text{sinc}(f) e^{j2\pi f(3)} \\ &= 2 \cos(6\pi f) \text{sinc}(f)\end{aligned}$$

(b) Using the convolution property of the Fourier transform,

$$T(f) = \mathcal{F}[\text{sinc}^3(t)] = \mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)] = \Lambda(f) \star \Pi(f).$$

Note that

$$\Pi(f) \star \Lambda(f) = \int_{-\infty}^{\infty} \Pi(\theta) \Lambda(f - \theta) d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(f - \theta) d\theta = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v) dv,$$

From which it follows that

$$\text{For } f \leq -\frac{3}{2}, \quad T(f) = 0$$

$$\text{For } -\frac{3}{2} < f \leq -\frac{1}{2}, \quad T(f) = \int_{-1}^{f+\frac{1}{2}} (v+1) dv = \left( \frac{1}{2}v^2 + v \right) \Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}$$

$$\begin{aligned}\text{For } -\frac{1}{2} < f \leq \frac{1}{2}, \quad T(f) &= \int_{f-\frac{1}{2}}^0 (v+1) dv + \int_0^{f+\frac{1}{2}} (-v+1) dv \\ &= \left( \frac{1}{2}v^2 + v \right) \Big|_{f-\frac{1}{2}}^0 + \left( -\frac{1}{2}v^2 + v \right) \Big|_0^{f+\frac{1}{2}} = -f^2 + \frac{3}{4}\end{aligned}$$

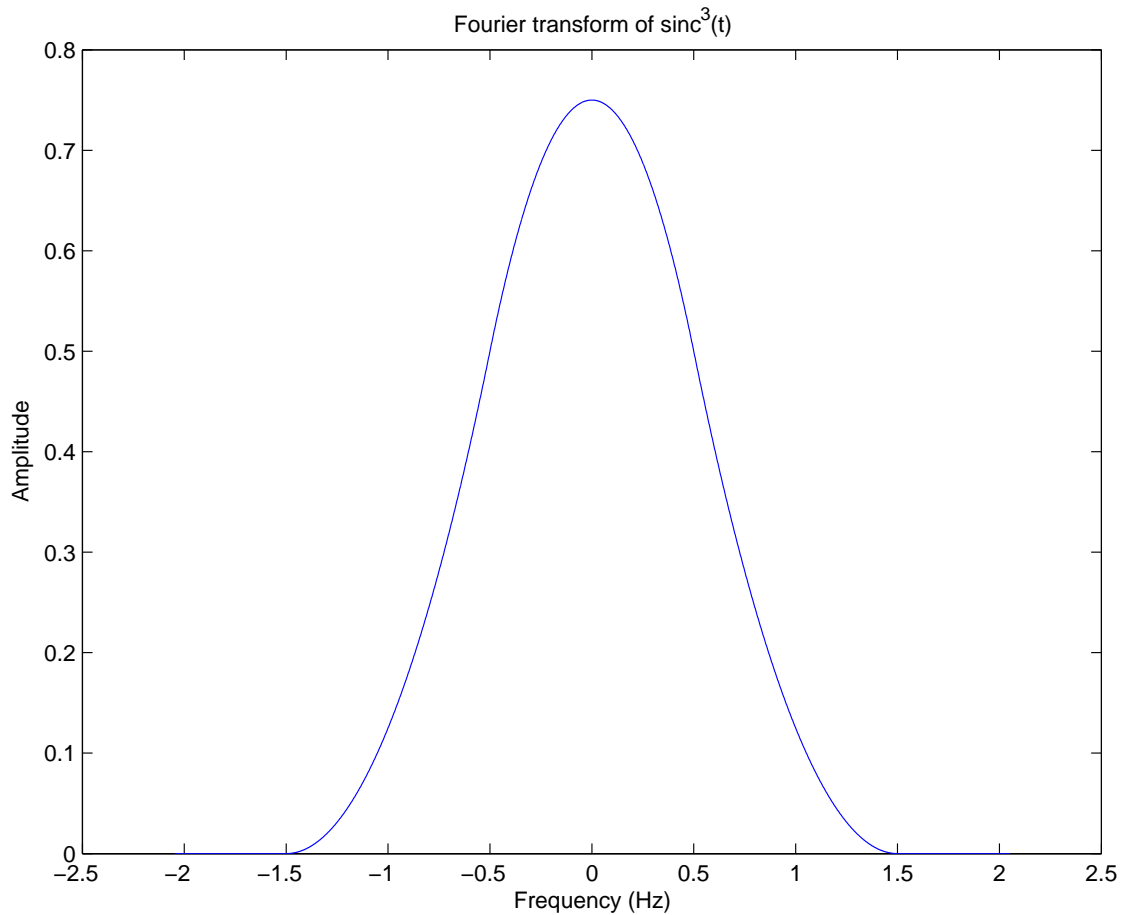
$$\text{For } \frac{1}{2} < f \leq \frac{3}{2}, \quad T(f) = \int_{f-\frac{1}{2}}^1 (-v+1) dv = \left( -\frac{1}{2}v^2 + v \right) \Big|_{f-\frac{1}{2}}^1 = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}$$

$$\text{For } \frac{3}{2} < f, \quad T(f) = 0$$

Thus,

$$T(f) = \mathcal{F}\{\text{sinc}^3(t)\} = \begin{cases} 0, & f \leq -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}, & -\frac{3}{2} < f \leq -\frac{1}{2} \\ -f^2 + \frac{3}{4}, & -\frac{1}{2} < f \leq \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}, & \frac{1}{2} < f \leq \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

A plot of  $T(f)$  is shown in the following figure, and was produced with Matlab script `proakis_salehi_2_10_4.m`, available in the web site of the class.



26. Matlab problems. These two problems needs the following three Matlab scripts: `homework2af04.m`, `rectpulse.m` and `homework2bf04.m`, available in the class web site.

- (a) The scripts `homework2af04.m` and `rectpulse.m` plot the amplitude spectrum of the Fourier transform  $X(f)$  of the signal

$$x(t) = \Pi\left(\frac{t}{\tau}\right).$$

Run the script `homework2af04.m`. You will be requested to enter the width  $\tau$  of the pulse. Use values of  $\tau$  equal to 0.1 and 0.2. Print or sketch the corresponding figures. Based on the scaling property, discuss the results.

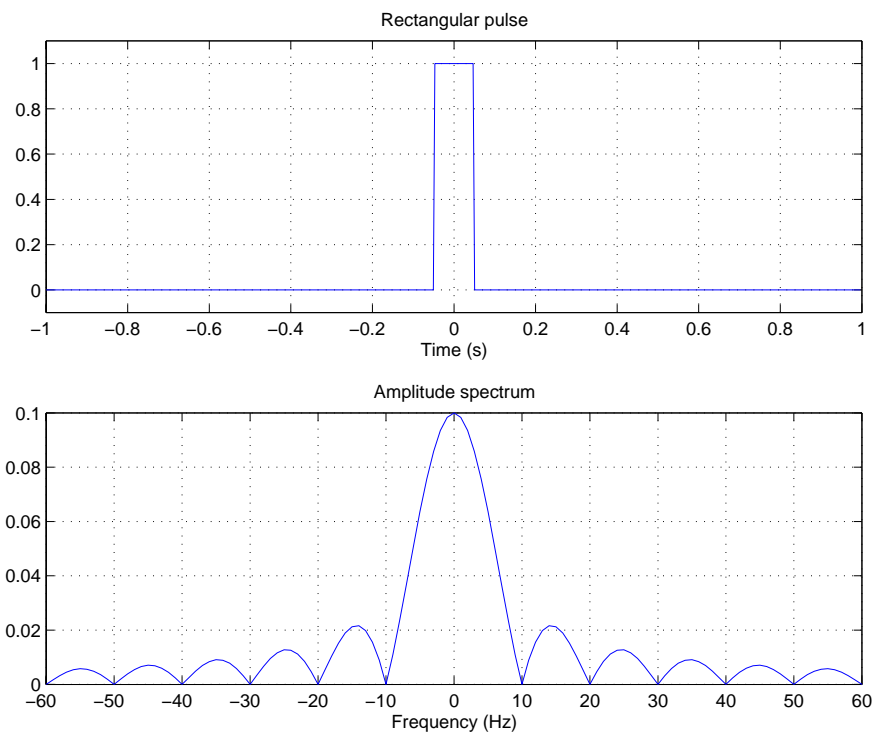
- (b) The scripts `homework2bf04.m` uses the *inverse fast Fourier transform* (IFFT) to compute numerically the signal associated with a spectrum consisting of pair of impulses:

$$X(f) = \frac{1}{2} \delta(f + Fc) + \frac{1}{2} \delta(f - Fc).$$

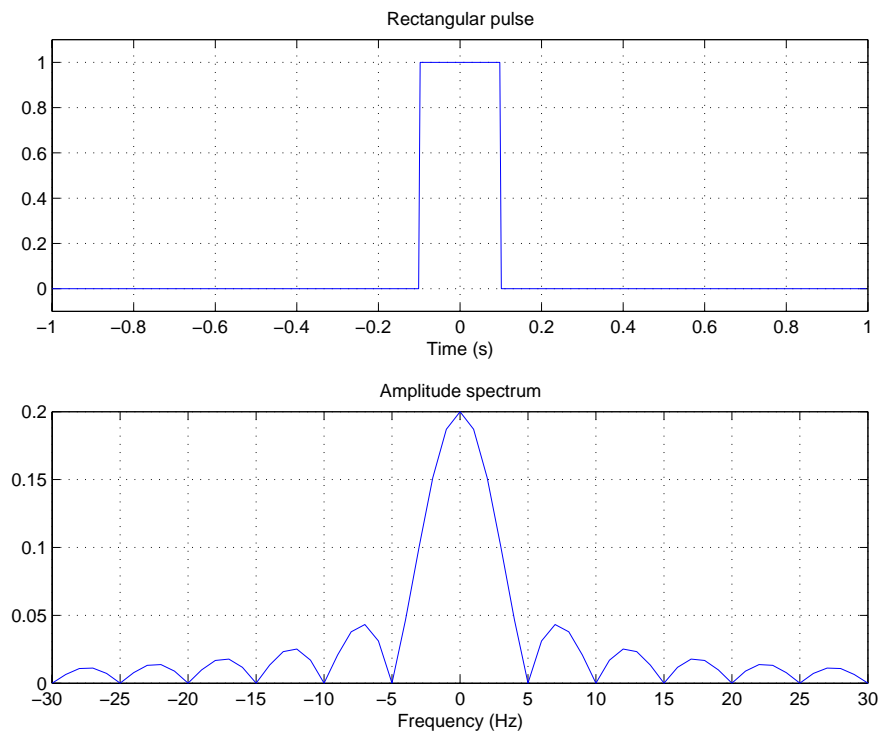
Run the script `homework2a.m` and print or sketch the corresponding figures.

**Solution:**

(a) Pulse width  $\tau = 0.1$ :



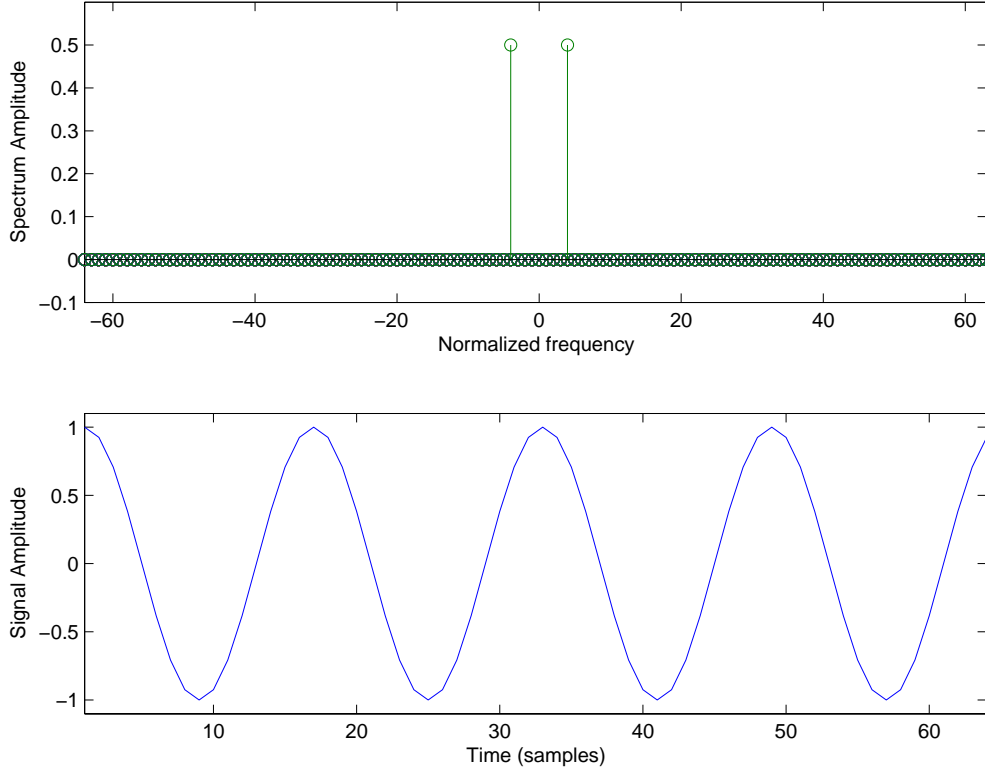
Pulse width  $\tau = 0.2$ :



The plots agree with the theoretical expression:

$$\mathcal{F} \left\{ \Pi \left( \frac{t}{\tau} \right) \right\} = \tau \operatorname{sinc}(\tau f).$$

(b)



27. Using the convolution theorem, show that

$$\operatorname{sinc}(\alpha t) \star \operatorname{sinc}(\beta t) = \frac{1}{\beta} \operatorname{sinc}(\alpha t), \quad \alpha \leq \beta.$$

**Solution:** Note that, for  $\alpha \leq \beta$ ,

$$\mathcal{F} \{ \operatorname{sinc}(\alpha t) \star \operatorname{sinc}(\beta t) \} = \frac{1}{\alpha} \Pi \left( \frac{f}{\alpha} \right) \cdot \frac{1}{\beta} \Pi \left( \frac{f}{\beta} \right) = \frac{1}{\beta} \left[ \frac{1}{\alpha} \Pi \left( \frac{f}{\alpha} \right) \right],$$

and

$$\mathcal{F}^{-1} \left\{ \frac{1}{\alpha} \Pi \left( \frac{f}{\alpha} \right) \right\} = \operatorname{sinc}(\alpha t),$$

As a result,

$$\operatorname{sinc}(\alpha t) \star \operatorname{sinc}(\beta t) = \frac{1}{\beta} \operatorname{sinc}(\alpha t).$$

28. Find the output  $y(t)$  of an LTI system with impulse response  $h(t) = e^{-\alpha t}u(t)$  when driven by the input  $x(t) = e^{-\beta t}u(t)$ . Treat the special case  $\alpha = \beta$  separately. Determine if  $y(t)$  is an energy signal or a power signal by finding the energy  $E$  or the power  $P$ .

**Solution:** Using the convolution theorem we obtain

$$\begin{aligned} Y(f) &= X(f)H(f) = \left(\frac{1}{\alpha + j2\pi f}\right)\left(\frac{1}{\beta + j2\pi f}\right) \\ &= \frac{1}{(\beta - \alpha)} \frac{1}{\alpha + j2\pi f} - \frac{1}{(\beta - \alpha)} \frac{1}{\beta + j2\pi f} \end{aligned}$$

Thus

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \frac{1}{(\beta - \alpha)}[e^{-\alpha t} - e^{-\beta t}]u_{-1}(t).$$

If  $\alpha = \beta$  then  $X(f) = H(f) = \frac{1}{\alpha + j2\pi f}$ . In this case

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \mathcal{F}^{-1}\left[\left(\frac{1}{\alpha + j2\pi f}\right)^2\right] = te^{-\alpha t}u_{-1}(t)$$

The signal is of the energy type with energy

$$\begin{aligned} E_y &= \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |y(t)|^2 dt = \lim_{T \rightarrow \infty} \int_0^{\frac{T}{2}} \frac{1}{(\beta - \alpha)^2} (e^{-\alpha t} - e^{-\beta t})^2 dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{(\beta - \alpha)^2} \left[ -\frac{1}{2\alpha} e^{-2\alpha t} \Big|_0^{T/2} - \frac{1}{2\beta} e^{-2\beta t} \Big|_0^{T/2} + \frac{2}{(\alpha + \beta)} e^{-(\alpha + \beta)t} \Big|_0^{T/2} \right] \\ &= \frac{1}{(\beta - \alpha)^2} \left[ \frac{1}{2\alpha} + \frac{1}{2\beta} - \frac{2}{\alpha + \beta} \right] = \frac{1}{2\alpha\beta(\alpha + \beta)} \end{aligned}$$

29. Can the response of an LTI system to the input  $x(t) = \text{sinc}(t)$  be  $y(t) = \text{sinc}^2(t)$ ? Justify your answer.

**Solution:** The answer is no. Let the response of the LTI system be  $h(t)$  with Fourier transform  $H(f)$ . Then, from the convolution theorem we obtain

$$Y(f) = H(f)X(f) \implies \Lambda(f) = \Pi(f)H(f)$$

This is impossible since  $\Pi(f) = 0$  for  $|f| > \frac{1}{2}$  whereas  $\Lambda(f) \neq 0$  for  $\frac{1}{2} < |f| \leq 1$ .

30. Consider the periodic signals

$$\begin{aligned} \text{(a)} \quad x_1(t) &= \sum_{n=-\infty}^{\infty} \Lambda(t - 2n) \\ \text{(b)} \quad x_2(t) &= \sum_{n=-\infty}^{\infty} \Lambda(t - n) \end{aligned}$$

Find the Fourier series coefficients without any integrals, by using a table of Fourier transforms (such as Table 2.1 in the textbook) and the relation

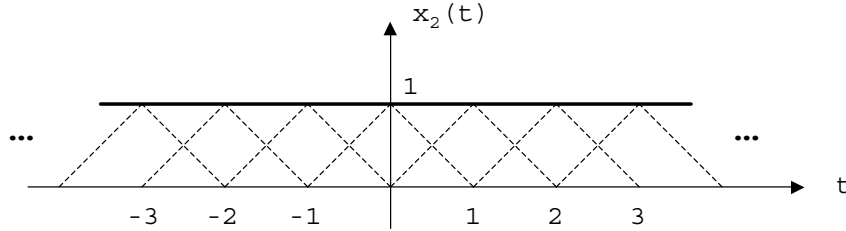
$$x_n = \frac{1}{T_0} X_{T_0} \left( \frac{n}{T_0} \right).$$

**Solution:**

(1)  $X_{T_0}(f) = \text{sinc}^2(f)$ , and  $T_0 = 2$ . Therefore,

$$x_n = \frac{1}{T_0} \text{sinc}^2\left(\frac{n}{T_0}\right) = \frac{1}{2} \text{sinc}^2\left(\frac{n}{2}\right).$$

(2) Note that  $x_2(t) = 1$ , as shown in the figure below:



It follows that  $X_2(f) = \delta(f)$ . The signal can also be considered as periodic with period  $T_0 = 1$  and therefore  $x_n = \delta(n)$ . In other words,  $x_0 = 1$  and  $x_n = 0, \forall n \neq 0$ .

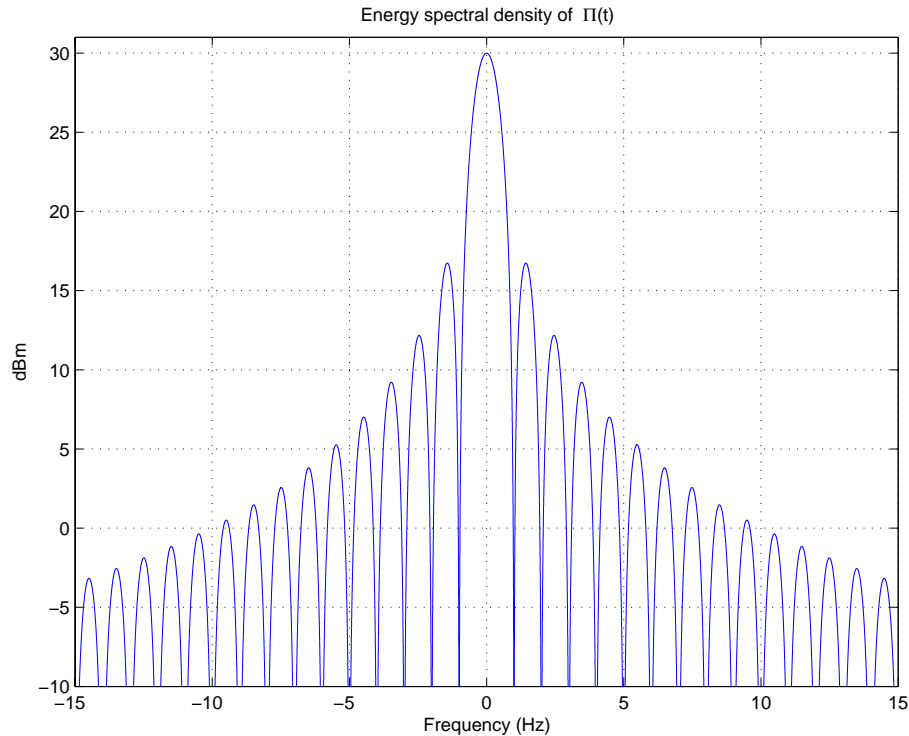
### 31. MATLAB problem.

Download and execute the Matlab script `homework3f04.m` from the web site of the class. The script finds the 50% (or 3-dB) energy bandwidth,  $B_{3\text{-dB}}$ , and the 95% energy bandwidth,  $B_{95}$ , of a rectangular pulse

$$x(t) = \Pi(t),$$

from its energy spectral density,  $\mathcal{G}(f) = \text{sinc}^2(f)$ . Give the values of  $B_{3\text{-dB}}$  and  $B_{95}$ , and print or sketch  $\mathcal{G}(f)$  in dBm, where dBm is with reference to  $10^{-3}$  Joule/Hz.

**Solution:**  $B_{3\text{-dB}} = 0.268311$  Hz and  $B_{95} = 1.668457$  Hz.



### 32. MATLAB problem

Based on the script `homework3f04.m` of the previous problem, write a Matlab script to find numerically the energy  $E_1$  contained in the first “lobe” of the energy spectral density, that is,

$$E_1 = \int_{-1}^1 \mathcal{G}(f) df,$$

#### Solution:

$E_1 = 0.902823$  Joules. This was produced by the following script:

```
% Name: homework3_2.m
% For the EE160 students of San Jose State University in Fall 2004
N = 4096;
f = -1:1/N:1;
G = sinc(f).^2;
E = sum(G)/N;
fprintf('The energy in the main lobe of G(g) is %8.6f Joules\n', E);
```

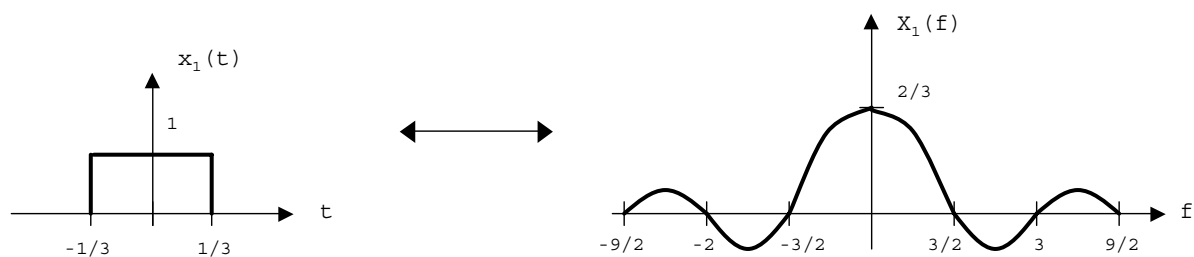
### 33. Sketch carefully the following signals and their Fourier transform

(a)  $x_1(t) = \Pi\left(\frac{3t}{2}\right)$ .

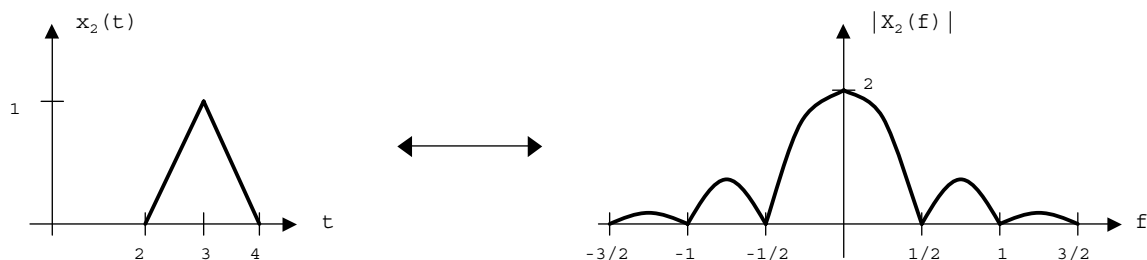
(b)  $x_2(t) = \Lambda\left(\frac{1}{2}(t-3)\right)$ .

#### Solution:

(a)  $X_1(f) = \frac{2}{3} \text{sinc}\left(\frac{2}{3}f\right)$ .



(b)  $X_2(f) = 2 \text{sinc}^2(2f) e^{-j6\pi f}$ .

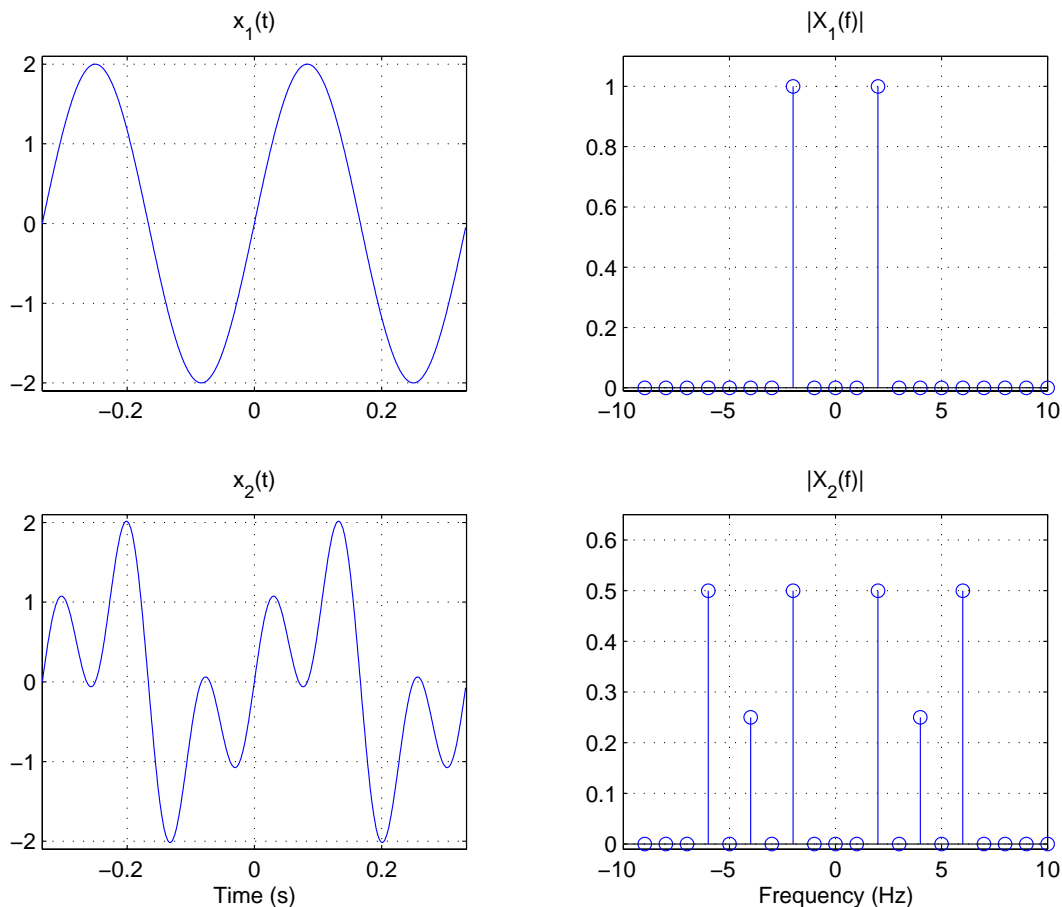




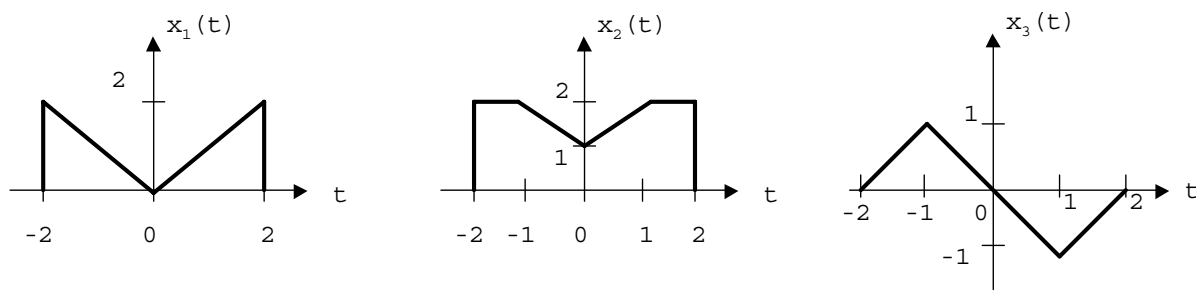
34. MATLAB problem.

Download and execute the Matlab script `homework4f04.m` from the web site of the class. The script illustrates two signals in the time domain and their corresponding Fourier transforms. This serves to verify that the time variation is proportional to the bandwidth. Sketch or print the plots.

**Solution:**



35. Determine the Fourier transform of the signals shown below.



**Solution:**

(a) Write  $x_1(t) = 2 \Pi(\frac{t}{4}) - 2 \Lambda(\frac{t}{2})$ . Then

$$X_1(f) = \mathcal{F} \left[ 2 \Pi \left( \frac{t}{4} \right) \right] - \mathcal{F} \left[ 2 \Lambda \left( \frac{t}{2} \right) \right] = 8 \operatorname{sinc}(4f) - 4 \operatorname{sinc}^2(2f)$$

(b) Write  $x_2(t) = 2 \Pi(\frac{t}{4}) - \Lambda(t)$ . Then

$$X_2(f) = 8 \operatorname{sinc}(4f) - \operatorname{sinc}^2(f)$$

(d) Note that  $x_3(t) = \Lambda(t+1) - \Lambda(t-1)$ . Then

$$X_3(f) = \operatorname{sinc}^2(f)e^{j2\pi f} - \operatorname{sinc}^2(f)e^{-j2\pi f} = 2j \operatorname{sinc}^2(f) \sin(2\pi f)$$

36. Use the convolution theorem to show that

$$\operatorname{sinc}(t) \star \operatorname{sinc}(t) = \operatorname{sinc}(t)$$

**Solution:**

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)] \cdot \mathcal{F}[y(t)] = X(f) \cdot Y(f)$$

Thus

$$\begin{aligned} \operatorname{sinc}(t) \star \operatorname{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\operatorname{sinc}(t) \star \operatorname{sinc}(t)]] \\ &= \mathcal{F}^{-1}[\mathcal{F}[\operatorname{sinc}(t)] \cdot \mathcal{F}[\operatorname{sinc}(t)]] \\ &= \mathcal{F}^{-1}[\Pi(f) \cdot \Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\ &= \operatorname{sinc}(t) \end{aligned}$$

37. Using the Fourier transform, evaluate the following integrals:

(a)  $\int_0^\infty e^{-\alpha t} \operatorname{sinc}(t) dt$

(b)  $\int_0^\infty e^{-\alpha t} \operatorname{sinc}^2(t) dt$

(c)  $\int_0^\infty e^{-\alpha t} \cos(\beta t) dt$

**Solution:**

(a)

$$\begin{aligned} \int_0^\infty e^{-\alpha t} \operatorname{sinc}(t) dt &= \int_{-\infty}^\infty e^{-\alpha t} u_{-1}(t) \operatorname{sinc}(t) dt \\ &= \int_{-\infty}^\infty \frac{1}{\alpha + j2\pi f} \Pi(f) df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\alpha + j2\pi f} df \\ &= \frac{1}{j2\pi} \ln(\alpha + j2\pi f) \Big|_{-1/2}^{1/2} = \frac{1}{j2\pi} \ln\left(\frac{\alpha + j\pi}{\alpha - j\pi}\right) = \frac{1}{\pi} \tan^{-1} \frac{\pi}{\alpha} \end{aligned}$$

(b)

$$\begin{aligned}
\int_0^\infty e^{-\alpha t} \text{sinc}^2(t) dt &= \int_{-\infty}^\infty e^{-\alpha t} u_{-1}(t) \text{sinc}^2(t) dt \\
&= \int_{-\infty}^\infty \frac{1}{\alpha + j2\pi f} \Lambda(f) df \\
&= \int_{-1}^0 \frac{f+1}{\alpha + j\pi f} df + \int_0^1 \frac{-f+1}{\alpha + j\pi f} df
\end{aligned}$$

But  $\int \frac{x}{a+bx} dx = \frac{x}{b} - \frac{a}{b^2} \ln(a+bx)$  so that

$$\begin{aligned}
\int_0^\infty e^{-\alpha t} \text{sinc}^2(t) dt &= \left( \frac{f}{j2\pi} + \frac{\alpha}{4\pi^2} \ln(\alpha + j2\pi f) \right) \Big|_{-1}^0 \\
&\quad - \left( \frac{f}{j2\pi} + \frac{\alpha}{4\pi^2} \ln(\alpha + j2\pi f) \right) \Big|_0^1 + \frac{1}{j2\pi} \ln(\alpha + j2\pi f) \Big|_{-1}^1 \\
&= \frac{1}{\pi} \tan^{-1}\left(\frac{2\pi}{\alpha}\right) + \frac{\alpha}{2\pi^2} \ln\left(\frac{\alpha}{\sqrt{\alpha^2 + 4\pi^2}}\right)
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^\infty e^{-\alpha t} \cos(\beta t) dt &= \int_{-\infty}^\infty e^{-\alpha t} u_{-1}(t) \cos(\beta t) dt \\
&= \frac{1}{2} \int_{-\infty}^\infty \frac{1}{\alpha + j2\pi f} \left( \delta\left(f - \frac{\beta}{2\pi}\right) + \delta\left(f + \frac{\beta}{2\pi}\right) \right) df \\
&= \frac{1}{2} \left[ \frac{1}{\alpha + j\beta} + \frac{1}{\alpha - j\beta} \right] = \frac{\alpha}{\alpha^2 + \beta^2}
\end{aligned}$$

## Sampling of lowpass signals

38. The signal  $x(t) = A \text{sinc}(1000t)$  be sampled with a sampling frequency of 2000 samples per second. Determine the most general class of reconstruction filters for the perfect reconstruction of  $x(t)$  from its samples.

**Solution:**

$$x(t) = A \text{sinc}(1000\pi t) \iff X(f) = \frac{A}{1000} \Pi\left(\frac{f}{1000}\right)$$

Thus the bandwidth  $W$  of  $x(t)$  is  $1000/2 = 500$ . Since we sample at  $f_s = 2000$  there is a gap between the image spectra equal to

$$2000 - 500 - W = 1000$$

The reconstruction filter should have a bandwidth  $W'$  such that  $500 < W' < 1500$ . A filter that satisfy these conditions is

$$H(f) = T_s \Pi\left(\frac{f}{2W'}\right) = \frac{1}{2000} \Pi\left(\frac{f}{2W'}\right)$$

and the more general reconstruction filters have the form

$$H(f) = \begin{cases} \frac{1}{2000} & |f| < 500 \\ \text{arbitrary} & 500 < |f| < 1500 \\ 0 & |f| > 1500 \end{cases}$$

39. The lowpass signal  $x(t)$  with a bandwidth of  $W$  is sampled at intervals of  $T_s$  seconds, and the signal

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)p(t - nT_s)$$

is generated, where  $p(t)$  is an arbitrary pulse (not necessarily limited to the interval  $[0, T_s]$ ).

- Find the Fourier transform of  $x_p(t)$ .
- Find the conditions for perfect reconstruction of  $x(t)$  from  $x_p(t)$ .
- Determine the required reconstruction filter.

**Solution:**

- 

$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)p(t - nT_s) \\ &= p(t) \star \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \\ &= p(t) \star x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \end{aligned}$$

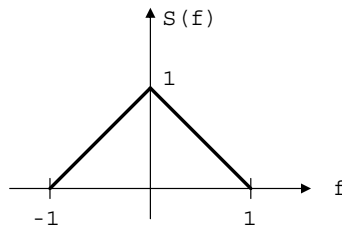
Thus

$$\begin{aligned} X_p(f) &= P(f) \cdot \mathcal{F} \left[ x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] \\ &= P(f)X(f) \star \mathcal{F} \left[ \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] \\ &= P(f)X(f) \star \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}) \\ &= \frac{1}{T_s} P(f) \sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s}) \end{aligned}$$

- In order to avoid aliasing  $\frac{1}{T_s} > 2W$ . Furthermore the spectrum  $P(f)$  should be invertible for  $|f| < W$ .
- $X(f)$  can be recovered using the reconstruction filter  $\Pi(\frac{f}{2W_{\Pi}})$  with  $W < W_{\Pi} < \frac{1}{T_s} - W$ . In this case

$$X(f) = X_p(f)T_s P^{-1}(f)\Pi(\frac{f}{2W_{\Pi}})$$

40. Consider a signal  $s(t)$  whose Fourier transform is given below:



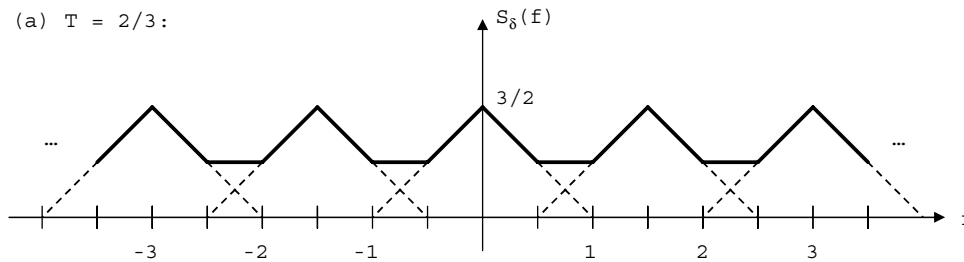
Sketch carefully the Fourier transform  $S_\delta(f)$  of the sampled signal

$$\begin{aligned} s_\delta(t) &= s(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} s(nT) \delta(t - nT) \end{aligned}$$

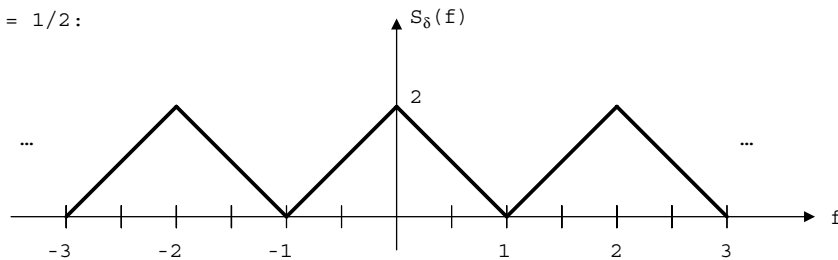
for (a)  $T = 2/3$  and (b)  $T = 1/2$ . For each case, only if possible, specify a filter characteristic that allows a complete reconstruction of  $s(t)$  from  $s_\delta(t)$ .

**Solution:**

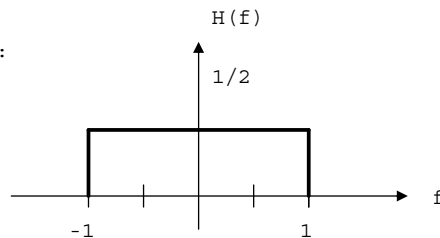
(a)  $T = 2/3$ :



(b)  $T = 1/2$ :



Reconstruction filter:  
(only for  $T=1/2$ )

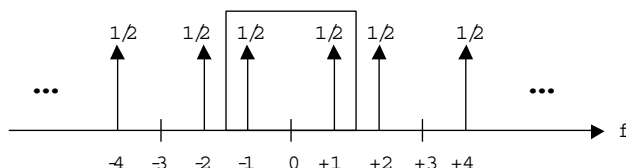


41. A sinusoidal signal of frequency 1 Hz is to be sampled periodically.

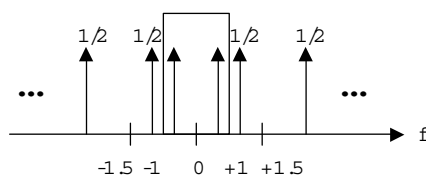
- Find the maximum allowable time interval between samples.
- Samples are taken at  $1/3$  second intervals. Show graphically, to your satisfaction, that no other sine waveform with bandwidth less than 1.5 Hz can be represented by these samples.
- Samples are taken at  $2/3$  second intervals. Show graphically these samples may represent another sine waveform of frequency less than 1.5 Hz.

**Solution:**

- (a)  $T_s = 1/2$ , i.e., the inverse of the Nyquist rate which in this case is 2 Hz.  
(b) At  $f_s = 3$  Hz, the sampled signal spectrum consists of nonoverlapping copies of the signal:

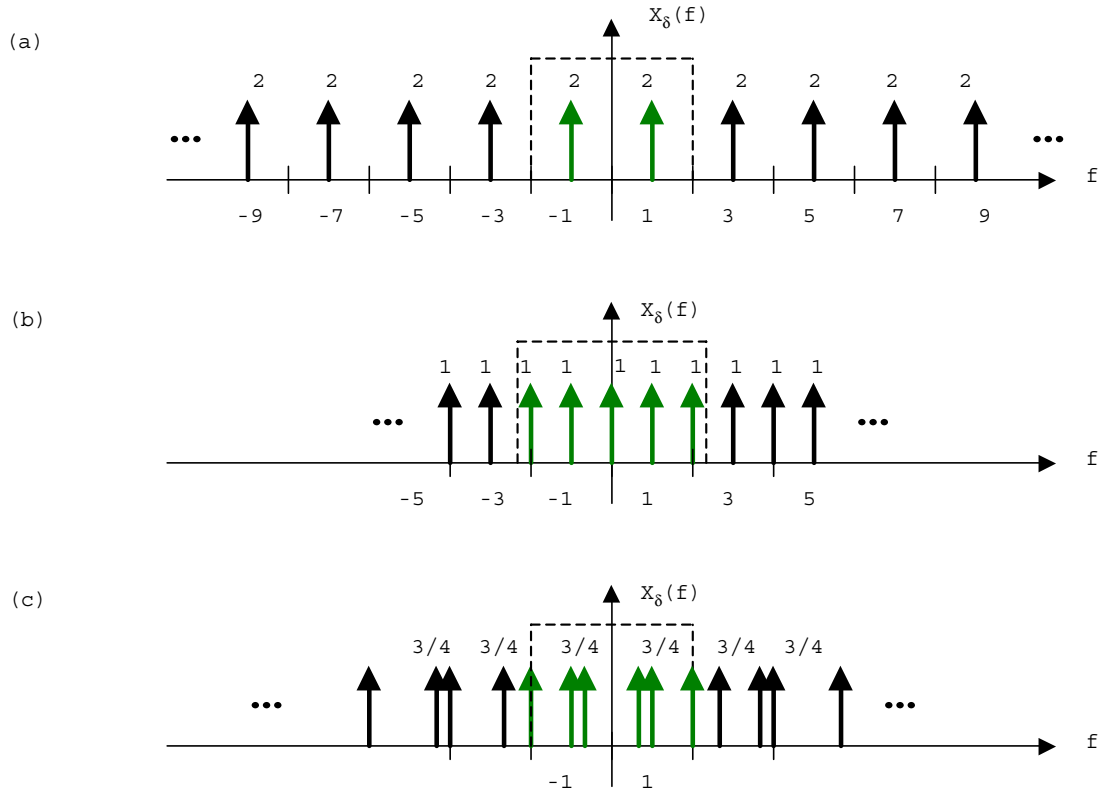
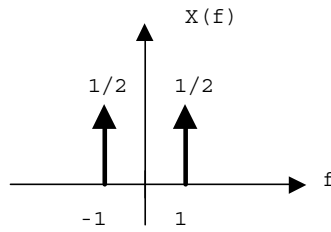


- (c) In this case,  $f_s = 3/2$  Hz which is less than the Nyquist rate. The copies of the signal spectrum overlap and the samples can be those of a signal with lower frequency.



42. The signal  $x(t) = \cos(2\pi t)$  is ideally sampled with a train of impulses. Sketch the spectrum  $X_\delta(f)$  of the sampled signal, and find the reconstructed signal  $\hat{x}(t)$ , for the following values of sampling period  $T_s$  and ideal lowpass reconstruction filter bandwidth  $W'$ :
- (a)  $T_s = 1/4$ ,  $W' = 2$ .  
(b)  $T_s = 1$ ,  $W' = 5/2$ .  
(c)  $T_s = 2/3$ ,  $W' = 2$ .

**Solution:** The spectra of the signal  $X(f)$ , and that of the sampled signal  $X_\delta(f)$ , for each case are shown in the figure below:



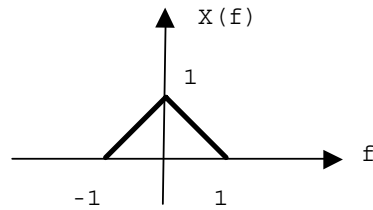
The reconstructed signals for each value of sampling period  $T_s$  and ideal lowpass reconstruction filter bandwidth  $W'$  are:

- (a)  $\hat{x}(t) = \cos(2\pi t)$ .
- (b)  $\hat{x}(t) = 1 + 2 \cos(2\pi t) + 2 \cos(4\pi t)$ .
- (c)  $\hat{x}(t) = \cos(\pi t) + \cos(2\pi t) + \cos(4\pi t)$ .

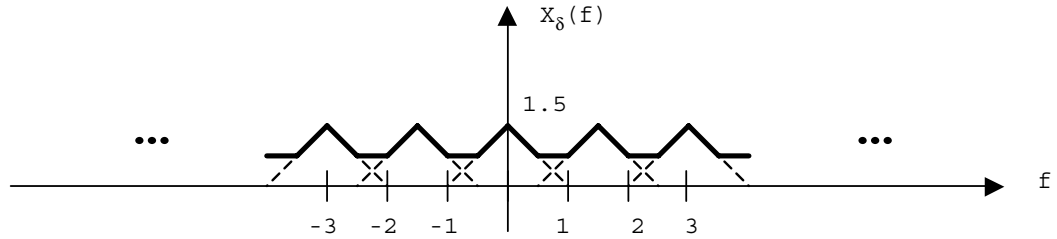
43. The signal  $x(t) = \text{sinc}^2(t)$  is ideally sampled with a train of impulses. Sketch the spectrum  $X_\delta(f)$  of the sampled signal, for the sampling periods below. For those values of  $T_s$  for which reconstruction is possible, specify the range of the cutoff frequency  $W'$  of the ideal reconstruction filter.

- (a)  $T_s = 2/3$ .
- (b)  $T_s = 1$ .
- (c)  $T_s = 1/4$ .

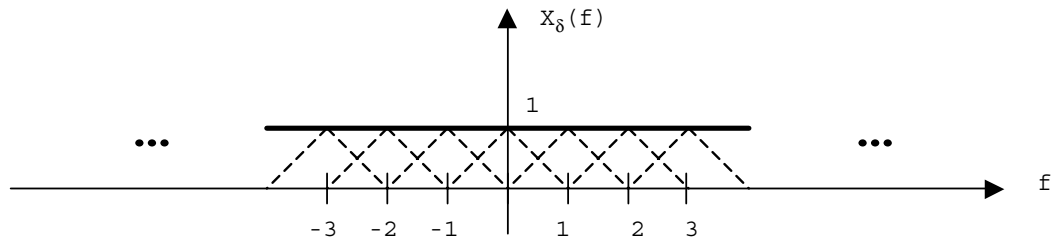
**Solution:** The spectra of the signal  $X(f)$ , and that of the sampled signal  $X_\delta(f)$ , for each case are shown in the figure below:



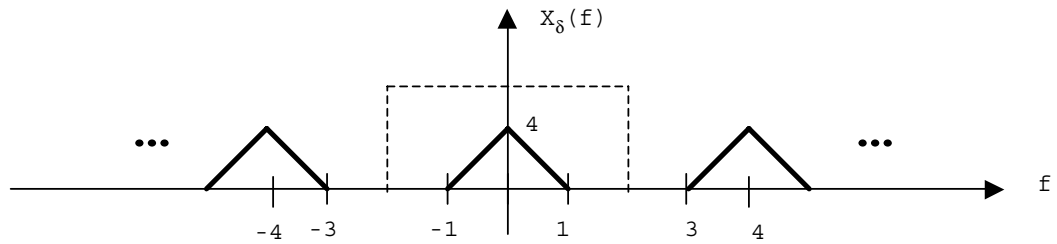
(a)



(b)

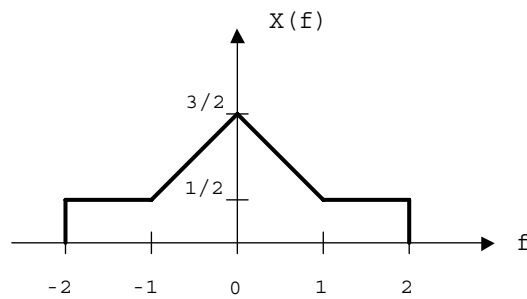


(c)



The value of  $T_s$  for which reconstruction is possible, is (c)  $T_s = 1/4$ . The range of the cutoff frequency  $W'$  of the ideal reconstruction filter is  $1 < W' < 3$ .

44. A lowpass signal has spectrum as shown below.



This signal is sampled at  $f_s$  samples/second with impulses and reconstructed using an ideal lowpass filter (LPF) of bandwidth  $W = 2$  and amplitude  $1/f_s$ . Let  $\tilde{x}(t)$  denote the output of

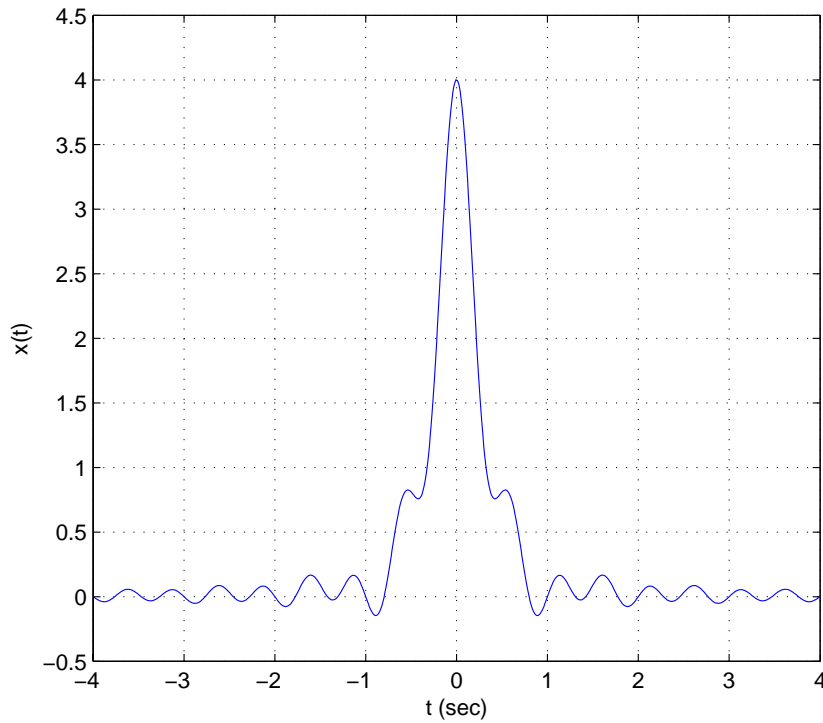


the LPF.

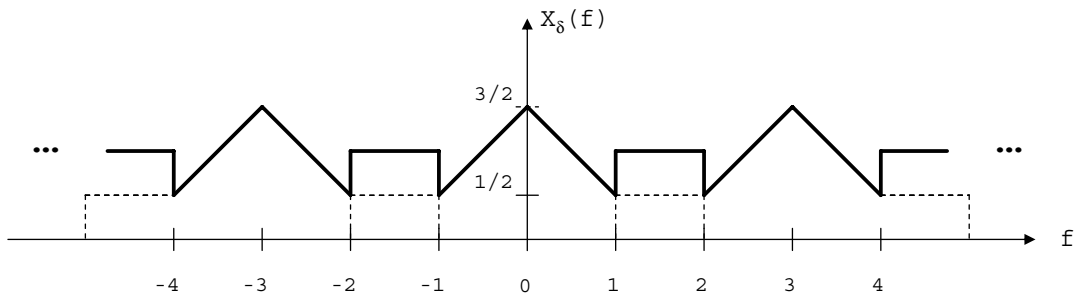
- Give an expression for  $x(t)$  and sketch the waveform.
- The sampling frequency is  $f_s = 3$ . Sketch the spectra of the sampled signal,  $X_\delta(f)$  and that of the recovered signal  $\tilde{X}(f)$ . Also, sketch the reconstructed waveform  $\tilde{x}(t)$ .
- Repeat part (b) with  $f_s = 4$ .

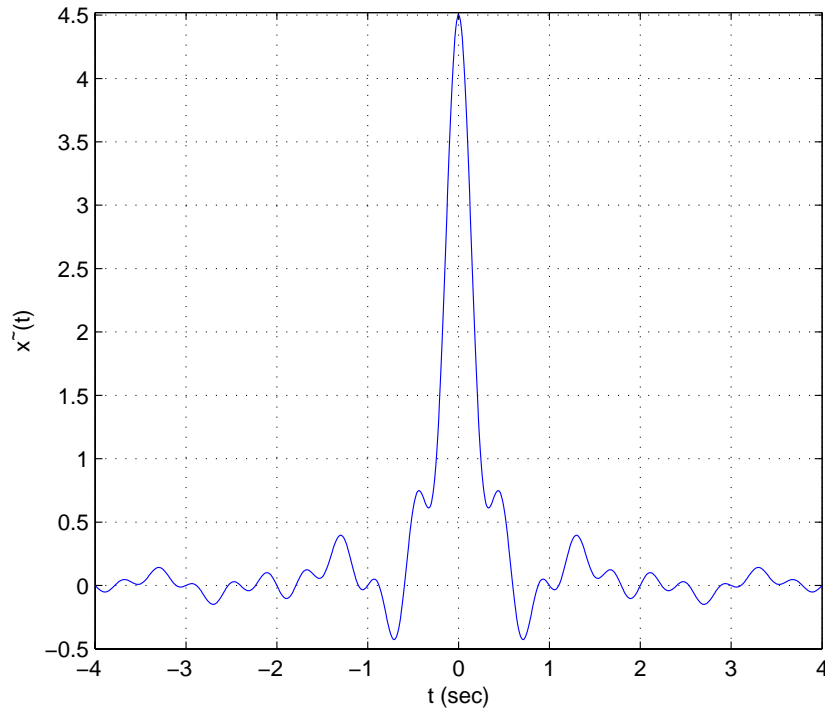
**Solution:**

- $x(t) = 2 \operatorname{sinc}(4t) + \operatorname{sinc}^2(t)$ .

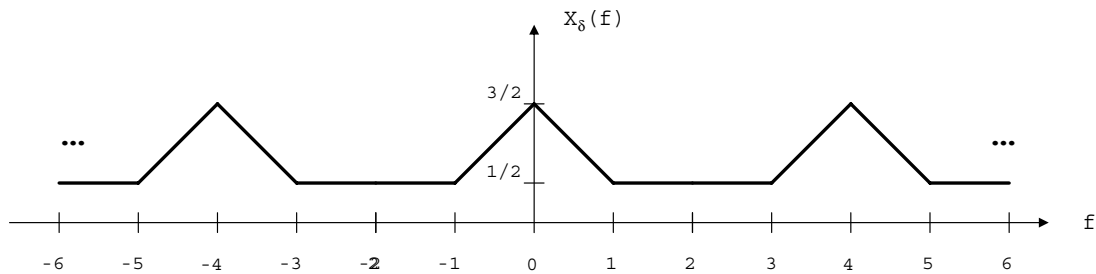


- $\tilde{x}(t) = \operatorname{sinc}(2t) + \operatorname{sinc}^2(t) + 2 \operatorname{sinc}(t) \cos(3\pi t)$ .





- (c) As shown by the sampled spectrum below, there is no overlap between the shifted copies of  $X(f)$ . Therefore,  $\tilde{x}(t) = x(t)$ .



45. A compact disc (CD) records audio signals digitally using PCM. Assume the audio signal bandwidth to be 15 KHz.

- What is the Nyquist rate?
- If the Nyquist samples are quantized to  $L = 65,536$  levels and then binary coded, determine the number of bits required to encode a sample.
- Assuming that the signal is sinusoidal and that the maximum signal amplitude is 1 volt, determine the quantization step and the signal-to-quantization noise ratio.
- Determine the number of bits per second (bit/s) required to encode the audio signal.
- For practical reasons, signals are sampled at above the Nyquist rate, as discussed in class. Practical CDs use 44,000 samples per second. For  $L = 65,536$  determine the number of bits per second required to encode the signal and the minimum bandwidth required to transmit the encoded signal.

**Solution:**

(a)  $f_s = 30000$  samples/s

(b)  $\log_2(L) = 16$  bits

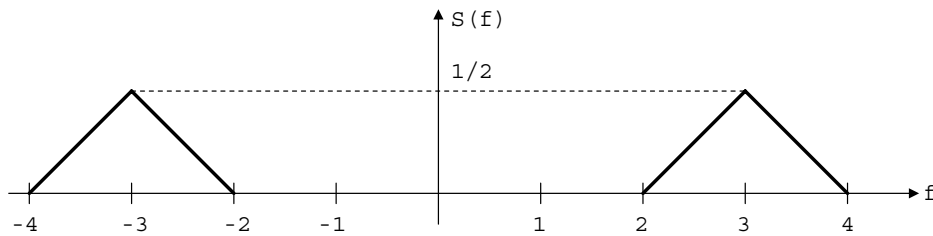
(c)  $\Delta = 2/2^{16} = 2^{-15}$  volts

(d)  $16f_s = 480000$  bits/s

(e)  $16 \times 44000 = 704000$  bits/s and  $B_T = 704000/2 = 352000$  Hz.

## Bandpass signals

46. Consider a signal  $s(t)$  whose Fourier transform is given below:



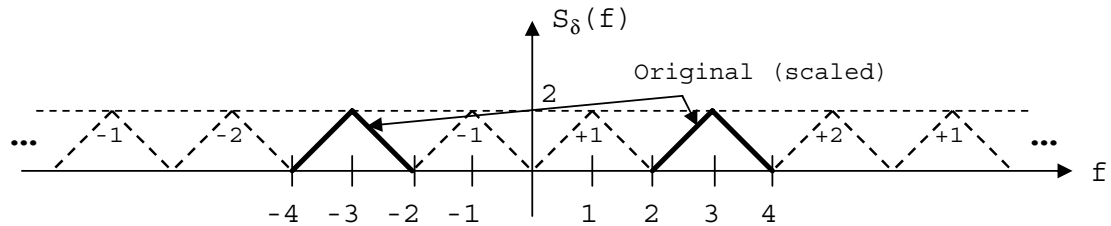
Sketch carefully the Fourier transform  $S_\delta(f)$  of the sampled signal

$$\begin{aligned} s_\delta(t) &= s(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} s(nT) \delta(t - nT) \end{aligned}$$

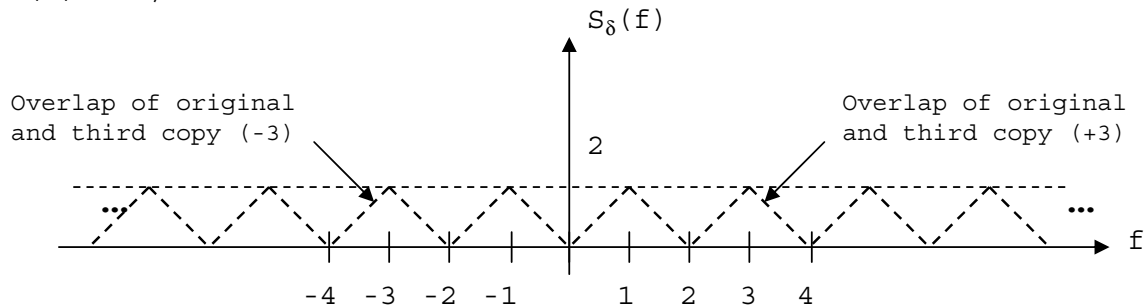
for (a)  $T = 1/4$  and (b)  $T = 1/2$ . For each case, only if possible, specify a filter characteristic that allows a complete reconstruction of  $s(t)$  from  $s_\delta(t)$ .

**Solution:**

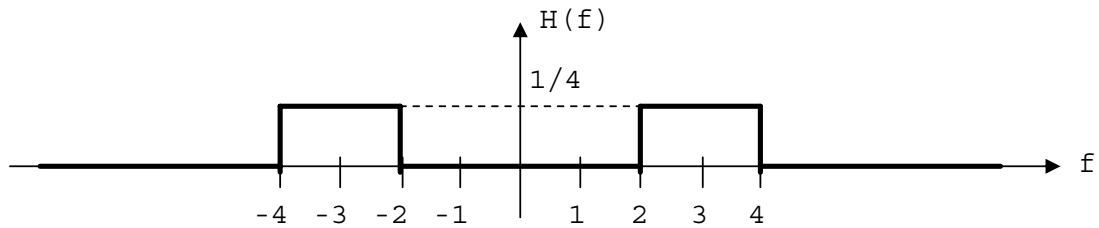
(a)  $T=1/4$ :



(b)  $T=1/2$ :



Reconstruction filter:



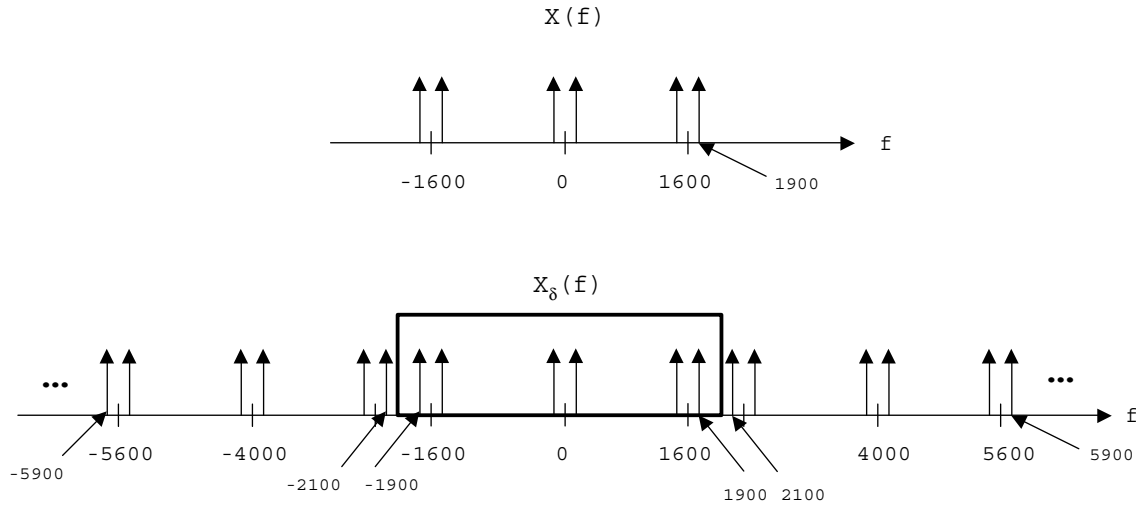
47. Determine the range of permissible cutoff frequencies for an ideal low pass filter used to reconstruct the signal

$$x(t) = 10 \cos(600\pi t) \cos^2(1600\pi t),$$

which is sampled at 4000 samples per second. Sketch  $X(f)$  and  $X_\delta(f)$ . Find the minimum allowable sampling frequency.

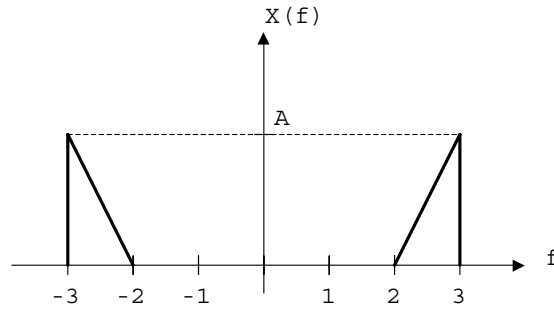
**Solution:**

The cutoff frequency of the reconstruction filter can be in the range between  $W = 1900$  Hz and  $f_s - W = 2100$  Hz.



The minimum (Nyquist) sampling frequency is  $2W = 3800$  Hz.

48. Given the bandpass signal spectrum shown in the figure below, sketch the spectra for the following sampling rates  $f_s$  and indicate which ones are suitable for the reconstruction of the signal from its samples: (a)  $2B$  (b)  $2.5B$  (c)  $3B$  (d)  $4B$  (e)  $5B$  (f)  $6B$ .



### Solution:

For bandpass sampling and recovery, all but (b) and (e) will work theoretically, although an ideal filter with bandwidth exactly equal to the unsampled signal bandwidth is necessary. For lowpass sampling and recovery, only (f) will work.

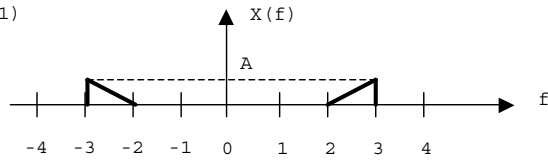
In the figure next page, the case (a)  $f_s = 2B$ , with  $B = 1$  for convenience, is illustrated. The terms used are  $n = 0, \pm 1, \pm 2, \pm 3$  in the expression of the sampled spectrum:

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - n f_s) = 2 \sum_{n=-\infty}^{\infty} X(f - 2n).$$

Spectra for higher values of  $n$  do not overlap with the spectrum of the original signal and are therefore not shown.

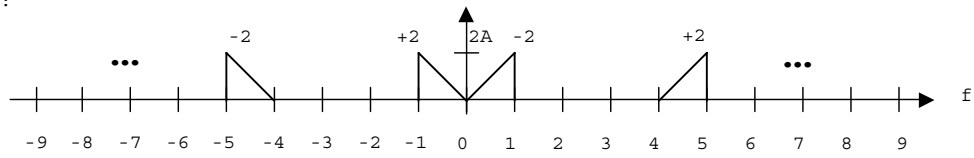
Case  $f_s = 2B$ :

( $B=1$ )



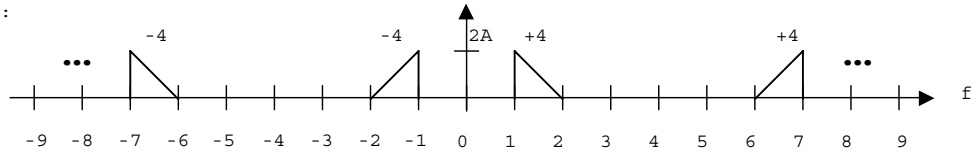
$n=0$

$X(f-2), X(f+2)$ :



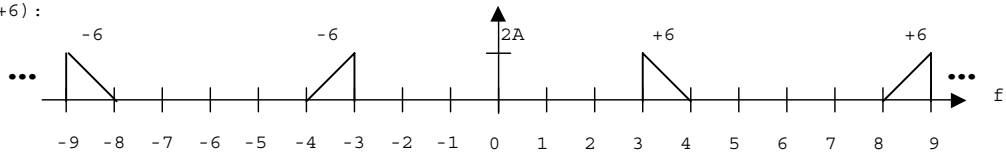
$n=1, -1$

$X(f-4), X(f+4)$ :

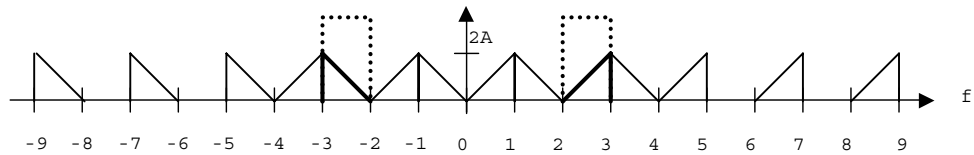


$n=2, -2$

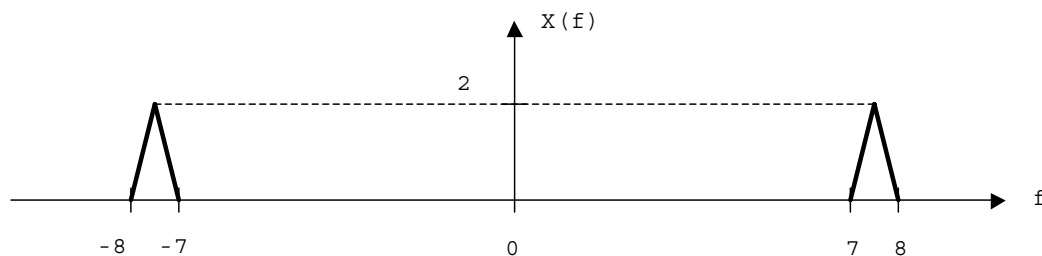
$X(f-6), X(f+6)$ :



$n=3, -3$



49. (Downconversion by bandpass sampling) Consider the bandpass signal  $x(t)$  whose spectrum is shown below.



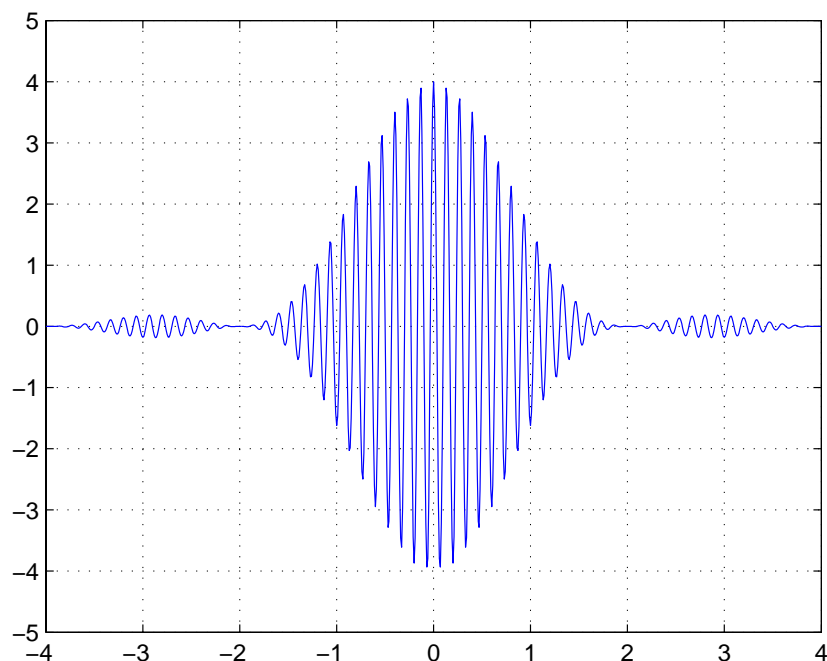
This signal is sampled at  $f_s = 4$  samples/second with ideal impulses.

- Sketch the signal  $x(t)$ .
- Sketch the spectrum  $X_\delta(f)$  of the sampled signal.
- Sketch the output  $\tilde{x}(t)$  of an ideal bandpass filter (BPF) of bandwidth  $B = 1$  and amplitude  $1/f_s$ , centered at  $f_0 = 7.5$ .
- The sampled signal is now passed through an ideal *lowpass* filter (LPF) of bandwidth  $W = 1$  and amplitude  $1/f_s$ . Show that the output  $\tilde{x}_d(t)$  is also a bandpass signal, equivalent to the original signal, but with a lower value of center frequency  $f'_0$ . Find the value of  $f'_0$  and sketch the signal  $\tilde{x}_d(t)$ .

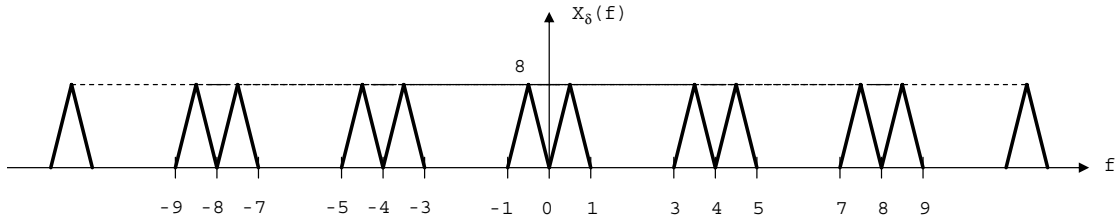
**Solution:**

- (a) The signal is  $x(t) = 4 \operatorname{sinc}^2\left(\frac{t}{2}\right) \cos(15\pi t)$ , which is plotted below via the Matlab script:

```
t=-4:0.01:4; plot(t,4*sinc(t./2).^2.*cos(15*pi.*t))
```

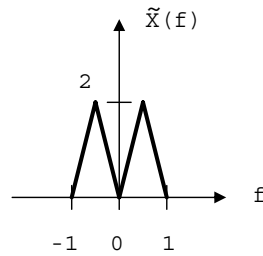


(b)



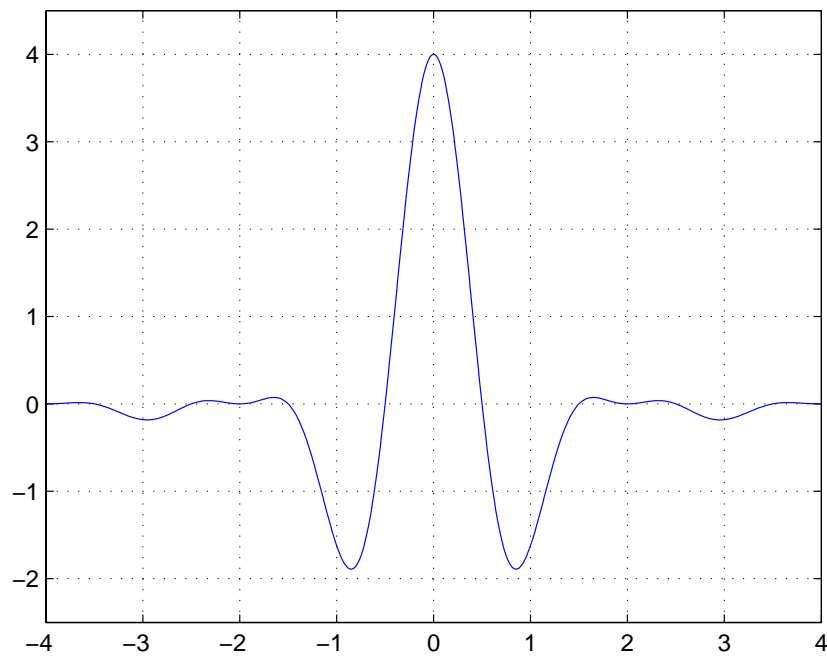
(c) There is no overlap between copies of  $X(f)$  and therefore  $\tilde{x}(t) = x(t)$ .

(d) The spectrum at the output of the LPF is:



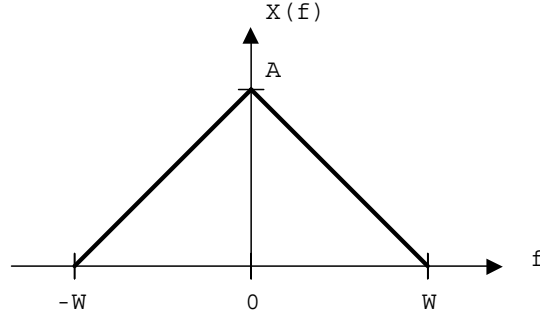
Therefore, the output signal is given by  $x(t) = 4 \operatorname{sinc}^2\left(\frac{t}{2}\right) \cos(\pi t)$ , which is plotted below via the Matlab script:

```
t=-4:0.01:4; plot(t,4*sinc(t./2).^2.*cos(pi.*t))
```





50. Assume that the Fourier transform of a signal  $x(t)$  is real and has the shape shown in the figure below:



Determine and plot the spectrum of each of the following signals, where  $\hat{x}(t)$  denotes the Hilbert transform of  $x(t)$ ,

- (a)  $x_1(t) = \frac{3}{4} x(t) + \frac{1}{4} j \hat{x}(t)$
- (b)  $x_2(t) = \left[ \frac{3}{4} x(t) + \frac{3}{4} j \hat{x}(t) \right] e^{j2\pi f_0 t}, \quad f_0 \gg W$
- (c)  $x_3(t) = \left[ \frac{3}{4} x(t) + \frac{1}{4} j \hat{x}(t) \right] e^{j2\pi W t}$
- (d)  $x_3(t) = \left[ \frac{3}{4} x(t) - \frac{1}{4} j \hat{x}(t) \right] e^{j\pi W t}$

**Solution:**

- (a) Note that  $\mathcal{F}\{j\hat{x}(t)\} = j[-j \operatorname{sgn}(f)]X(f)$ . As a result,

$$\begin{aligned} X_1(f) &= \frac{3}{4}X(f) + \frac{1}{4}j[-j \operatorname{sgn}(f)]X(f) \\ &= \left[ \frac{3}{4} + \frac{1}{4} \operatorname{sgn}(f) \right] X(f) \\ &= \begin{cases} \frac{1}{2}X(f), & f < 0; \\ X(f), & f \geq 0. \end{cases} \end{aligned}$$

- (b) In this case,

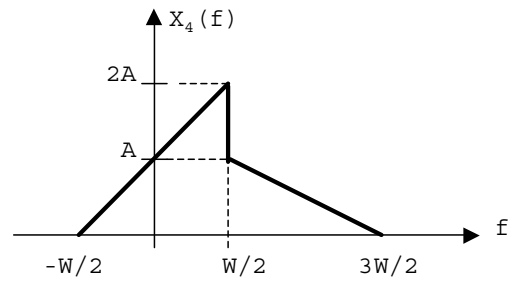
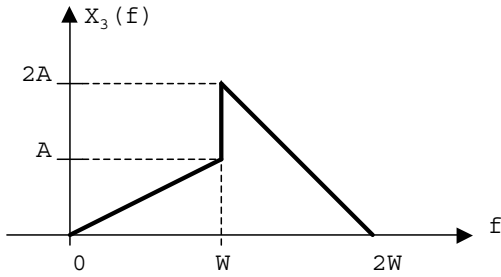
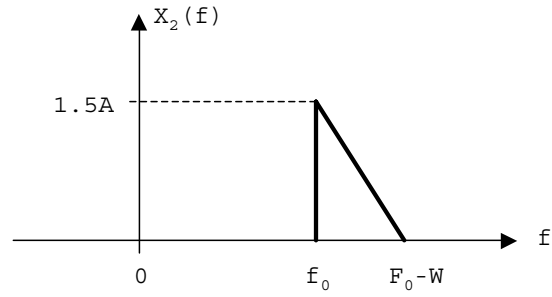
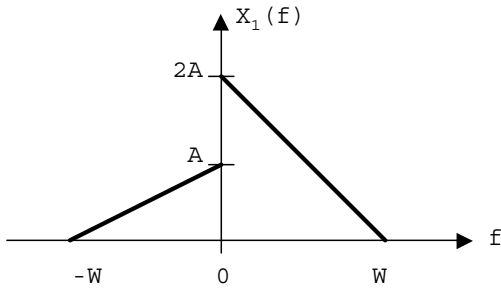
$$\begin{aligned} X_2(f) &= \frac{3}{4} [1 + \operatorname{sgn}(f - f_0)] X(f - f_0) \\ &= \begin{cases} 0, & f < f_0; \\ \frac{3}{2}X(f - f_0), & f \geq f_0. \end{cases} \end{aligned}$$

- (c) The answer here is the same as in part (a), with the difference of a shift to the right in the frequency domain by  $W$ ,

$$X_3(f) = \begin{cases} \frac{1}{2}X(f - W), & f < W; \\ X(f), & f \geq W. \end{cases}$$

- (d) Here,

$$X_4(f) = \left[ \frac{3}{4} - \frac{1}{4} \operatorname{sgn}\left(f - \frac{W}{2}\right) \right] X\left(f - \frac{W}{2}\right) \quad (4)$$



51. Consider the signal

$$x(t) = 2W \operatorname{sinc}(2Wt) \cos(2\pi f_0 t)$$

- Obtain and sketch the spectrum of the analytical signal  $x_p(t) = x(t) + j\hat{x}(t)$
- Obtain and sketch the spectrum of the complex envelope (or complex baseband representation)  $\tilde{x}(t)$
- Find the complex envelope  $\tilde{x}(t)$

**Solution:**

- The spectrum of the analytical signal is

$$X_p(f) = X(f) + j[-j \operatorname{sgn}(f)]X(f) = [1 + \operatorname{sgn}(f)]X(f),$$

where  $X(f)$  is the Fourier transform of  $x(t)$ , given by

$$X(f) = \frac{1}{2} \left[ \Pi\left(\frac{f+f_0}{2W}\right) + \Pi\left(\frac{f-f_0}{2W}\right) \right].$$

Consequently,

$$X_p(f) = \Pi\left(\frac{f-f_0}{2W}\right), \quad f_0 > 2W,$$

a rectangular pulse of width  $2W$  centered at  $f = f_0$ .

- The complex envelope  $x_p(t)$  is

$$x_p(t) = \tilde{x}(t)e^{j2\pi f_0 t}.$$

Therefore,  $\tilde{x}(t) = x_p(t)e^{-j2\pi f_0 t}$ , and

$$\tilde{X}(f) \triangleq \mathcal{F}\{\tilde{x}(t)\} = [X_p(f)]_{f \rightarrow f+f_0} = \Pi\left(\frac{f}{2W}\right),$$

a rectangular pulse of width  $2W$  centered at  $f = 0$ .

(c) The complex envelope is given by

$$\tilde{x}(t) = \mathcal{F}^{-1}\{\tilde{X}(f)\} = 2W \text{sinc}(2Wt).$$

52. The signal

$$x(t) = \Pi\left(\frac{t}{\tau}\right) \cos[2\pi(f_0 - \Delta f)t], \quad \Delta f \ll f_0$$

is applied at the input of a filter (LTI system) with impulse response

$$h(t) = \alpha e^{-\alpha t} \cos(2\pi f_0 t) u(t).$$

Find the output signal  $y(t)$  using complex envelope techniques.

**Solution:** For  $t < -\tau/2$ , the output is zero. For  $|t| \leq \tau/2$ , the result is

$$y(t) = \frac{\alpha/2}{\sqrt{\alpha^2 + (2\pi\Delta f)^2}} \left\{ \cos[2\pi(f_0 + \Delta f)t - \theta] - e^{-\alpha(t+\tau/2)} \cos[2\pi(f_0 + \Delta f)t + \theta] \right\},$$

and for  $t > \tau/2$ , the output is

$$y(t) = \frac{(\alpha/2)e^{-\alpha t}}{\sqrt{\alpha^2 + (2\pi\Delta f)^2}} \left\{ e^{\alpha\tau/2} \cos[2\pi(f_0 + \Delta f)t - \theta] - e^{-\alpha\tau/2} \cos[2\pi(f_0 + \Delta f)t + \theta] \right\}.$$

53. The bandpass signal  $x(t) = \text{sinc}(t) \cos(2\pi f_0 t)$  is passed through a bandpass filter with impulse response  $h(t) = \text{sinc}^2(t) \sin(2\pi f_0 t)$ , Using the lowpass equivalents of both input and impulse response, find the lowpass equivalent of the output and from it find the output  $y(t)$ .

**Solution:**

$$\begin{aligned} x(t) = \text{sinc}(t) \cos(2\pi f_0 t) &\iff X(f) = \frac{1}{2}\Pi(f + f_0) + \frac{1}{2}\Pi(f - f_0) \\ h(t) = \text{sinc}^2(t) \sin(2\pi f_0 t) &\iff H(f) = -\frac{1}{2j}\Lambda(f + f_0) + \frac{1}{2j}\Lambda(f - f_0) \end{aligned}$$

The lowpass equivalents are

$$\begin{aligned} X_l(f) &= 2u(f + f_0)X(f + f_0) = \Pi(f) \\ H_l(f) &= 2u(f + f_0)H(f + f_0) = \frac{1}{j}\Lambda(f) \\ Y_l(f) &= \frac{1}{2}X_l(f)H_l(f) = \begin{cases} \frac{1}{2j}(f + 1) & -\frac{1}{2} < f \leq 0 \\ \frac{1}{2j}(-f + 1) & 0 \leq f < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Taking the inverse Fourier transform of  $Y_l(f)$  we can find the lowpass equivalent response of

the system. Thus,

$$\begin{aligned}
y_l(t) &= \mathcal{F}^{-1}[Y_l(f)] \\
&= \frac{1}{2j} \int_{-\frac{1}{2}}^0 (f+1)e^{j2\pi ft} df + \frac{1}{2j} \int_0^{\frac{1}{2}} (-f+1)e^{j2\pi ft} df \\
&= \frac{1}{2j} \left[ \frac{1}{j2\pi t} f e^{j2\pi ft} + \frac{1}{4\pi^2 t^2} e^{j2\pi ft} \right] \Big|_{-\frac{1}{2}}^0 + \frac{1}{2j} \frac{1}{j2\pi t} e^{j2\pi ft} \Big|_{-\frac{1}{2}}^0 \\
&\quad - \frac{1}{2j} \left[ \frac{1}{j2\pi t} f e^{j2\pi ft} + \frac{1}{4\pi^2 t^2} e^{j2\pi ft} \right] \Big|_0^{\frac{1}{2}} + \frac{1}{2j} \frac{1}{j2\pi t} e^{j2\pi ft} \Big|_0^{\frac{1}{2}} \\
&= j \left[ -\frac{1}{4\pi t} \sin \pi t + \frac{1}{4\pi^2 t^2} (\cos \pi t - 1) \right]
\end{aligned}$$

The output of the system  $y(t)$  can now be found from  $y(t) = \text{Re}[y_l(t)e^{j2\pi f_0 t}]$ . Thus

$$\begin{aligned}
y(t) &= \text{Re} \left[ \left( j \left[ -\frac{1}{4\pi t} \sin \pi t + \frac{1}{4\pi^2 t^2} (\cos \pi t - 1) \right] \right) (\cos 2\pi f_0 t + j \sin 2\pi f_0 t) \right] \\
&= \left[ \frac{1}{4\pi^2 t^2} (1 - \cos \pi t) + \frac{1}{4\pi t} \sin \pi t \right] \sin 2\pi f_0 t
\end{aligned}$$

Note: An alternative solution is covered in class.

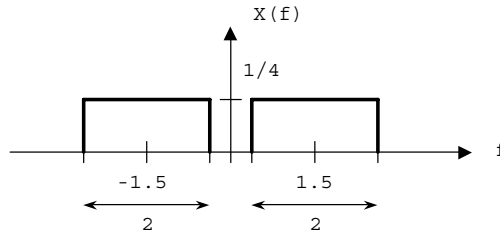
54. A bandpass signal is given by

$$x(t) = \text{sinc}(2t) \cos(3\pi t).$$

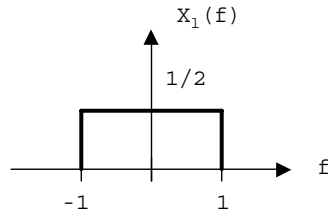
- Is the signal narrowband or wideband? Justify your answer.
- Find the complex baseband equivalent  $x_\ell(t)$  and sketch carefully its spectrum.
- Give an expression for the Hilbert transform of  $x(t)$ .

**Solution:**

- The Fourier transform of the signal is  $X(f) = \frac{1}{2} \left[ \frac{1}{2} \Pi \left( \frac{f+3/2}{2} \right) + \frac{1}{2} \Pi \left( \frac{f-3/2}{2} \right) \right]$ . The following sketch shows that the signal is wideband, as  $B = 2$  and  $f_0 = 3/2$ .



- From the quadrature modulator expression  $x(t) = x_c(t) \cos(2\pi f_0 t) - x_s(t) \sin(2\pi f_0 t)$ , it follows that  $x_s(t) = 0$  and therefore  $x_\ell(t) = x_c(t) = \text{sinc}(2t)$ . The corresponding spectrum is sketched below:



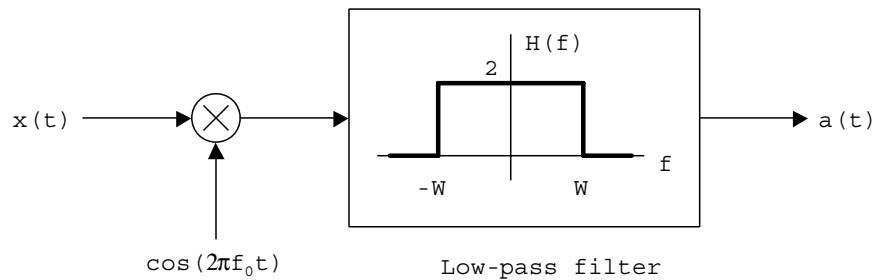
- (c) Use the expression  $\hat{x}(t) = x_c(t) \sin(2\pi f_0 t) + x_s(t) \cos(2\pi f_0 t)$ , from which it follows that  $\hat{x}(t) = \text{sinc}(2t) \sin(3\pi t)$ .

55. As shown in class, the in-phase and quadrature components,  $x_c(t)$  and  $x_s(t)$ , respectively, of the complex baseband (or lowpass) equivalent  $x_\ell(t)$  of a bandpass signal  $x(t)$  can be obtained as

$$\begin{bmatrix} x_c(t) \\ x_s(t) \end{bmatrix} = \begin{bmatrix} \cos(2\pi f_0 t) & \sin(2\pi f_0 t) \\ -\sin(2\pi f_0 t) & \cos(2\pi f_0 t) \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix},$$

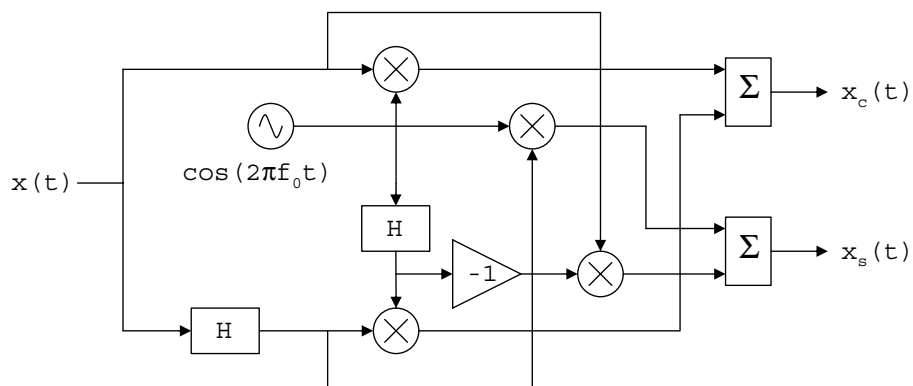
where  $\hat{x}(t)$  is the Hilbert transform of  $x(t)$ .

- (a) Sketch a block diagram of a system — using  $H$  to label the block that performs the Hilbert transform — that has as input  $x(t)$  and as outputs  $x_c(t)$  and  $x_s(t)$ .
- (b) (Amplitude modulation) Let  $x(t) = a(t) \cos(2\pi f_0 t)$ . Assume that the bandwidth  $W$  of the signal  $a(t)$  is such that  $W \ll f_0$ . Show that  $a(t)$  can be recovered with the following system

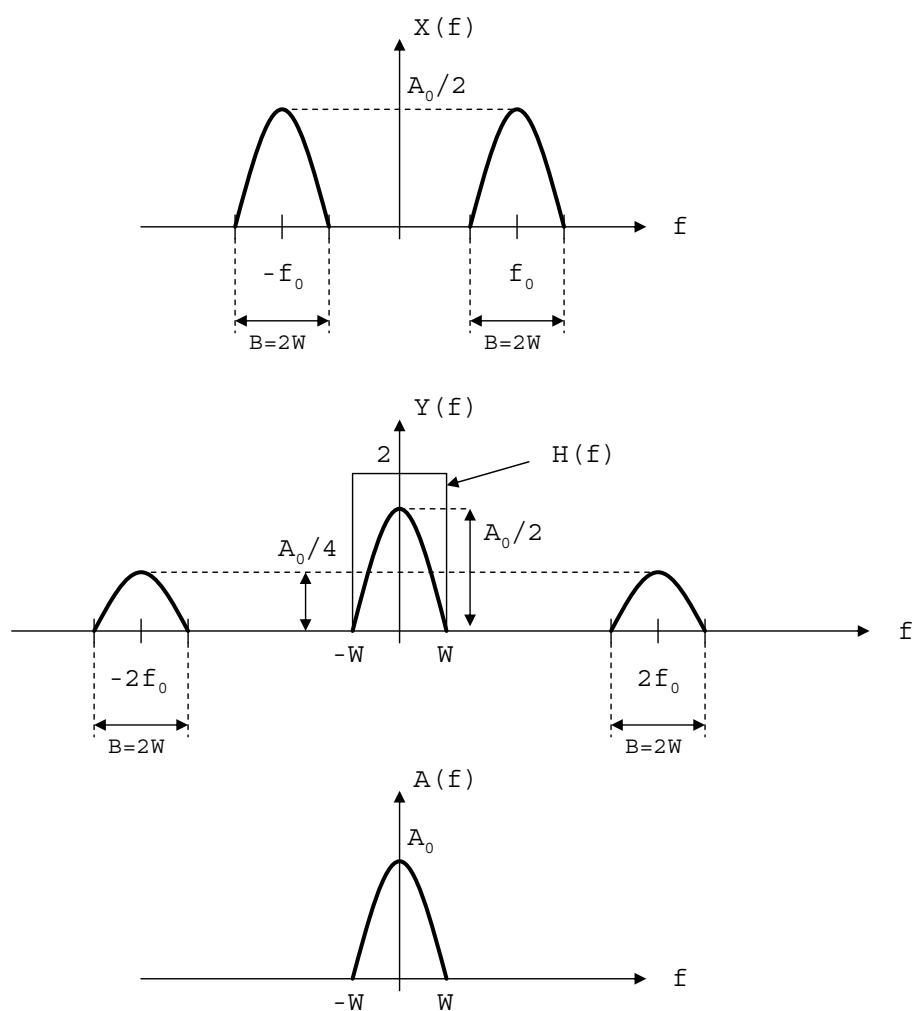


**Solution:**

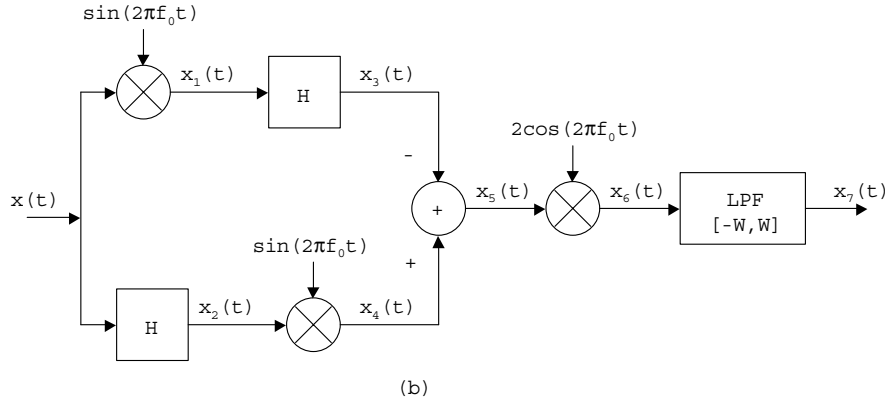
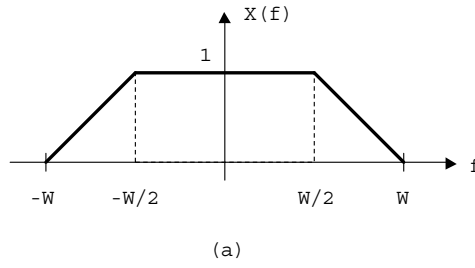
- (a)



- (b) Use the modulation property of the Fourier transform. Let  $y(t)$  denote the mixer output. The spectra are shown in the figure below.



56. A lowpass signal  $x(t)$  has a Fourier transform shown in the figure (a) below.



The signal is applied to the system shown in figure (b). The blocks marked  $H$  represent Hilbert transform blocks and it is assumed that  $W \ll f_0$ . Determine the signals  $x_i(t)$  and plot  $X_i(f)$ , for  $1 \leq i \leq 7$ .

(Hint: Use the fact that  $x(t) \widehat{\sin(2\pi f_0 t)} = -x(t) \cos(2\pi f_0 t)$  and  $x(t) \widehat{\cos(2\pi f_0 t)} = x(t) \sin(2\pi f_0 t)$ , when the bandwidth  $W$  of  $x(t)$  is much smaller than  $f_0$ .)

**Solution:** This is an example of single sideband (SSB) amplitude modulation (AM).

$$\begin{aligned} x_1(t) &= x(t) \sin(2\pi f_0 t) \\ X_1(f) &= -\frac{1}{2j} X(f + f_0) + \frac{1}{2j} X(f - f_0) \end{aligned}$$

$$\begin{aligned} x_2(t) &= \hat{x}(t) \\ X_2(f) &= -j \operatorname{sgn}(f) X(f) \end{aligned}$$

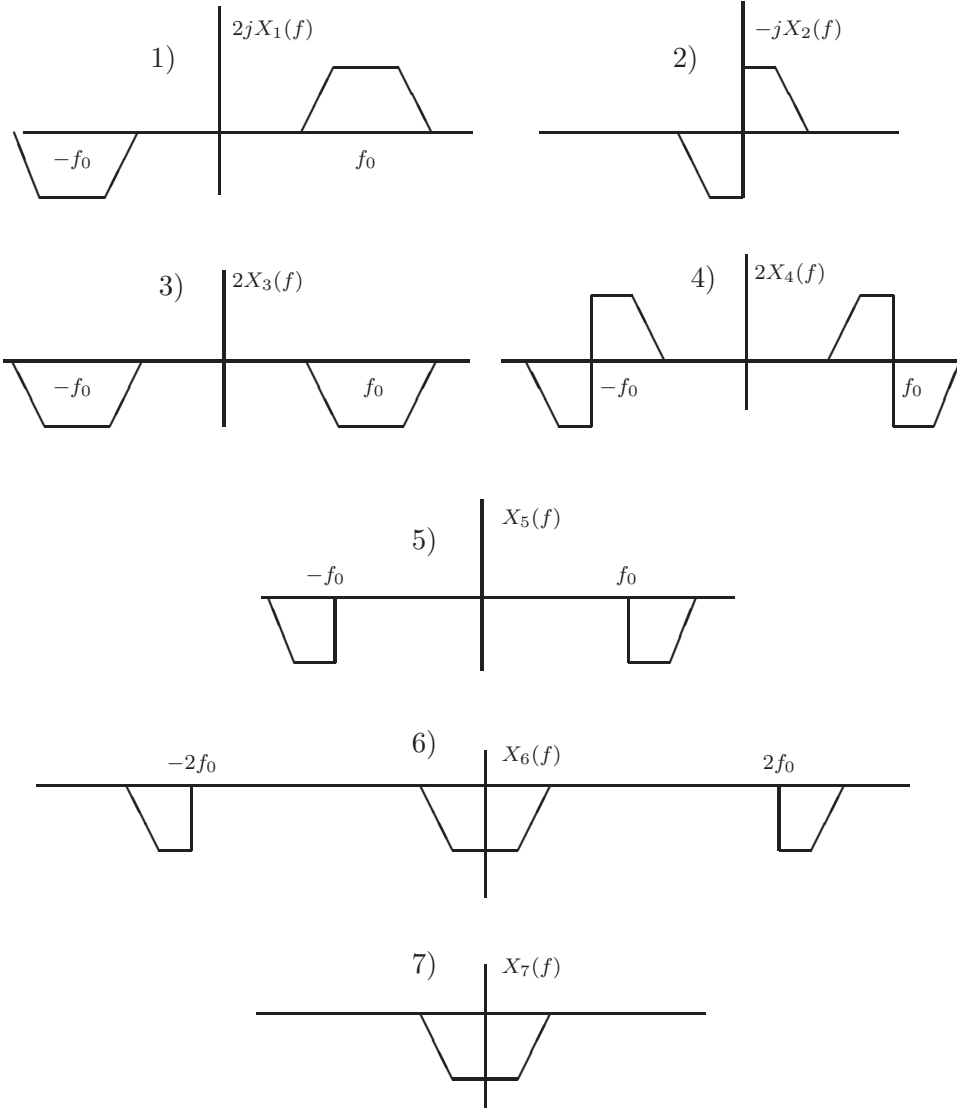
$$\begin{aligned} x_3(t) &= \hat{x}_1(t) = x(t) \widehat{\sin(2\pi f_0 t)} = -x(t) \cos(2\pi f_0 t) \\ X_3(f) &= -\frac{1}{2} X(f + f_0) - \frac{1}{2} X(f - f_0) \end{aligned}$$

$$\begin{aligned} x_4(t) &= x_2(t) \sin(2\pi f_0 t) = \hat{x}(t) \sin(2\pi f_0 t) \\ X_4(f) &= -\frac{1}{2j} \hat{X}(f + f_0) + \frac{1}{2j} \hat{X}(f - f_0) \\ &= -\frac{1}{2j} [-j \operatorname{sgn}(f + f_0) X(f + f_0)] + \frac{1}{2j} [-j \operatorname{sgn}(f - f_0) X(f - f_0)] \\ &= \frac{1}{2} \operatorname{sgn}(f + f_0) X(f + f_0) - \frac{1}{2} \operatorname{sgn}(f - f_0) X(f - f_0) \end{aligned}$$

$$\begin{aligned}
x_5(t) &= \hat{x}(t) \sin(2\pi f_0 t) + x(t) \cos(2\pi f_0 t) \\
X_5(f) &= X_4(f) - X_3(f) = \frac{1}{2}X(f + f_0)(\text{sgn}(f + f_0) - 1) - \frac{1}{2}X(f - f_0)(\text{sgn}(f - f_0) + 1)
\end{aligned}$$

$$\begin{aligned}
x_6(t) &= [\hat{x}(t) \sin(2\pi f_0 t) + x(t) \cos(2\pi f_0 t)] 2 \cos(2\pi f_0 t) \\
X_6(f) &= X_5(f + f_0) + X_5(f - f_0) \\
&= \frac{1}{2}X(f + 2f_0)(\text{sgn}(f + 2f_0) - 1) - \frac{1}{2}X(f)(\text{sgn}(f) + 1) \\
&\quad + \frac{1}{2}X(f)(\text{sgn}(f) - 1) - \frac{1}{2}X(f - 2f_0)(\text{sgn}(f - 2f_0) + 1) \\
&= -X(f) + \frac{1}{2}X(f + 2f_0)(\text{sgn}(f + 2f_0) - 1) - \frac{1}{2}X(f - 2f_0)(\text{sgn}(f - 2f_0) + 1)
\end{aligned}$$

$$\begin{aligned}
x_7(t) &= x_6(t) \star 2W \text{sinc}(2Wt) = -x(t) \\
X_7(f) &= X_6(f) \Pi\left(\frac{f}{2W}\right) = -X(f)
\end{aligned}$$





## Analog amplitude-modulation (AM) systems

57. The message signal  $m(t) = 2 \cos(400t) + 4 \sin(500t + \frac{\pi}{3})$  modulates the carrier signal  $c(t) = A \cos(8000\pi t)$ , using DSB amplitude modulation. Find the time domain and frequency domain representation of the modulated signal and plot the spectrum (Fourier transform) of the modulated signal. What is the power content of the modulated signal?

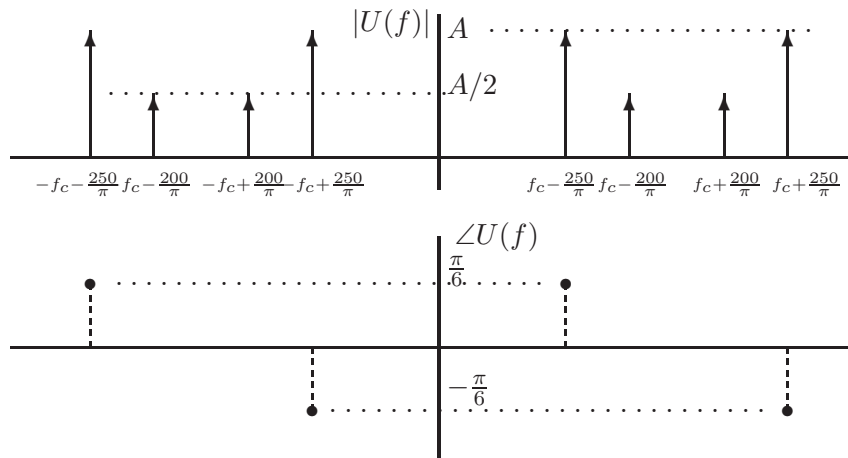
**Solution:** The modulated signal is

$$\begin{aligned}
 u(t) &= m(t)c(t) = Am(t) \cos(2\pi 4 \times 10^3 t) \\
 &= A \left[ 2 \cos(2\pi \frac{200}{\pi} t) + 4 \sin(2\pi \frac{250}{\pi} t + \frac{\pi}{3}) \right] \cos(2\pi 4 \times 10^3 t) \\
 &= A \cos(2\pi(4 \times 10^3 + \frac{200}{\pi})t) + A \cos(2\pi(4 \times 10^3 - \frac{200}{\pi})t) \\
 &\quad + 2A \sin(2\pi(4 \times 10^3 + \frac{250}{\pi})t + \frac{\pi}{3}) - 2A \sin(2\pi(4 \times 10^3 - \frac{250}{\pi})t - \frac{\pi}{3})
 \end{aligned}$$

Taking the Fourier transform of the previous relation, we obtain

$$\begin{aligned}
 U(f) &= A \left[ \delta(f - \frac{200}{\pi}) + \delta(f + \frac{200}{\pi}) + \frac{2}{j} e^{j\frac{\pi}{3}} \delta(f - \frac{250}{\pi}) - \frac{2}{j} e^{-j\frac{\pi}{3}} \delta(f + \frac{250}{\pi}) \right] \\
 &\quad * \frac{1}{2} [\delta(f - 4 \times 10^3) + \delta(f + 4 \times 10^3)] \\
 &= \frac{A}{2} \left[ \delta(f - 4 \times 10^3 - \frac{200}{\pi}) + \delta(f - 4 \times 10^3 + \frac{200}{\pi}) \right. \\
 &\quad + 2e^{-j\frac{\pi}{6}} \delta(f - 4 \times 10^3 - \frac{250}{\pi}) + 2e^{j\frac{\pi}{6}} \delta(f - 4 \times 10^3 + \frac{250}{\pi}) \\
 &\quad + \delta(f + 4 \times 10^3 - \frac{200}{\pi}) + \delta(f + 4 \times 10^3 + \frac{200}{\pi}) \\
 &\quad \left. + 2e^{-j\frac{\pi}{6}} \delta(f + 4 \times 10^3 - \frac{250}{\pi}) + 2e^{j\frac{\pi}{6}} \delta(f + 4 \times 10^3 + \frac{250}{\pi}) \right]
 \end{aligned}$$

The figure below shows the magnitude and the phase of the spectrum  $U(f)$ .



To find the power content of the modulated signal we write  $u^2(t)$  as

$$\begin{aligned} u^2(t) &= A^2 \cos^2\left(2\pi\left(4 \times 10^3 + \frac{200}{\pi}\right)t\right) + A^2 \cos^2\left(2\pi\left(4 \times 10^3 - \frac{200}{\pi}\right)t\right) \\ &\quad + 4A^2 \sin^2\left(2\pi\left(4 \times 10^3 + \frac{250}{\pi}\right)t + \frac{\pi}{3}\right) + 4A^2 \sin^2\left(2\pi\left(4 \times 10^3 - \frac{250}{\pi}\right)t - \frac{\pi}{3}\right) \\ &\quad + \text{terms of cosine and sine functions in the first power} \end{aligned}$$

Hence,

$$P = \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t) dt = \frac{A^2}{2} + \frac{A^2}{2} + \frac{4A^2}{2} + \frac{4A^2}{2} = 5A^2$$

58. In a DSB AM system, the carrier is  $c(t) = A \cos(2\pi f_c t)$  and the message signal is given by  $m(t) = \text{sinc}(t) + \text{sinc}^2(t)$ . Find the frequency domain representation and the bandwidth of the modulated signal.

**Solution:**

$$u(t) = m(t)c(t) = A(\text{sinc}(t) + \text{sinc}^2(t)) \cos(2\pi f_c t)$$

Taking the Fourier transform of both sides, we obtain

$$\begin{aligned} U(f) &= \frac{A}{2} [\Pi(f) + \Lambda(f)] \star (\delta(f - f_c) + \delta(f + f_c)) \\ &= \frac{A}{2} [\Pi(f - f_c) + \Lambda(f - f_c) + \Pi(f + f_c) + \Lambda(f + f_c)] \end{aligned}$$

$\Pi(f - f_c) \neq 0$  for  $|f - f_c| < \frac{1}{2}$ , whereas  $\Lambda(f - f_c) \neq 0$  for  $|f - f_c| < 1$ . Hence, the bandwidth of the modulated signal is 2.

59. A DSB-modulated signal  $u(t) = A m(t) \cos(2\pi f_c t)$  is mixed (multiplied) with a local carrier  $x_L(t) = \cos(2\pi f_c t + \theta)$  and the output is passed through a lowpass filter (LPF) with bandwidth equal to the bandwidth of the message signal  $m(t)$ . Denote the power of the signal at the output of the LPF by  $P_{\text{out}}$  and the power of the modulated signal by  $P_U$ . Plot the ratio  $\frac{P_{\text{out}}}{P_U}$ , as a function of  $\theta$ , for  $0 \leq \theta \leq \pi$ .

**Solution:** The mixed signal  $y(t)$  is given by

$$\begin{aligned} y(t) &= u(t) \cdot x_L(t) = A m(t) \cos(2\pi f_c t) \cos(2\pi f_c t + \theta) \\ &= \frac{A}{2} m(t) [\cos(2\pi 2f_c t + \theta) + \cos(\theta)] \end{aligned}$$

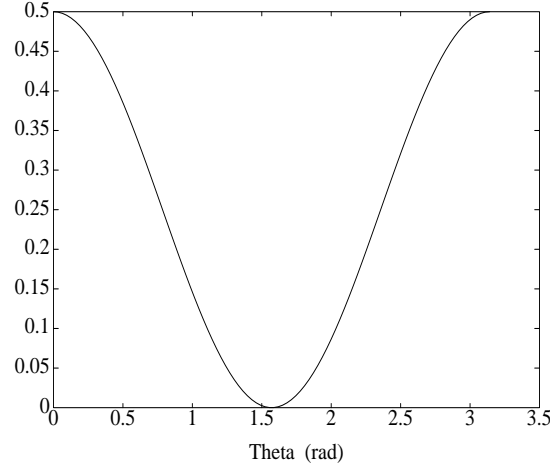
The lowpass filter will cut-off the frequencies above  $W$ , where  $W$  is the bandwidth of the message signal  $m(t)$ . Thus, the output of the lowpass filter is

$$z(t) = \frac{A}{2} m(t) \cos(\theta)$$

If the power of  $m(t)$  is  $P_M$ , then the power of the output signal  $z(t)$  is  $P_{\text{out}} = P_M \frac{A^2}{4} \cos^2(\theta)$ . The power of the modulated signal  $u(t) = A m(t) \cos(2\pi f_c t)$  is  $P_U = \frac{A^2}{2} P_M$ . Hence,

$$\frac{P_{\text{out}}}{P_U} = \frac{1}{2} \cos^2(\theta)$$

A plot of  $\frac{P_{\text{out}}}{P_U}$  for  $0 \leq \theta \leq \pi$  is given in the next figure.



60. The output signal from an AM modulator is

$$u(t) = 5 \cos(1800\pi t) + 20 \cos(2000\pi t) + 5 \cos(2200\pi t).$$

- Determine the message signal  $m(t)$  and the carrier  $c(t)$ . (Hint: Look at the spectrum of  $u(t)$ .)
- Determine the modulation index.
- Determine the ratio of the power in the sidebands to the power in the carrier.

**Solution:**

(a)

$$\begin{aligned} u(t) &= 5 \cos(1800\pi t) + 20 \cos(2000\pi t) + 5 \cos(2200\pi t) \\ &= 20 \left(1 + \frac{1}{2} \cos(200\pi t)\right) \cos(2000\pi t) \end{aligned}$$

The modulating signal is  $m(t) = \cos(2\pi 100t)$  whereas the carrier signal is  $c(t) = 20 \cos(2\pi 1000t)$ .

- Since  $-1 \leq \cos(2\pi 100t) \leq 1$ , we immediately have that the modulation index is  $\alpha = \frac{1}{2}$ .
- The power of the carrier component is  $P_{\text{carrier}} = \frac{400}{2} = 200$ , whereas the power in the sidebands is  $P_{\text{sidebands}} = \frac{400\alpha^2}{2} = 50$ . Hence,

$$\frac{P_{\text{sidebands}}}{P_{\text{carrier}}} = \frac{50}{200} = \frac{1}{4}$$

61. An SSB AM signal is generated by modulating an 800 kHz carrier by the message signal  $m(t) = \cos(2000\pi t) + 2\sin(2000\pi t)$ . Assume that the amplitude of the carrier is  $A_c = 100$ .

- Determine the Hilbert transform of the message signal,  $\hat{m}(t)$ .

- (b) Find the time-domain expression for the lower sideband SSB (LSSB) AM signal.  
(c) Determine the spectrum of the LSSB AM signal.

**Solution:**

- (a) The Hilbert transform of  $\cos(2\pi 1000t)$  is  $\sin(2\pi 1000t)$ , whereas the Hilbert transform of  $\sin(2\pi 1000t)$  is  $-\cos(2\pi 1000t)$ . Thus

$$\hat{m}(t) = \sin(2\pi 1000t) - 2 \cos(2\pi 1000t)$$

- (b) The expression for the LSSB AM signal is

$$u_l(t) = A_c m(t) \cos(2\pi f_c t) + A_c \hat{m}(t) \sin(2\pi f_c t)$$

Substituting  $A_c = 100$ ,  $m(t) = \cos(2\pi 1000t) + 2 \sin(2\pi 1000t)$  and  $\hat{m}(t) = \sin(2\pi 1000t) - 2 \cos(2\pi 1000t)$  in the previous, we obtain

$$\begin{aligned} u_l(t) &= 100 [\cos(2\pi 1000t) + 2 \sin(2\pi 1000t)] \cos(2\pi f_c t) \\ &+ 100 [\sin(2\pi 1000t) - 2 \cos(2\pi 1000t)] \sin(2\pi f_c t) \\ &= 100 [\cos(2\pi 1000t) \cos(2\pi f_c t) + \sin(2\pi 1000t) \sin(2\pi f_c t)] \\ &+ 200 [\cos(2\pi f_c t) \sin(2\pi 1000t) - \sin(2\pi f_c t) \cos(2\pi 1000t)] \\ &= 100 \cos(2\pi(f_c - 1000)t) - 200 \sin(2\pi(f_c - 1000)t) \end{aligned}$$

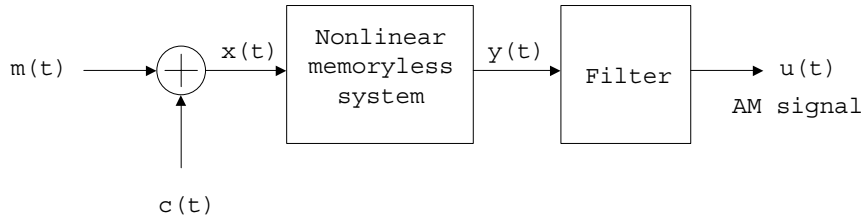
- (c) Taking the Fourier transform of the previous expression we obtain

$$\begin{aligned} U_l(f) &= 50 (\delta(f - f_c + 1000) + \delta(f + f_c - 1000)) \\ &+ 100j (\delta(f - f_c + 1000) - \delta(f + f_c - 1000)) \\ &= (50 + 100j)\delta(f - f_c + 1000) + (50 - 100j)\delta(f + f_c - 1000) \end{aligned}$$

Hence, the magnitude spectrum is given by

$$\begin{aligned} |U_l(f)| &= \sqrt{50^2 + 100^2} (\delta(f - f_c + 1000) + \delta(f + f_c - 1000)) \\ &= 10\sqrt{125} (\delta(f - f_c + 1000) + \delta(f + f_c - 1000)) \end{aligned}$$

62. The system shown in the figure below can be used to generate an AM signal.



The carrier is  $c(t) = \cos(2\pi f_0 t)$  and the modulating signal  $m(t)$  has zero mean and its maximum absolute value is  $A_m = \max |m(t)|$ . The nonlinear device has a quadratic input-output characteristic given by

$$y(t) = a x(t) + b x^2(t).$$

- (a) Give an expression of  $y(t)$  in terms of  $m(t)$  and  $c(t)$ .

- (b) Specify the filter characteristics such that an AM signal is obtain at its output.  
(c) What is the modulation index?

**Solution:**

(a)

$$\begin{aligned}
 y(t) &= ax(t) + bx^2(t) \\
 &= a(m(t) + \cos(2\pi f_0 t)) + b(m(t) + \cos(2\pi f_0 t))^2 \\
 &= am(t) + bm^2(t) + a \cos(2\pi f_0 t) \\
 &\quad + b \cos^2(2\pi f_0 t) + 2bm(t) \cos(2\pi f_0 t)
 \end{aligned}$$

- (b) The filter should reject the low frequency components, the terms of double frequency and pass only the signal with spectrum centered at  $f_0$ . Thus the filter should be a BPF with center frequency  $f_0$  and bandwidth  $W$  such that  $f_0 - W_M > f_0 - \frac{W}{2} > 2W_M$  where  $W_M$  is the bandwidth of the message signal  $m(t)$ .  
(c) The AM output signal can be written as

$$u(t) = a(1 + \frac{2b}{a}m(t)) \cos(2\pi f_0 t)$$

Since  $A_m = \max[|m(t)|]$  we conclude that the modulation index is

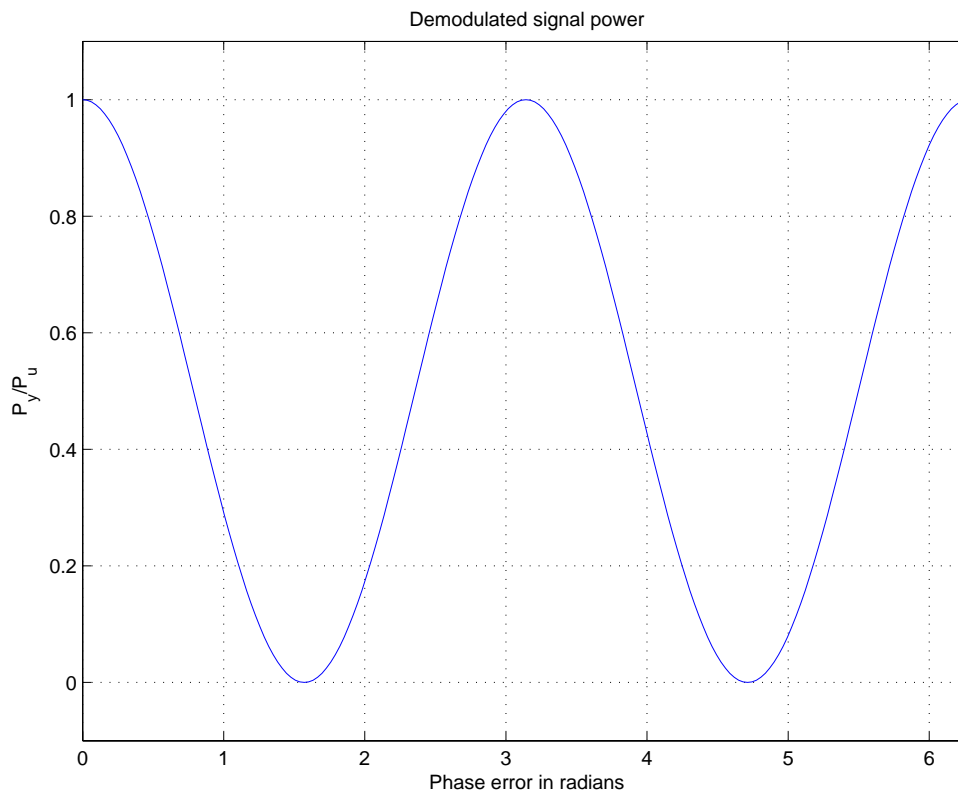
$$\alpha = \frac{2bA_m}{a}$$

63. Consider a message signal  $m(t) = \cos(2\pi t)$ . The carrier frequency is  $f_c = 5$ .

- (a) (Suppressed carrier AM) Plot the power  $P_u$  of the modulated signal as a function of the phase difference (between transmitter and receiver)  $\Delta\phi \triangleq \phi_c - \phi_r$ .  
(b) (Conventional AM) With a modulation index  $a = 0.5$  (or 50%), sketch carefully the modulated signal  $u(t)$ .

**Solution:**

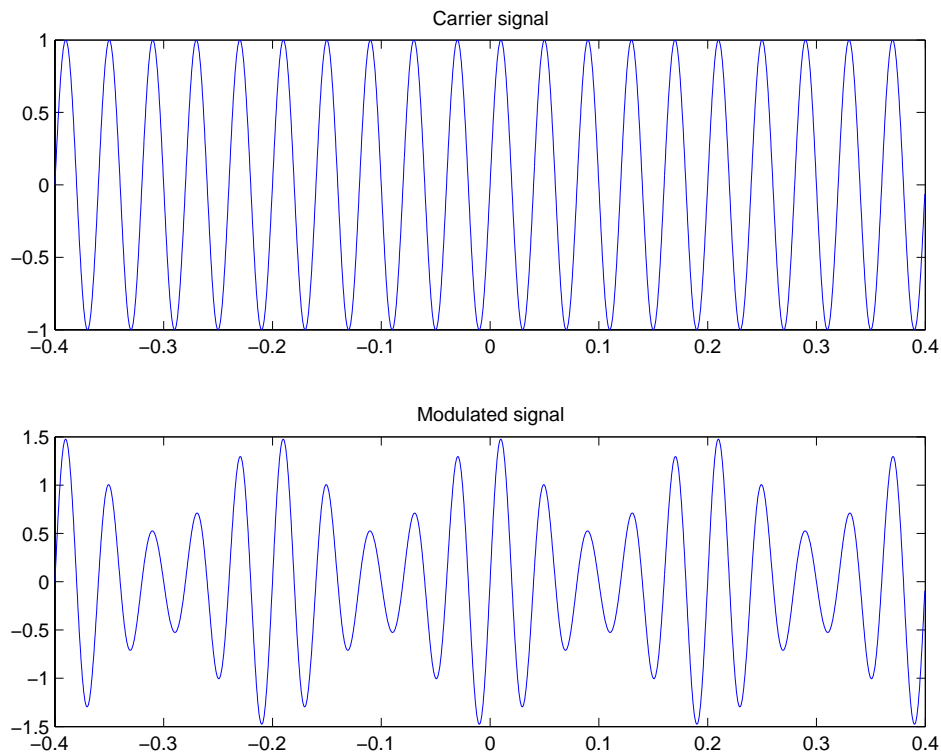
- (a) The demodulated signal power is given by  $P_y = P_u \cos^2(\Delta\phi)$ , which has maximum value  $P_y = P_u$  for  $\Delta\phi = 0, \pi$  and minimum value  $P_y = 0$  for  $\Delta\phi = \pi/2, 3\pi/2$ . This is shown below, which is a plot of  $P_y/P_u$  as a function of  $\Delta\phi$ .



(b) The modulated signal is

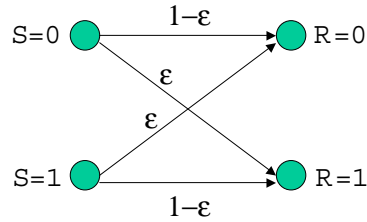
$$u(t) = A_c [1 + 0.5 \cos(2\pi t)] \cos(10\pi t),$$

and plotted in the figure below, together with the carrier  $\cos(10\pi t)$  (top graph). Both plots are normalized with respect to the carrier amplitude  $A_c$ .



## Probability and random signals

64. A (random) binary source produces  $S = 0$  and  $S = 1$  with probabilities 0.3 and 0.7, respectively. The output of the source  $S$  is transmitted over a noisy (binary symmetric) channel with a probability of error (converting a 0 into a 1, or a 1 into a 0) of  $\epsilon = 0.2$ .



- (a) Find the probability that  $R = 1$ . (Hint: Total probability theorem.)  
 (b) Find the (a-posteriori) probability that  $S = 1$  was produced by the source given that  $R = 1$  is observed. (Hint: Bayes rule.)

**Solution:**

(a)

$$\begin{aligned} P(R = 1) &= P(R = 1|S = 1)P(S = 1) + P(R = 1|S = 0)P(S = 0) \\ &= 0.8 \cdot 0.7 + 0.2 \cdot 0.3 = 0.62 \end{aligned}$$

where we have used  $P(S = 1) = .7$ ,  $P(S = 0) = .3$ ,  $P(R = 1|S = 0) = \epsilon = 0.2$  and  $P(R = 1|S = 1) = 1 - \epsilon = 1 - 0.2 = .8$

(b)

$$P(S = 1|R = 1) = \frac{P(S = 1, R = 1)}{P(R = 1)} = \frac{P(R = 1|S = 1)P(S = 1)}{P(R = 1)} = \frac{0.8 \cdot 0.7}{0.62} = 0.9032$$

65. A random variable  $X$  has a PDF  $f_X(x) = \Lambda(x)$ . Find the following:

- (a) The CDF of  $X$ ,  $F_X(x)$ .  
 (b)  $\Pr\{X > \frac{1}{2}\}$ .  
 (c)  $\Pr\{X > 0|X < \frac{1}{2}\}$ .  
 (d) The conditional PDF  $f_X(x|X > \frac{1}{2})$ .

**Solution:**

(a)

$$\begin{aligned} x < -1 &\Rightarrow F_X(x) = 0 \\ -1 \leq x \leq 0 &\Rightarrow F_X(x) = \int_{-1}^x (v+1)dv = \left(\frac{1}{2}v^2 + v\right)\Big|_{-1}^x = \frac{1}{2}x^2 + x + \frac{1}{2} \\ 0 \leq x \leq 1 &\Rightarrow F_X(x) = \int_{-1}^0 (v+1)dv + \int_0^x (-v+1)dv = -\frac{1}{2}x^2 + x + \frac{1}{2} \\ x \geq 1 &\Rightarrow F_X(x) = 1 \end{aligned}$$

(b)

$$p(X > \frac{1}{2}) = 1 - F_X(\frac{1}{2}) = 1 - \frac{7}{8} = \frac{1}{8}$$

(c)

$$p(X > 0 | X < \frac{1}{2}) = \frac{p(X > 0, X < \frac{1}{2})}{p(X < \frac{1}{2})} = \frac{F_X(\frac{1}{2}) - F_X(0)}{1 - p(X > \frac{1}{2})} = \frac{3}{7}$$

(d) We find first the CDF

$$F_X(x | X > \frac{1}{2}) = p(X \leq x | X > \frac{1}{2}) = \frac{p(X \leq x, X > \frac{1}{2})}{p(X > \frac{1}{2})}$$

If  $x \leq \frac{1}{2}$  then  $p(X \leq x | X > \frac{1}{2}) = 0$  since the events  $E_1 = \{X \leq \frac{1}{2}\}$  and  $E_2 = \{X > \frac{1}{2}\}$  are disjoint. If  $x > \frac{1}{2}$  then  $p(X \leq x | X > \frac{1}{2}) = F_X(x) - F_X(\frac{1}{2})$  so that

$$F_X(x | X > \frac{1}{2}) = \frac{F_X(x) - F_X(\frac{1}{2})}{1 - F_X(\frac{1}{2})}$$

Differentiating this equation with respect to  $x$  we obtain

$$f_X(x | X > \frac{1}{2}) = \begin{cases} \frac{f_X(x)}{1 - F_X(\frac{1}{2})} & x > \frac{1}{2} \\ 0 & x \leq \frac{1}{2} \end{cases}$$

66. A random process is given by  $X(t) = A + Bt$ , where  $A$  and  $B$  are independent random variables uniformly distributed in the interval  $[-1, 1]$ . Find:

(a) The mean function  $m_x(t)$ .

(b) The autocorrelation function  $R_x(t_1, t_2)$ .

(c) Is  $X(t)$  a stationary process?

**Solution:**

$$m_X(t) = E[A + Bt] = E[A] + E[B]t = 0$$

where the last equality follows from the fact that  $A, B$  are uniformly distributed over  $[-1, 1]$  so that  $E[A] = E[B] = 0$ .

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] = E[(A + Bt_1)(A + Bt_2)] \\ &= E[A^2] + E[AB]t_2 + E[BA]t_1 + E[B^2]t_1t_2 \end{aligned}$$

The random variables  $A, B$  are independent so that  $E[AB] = E[A]E[B] = 0$ . Furthermore

$$E[A^2] = E[B^2] = \int_{-1}^1 x^2 \frac{1}{2} dx = \frac{1}{6} x^3 \Big|_{-1}^1 = \frac{1}{3}$$

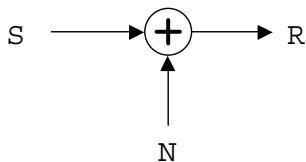
Thus

$$R_X(t_1, t_2) = \frac{1}{3} + \frac{1}{3} t_1 t_2$$

and the process is not stationary.



67. A simple binary communication system model, with *additive Gaussian noise*, is the following:  $S$  is a binary random variable, representing the message bit sent, taking values  $-1$  and  $+1$  with equal probability. Additive noise is represented by a Gaussian random variable  $N$  of zero mean and variance  $\sigma^2$ . The received value is a random variable  $R = S + N$ . This is illustrated in the figure below:



Find the probability density function (pdf) of  $R$ .

(Hint: The pdf of  $R$  is equal to the convolution of the pdf's of  $S$  and  $N$ . Remember to use the **pdf** of  $S$ , which consists of two impulses. The same result can be obtained by first conditioning on a value of  $S$  and then integrating over the pdf of  $S$ .)

**Solution:** Note that  $S$  is a discrete random variable. Therefore, it can be specified by its *probability mass function* (PMF),  $P[S = +1] = P[S = -1] = 1/2$ , or by a PDF that consists of two impulses, each of weight  $1/2$ , centered at  $\pm 1$ .

Given a value of  $S = s$ , the conditional PDF of  $R$  is

$$p_{R|S}(r) = p_N(r - s) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(r - s)^2\right).$$

Then, applying the total probability theorem, the PDF of  $R$  is obtained as

$$\begin{aligned} p_R(r) &= p_{R|S=+1}(r) P[S = +1] + p_{R|S=-1}(r) P[S = -1] \\ &= \frac{1}{2} p_N(r - 1) + \frac{1}{2} p_N(r + 1) \\ &= \frac{1}{2\sqrt{2\pi\sigma^2}} \left[ \exp\left(-\frac{1}{2\sigma^2}(r - 1)^2\right) + \exp\left(-\frac{1}{2\sigma^2}(r + 1)^2\right) \right]. \end{aligned}$$

68. The joint PDF of two random variables  $X$  and  $Y$  can be expressed as

$$p_{X,Y}(x, y) = \frac{c}{\pi} \exp\left(-\frac{1}{2} \left[ \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 \right]\right)$$

- Are  $X$  and  $Y$  independent?
- Find the value of  $c$ .
- Compute the probability of the event  $\{0 < X \leq 3, 0 < Y \leq 2\}$ .

[Hint: Use the Gaussian  $Q$ -function.]

**Solution:**

- The given joint PDF can be factored as the product of two functions  $f(x)$  and  $g(y)$ . As a consequence,  $X$  and  $Y$  are independent.

- (b)  $p_{X,Y}(x, y)$  has the form of the joint PDF of two zero-mean independent Gaussian random variables (i.e.,  $\rho = 0$ ) with variances  $\sigma_X^2 = 9$  and  $\sigma_Y^2 = 4$ :

$$p_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3 \cdot 2} \exp \left( -\frac{1}{2} \left[ \left( \frac{x}{3} \right)^2 + \left( \frac{y}{2} \right)^2 \right] \right),$$

from which it follows that  $c = 1/12$ .

- (c) Let  $E = \{0 < X \leq 3, 0 < Y \leq 2\}$ . Then

$$P[E] = \left( Q(0) - Q\left(\frac{3}{\sigma_X}\right) \right) \left( Q(0) - Q\left(\frac{2}{\sigma_Y}\right) \right) = (Q(0) - Q(1))^2 = \left( \frac{1}{2} - Q(1) \right)^2 = 0.1165$$

69. A 4-level quantizer is defined by the following input-output relation:

$$y = g(x) = \begin{cases} +3, & x \geq 2; \\ +1, & 0 \leq x < 2; \\ -1, & -2 \leq x < 0; \\ -3, & x < -2. \end{cases}$$

Let the input be a Gaussian random variable  $X$ , of zero mean and unit variance,  $N(0; 1)$ . You are asked to compute the probability mass function (PMF) of the output  $Y = g(X)$ , i.e.,  $P[Y = y]$  for  $y \in \{\pm 1, \pm 3\}$ . Express your result in terms of the Gaussian  $Q$ -function.

**Solution:** The PMF of  $Y$  is given by:

$$\begin{aligned} P[Y = +3] &= P[Y = -3] = Q(2) & (= 2.275 \times 10^{-2}) \\ P[Y = +1] &= P[Y = -1] = Q(0) - Q(2) & \left( = \frac{1}{2} - 2.275 \times 10^{-2} = 0.47725 \right) \end{aligned}$$

70. Let  $X$  be an r.v. of mean  $m_X \triangleq E\{X\}$  and variance  $\sigma_X^2 \triangleq E\{(X - m_X)^2\}$ . Find the mean  $m_Y$  and variance  $\sigma_Y^2$ , in terms of  $m_X$  and  $\sigma_X^2$ , of the r.v.  $Y = X + a$ , where  $a$  is a constant.

**Solution:**  $m_Y = E\{Y\} = E\{X + a\} = E\{X\} + a = m_X + a$ , and  $\sigma_Y^2 = E\{Y^2\} - m_Y^2 = \sigma_X^2$ .

71. Let  $G$  be a Gaussian r.v. of mean  $m = 0$  and variance  $\sigma^2 = 1$ . Find the mean  $m_N$  and variance  $\sigma_N^2$  of the r.v.  $N = aG$ , where  $a$  is a constant. This transformation is used in computer simulations to generate samples of an AWGN process. If it is desired to generate samples with  $m_N = 0$  and variance  $\sigma_N^2 = N_0/2$ , what is the value of  $a$ ?

**Solution:**  $m_N = E\{N\} = aE\{G\} = am = 0$ , and  $\sigma_N^2 = E\{N^2\} = a^2E\{G^2\} = a^2\sigma^2 = a^2$ . If  $\sigma_N^2 = N_0/2$ , then  $a^2 = N_0/2$  and therefore  $a = \sqrt{N_0/2}$ .

72. Let  $X$  be a uniform r.v. over the unit interval  $[0, 1]$ . Show that  $P[X \leq a] = a$ , where  $0 < a \leq 1$ . This fact is used in computer simulations to generate random bits.

**Solution:** The CDF of  $X$  is

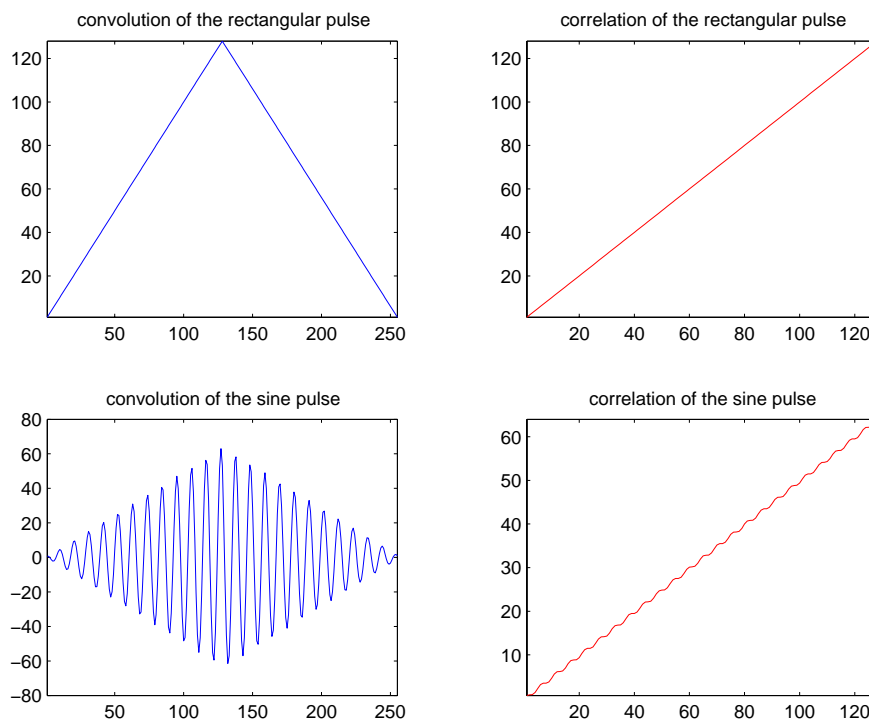
$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

Therefore  $P[X \leq a] = F_X(a) = a$ , for  $0 < a \leq 1$ .

## Optimum receivers for data transmission over AWGN channels

73. Verify that the output of a correlator and a matched filter are maximum at  $t = T$ , even though they may differ at other times. Download the MATLAB script `corr_vs_MF.m` from the web site. Execute the script and print or sketch the four graphs. Explain why this is the case, by examining the output of the matched filter expressed as a convolution integral and making a change of variables.

**Solution:**



Matched filter and correlator outputs for a rectangular pulse and a sine pulse.

To see why the outputs have the same value at  $t = T$ , write the convolution integral of the matched filter:

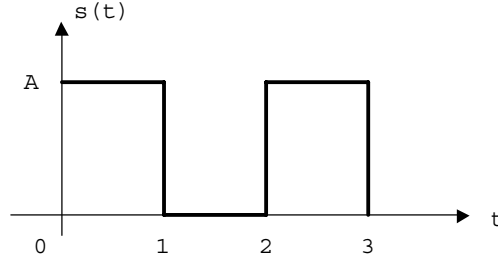
$$\begin{aligned}
 v(T) &= h(t) * s(t)|_{t=T} \\
 &= \int_0^T h(\tau) s(T - \tau) d\tau = \int_0^T s(T - \tau) s(T - \tau) d\tau \\
 &= \int_0^T s(u)^2 du,
 \end{aligned}$$

with  $u = T - \tau$ . Therefore, the output of the matched filter sampled at  $t = T$  equals the output of a correlator over the interval  $[0, T]$ , with  $s(t)$  as the reference signal.

74. The received signal in a binary communication system that employs antipodal signals is

$$r(t) = s(t) + n(t),$$

where  $s(t)$  is shown in the figure below and  $n(t)$  is AWGN with power spectral density  $N_0/2$  W/Hz.



- Sketch carefully the impulse response of the filter matched to  $s(t)$
- Sketch carefully the output of the matched filter when the input is  $s(t)$
- Determine the variance of the noise at the output of the matched filter at  $t = 3$
- Determine the probability of error as a function of  $A$  and  $N_0$

**Solution:**

- (a) The impulse response of the filter matched to  $s(t)$  is

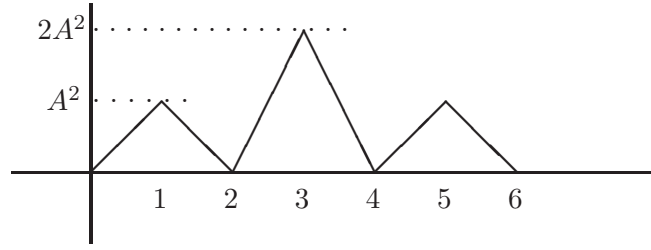
$$h(t) = s(T - t) = s(3 - t) = s(t)$$

where we have used the fact that  $s(t)$  is even with respect to the  $t = \frac{T}{2} = \frac{3}{2}$  axis.

- (b) The output of the matched filter is

$$\begin{aligned}
 y(t) &= s(t) \star s(t) = \int_0^t s(\tau) s(t - \tau) d\tau \\
 &= \begin{cases} 0 & t < 0 \\ A^2 t & 0 \leq t < 1 \\ A^2(2 - t) & 1 \leq t < 2 \\ 2A^2(t - 2) & 2 \leq t < 3 \\ 2A^2(4 - t) & 3 \leq t < 4 \\ A^2(t - 4) & 4 \leq t < 5 \\ A^2(6 - t) & 5 \leq t < 6 \\ 0 & 6 \leq t \end{cases}
 \end{aligned}$$

A sketch of  $y(t)$  is depicted in the next figure



(c) At the output of the matched filter and for  $t = T = 3$  the noise is

$$\begin{aligned} n_T &= \int_0^T n(\tau)h(T-\tau)d\tau \\ &= \int_0^T n(\tau)s(T-(T-\tau))d\tau = \int_0^T n(\tau)s(\tau)d\tau \end{aligned}$$

The variance of the noise is

$$\begin{aligned} \sigma_{n_T}^2 &= E \left[ \int_0^T \int_0^T n(\tau)n(v)s(\tau)s(v)d\tau dv \right] \\ &= \int_0^T \int_0^T s(\tau)s(v)E[n(\tau)n(v)]d\tau dv \\ &= \frac{N_0}{2} \int_0^T \int_0^T s(\tau)s(v)\delta(\tau-v)d\tau dv \\ &= \frac{N_0}{2} \int_0^T s^2(\tau)d\tau = N_0A^2 \end{aligned}$$

(d) For antipodal equiprobable signals the probability of error is

$$P(e) = Q \left[ \sqrt{\left(\frac{S}{N}\right)_o} \right]$$

where  $\left(\frac{S}{N}\right)_o$  is the output SNR from the matched filter. Since

$$\left(\frac{S}{N}\right)_o = \frac{y^2(T)}{E[n_T^2]} = \frac{4A^4}{N_0A^2}$$

we obtain

$$P(e) = Q \left[ \sqrt{\frac{4A^2}{N_0}} \right]$$

75. Montecarlo simulation of a binary transmission system over an AWGN channel.

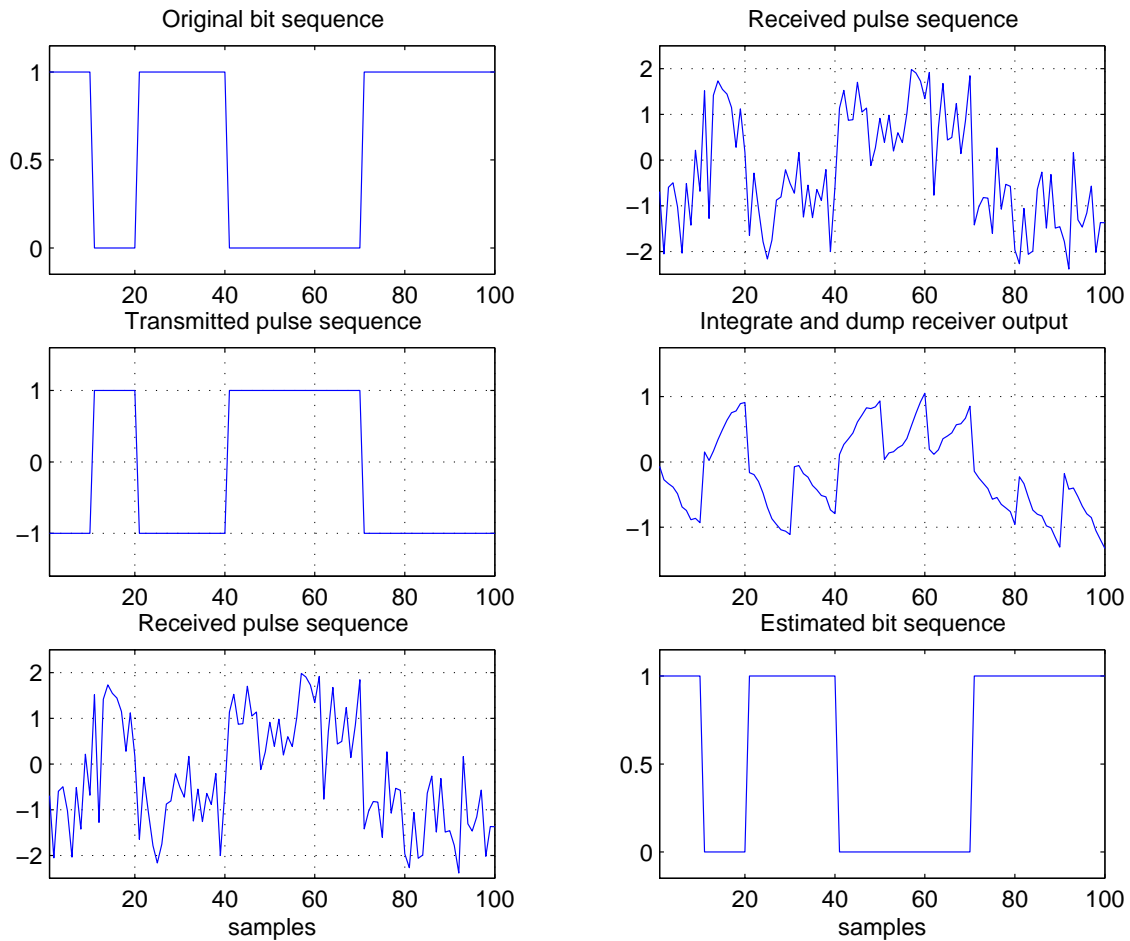
- (a) Download and execute the Matlab script `integrate_and_dump.m` from the web site of the class. Upon completion, a plot shows waveforms associated with the transmission of ten random bits over an AWGN channel, with a rectangular pulse shape and an integrate-and-dump receiver. Execute the script with your student ID, an amplitude  $a = 1$  and with  $N_0 = 1$ . Sketch or print the plot.

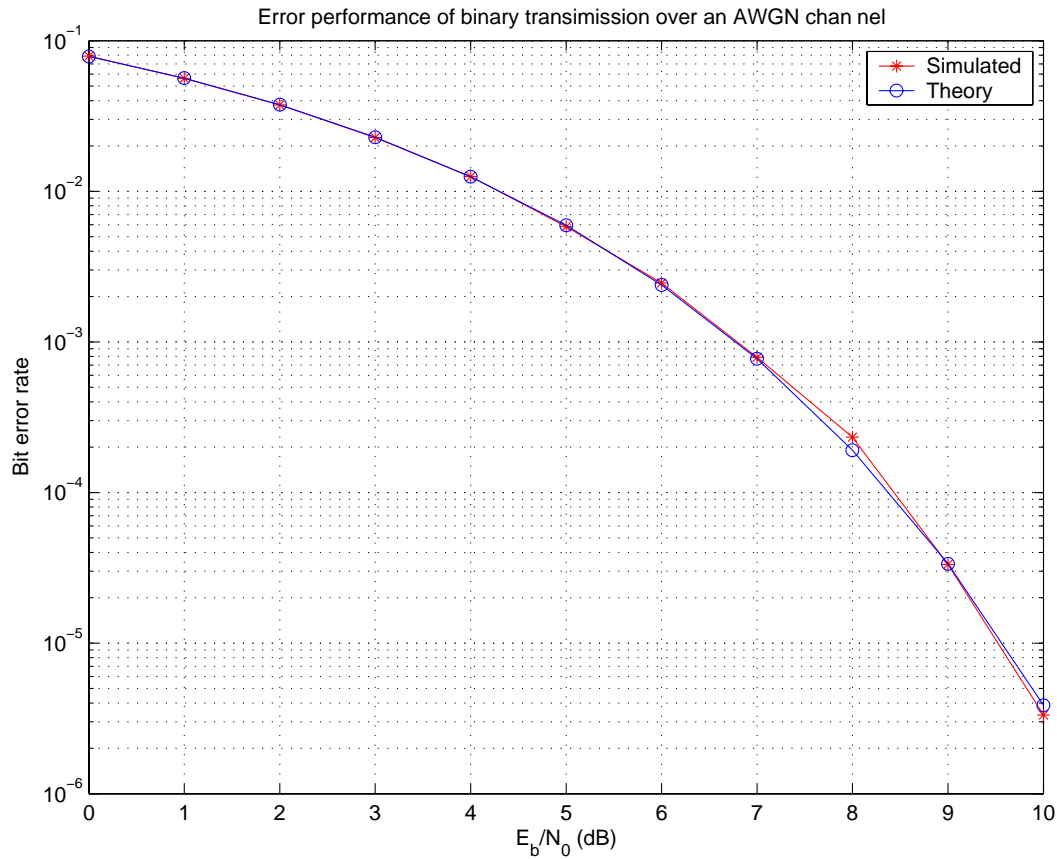
- (b) Download and execute the Matlab scripts `intdmp_simulation.m` and `Q.m` from the web site of the class. The first script simulates the transmission of random bits over an ideal AWGN channel and computes the bit error rate (BER) as a function of the signal-to-noise ratio (SNR),  $E_s/N_0$  in dB. The script will plot the simulated BER versus SNR as well as the theoretical expression for the bit error probability  $P[e] = Q(\sqrt{2E_s/N_0})$ . Execute the script with your student ID and sketch or print the resulting plot.

NOTE: The script may take several minutes to finish.

**Solution:**

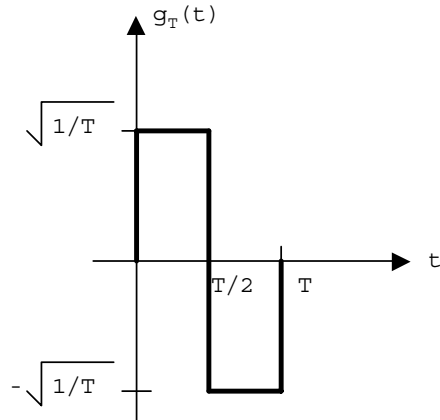
(a)





(b)

76. In computer communications at 10 Mbps using the Ethernet standard (IEEE 802.3), Manchester or bi-phase pulse formatting is used. Polar mapping is employed such that, in the interval  $[0, T]$ , a “0” is sent as  $s_0(t) = a g(t)$ , and a “1” is sent as  $s_1(t) = -a g(t)$ , where the pulse shape  $g_T(t)$  is sketched in the figure below:

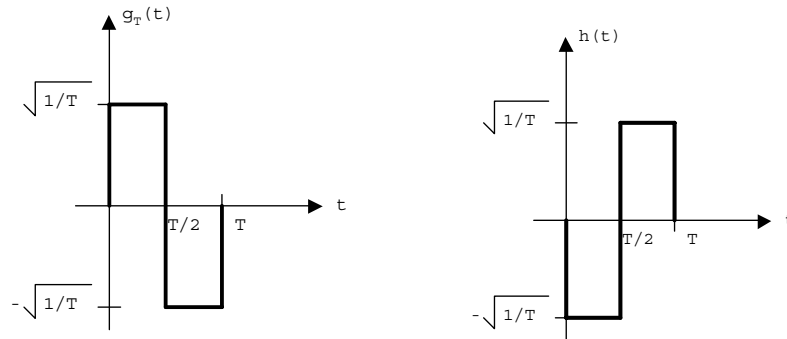


- Find the impulse response  $h(t)$  of the matched filter (MF) for  $g_T(t)$ .
- Assume that a “0” is sent and no noise is present. The input of the MF is  $r(t) = s_0(t)$ . Find the corresponding response  $y(t) = r(t) * h(t)$ , sampled at  $t = T$ . i.e.,  $Y = y(T)$ . What is the value of  $Y$  if a “1” is sent?
- A key advantage of the Manchester format is its capability to detect collisions. To see this, consider two bit sequences from two users. User 1 transmits the sequence “001101”

and user 2 the sequence “011011”. Sketch the corresponding transmitted sequences  $s_1(t)$  and  $s_2(t)$ , as well as the received sequence  $s_1(t) + s_2(t)$ . Comment on your results. Specifically, how is a collision detected?

**Solution:**

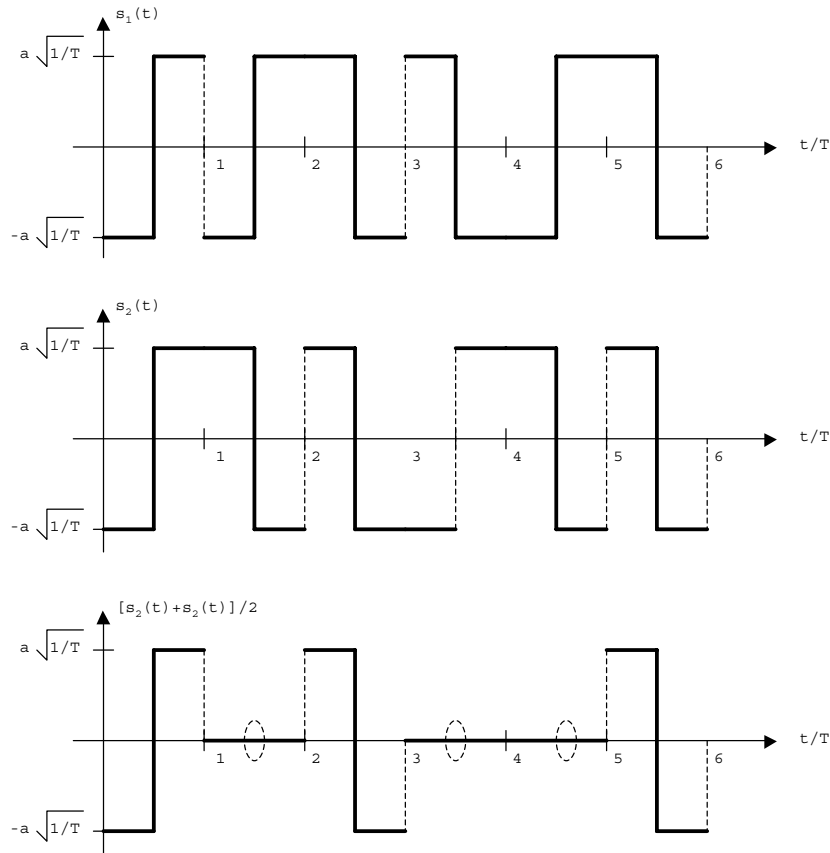
(a)



Pulse shape  $g_T(t)$  and impulse response of matched filter.

(b)  $Y = +a$ . If a “1” is sent, then  $Y = -a$ .

(c)



Collisions are detected in the received sequence  $s_1(t) + s_2(t)$  by the *absence of transitions* at times  $nT + T/2$ , for  $n = 1, 3, 4$ .

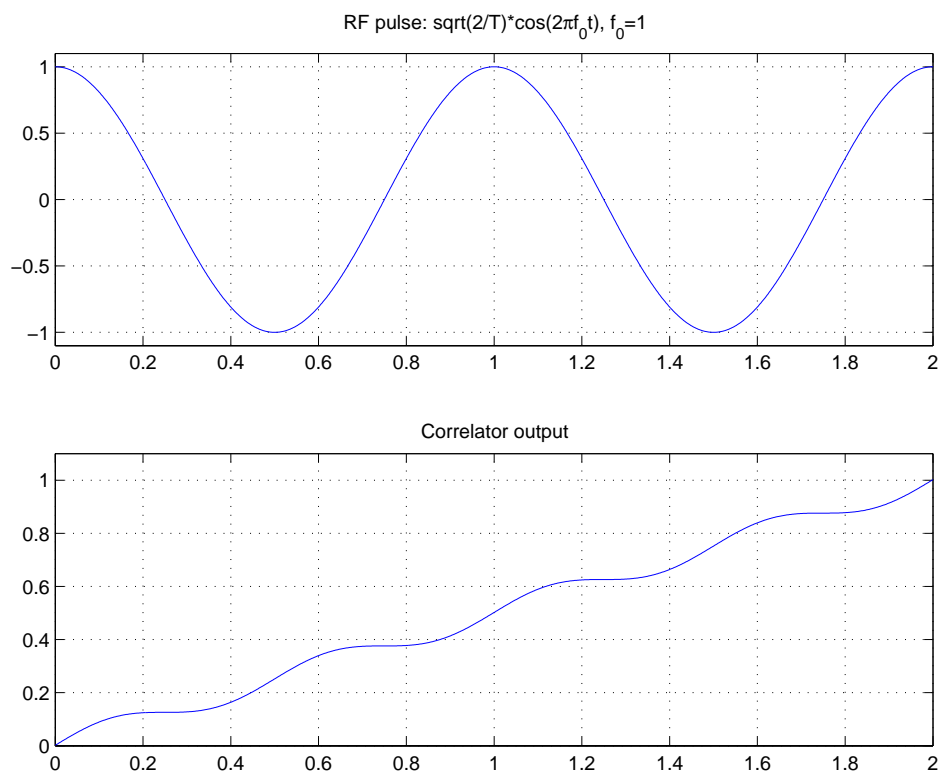


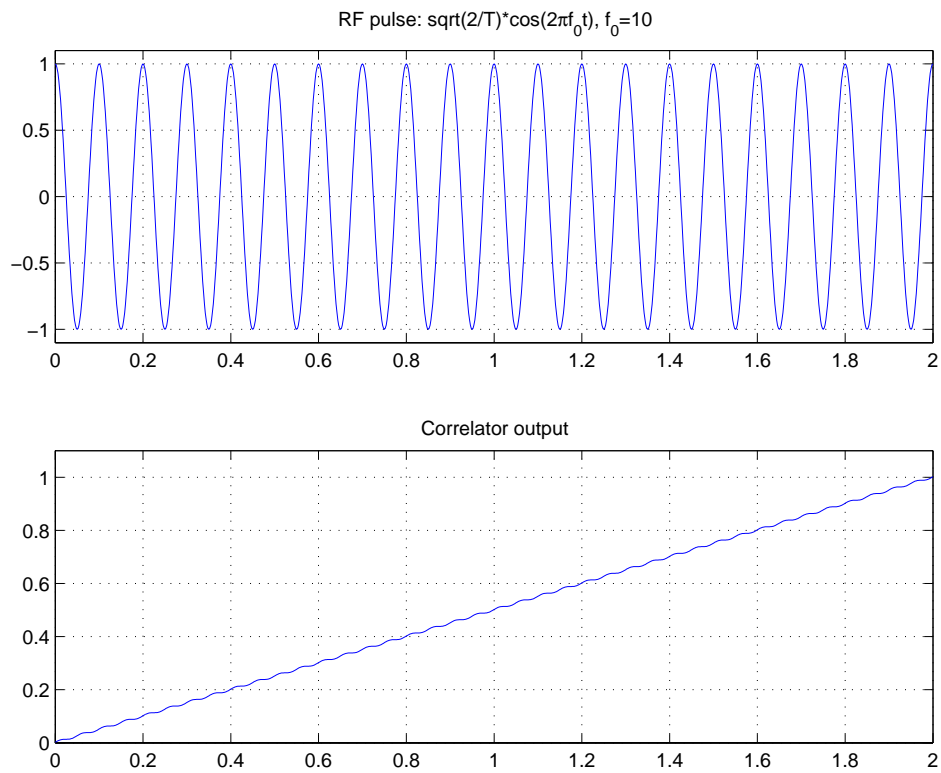
77. Download the Matlab script `rf_pulse_mfoutput.m` from the class website. This script computes the output of a correlator for an RF pulse,

$$g(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_0 t), \quad t \in [0, T],$$

where  $f_0 = n_0/T$ , and  $n_0$  is a positive integer. You are required to run the script and record the resulting plots for two values of  $n_0$ ,  $n_0 = 2$  and  $n_0 = 20$ . What conclusion can you draw from these plots?

**Solution:**



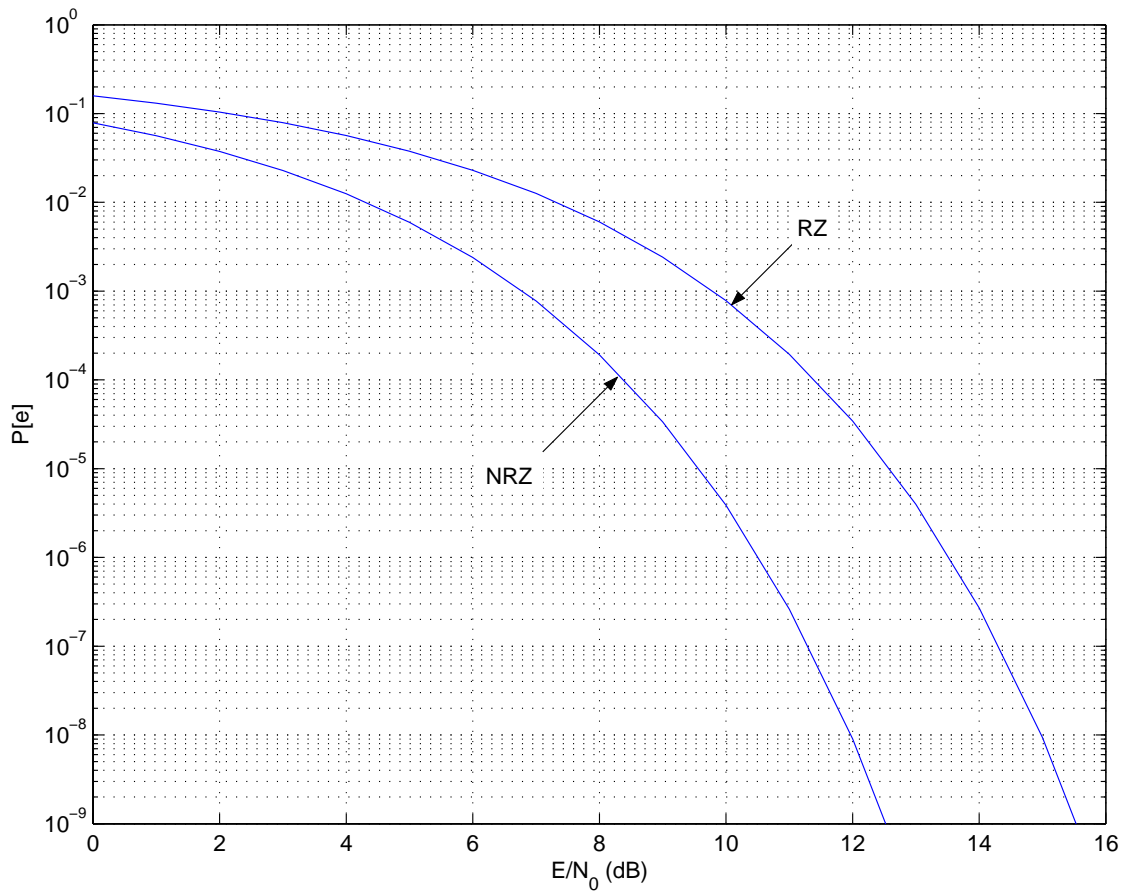


The MF output approaches a ramp as the value of the center frequency  $f_0$  increases.

78. Compare NRZ and RZ signaling techniques in terms of probability of a bit error  $P[e]$ . The bit rate  $R = 1/T$  and pulse amplitude  $a$  are fixed.
- Plot or sketch the curve of energy-to-noise ratio  $E/N_0$  (dB) versus probability of  $P[e]$ . (Hint: The energy of an RZ pulse is one half that of an NRZ pulse.)
  - How should the rate  $R$  be modified in the case of RZ signaling so that its  $P[e]$  is the same as NRZ signaling?

**Solution:**

-



- (b) To achieve the same  $P[e]$  as NRZ, for the same peak amplitude constraints, the rate of RZ should be 50% smaller than that of NRZ. To see this, consider as an example rectangular pulse signaling with polar mapping. Then  $E_{NRZ} = A^2T$  and  $E_{RZ} = A^2T/2$ . Therefore, the period  $T$  in RZ should be doubled.

79. Compare polar and unipolar bit mapping techniques in terms of  $P[e]$ . The peak pulse energy is fixed. Plot or sketch curve of energy-to-noise ratio  $E/N_0$  (dB) versus probability of a bit error for both techniques. (Hint: The distance between the signal points  $d_{12}$  is  $d_{12} = 2\sqrt{E}$  for polar and  $d_{12} = \sqrt{E}$  for unipolar.)

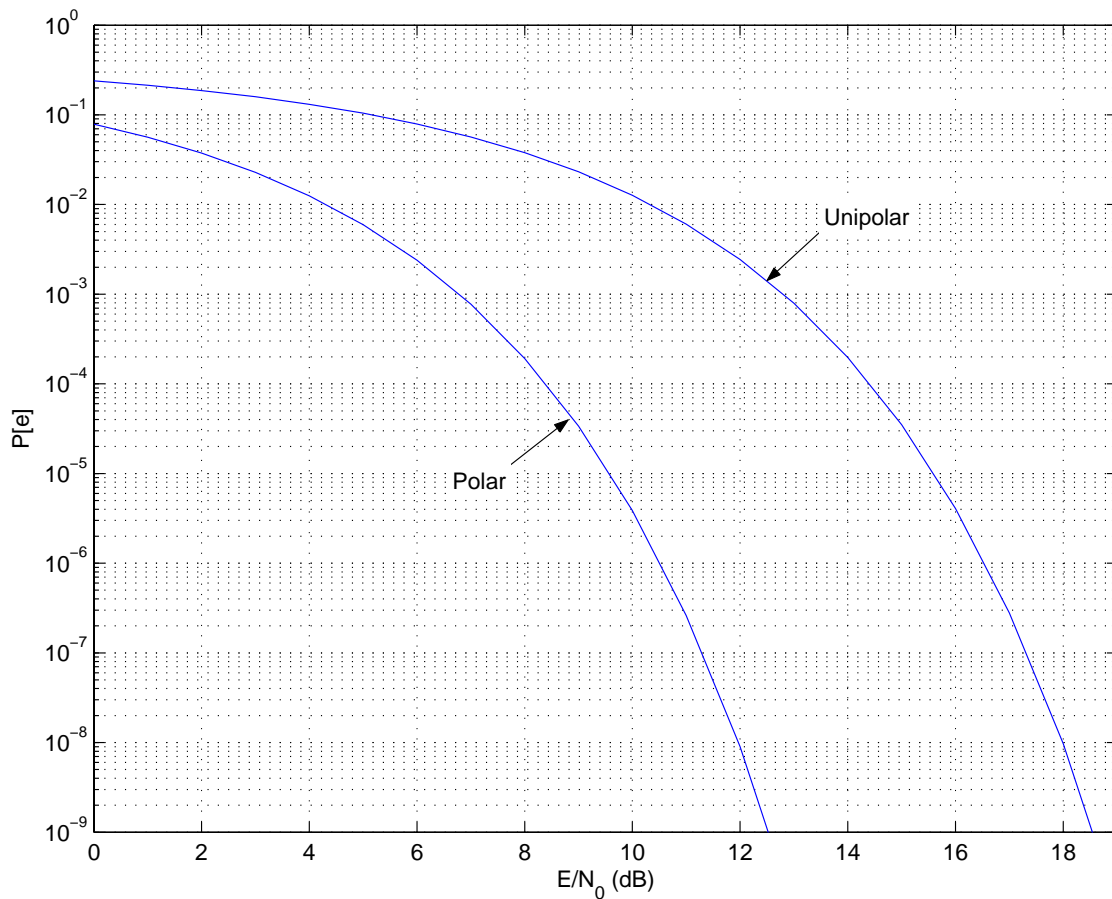
**Solution:** If the peak pulse energy  $E$  is fixed. Then for polar mapping

$$P[e] = Q\left(\sqrt{\frac{2E}{N_0}}\right),$$

and for unipolar mapping

$$P[e] = Q\left(\sqrt{\frac{E}{2N_0}}\right).$$

The figure below shows the plots of  $E/N_0$  in dB versus  $P[e]$ .



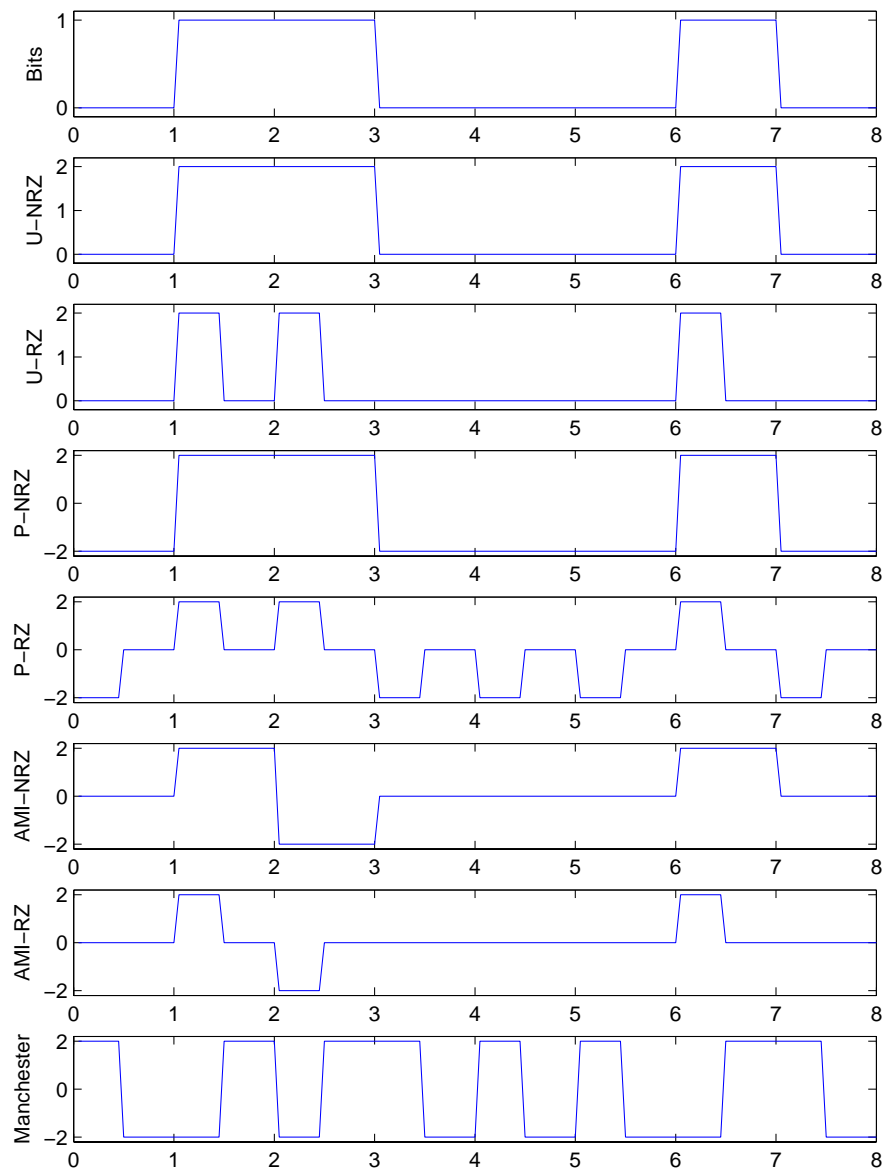
80. The bit stream  $\{b_n\} = 0, 1, 1, 0, 0, 0, 1, 0$  is to be sent through a channel (lowpass LTI system with large bandwidth). Assume that rectangular pulses of amplitude  $A$  are used and the bit rate is  $1/T$  bps. In polar mapping, use the rule:

$b_n$	$a_n$
0	$-A$
1	$+A$

Sketch the transmitted signal for each of the following line coding schemes:

- (a) Unipolar NRZ
- (b) Unipolar RZ
- (c) Polar NRZ
- (d) Polar RZ
- (e) AMI-NRZ (Assume that  $-A$  is the initial state).
- (f) AMI-RZ (Assume that  $-A$  is the initial state).
- (g) Manchester

**Solution:** (Line coding. Bit stream  $\{b_n\} = 0, 1, 1, 0, 0, 0, 1, 0$ .) The transmitted signals are shown in the figure below (amplitude  $A = 2$ ), and can be reproduced using the Matlab script `homework6s05.m` available in the class web site.



81. Compare the seven schemes in problem 1, in terms of average (DC) power and average (DC) amplitude level.

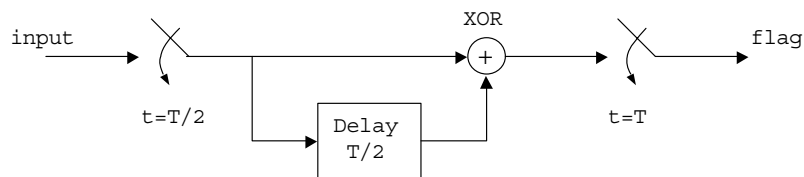
**Solution:**

Technique	$P_{\text{ave}}$	$\bar{a}_k$
U-NRZ	$\frac{A^2 T}{2}$	$\frac{A}{2}$
U-RZ	$\frac{A^2 T}{4}$	$\frac{A}{4}$
P-NRZ	$A^2 T$	0
P-RZ	$\frac{A^2 T}{2}$	0
AMI-NRZ	$\frac{A^2 T}{2}$	0
AMI-RZ	$\frac{A^2 T}{4}$	0
Manchester	$A^2 T$	0

82. Manchester coding has the desirable feature that it is possible to detect the presence of errors in the received signal. Explain how this is achieved. Sketch a block diagram of an error detection circuit.

**Solution:**

Manchester coding has the desirable feature that it is possible to detect the presence of errors in the received signal. This is done by checking that there is always a transition in the middle of a bit period. A simple block diagram on to achieve this is shown below:



The input is assumed to be sampled at a proper timing phase  $t = kT/2 + \tau$ , so that the XOR gate outputs a “1” at least every other sample (spaced by  $T/2$ ). The second sampler checks that there is a transition every bit. If this is the case, then the flag is always equal to “1”; otherwise, the flag is “0” and an error is detected.

A Simulink model based on this idea can be found in the web site of the class under the name `ETHERNET_ERROR_CHECK.mdl`. After a transition period, the system always outputs a “1” whenever the input is Manchester coded, and a “0” whenever there is no transition in (at least) the following bit.

### 83. Demodulation of binary antipodal signals

$$s_1(t) = -s_2(t) = \begin{cases} \sqrt{\frac{E_b}{T}}, & 0 \leq t \leq T; \\ 0, & \text{otherwise.} \end{cases}$$

can be accomplished by an integrate-and-dump receiver followed by a detector (threshold device).

- Determine the SNR at the output of the integrator sampled at  $t = T$ . (Hint: Show that the signal power at the output of the integrator, sampled at  $t = T$ , is  $E_s = E_b T$ . The noise power at the output of the integrator, sampled at  $t = T$ , is  $P_n = \frac{N_0}{2} T$ . The output SNR is then the ratio  $E_s/P_n$ .)
- Without any computation, what is the answer to part (a)? (See class notes. The integrator is the correlator of a rectangular pulse of width  $T$  and amplitude  $1/\sqrt{T}$ .)

#### Solution:

- The output of the integrator is

$$\begin{aligned} y(t) &= \int_0^t r(\tau) d\tau = \int_0^t [s_i(\tau) + n(\tau)] d\tau \\ &= \int_0^t s_i(\tau) d\tau + \int_0^t n(\tau) d\tau \end{aligned}$$

At time  $t = T$  we have

$$y(T) = \int_0^T s_i(\tau) d\tau + \int_0^T n(\tau) d\tau = \pm \sqrt{\frac{E_b}{T}} T + \int_0^T n(\tau) d\tau$$

The signal energy at the output of the integrator at  $t = T$  is

$$E_s = \left( \pm \sqrt{\frac{E_b}{T}} T \right)^2 = E_b T.$$

The noise power can be computed as follows:

$$\begin{aligned} P_n &= E \left[ \int_0^T \int_0^T n(\tau) n(v) d\tau dv \right] \\ &= \int_0^T \int_0^T E[n(\tau) n(v)] d\tau dv \\ &= \frac{N_0}{2} \int_0^T \int_0^T \delta(\tau - v) d\tau dv \end{aligned} \tag{5}$$

$$\begin{aligned} &= \frac{N_0}{2} \int_0^T \left\{ \int_0^T \delta(\tau - v) d\tau \right\} dv \\ &= \frac{N_0}{2} \int_0^T 1 dv = \frac{N_0}{2} T \end{aligned} \tag{6}$$

Hence, the output SNR is

$$\text{SNR} = \frac{E_s}{P_n} = \frac{2E_b}{N_0}$$

- (b) Without any computation, what is the answer to part (a)?

The signals can be expressed as

$$s_i(t) = \pm \sqrt{E_b} \psi(t), \quad i = 1, 2, \quad 0 < t \leq T,$$

where  $\psi(t)$  is a rectangular pulse of unit energy, i.e., amplitude  $1/\sqrt{T}$  and duration  $T$ . The filter matched to  $\psi(t)$  has impulse response  $h(t) = \psi(T - t)$ . Its output  $y(t)$  when  $s(t)$  is the input sampled at  $t = T$  is  $y(T) = Y = \sqrt{E_b} \cdot E_\psi = \sqrt{E_b}$ , and has energy  $E_b$ . Also, when the input is AWGN, the output is zero mean with variance  $N_0/2 \cdot E_\psi = N_0/2$ . Therefore,

$$\text{SNR} = \frac{2E_b}{N_0}$$

84. A binary communication system employs the signals

$$\begin{aligned} s_0 &= 0, & 0 \leq t \leq T; \\ s_1 &= A, & 0 \leq t \leq T, \end{aligned}$$

for transmission of the information. This is called *on-off signaling*. The demodulator cross-correlates the received signal  $r(t)$  with  $s_1(t)$  and sampled the output of the correlator at  $t = T$ .

- Determine the optimum detector for an AWGN channel and the optimum threshold, assuming the the signals are equally probable
- Find the probability of error as a function of the SNR. How does on-off signaling compare with antipodal signaling?

**Solution:**

- (a) The received signal may be expressed as

$$r(t) = \begin{cases} n(t) & \text{if } s_0(t) \text{ was transmitted} \\ A + n(t) & \text{if } s_1(t) \text{ was transmitted} \end{cases}$$

Assuming that  $s(t)$  has unit energy, then the sampled outputs of the correlators are

$$r = s_m + n, \quad m = 0, 1$$

where  $s_0 = 0$ ,  $s_1 = A\sqrt{T}$  and the noise term  $n$  is a zero-mean Gaussian random variable with variance

$$\begin{aligned} \sigma_n^2 &= E \left[ \frac{1}{\sqrt{T}} \int_0^T n(t) dt \frac{1}{\sqrt{T}} \int_0^T n(\tau) d\tau \right] \\ &= \frac{1}{T} \int_0^T \int_0^T E [n(t)n(\tau)] dt d\tau \\ &= \frac{N_0}{2T} \int_0^T \int_0^T \delta(t - \tau) dt d\tau = \frac{N_0}{2} \end{aligned}$$



The probability density function for the sampled output is

$$\begin{aligned} f(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\ f(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of  $s_0$  if

$$\text{PM}(\mathbf{r}, \mathbf{s}_0) = f(r|s_0) > f(r|s_1) = \text{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of  $s_1$ . The decision rule may be expressed as

$$\frac{\text{PM}(\mathbf{r}, \mathbf{s}_0)}{\text{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \begin{matrix} s_0 \\ \geq \\ s_1 \end{matrix} \quad 1$$

or equivalently

$$\begin{matrix} s_1 \\ r \geq \\ s_0 \end{matrix} \quad \frac{1}{2} A \sqrt{T}$$

The optimum threshold is  $\frac{1}{2} A \sqrt{T}$ .

(b) The average probability of error is

$$\begin{aligned} P(e) &= \frac{1}{2} P(e|s_0) + \frac{1}{2} P(e|s_1) \\ &= \frac{1}{2} \int_{\frac{1}{2} A \sqrt{T}}^{\infty} f(r|s_0) dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2} A \sqrt{T}} f(r|s_1) dr \\ &= \frac{1}{2} \int_{\frac{1}{2} A \sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2} A \sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} dr \\ &= \frac{1}{2} \int_{\frac{1}{2} \sqrt{\frac{2}{N_0}} A \sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \frac{1}{2} \int_{-\infty}^{-\frac{1}{2} \sqrt{\frac{2}{N_0}} A \sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= Q \left[ \frac{1}{2} \sqrt{\frac{2}{N_0}} A \sqrt{T} \right] = Q \left[ \sqrt{\text{SNR}} \right] \end{aligned}$$

where

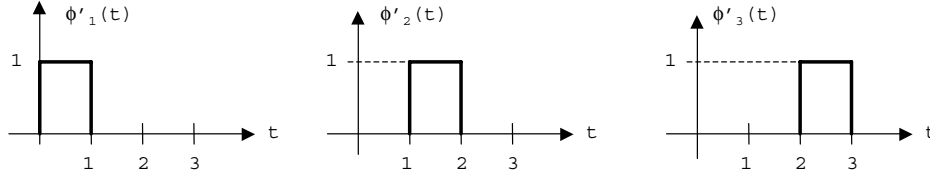
$$\text{SNR} = \frac{\frac{1}{2} A^2 T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

## Multi-dimensional signals

86. In class, the first three steps of the Gramm-Schmidt procedure for problem 7.6 in the textbook were completed.

- (a) Perform step 4 of the procedure and find the 3-dimensional basis,  $\Phi \triangleq \{\phi_1(t), \phi_2(t), \phi_3(t)\}$ , that contains the 4 signals  $s_m(t)$ ,  $m = 1, 2, 3, 4$ . Represent the signals as points  $\bar{s}_m$ ,  $m = 1, 2, 3, 4$ , in this three-dimensional signal space.
- (b) Verify that the distance to the origin (or norm) of the points  $\bar{s}_m$  in the signal space with basis  $\Phi$  of part (a) is the same as that of the signal points obtained with the basis  $\Phi' \triangleq \{\phi'_1(t), \phi'_2(t), \phi'_3(t)\}$ , where  $\phi'_n(t)$ ,  $n = 1, 2, 3$ , are illustrated in Fig. 1 below.



Orthonormal waveforms in basis  $\Phi'$ .

**Solution:**

- (a) In class it was shown that

$$\begin{aligned}\phi_1(t) &= \frac{1}{\sqrt{3}}, \quad 0 < t \leq 3, \\ \phi_2(t) &= \begin{cases} \sqrt{\frac{2}{3}}, & 0 < t \leq 1; \\ \frac{1}{2}\sqrt{\frac{2}{3}}, & 1 < t \leq 3, \end{cases} \\ \phi_3(t) &= 0.\end{aligned}$$

Compute the projections,  $c_{41}$  and  $c_{42}$ , of  $s_4(t)$  onto the signals  $\phi_1(t)$  and  $\phi_2(t)$ , respectively,

$$\begin{aligned}c_{41} &= \int_0^3 s_4(t)\phi_1(t)dt = \frac{4}{\sqrt{3}}, \\ c_{42} &= \int_0^3 s_4(t)\phi_2(t)dt = \sqrt{\frac{2}{3}}.\end{aligned}$$

Then form the difference signal

$$d_4(t) = s_4(t) - c_{41}\phi_1(t) - c_{42}\phi_2(t) = \begin{cases} 0, & 0 < t \leq 1; \\ 1, & 1 < t \leq 2; \\ -1, & 2 < t \leq 3. \end{cases}$$

The energy of  $d_4(t)$  is  $E_{d_4} = 2$  and it follows that

$$\phi_4(t) = \frac{d_4(t)}{\sqrt{E_{d_4}}} = \begin{cases} 0, & 0 < t \leq 1; \\ \frac{1}{\sqrt{2}}, & 1 < t \leq 2; \\ -\frac{1}{\sqrt{2}}, & 2 < t \leq 3. \end{cases}$$

The coordinates of the vector representation  $\bar{s}_m = (s_{m1} \ s_{m2} \ s_{m3})$  of the signal  $s_m(t)$ , with respect to the basis  $\Phi = \{\phi_1(t), \phi_2(t), \phi_4(t)\}$ ,  $1 \leq m \leq 4$ , are computed as

$$s_{mn} = \int_0^3 s_m(t) \phi_n(t) dt, \quad 1 \leq m \leq 4, \quad 1 \leq n \leq 3.$$

This results in

$$\begin{aligned} \bar{s}_1 &= \begin{bmatrix} 2\sqrt{3} & 0 & 0 \end{bmatrix} \\ \bar{s}_2 &= \begin{bmatrix} \frac{2}{\sqrt{3}} & 2\sqrt{\frac{2}{3}} & 0 \end{bmatrix} \\ \bar{s}_3 &= \begin{bmatrix} -\frac{4}{\sqrt{3}} & 2\sqrt{\frac{2}{3}} & 0 \end{bmatrix} \\ \bar{s}_4 &= \begin{bmatrix} \frac{4}{\sqrt{3}} & \sqrt{\frac{2}{3}} & \frac{2}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

The energies (squared distances to the origin) of the signals are computed from their vector representation as

$$E_m = \sum_{n=1}^3 |s_{mn}|^2, \quad 1 \leq m \leq 4.$$

It follows that

$$E_1 = 12, \quad E_2 = \frac{4+8}{3} = 4, \quad E_3 = \frac{16+8}{3} = 8, \quad E_4 = \frac{16+2}{3} + 2 = 8.$$

(b) In this case, by inspection, the vector representation of the signals is easy to obtain:

$$\begin{aligned} \bar{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\ \bar{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\ \bar{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\ \bar{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \end{aligned}$$

Note that  $s_3(t) = s_2(t) - s_1(t)$  and therefore the dimensionality of the signal space spanned by  $\Phi'$  is 3. It is easy to check that the signal energies obtained from this vector representation are the same as those obtained in part (a) above.

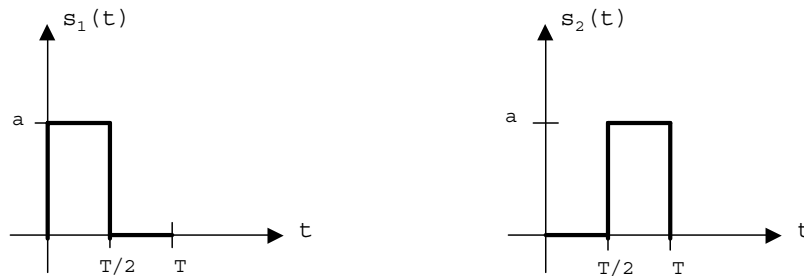
87. The signals  $\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$  and  $\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$ , for  $(k-1)T < t \leq kT$ , where  $k$  is an integer and  $T$  is the symbol duration, are used as basis functions for bandpass digital modulation.

Show that  $\phi_1(t)$  and  $\phi_2(t)$  are orthogonal over the interval  $[(k-1)T, kT]$ .

**Solution:** It is relatively easy to show that

$$\int_{(k-1)T}^{kT} \phi_1(t) \phi_2(t) dt = \frac{2}{T} \int_{(k-1)T}^{kT} \cos(2\pi f_c t) \sin(2\pi f_c t) dt = 0.$$

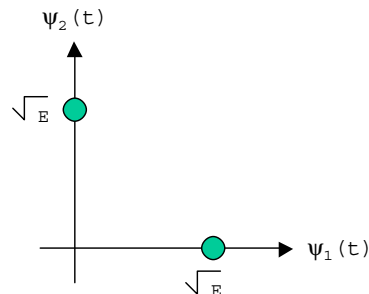
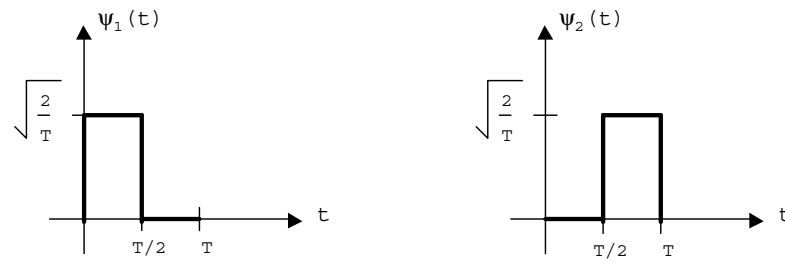
88. The figure below depicts two pulse shapes used in a binary orthogonal signaling scheme known as binary PPM (or 2-PPM). Transmission takes place over an AWGN channel with  $S_N(f) = \frac{N_0}{2}$  W/Hz.



- Find the value of the amplitude  $a$  in terms of the energy per bit  $E_b$  and the bit duration  $T$ .
- Determine the orthonormal basis signals  $\psi_1(t)$  and  $\psi_2(t)$  for 2-PPM and sketch the location of the signal points  $\bar{s}_1$  and  $\bar{s}_2$ , corresponding to signals  $s_1(t)$  and  $s_2(t)$ , in the two-dimensional signal space.
- Give an expression for the probability of a bit error. (Hint: Use the general expression  $P[\text{error}] = Q\left(\sqrt{d_{12}^2/2N_0}\right)$ .)

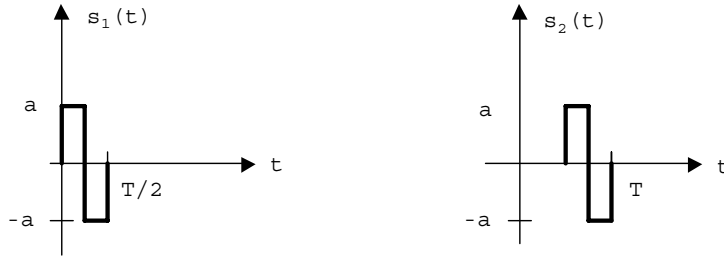
**Solution:**

- $a = \sqrt{\frac{2E_b}{T}}$ .
- 



(c)  $P[\text{error}] = Q\left(\sqrt{\frac{E_b}{N_0}}\right).$

89. The two pulses shown in the figure below are employed in a binary orthogonal signaling scheme to be used in a digital communication system.

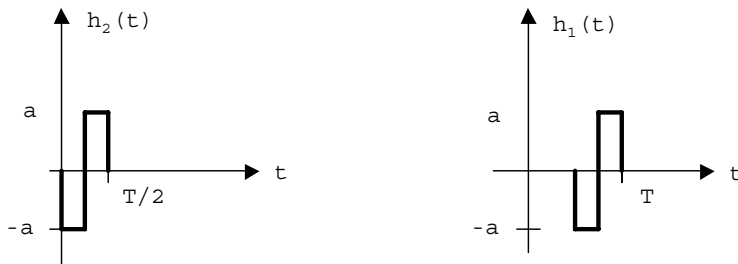


Pulses used in a binary orthogonal signaling scheme.

- Sketch carefully the impulse responses of a pair of matched filters for this system.
- Find the means and variances of the outputs of the matched filters, sampled at  $t = T$ , when  $s_1(t)$  is sent through an AWGN channel with noise energy  $N_0/2$ .
- A correlator receiver is employed. However, due to a timing error, the output is incorrectly sampled at  $t = 0.75T$ . What is the increase in the probability of a bit error,  $P_b$ , compared to that of an ideal system?

**Solution:**

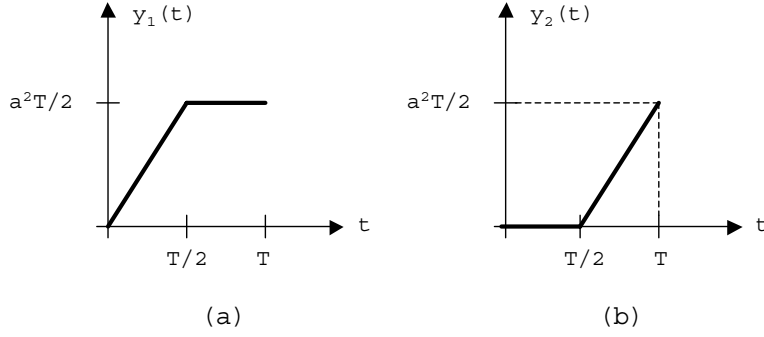
- (a) Matched filters:



- (b) When  $s_1(t)$  is sent,  $Y_1$  is a Gaussian r.v. of mean  $m_1 = E_1 = a^2T/2$  and variance  $\sigma_1^2 = \frac{N_0}{2}E_1 = \frac{N_0}{2}(a^2T/2)$ , and  $Y_2$  is a Gaussian r.v. of mean  $m_2 = 0$  and variance  $\sigma_2^2 = \frac{N_0}{2}E_2 = \frac{N_0}{2}(a^2T/2)$ .

Alternatively, defining the orthonormal functions  $\psi_1(t)$  and  $\psi_2(t)$ , we can write  $s_1(t) = a\sqrt{T/2}\psi_1(t)$ . In this case, the outputs of the filters matched to the orthonormal functions, given that  $s_1(t)$  is sent, are such that  $Y_1$  is a Gaussian r.v. of mean  $m_1 = a\sqrt{T/2}$  and variance  $\sigma_1^2 = \frac{N_0}{2}$ , and  $Y_2$  is a Gaussian r.v. of mean  $m_2 = 0$  and variance  $\sigma_2^2 = \frac{N_0}{2}$ .

(c) Correlator outputs:



(a) First output when  $s_1(t)$  sent. (b) Second output when  $s_2(t)$  sent.

When sampled at  $t = 0.75T$ , the mean and variance of  $Y_1$ , given that  $s_1(t)$  is sent, remain at  $m_1 = \frac{a^2T}{2}$  and  $\sigma_1^2 = \frac{N_0}{2} \left( \frac{a^2T}{2} \right)$ . However, when  $s_2(t)$  is sent, the mean and variance of  $Y_2$  change to  $m_2 = (0.5) \left( \frac{a^2T}{2} \right) = \frac{a^2T}{4}$  and  $\sigma_1^2 = (0.5) \frac{N_0}{2} \left( \frac{a^2T}{2} \right) = \frac{N_0}{2} \left( \frac{a^2T}{4} \right)$ . From this observation, it follows that

$$P[\text{error} | \bar{s}_1 \text{ sent}] = Q \left( \sqrt{\frac{E}{N_0}} \right) = Q \left( \sqrt{\frac{a^2T}{2N_0}} \right), \text{ and}$$

$$P[\text{error} | \bar{s}_2 \text{ sent}] = Q \left( \sqrt{\frac{E}{2N_0}} \right) = Q \left( \sqrt{\frac{a^2T}{4N_0}} \right).$$

As a result, the average probability of a bit error becomes

$$P_b = \frac{1}{2} \left[ Q \left( \sqrt{\frac{a^2T}{2N_0}} \right) + Q \left( \sqrt{\frac{a^2T}{4N_0}} \right) \right].$$

In conventional binary orthogonal signaling,

$$P_b = Q \left( \sqrt{\frac{a^2T}{2N_0}} \right).$$

The increase in the probability of a bit error, due to the timing error, is

$$\Delta_{P_b} = \frac{1}{2} \left[ Q \left( \sqrt{\frac{a^2T}{4N_0}} \right) - Q \left( \sqrt{\frac{a^2T}{2N_0}} \right) \right].$$

## Pulse-amplitude modulation (PAM)

90. A binary PAM communication system is used to transmit data over an AWGN channel. The prior probabilities for the bits are  $\Pr\{a_m = 1\} = 1/3$  and  $\Pr\{a_m = -1\} = 2/3$ .

- (a) Determine the optimum maximum-likelihood decision rule for the detector  
(b) Find the average probability of a bit error

**Solution:**

- (a) The optimum threshold is given by

$$\eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$$

- (b) The average probability of error is ( $\eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$ )

$$\begin{aligned} P(e) &= p(a_m = -1) \int_{\eta}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &\quad + p(a_m = 1) \int_{-\infty}^{\eta} \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &= \frac{2}{3} Q \left[ \frac{\eta + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}} \right] + \frac{1}{3} Q \left[ \frac{\sqrt{\mathcal{E}_b} - \eta}{\sqrt{N_0/2}} \right] \\ &= \frac{2}{3} Q \left[ \frac{\sqrt{2N_0/\mathcal{E}_b} \ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}} \right] + \frac{1}{3} Q \left[ \sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b} \ln 2}{4} \right] \end{aligned}$$

91. Determine the average energy of a set of  $M$  PAM signals of the form

$$s_m(t) = s_m \phi(t), \quad m = 1, 2, \dots, M, \quad 0 \leq t \leq T,$$

where

$$s_m = \sqrt{E_g} A_m, \quad m = 1, 2, \dots, M.$$

The signals are equally probable with amplitudes that are symmetric about zero and are uniformly spaced with distance  $d$  between adjacent amplitudes.

**Solution:** The amplitudes  $A_m$  take the values

$$A_m = (2m - 1 - M) \frac{d}{2}, \quad m = 1, \dots, M$$

Hence, the average energy is

$$\begin{aligned} E_{ave} &= \frac{1}{M} \sum_{m=1}^M s_m^2 = \frac{d^2}{4M} E_g \sum_{m=1}^M (2m - 1 - M)^2 \\ &= \frac{d^2}{4M} E_g \sum_{m=1}^M [4m^2 + (M+1)^2 - 4m(M+1)] \\ &= \frac{d^2}{4M} E_g \left( 4 \sum_{m=1}^M m^2 + M(M+1)^2 - 4(M+1) \sum_{m=1}^M m \right) \\ &= \frac{d^2}{4M} E_g \left( 4 \frac{M(M+1)(2M+1)}{6} + M(M+1)^2 - 4(M+1) \frac{M(M+1)}{2} \right) \\ &= \frac{M^2 - 1}{3} \frac{d^2}{4} E_g. \end{aligned}$$

Moreover, note that if  $a = d/2$  and  $E_g = 1$ , as we did in class, then

$$E_{ave} = \frac{M^2 - 1}{3} a^2,$$

and therefore

$$a = \sqrt{\frac{3 E_{ave}}{M^2 - 1}}.$$

92. A speech signal is sampled at a rate of 8 KHz, logarithmically compressed and encoded into a PCM format using 8 bits per sample. The PCM data is transmitted through an AWGN baseband channel via  $M$ -level PAM signaling. Determine the required transmission bandwidth when (a)  $M = 4$ , (b)  $M = 8$  and (c)  $M = 16$ . (Assume rectangular pulses and the zero-to-null definition of bandwidth.)

**Solution:** The bandwidth required for transmission of an  $M$ -ary PAM signal is

$$W = \frac{R_b}{2 \log_2 M} \text{ Hz}$$

The bit rate can be obtained from the data given as

$$R_b = 8 \times 10^3 \frac{\text{samples}}{\text{sec}} \times 8 \frac{\text{bits}}{\text{sample}} = 64 \times 10^3 \frac{\text{bits}}{\text{sec}}$$

As a result,

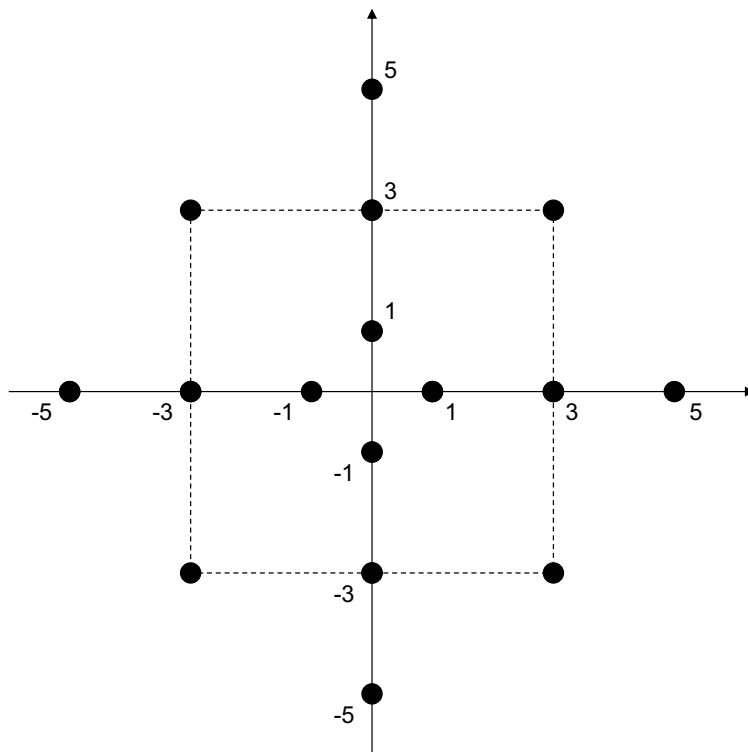
$$W = \begin{cases} 16 \text{ KHz}, & M = 4; \\ 10.667 \text{ KHz}, & M = 8; \\ 8 \text{ KHz}, & M = 16. \end{cases}$$

93. HOMEWORK 9, Spring 2005. Problems 1 and 3

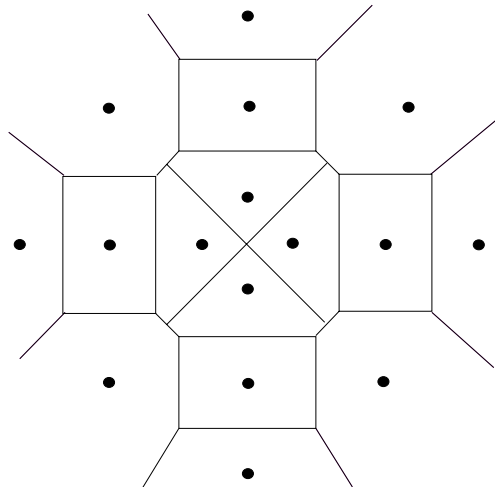
## Quadrature-amplitude modulation (QAM)

94. The 16-QAM signal constellation shown in the figure below is an international standard for (analog) telephone-line modems known as V.29. Determine the decision boundaries for the detector, assuming equiprobable signals and high SNR so that errors only occur between adjacent points.





**Solution:** The optimum decision boundary of a point is determined by the perpendicular bisectors of each line segment connecting the point with its neighbors. The decision regions for the V.29 constellation are sketched in the next figure:



95. A digital communication system transmits data using QAM signaling over a voice-band telephone channel at a rate 2400 symbols/s (baud). The additive noise is assumed to be white and Gaussian. You are asked to determine the energy-per-bit-to-noise ratio  $E_b/N_0$  required to achieve an error probability of  $10^{-5}$  for a bit rate equal to:

- (a) 4800 bits/s
- (b) 9600 bits/s
- (c) 19200 bits/s

(d) 31200 bits/s

(e) What conclusions can you draw from parts (a) to (d)?

(Hint: Use the following expression for the error probability of an  $M$ -ary QAM system:

$$P_M \approx 4 Q \left( \sqrt{\frac{3\ell E_b}{(M-1)N_0}} \right),$$

where  $M = 2^\ell$  and  $\ell$  is the *number of bits per symbol*. Also, if needed, use the following approximation of the Gaussian  $Q$ -function:

$$Q(x) \approx \frac{1}{2} e^{-\frac{x^2}{2}}.$$

**Solution:**

We assume that ideal Nyquist signaling is employed so that the bandwidth of the bandpass signal is equal to the baud rate (symbols/second), i.e.,  $1/T$  Hz.

(a) The number of bits per symbol is

$$\ell = \frac{4800}{W} = \frac{4800}{2400} = 2.$$

Thus, a 4-QAM constellation is used for transmission. Using the given approximation of the probability of error for an  $M$ -ary QAM system, with  $P_M = 10^{-5}$  and  $\ell = 2$  we obtain

$$4 Q \left( \sqrt{\frac{2E_b}{N_0}} \right) = 10^{-5} \implies \frac{E_b}{N_0} = 10.58 = 10.24 \text{ dB},$$

where Table 5.1, on page 220 of the textbook has been used.

(b) If the bit rate of transmission is 9600 bps, then

$$\ell = \frac{9600}{2400} = 4.$$

In this case a 16-QAM constellation is used and

$$4 Q \left( \sqrt{\frac{4 \times E_b}{15 \times N_0}} \right) = 10^{-5} \implies \frac{E_b}{N_0} = 26.45 = 14.22 \text{ dB}.$$

(c) If the bit rate of transmission is 19200 bps, then

$$k = \frac{19200}{2400} = 8$$

In this case a 256-QAM constellation is used and

$$\frac{E_b}{N_0} = 224.82 = 23.52 \text{ dB}.$$

(d) If the bit rate of transmission is 31200 bps, then

$$k = \frac{31200}{2400} = 13$$

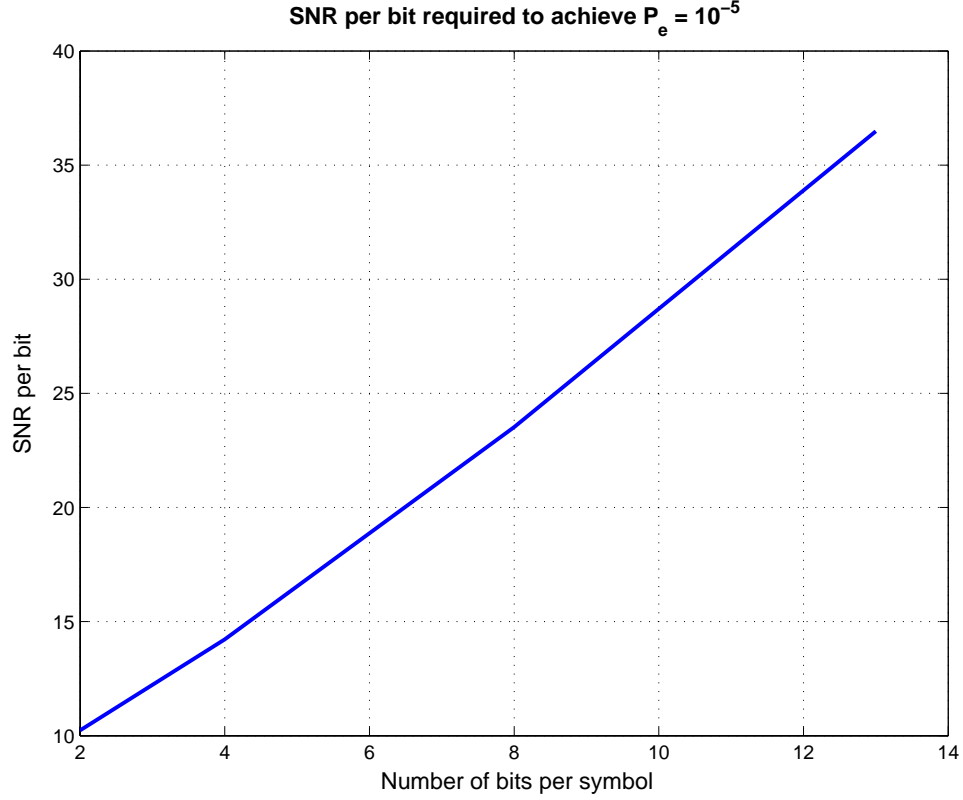
In this case an 8192-QAM constellation is used and

$$\frac{E_b}{N_0} = 4444.14 = 36.48 \text{ dB}.$$

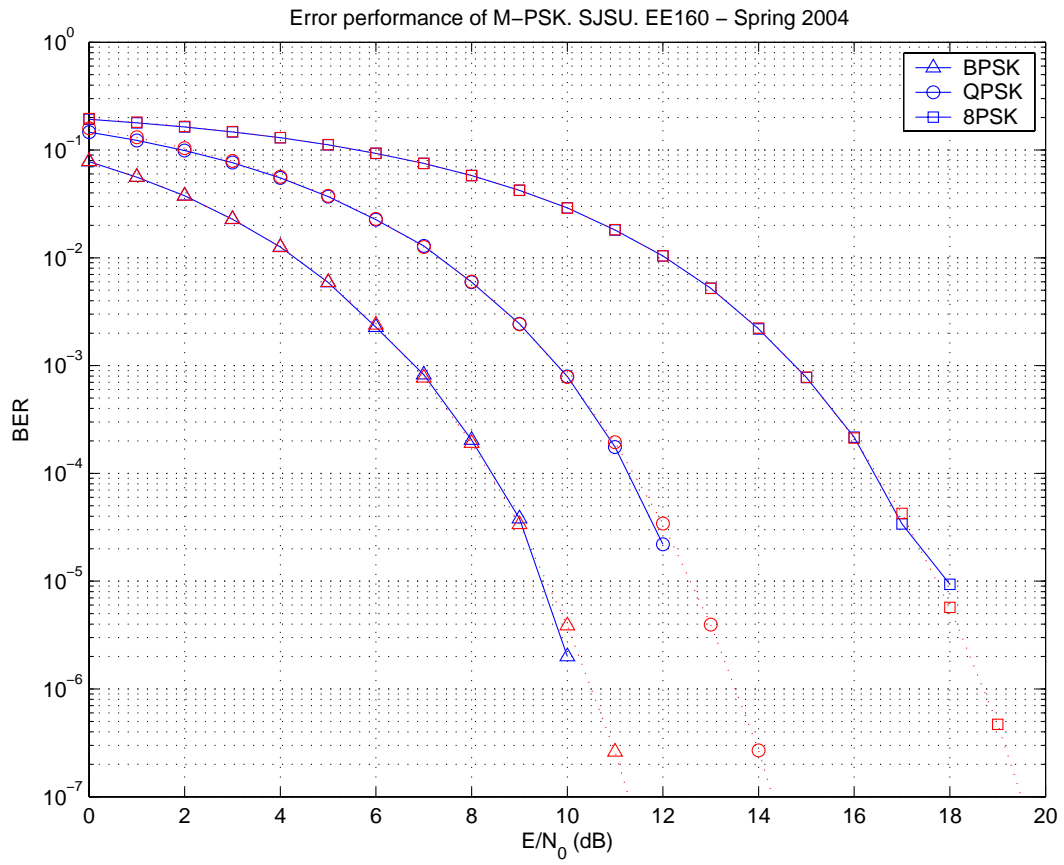
- (e) The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts (a)-(d):

$\ell$	2	4	8	13
SNR per bit (dB)	10.24	14.22	23.52	36.48

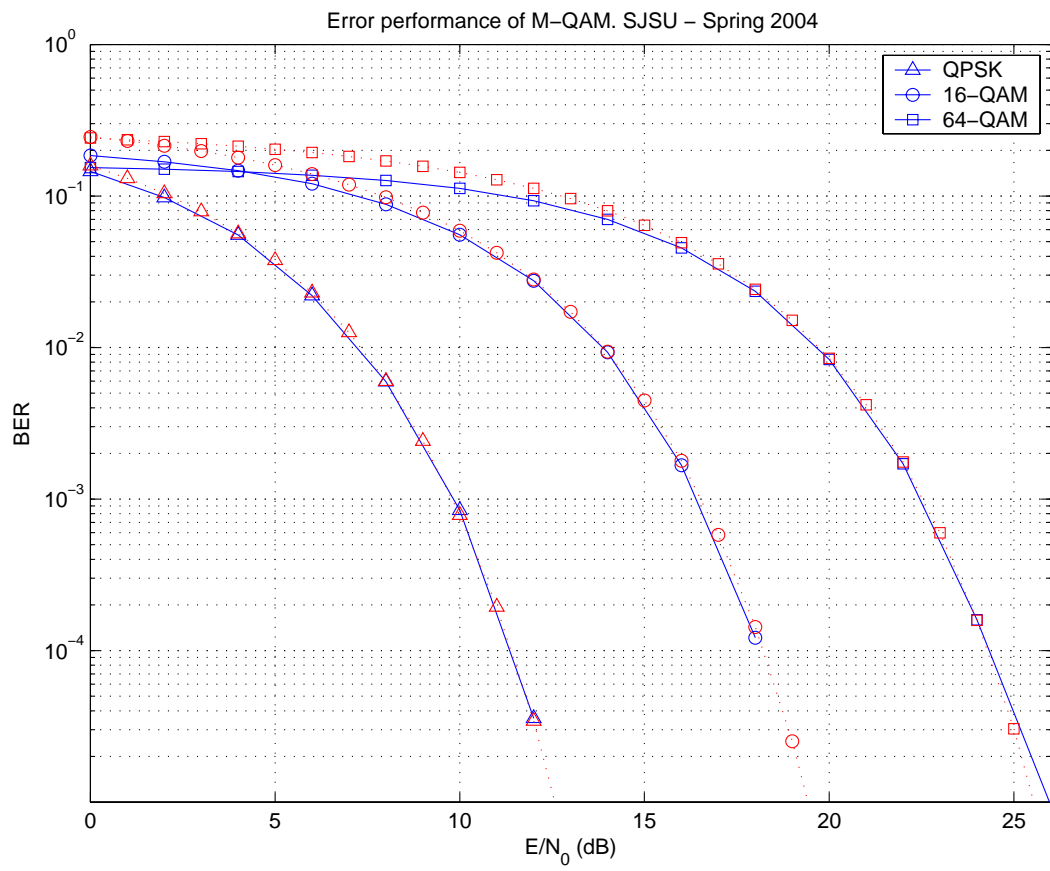
As observed in the next plot, there is an increase in average transmitted power of approximately *2.5 dB per additional bit per symbol*:



96. Plots of the approximated probability of a bit error  $P_b$  are compared with the simulated BER, for  $M$ -PSK and  $M$ -QAM, in Figs. 1 and 2, respectively, with Matlab scripts `digmodMPSK.m` and `digmodMQAM.m`. The approximated  $P_b$  values were produced by the Matlab scripts `digmodMPSK_approx.m` and `digmodMQAM_approx.m` posted in the web site.



Simulation (blue solid line) versus approximated  $P_b$  (red dotted line) for  $M$ -PSK.



Simulation (blue solid line) versus approximated  $P_b$  (red dotted line) for  $M$ -QAM.

## Phase-shift keying (PSK) modulation

98. Binary PSK (BPSK) is used for data transmission over an AWGN channel with power spectral density  $N_0/2 = 10^{-10}$  W/Hz. The transmitted signal energy is  $E_b = A^2T/2$ , where  $T$  is the bit duration and  $A$  is the signal amplitude. Determine the value of  $A$  needed to achieve an error probability of  $10^{-6}$ , if the data rate is:

- (a) 10 Kbit/s
- (b) 100 Kbit/s
- (c) 1 Mbit/s

**Solution:** For binary phase modulation, the error probability is

$$P_2 = Q \left[ \sqrt{\frac{2\mathcal{E}_b}{N_0}} \right] = Q \left[ \sqrt{\frac{A^2T}{N_0}} \right]$$

With  $P_2 = 10^{-6}$  we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \implies A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is  $T = 10^{-4}$  and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is  $10^5$  bps and  $10^6$  bps, the required amplitude of the signal is  $A = 2.12 \times 10^{-2}$  and  $A = 6.703 \times 10^{-2}$  respectively.

99. A QPSK modulated signal can be written as

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \theta_i), \quad (k-1)T < t \leq kT$$

where

$$\theta_i = (i-1)\frac{\pi}{2}, \quad i = 1, 2, 3, 4.$$

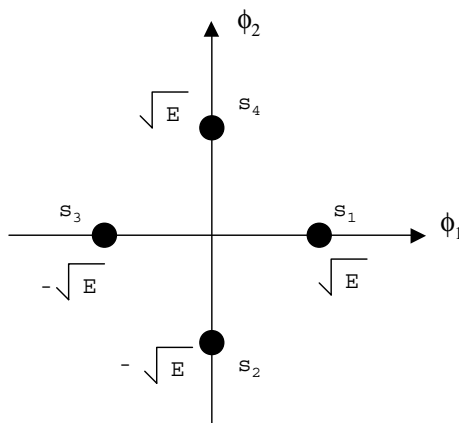
- (a) Express  $s_i(t)$  in terms of the basis functions  $\phi_1(t)$  and  $\phi_2(t)$  defined in Problem 1. (Hint: Use a well-known trigonometric identity.)
- (b) Represent the signals  $s_i(t)$ ,  $i = 1, 2, 3, 4$ , as vectors in the  $\phi_1\phi_2$ -plane (or IQ-plane).
- (c) Sketch carefully the decision regions  $Z_i$ ,  $i = 1, 2, 3, 4$ , in the  $\phi_1\phi_2$ -plane.

**Solution:**

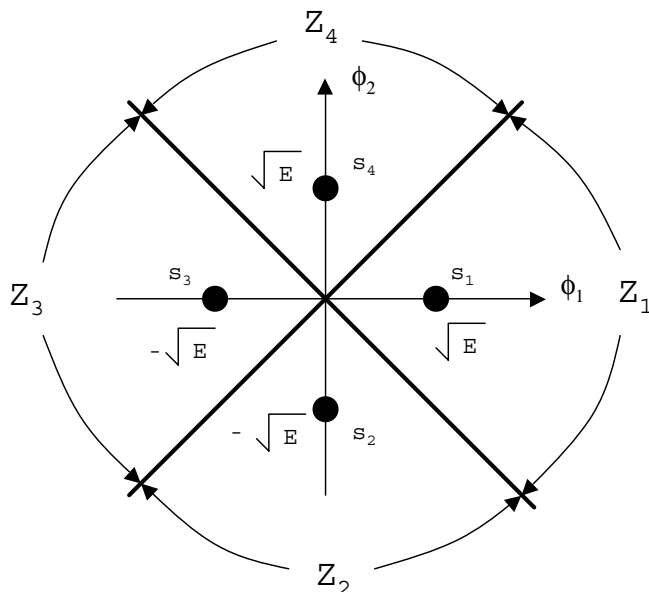
- (a) Using the trigonometric identity  $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ , we have that

$$\begin{aligned}
 s_i(t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \theta_i) \\
 &= \sqrt{\frac{2E}{T}} [\cos(\theta_i) \cos(2\pi f_c t) - \sin(\theta_i) \sin(2\pi f_c t)] \\
 &= \sqrt{E} \cos(\theta_i) \phi_1(t) - \sqrt{E} \sin(\theta_i) \phi_2(t)
 \end{aligned}$$

- (b) Representation in the  $\phi_1\phi_2$ -plane (note clockwise numbering of signals):



- (c) Decision regions:



100. Using a computer environment (e.g., Matlab), compare the exact probability of a symbol error,  $P[\epsilon]$ , with the union bound. What is the minimum value of  $E/N_0$  (dB) after which the union bound is tight?

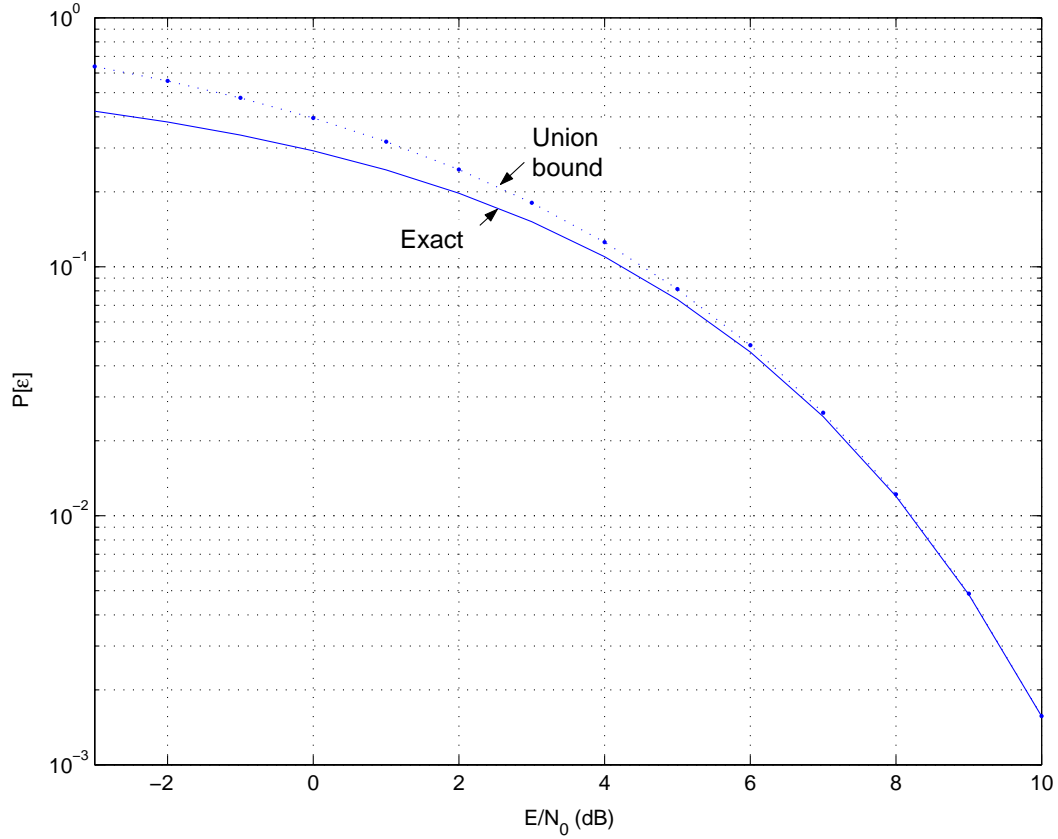
**Solution:** With a Matlab script, we may compare the exact expression

$$P[\epsilon] = 2 Q \left( \sqrt{\frac{E}{N_0}} \right) - \left[ Q \left( \sqrt{\frac{E}{N_0}} \right) \right]^2,$$

with the union bound

$$P[\epsilon] = 2 Q \left( \sqrt{\frac{E}{N_0}} \right) + Q \left( \sqrt{\frac{2E}{N_0}} \right).$$

The plots are shown in the graph below:



Note that the bound is tight after a minimum value of approximately  $(E/N_0)_{\min} = 9$  dB.

The Matlab script used to compute the points in the graph is the following:

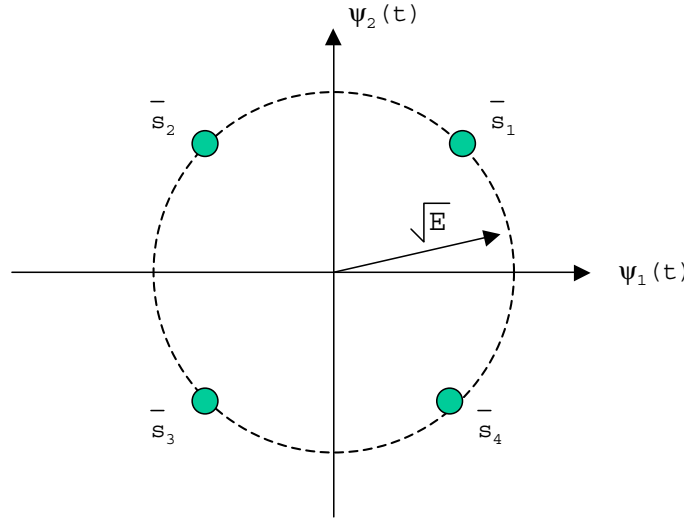
```
esno=[-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10];
esnor=10.^(esno/10);
pexact=2*erfc(sqrt(esnor)/sqrt(2))/2 - (erfc(sqrt(esnor)/sqrt(2))/2).^2;
pebnd=2*erfc(sqrt(esnor)/sqrt(2))/2 + erfc(sqrt(2*esnor)/sqrt(2))/2;
semilogy(esno,pexact);
hold on;
semilogy(esno,pebnd);
```



101. HOMEWORK 7, Spring 2004. Problems 2 and 3

102. Consider a QPSK communication system over an AWGN channel with  $S_N(f) = \frac{N_0}{2}$  W/Hz. Bit values 0 and 1 are assumed to be equally likely.

- (a) Using the fact that  $Q(x) < Q(y)$  for  $x > y$ , show that the probability of a *bit error*,  $P_b$  with Gray-mapped QPSK modulation is approximately the same as that with BPSK modulation, as a function of the energy per bit-to-noise ratio  $E_b/N_0$ . (Hint:  $P[\text{error}] \approx Q\left(\sqrt{E/N_0}\right)$ , and  $E = 2 E_b$ .)
- (b) Using the signals  $\psi_1(t)$  and  $\psi_2(t)$  from problem 2, sketch carefully the signals  $s_i(t)$  corresponding to the points  $\bar{s}_i$ , for  $i = 1, 2, 3, 4$ , in the QPSK constellation shown in the figure below.



**Solution:**

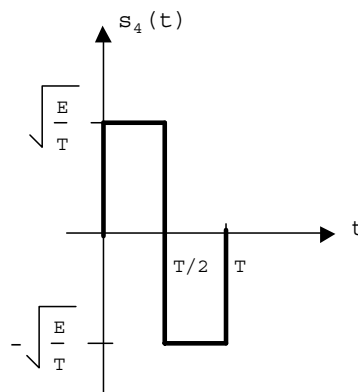
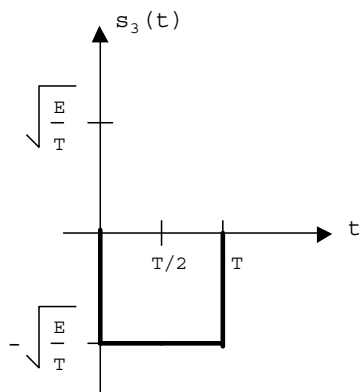
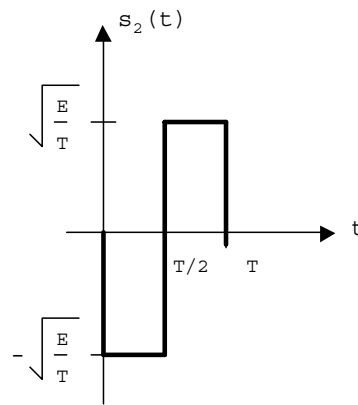
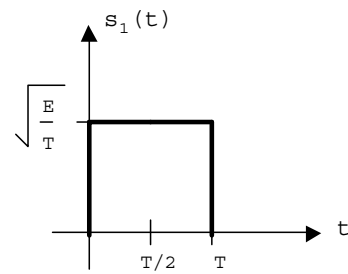
- (a) Since  $Q(x) < Q(y)$  for  $x > y$ ,  $P[\text{error}]$  is dominated by the probability of transmitting a signal point and making a decision favorable to one of two possible nearest points. The distance between two nearest signal points is  $d_{12} = 2\sqrt{E}/2 = \sqrt{2E}$ . From this it follows that

$$P[\text{error}] \approx 2 Q\left(\sqrt{\frac{E}{N_0}}\right).$$

Since each QPSK signal carries two bits,  $E = 2E_b$ . In addition, with Gray mapping,  $P_b = P[\text{error}]/2$ . As a result,

$$P_b \approx Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

- (b)

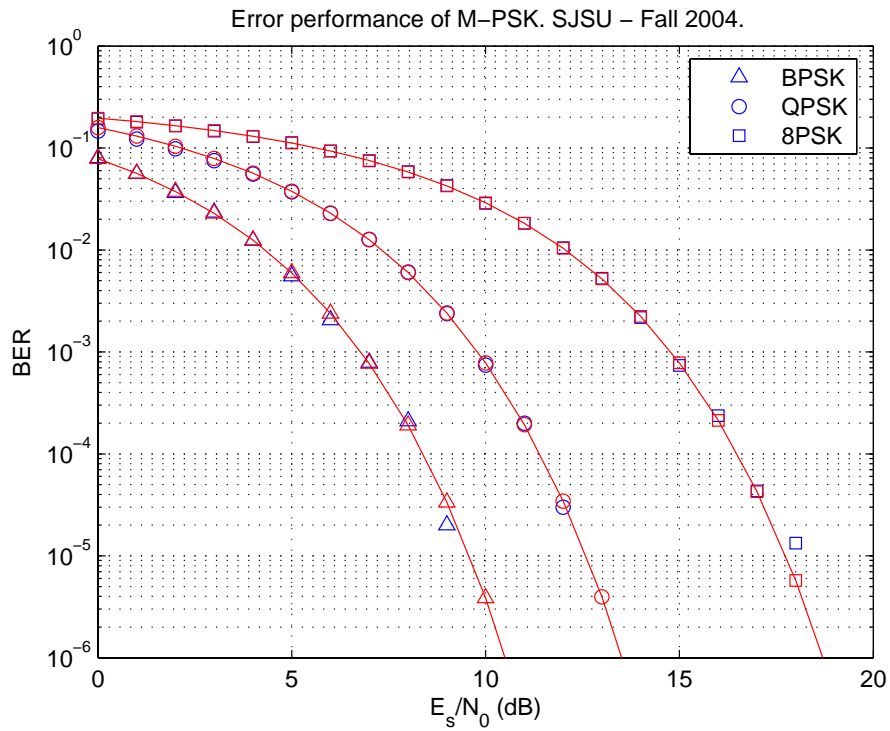


### 103. MATLAB experiment

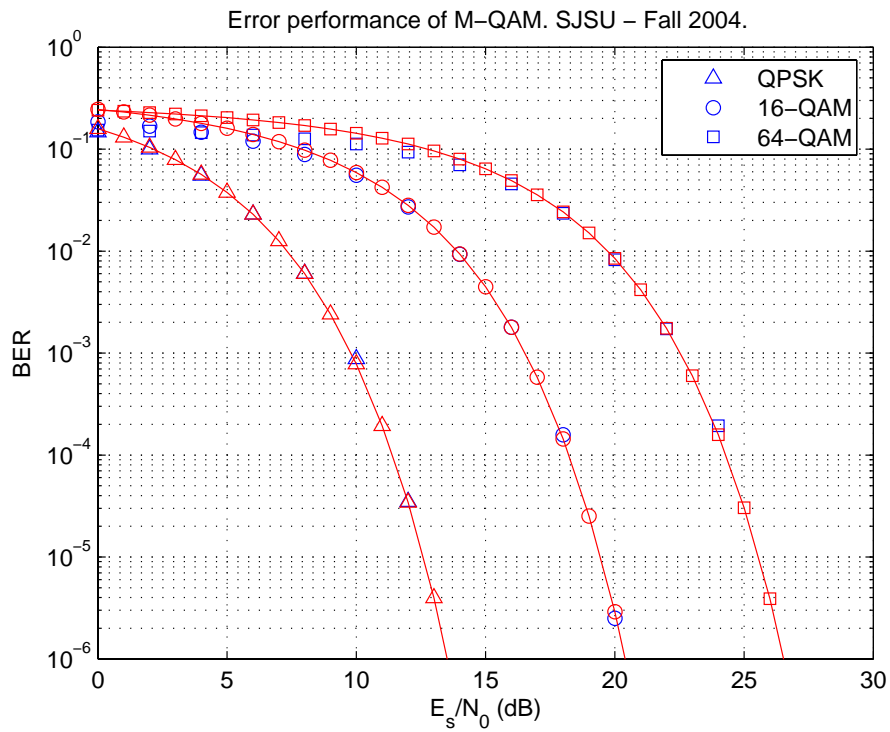
Simulate the error performance of M-PSK ( $M = 2, 4, 8$ ) and M-QAM ( $M = 4, 16, 64$ ) digital modulation schemes. The scripts `digmodMQAM.m` and `digmodMPSK.m` are available in the web site of the course, in section MATLAB EXAMPLES, under the heading Simulation of M-PSK and M-QAM modulation over AWGN channels.

- Run the script, sketch or plot the resulting curves and include them in your solution.
- Compare the simulated error performance of QPSK and 16-QAM against the approximated expressions given in the book. You will need to write a script to plot the approximated expressions and the simulation results in the same graph. Refer to the notes on digital modulation in the web site for the expressions.

**Solution:** The figures below show plots of simulated (symbols) and theoretical (symbols and line) BER for  $M$ -PSK and  $M$ -QAM, respectively. For BER values below  $10^{-2}$ , the theoretical expressions match the simulation results very closely.

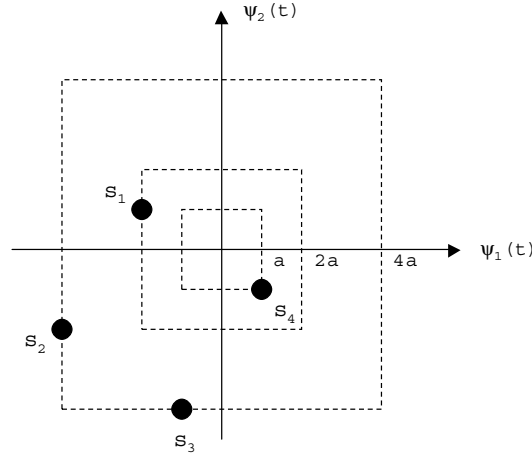


Simulated and theoretical BER performance of  $M$ -PSK modulation.



Simulated and theoretical BER performance of  $M$ -QAM modulation.

104. A digital communication system is designed using bandpass QPSK modulation with Gray mapping. Due to a DC offset and phase error in the modulator, the constellation is translated and rotated as shown in the figure below.



A translated and rotated QPSK constellation.

- Compute the average energy of the signals and sketch carefully the decision regions.
- Find the probability of a bit error  $P_b$  of this system.
- If a demodulator for conventional QPSK is employed, what is the resulting value of  $P_b$ ?

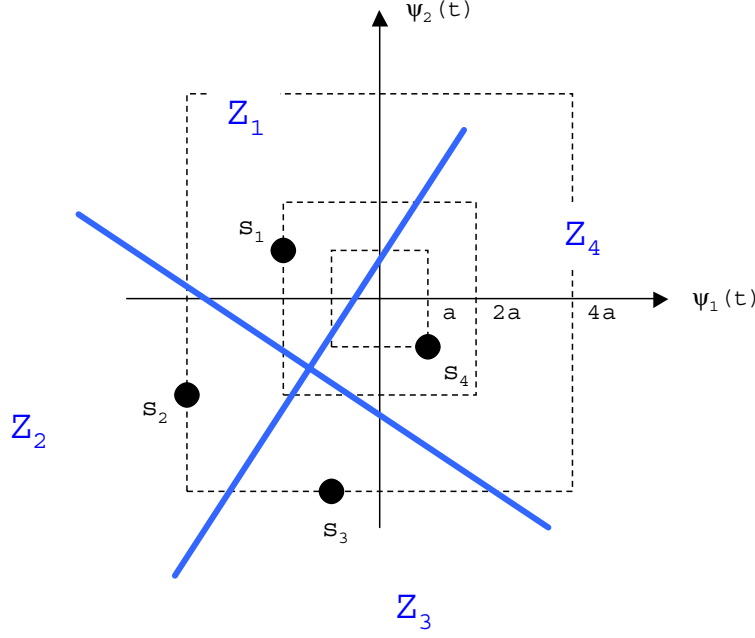
**Solution:**

- Average energy:

$$\begin{aligned}
 E &= \frac{1}{4} \sum_{i=1}^4 E_i \\
 &= \frac{1}{4} \{ [(-2a)^2 + (a)^2] + [(-4a)^2 + (-2a)^2] + [(-a)^2 + (-4a)^2] + [(a)^2 + (-a)^2] \} \\
 &= \frac{1}{4} \{ 44a^2 \} = 11a^2.
 \end{aligned}$$

From which it follows that  $a = \sqrt{E/11}$ .

The decision regions are shown in the figure below:



Decision regions of a translated and rotated QPSK constellation.

(b) Note that

$$d_{12}^2 = d_{23}^2 = d_{34}^2 = d_{41}^2 = 4a^2 + 9a^2 = 13a^2.$$

As a result,

$$P[\text{error}] \approx 2Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = 2Q\left(\sqrt{\frac{13E}{22N_0}}\right),$$

and

$$P_b \approx Q\left(\sqrt{\frac{26E_b}{22N_0}}\right).$$

Compared to conventional QPSK, there is a loss of  $10 \log_{10} \left(\frac{44}{26}\right) = 2.28$  dB in SNR. This is due to the DC offset. That is, the average of the signal points is not the origin, resulting in an increase of the average probability of a bit error.

(c) In conventional QPSK, the decision regions  $Z'_i$ ,  $i = 1, 2, 3, 4$ , are the quadrants of the  $Y_1/Y_2$ -plane. Consequently, the probability of error is given by

$$P[\text{error}] = \frac{1}{4} \sum_{i=1}^4 P[\text{error}|\bar{s}_i],$$

where

$$P[\text{error}|\bar{s}_1] = P[\bar{Y} \notin Z'_1|\bar{s}_1] = 1 - Q\left(\sqrt{\frac{8E}{11N_0}}\right) \left[1 - Q\left(\sqrt{\frac{2E}{11N_0}}\right)\right],$$

$$P[\text{error}|\bar{s}_2] = P[\bar{Y} \notin Z'_2|\bar{s}_2] = 1 - Q\left(\sqrt{\frac{8E}{11N_0}}\right) \left[1 - Q\left(\sqrt{\frac{32E}{11N_0}}\right)\right],$$

$$P[\text{error}|\bar{s}_3] = P[\bar{Y} \notin Z'_3|\bar{s}_3] \approx Q\left(\sqrt{\frac{2E}{11N_0}}\right) + Q\left(\sqrt{\frac{32E}{11N_0}}\right),$$

$$P[\text{error}|\bar{s}_4] = P[\bar{Y} \notin Z'_4|\bar{s}_4] \approx 2Q\left(\sqrt{\frac{2E}{11N_0}}\right),$$

where the fact that  $Q(x) > Q(y)$ , for  $x < y$ , has been used. It follows that

$$P[\text{error}] \approx \frac{1}{4} \left\{ 2 + 3Q\left(\sqrt{\frac{2E}{11N_0}}\right) \right\},$$

and therefore, for QPSK modulation (i.e., 2 bits per symbol), the desired result is

$$P_b \approx \frac{1}{8} \left\{ 2 + 3Q\left(\sqrt{\frac{4E_b}{11N_0}}\right) \right\}.$$

Finally, note that at high SNR values, bit error events are dominated by error events associated with the transmission of signal points  $\bar{s}_1$  and  $\bar{s}_2$ , which are located in the wrong quadrants. Therefore,

$$P_b \approx \frac{1}{4}, \quad \text{for high values of } E_b/N_0.$$

In this case, we say that there is an “error floor”. The value of  $P_b$  is never less than 0.25, regardless of how high the value of  $E_b/N_0$  is!!!

## Frequency-shift keying (FSK) modulation

105. The transmitter of a BFSK communication system sends an RF rectangular pulse  $s_m(t)$ , for  $m = 1, 2$ , in the interval  $0 \leq t \leq T$ , and in correspondence to the value of a source bit  $M \in \{0, 1\}$ , as follows:

$$\begin{aligned} M = 0 & \longrightarrow s_1(t) = a \cos(2\pi f_1 t), \\ M = 1 & \longrightarrow s_2(t) = a \cos(2\pi f_2 t), \end{aligned}$$

where  $a$  is the amplitude and the frequency separation is  $f_2 - f_1 = \frac{1}{2T}$ . Source bits take values 0 and 1 with equal probability. Transmission takes place over an AWGN channel with  $S_N(f) = \frac{N_0}{2}$  W/Hz.

- (a) Find the probability of a bit error,  $P[\text{error}]$ , in terms of the amplitude  $a$  and  $N_0$ .
- (b) It is desired to transmit bits at a rate of 1 Mbps with  $P[\text{error}] = 10^{-5}$ . The receiver introduces AWGN with  $N_0 = 10^{-10}$ . Determine the minimum value  $a_{\min}$  of the amplitude of the RF rectangular pulses.

- (c) It is desired to increase the rate to 2 Mbps. The other conditions are the same as in part (b). What is the minimum value  $a_{\min}$  of the amplitude of the RF rectangular pulses? Express in dB the additional power required compared to part (b).

**Solution:**

- (a) Note that  $s_i(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_i t)$ , for  $i = 1, 2$ . In other words,  $a = \sqrt{\frac{2E}{T}}$ . Alternatively, the energy of  $s_1(t)$  (or  $s_2(t)$ ) is equal to  $E = \frac{a^2 T}{2}$ . It follows that

$$P[\text{error}] = Q\left(\sqrt{\frac{E}{N_0}}\right) = Q\left(\sqrt{\frac{a^2 T}{2N_0}}\right).$$

- (b) We have that  $T = 10^{-6}$  and  $N_0 = 10^{-10}$ . Therefore

$$P[\text{error}] = 10^{-5} = Q\left(\sqrt{\frac{a^2 T}{2N_0}}\right) = Q\left(\sqrt{\frac{a^2 (10^{-6})}{2 (10^{-10})}}\right).$$

From the Matlab function `invQ.m` given in the web site, we have that  $Q(x) = 10^{-5}$ , for  $x = Q^{-1}(10^{-5}) = 4.2649$ . Therefore,

$$\sqrt{\frac{a^2 (10^4)}{2}} = 4.2649$$

and from this it follows that  $a_{\min}^2 = 3.64 \times 10^{-3} \text{ V}^2$ , or  $a_{\min} = 60.3 \text{ mV}$ .

- (c) The bit period is reduced by  $1/2$ . Therefore  $a_{\min} = \sqrt{2} (60.3) \text{ mV} = 85.3 \text{ mV}$ . Compared to part (b), the additional signal power required to achieve a probability of a bit error equal to  $10^{-5}$  is 3 dB.

106. Find  $z = E_b/N_0$  required to give  $P_b = 10^{-5}$  for the following coherent digital modulation schemes: (a) on-off keying (ASK), (b) BPSK, (c) BFSK and (d) BPSK with a phase error of 5 deg.

**Solution:**

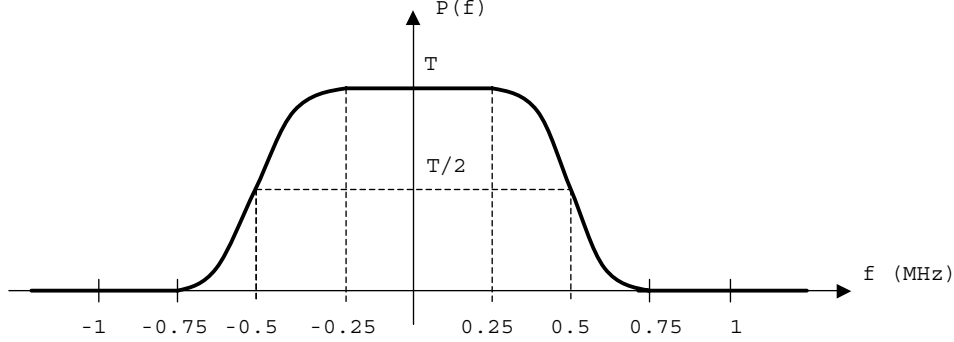
- (a) For ASK,  $P_E = Q(\sqrt{z}) = 10^{-5}$  gives  $z = 18.19$  or  $z = 12.6 \text{ dB}$ .  
(b) For BPSK,  $P_E = Q(\sqrt{2z}) = 10^{-5}$  gives  $z = 9.1$  or  $z = 9.59 \text{ dB}$ .  
(c) Binary FSK is the same as ASK.  
(d) The degradation of BPSK with a phase error of 5 degrees is

$$D_{\text{const}} = -20 \log_{10}[\cos(5^\circ)] = 0.033 \text{ dB},$$

so the required SNR to give a bit error probability of  $10^{-5}$  is  $9.59 + 0.033 = 9.623 \text{ dB}$ .

## PAM over bandlimited channels

107. Binary data at a rate of  $4 \times 10^6$  bits/s are transmitted using  $M$ -PAM over a bandlimited channel, using the raised cosine roll-off characteristic shown in the figure below:



Raised cosine roll-off characteristic.

- (a) Determine the roll-off factor  $\alpha$  and the value of  $M$  of this system.
- (b) Signals are transmitted over a bandlimited AWGN channel. Determine the minimum required bit energy-to-noise ratio,  $E_b/N_0$ , to achieve a probability of a bit error less than  $10^{-3}$ . Express your result in decibels (dB).

**Solution:**

- (a)  $R_b = 4$  Mbits/s. Note that  $1/2T = 0.5$  MHz, the point of symmetry of the raised cosine. As a result,  $1/T = 1$  MHz. Since  $R_b = 1/T_b = \log_2 M (1/T)$ , it follows that  $\log_2 M = 4$  and therefore  $M = 16$ .

To find the roll-off factor, note that  $2\alpha(1/2T) = 0.75 - 0.25 = 0.5$  MHz (the region of the cosine characteristic) and therefore, with  $1/2T = 0.5$  MHz, we obtain  $\alpha = 0.5$ .

- (b) The probability of a bit error for  $M$ -PAM is

$$P[\epsilon] = \frac{M-1}{M} Q \left( \sqrt{\frac{6 \log_2 M}{M^2 - 1} \frac{E_b}{N_0}} \right).$$

For  $M = 16$ ,

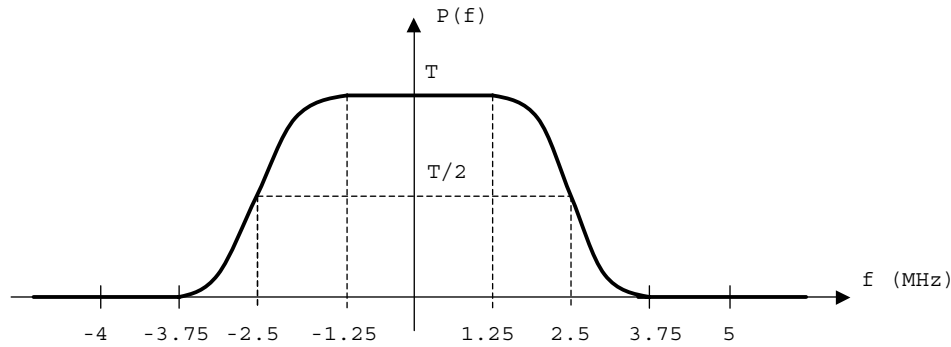
$$P[\epsilon] = \frac{15}{16} Q \left( \sqrt{\frac{24}{255} \left( \frac{E_b}{N_0} \right)_{\min}} \right) < 1 \times 10^{-3}$$

From the table of the  $Q$ -function, we have  $Q(3.07) = 0.0011$ . Therefore,

$$\sqrt{\frac{8}{85} \left( \frac{E_b}{N_0} \right)_{\min}} = 3.07 \quad \longrightarrow \quad \left( \frac{E_b}{N_0} \right)_{\min} = \frac{85}{8} (3.07)^2 = 100.14 = 20 \text{ dB}$$



108. Binary data are transmitted using 8-PAM over a bandlimited channel, using the raised cosine roll-off characteristic shown in the figure below:



Raised cosine roll-off characteristic.

- Determine the roll-off factor  $\alpha$  and the bit rate, in bits/s, of this system.
- Signals are transmitted over a bandlimited AWGN channel. Determine the minimum probability of a bit error if the bit energy-to-noise ratio is limited to a maximum of 20 dB.

**Solution:**

- $M = 8$ ,  $\log_2 M = 3$ . Note that  $1/2T = 2.5$  MHz, the point of symmetry of the raised cosine. As a result,  $1/T = 5$  MHz and the bit rate is therefore  $R_b = \log_2 M(1/T) = 15$  Mbits/s.

To find the roll-off factor, note that  $2\alpha(1/2T) = 3.75 - 1.25 = 2.5$  MHz (the region of the cosine characteristic) and therefore, with  $1/2T = 2.5$  MHz, we obtain  $\alpha = 0.5$ .

- The probability of a bit error for  $M$ -PAM is

$$P[\epsilon] = \frac{M-1}{M} Q \left( \sqrt{\frac{6 \log_2 M}{M^2 - 1} \frac{E_b}{N_0}} \right).$$

For a bit energy-to-noise ratio equal to 20 dB, we have that

$$10 \log_{10} \left( \frac{E_b}{N_0} \right) = 20 \quad \longrightarrow \quad \frac{E_b}{N_0} = 100.$$

Consequently, the minimum bit error probability is

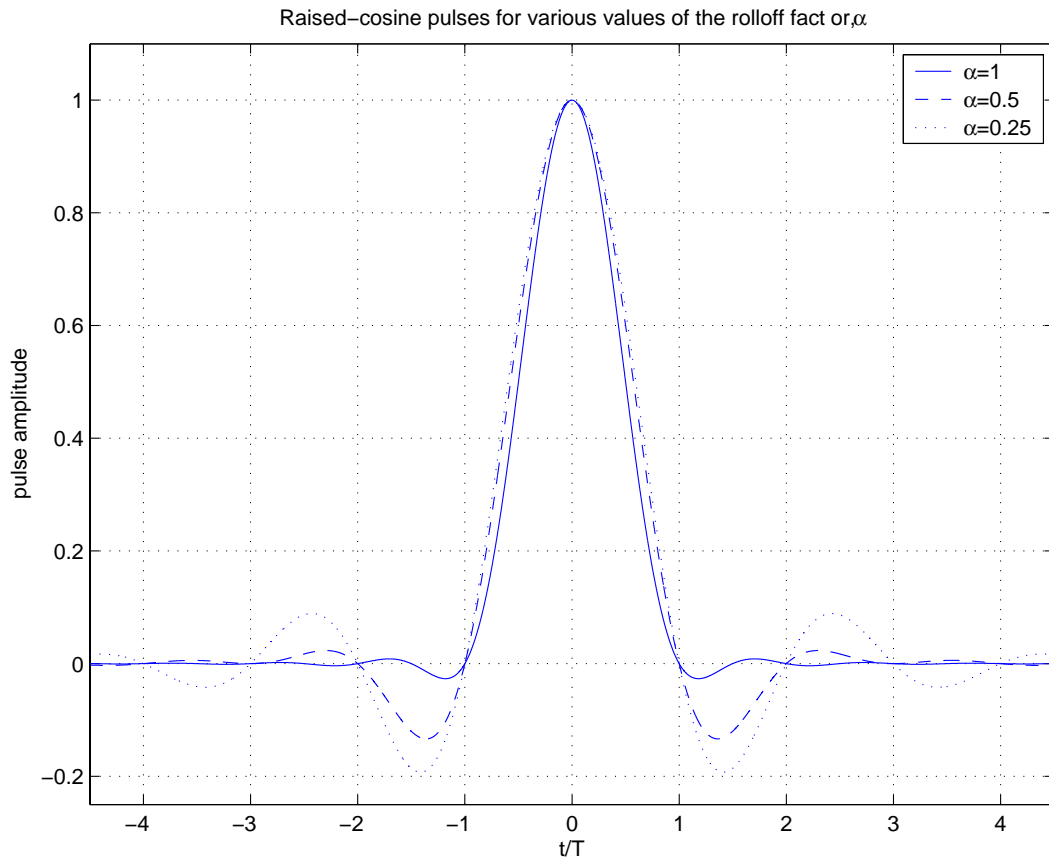
$$P[\epsilon] = \frac{7}{8} Q \left( \sqrt{\frac{6(3)}{63}} (100) \right) = \frac{7}{8} Q(5.345) = 3.96 \times 10^{-8}$$

109. HOMEWORK 4 Summer 2004, problems 1-4

## The Nyquist criterion for ISI-free transmission

110. Plot the raised-cosine pulse for different values of rolloff factor,  $\alpha$ . For this purpose, download the script `plot_raised_cosine_pulse.m` from the web site, execute it and sketch or print the resulting waveforms.

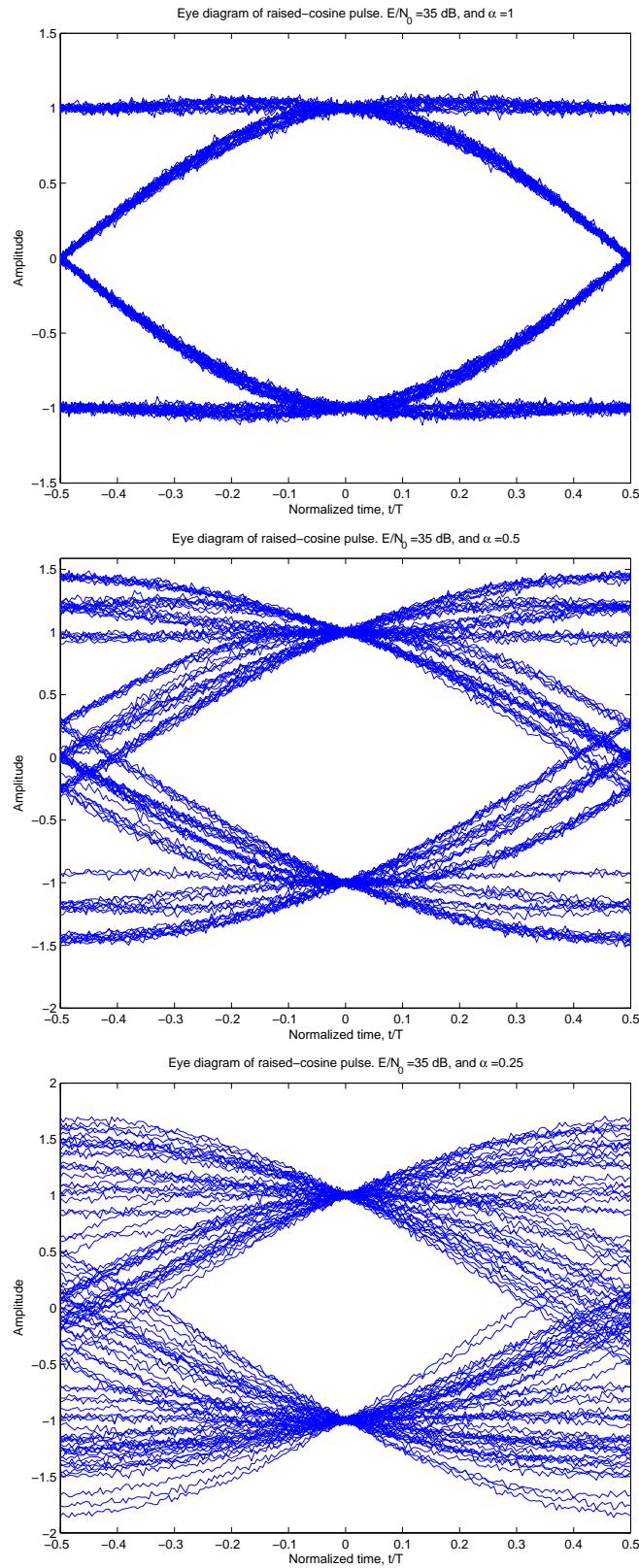
**Solution:**



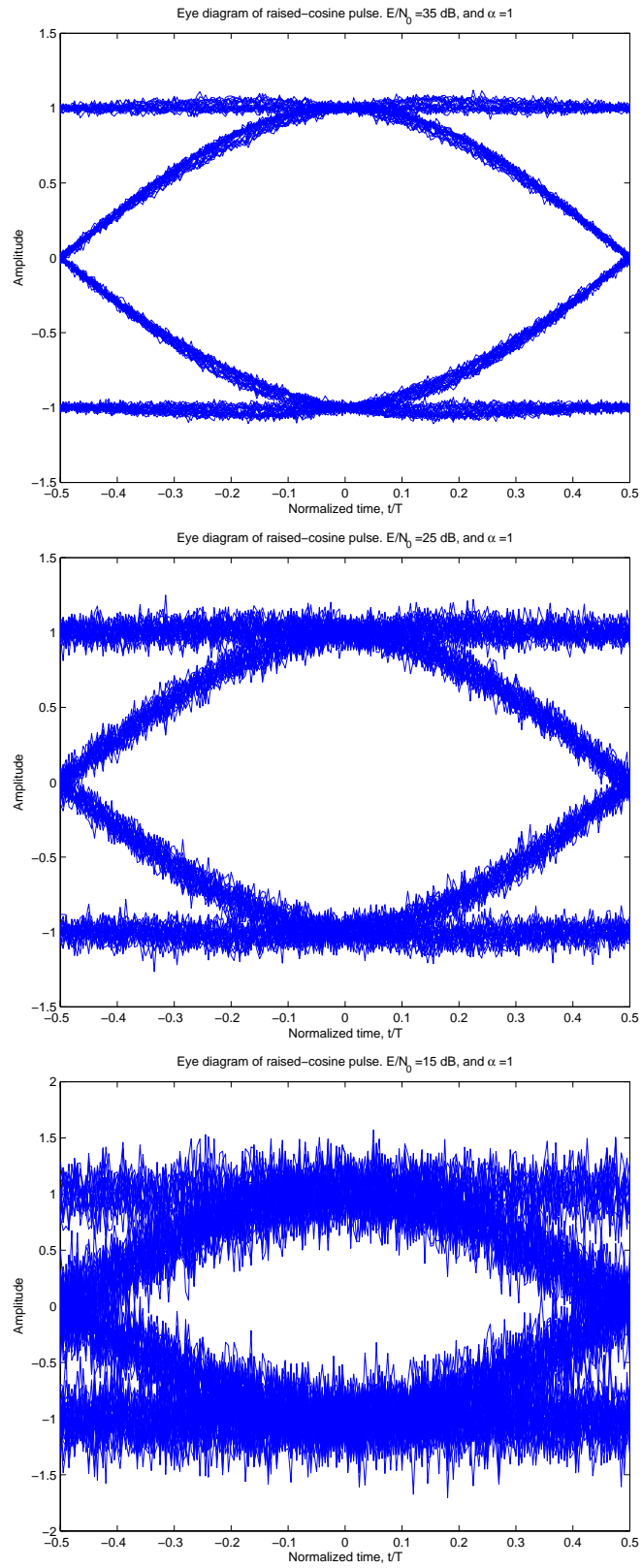
Raised-cosine pulse for different rolloff factors.

111. The eye diagram is useful in analyzing pulse shapes used in communication over bandlimited channels. You will experiment with this tool using MATLAB. Download the script `ayayay.m` from the web site.
- Run the script for the following pairs of values  $E/N_0$  (dB) and  $\alpha$ : [35, 1], [35, 0.5] and [35, 0.25]. Print or sketch the resulting eye diagrams and comment on the effects of  $\alpha$ .
  - Run the script for the following pairs of values of  $E/N_0$  (dB) and  $\alpha$ : [35, 1], [25, 1] and [15, 1]. Print or sketch the resulting eye diagrams and comment on the effects of  $E/N_0$ .

**Solution:**



- (a) It is observed from the eye diagrams that the timing error margin increases with the value of  $\alpha$ . On the other hand, the occupied bandwidth grows linearly with the rolloff factor. Thus there is a tradeoff between timing error margin and occupied bandwidth.



(b) The eye opening and timing error margin are reduced with increasing noise levels (decreasing SNR values).

## PCM, quantization and delta modulation

112. A message signal  $m(t)$  is transmitted by binary PCM without compression. Let the signal-to-quantization noise (SNR<sub>q</sub>) required be at least 47 dB. Determine the minimum number of quantization levels  $L$  required, assuming that  $m(t)$  is sinusoidal. With this value of  $L$ , determine the SNR<sub>q</sub>.

**Solution:**  $1.8 + 6v \geq 47$ . Therefore,  $v = 8$  bits and  $L = 256$ . This gives SNR<sub>q</sub> = 49.8 dB.

113. In a delta modulation system, a sinusoidal signal  $m(t) = \sin(6000\pi t)$  is sampled at 30,000 samples/sec. Determine the minimum step size  $\Delta$  to avoid slope overload.

**Solution:**  $\left| \frac{dm(t)}{dt} \right|_{\max} = 6000\pi$ . As a result,  $\Delta \geq 6000\pi T_s = \pi/5$ .

114. Find the maximum amplitude of a 1 KHz sinusoidal signal input to a delta modulator that will prevent slope overload, when the sampling rate is 10,000 samples/sec and the step size is  $\Delta = 0.1$ .

**Solution:**  $\frac{1}{T_s} = 10000$ .  $\Delta = 0.1$ . This implies that

$$\begin{aligned} \frac{\Delta}{T_s} &\geq 2000\pi \cdot A \\ 1000 &\geq 2000\pi \cdot A \end{aligned}$$

and consequently,

$$A_{\max} = \frac{1}{2\pi}.$$

115. Let  $m(t) = A_m \cos(2\pi f_m t)$  be the input to a delta modulator with parameters  $\Delta$  (step size) and  $T_s$  (sampling period). Show that the minimum sampling frequency  $f_{s,\min}$ , needed to avoid slope overload distortion, is given by

$$f_{s,\min} = \frac{2\pi f_m A_m}{\Delta}.$$

**Solution:** Note that

$$\left| \frac{dm(t)}{dt} \right|_{\max} = |-2\pi f_m A_m \sin(2\pi f_m t)|_{\max} = 2\pi f_m A_m.$$

Therefore, to avoid slope overload:

$$\frac{\Delta}{T_s} \geq 2\pi f_m A_m,$$

from which it follows that

$$f_s \triangleq \frac{1}{T_s} \geq \frac{2\pi f_m A_m}{\Delta}.$$

116. HOMEWORK 7 Spring 2004, problems 1 and 2

## Error correcting coding

117. Data are encoded with a binary (7,4) Hamming code and sent over a noisy channel. A received vector is  $r = (1101111)$ . Determine the syndrome  $s$ , the corresponding error vector  $e$  and the estimated code vector  $v$ .

**Solution:** The syndrome is:

$$s = r^T H = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Observe that the syndrome  $s$  is equal to the third column of  $H$ . The corresponding error vector is  $e = (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0)$ , and the estimated code vector  $v = r \oplus e = (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1)$ .

118. Write a parity-check matrix of a binary (15,11) Hamming code.

**Solution:** Any binary matrix with 15 columns consisting of all non-zero combinations of  $m = 15 - 11 = 4$  bits can be taken as a parity check matrix of a binary (15,11) Hamming code. An example is shown below:

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

119. Consider the application of a (7,4) Hamming code in a digital communication system.

- (a) If the received word is  $\bar{r} = 0101101$ , what is the decoded message  $\hat{m}$ ?
- (b) This code is applied to a BPSK system operating at an SNR of 8 dB. Estimate the probability of a bit error after decoding.
- (c) (Bonus) Design the complete digital circuits, in terms of flip-flops and logic gates, of an encoder and a decoder for this error correcting code.

**Solution:**

- (a) Assuming systematic encoding and the generator matrix

$$G = (I_4 \ A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

the parity-check matrix is given by

$$H = (A^T \ I_3) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The syndrome  $\bar{s}$  corresponding to the received word is computed as follows

$$\bar{s} = \bar{r}H^T = (0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (1 \ 0 \ 0).$$

The syndrome  $\bar{s}$  is equal to the transpose of the fifth column of the parity-check matrix  $H$ . Consequently, the most likely event is a single error in the fifth position. That is,  $\bar{e} = (0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0)$ , and the estimated codeword is:

$$\hat{c} = \bar{r} \oplus \bar{e} = (0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1) \oplus (0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0) = (0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1).$$

- (b) We have BPSK modulation with  $10\log_{10} \frac{E}{N_0} = 8$  dB or  $\frac{E}{N_0} = 6.3$ . The probability of error without coding is

$$\epsilon = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q(3.55) = 2.33 \times 10^{-4}.$$

With coding,

$$P_b \approx 3\epsilon^2 = 1.63 \times 10^{-7}.$$

(c) Digital logic design of encoder and decoder of the binary (7,4) Hamming code.

**1. Encoder** Let  $\bar{m}$  represent the message bits. Then

$$\begin{aligned}\bar{c} &= (c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6 \ c_7) \\ &= \bar{m}G = (m_1 \ m_2 \ m_3 \ m_4) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},\end{aligned}$$

and it follows that the coded bits are

$$\begin{aligned}c_1 &= m_1 \\ c_2 &= m_2 \\ c_3 &= m_3 \\ c_4 &= m_4 \\ c_5 &= m_2 \oplus m_3 \oplus m_4 \\ c_6 &= m_1 \oplus m_2 \oplus m_4 \\ c_7 &= m_1 \oplus m_3 \oplus m_4\end{aligned}$$

**2. Decoder** The syndrome  $\bar{s}$  is computed as

$$\begin{aligned}\bar{s} &= (s_1 \ s_2 \ s_3) \\ &= \bar{r}H^T = (r_1 \ r_2 \ r_3 \ r_4 \ r_5 \ r_6 \ r_7) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

and it follows that the syndrome bits are

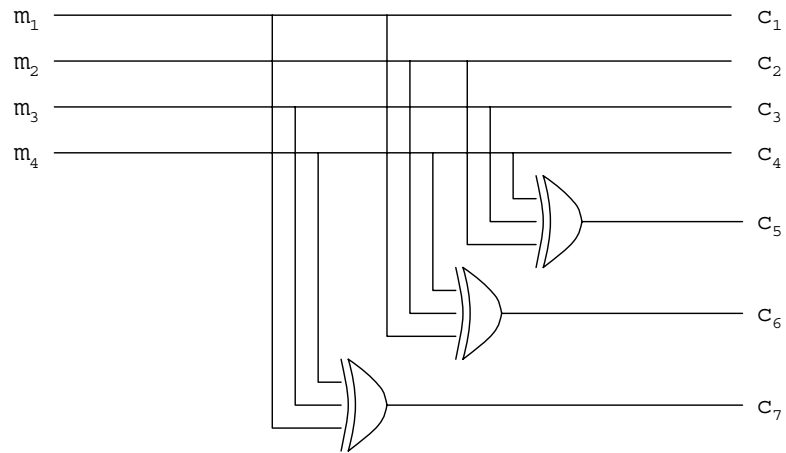
$$\begin{aligned}s_1 &= r_2 \oplus r_3 \oplus r_4 \oplus r_5 \\ s_2 &= r_1 \oplus r_2 \oplus r_4 \oplus r_6 \\ s_3 &= r_1 \oplus r_3 \oplus r_4 \oplus r_7\end{aligned}$$

The syndrome  $\bar{s}$  associated with a single-error pattern  $\bar{e} = e_1 \ e_2 \ e_3 \ e_4 \ e_5 \ e_6 \ e_7$  is equal to the transpose of the column of the parity-check matrix  $H$  in the position where the error occurs. Therefore,

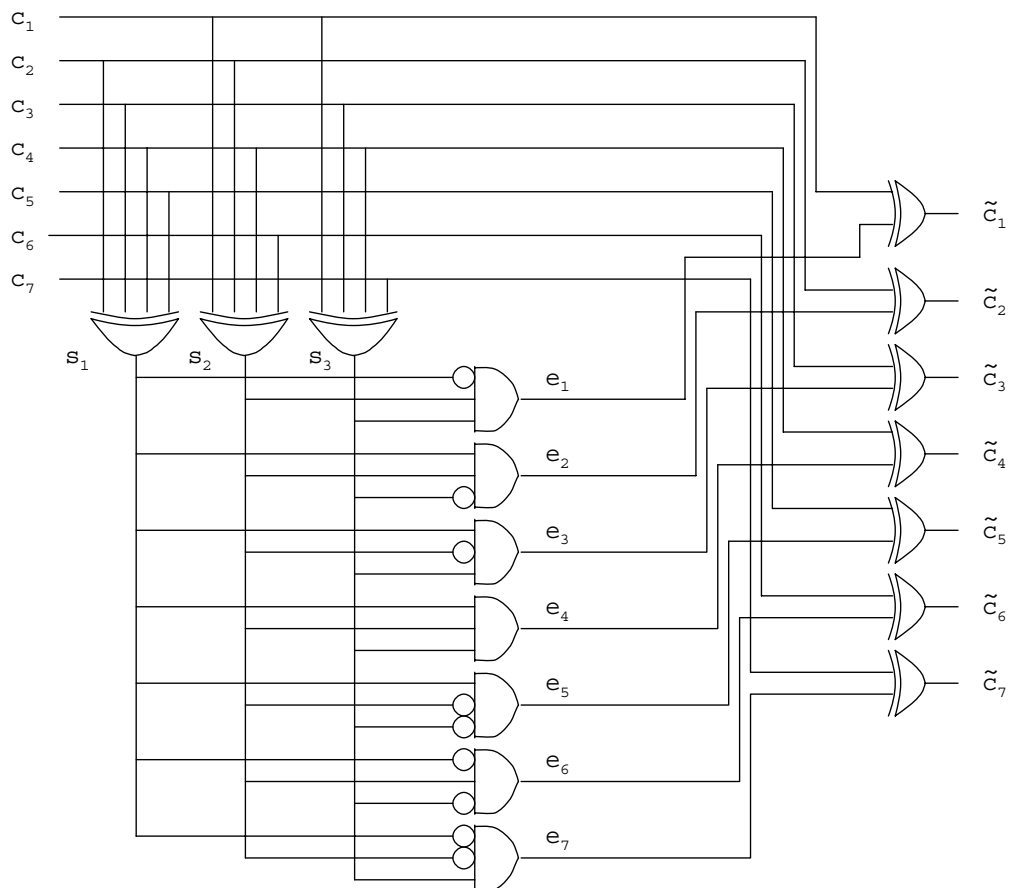
$$\begin{aligned}e_1 &= \bar{s}_1 s_2 s_3 \\ e_2 &= s_1 s_2 \bar{s}_3 \\ e_3 &= s_1 \bar{s}_2 s_3 \\ e_4 &= s_1 s_2 s_3 \\ e_5 &= s_1 \bar{s}_2 \bar{s}_3 \\ e_6 &= \bar{s}_1 s_2 \bar{s}_3 \\ e_7 &= \bar{s}_1 \bar{s}_2 s_3\end{aligned}$$



The circuits for encoding and decoding the (7,4) Hamming code are shown below.



Encoder of a (7,4) Hamming code.



Decoder of a (7,4) Hamming code.

## 120. MATLAB experiment

Download the Matlab script `homework10f04.m` from the website. Record the output and verify it. That is, verify that, for the given message and error vectors, the codeword, the received word and the syndrome are correct. (Note that in Matlab the position of the identity matrices in the generator and parity-check matrices of the code is different from that given in class.)

### Solution:

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12/15/04 12:25 AM MATLAB Command Window 1 of 2

---

To get started, select MATLAB Help or Demos from the Help menu.

```
>> homework10
```

```
Enter your student ID (tower card) number: 123456789
```

```
G =
```

```

1    1    0    1    0    0    0
0    1    1    0    1    0    0
1    1    1    0    0    1    0
1    0    1    0    0    0    1
```

```
H =
```

```

1    0    0    1    0    1    1
0    1    0    1    1    1    0
0    0    1    0    1    1    1
```

```
trt =
```

```

0    0    0    0    0    0    0
0    0    1    0    0    0    0
0    1    0    0    0    0    0
0    0    0    0    1    0    0
1    0    0    0    0    0    0
0    0    0    0    0    0    1
0    0    0    1    0    0    0
0    0    0    0    0    1    0
```

```
msg =
```

```

1    0    1    1
```

```
code =
```

```

1    0    0    1    0    1    1
```

```
err =
```

```

0    1    0    0    0    0    0
```

```
recd =
```

```

1    1    0    1    0    1    1
```

```
Syndrome = 2 (decimal), 0 1 0 (binary)
```

```
errvect =
```

```
0 1 0 0 0 0 0
```

```
correctedcode =
```

```
1 0 0 1 0 1 1
```

121. Find the capacity of the cascade connection of  $n$  binary-symmetric channels with the same crossover probability  $\epsilon$ . What is the capacity when the number of channels goes to infinity?

**Solution:** The overall channel is a binary symmetric channel with crossover probability  $p$ . To find  $p$  note that an error occurs if an odd number of channels produce an error. Thus,

$$p = \sum_{k=\text{odd}} \binom{n}{k} \epsilon^k (1 - \epsilon)^{n-k}$$

which is equal to

$$p = \frac{1}{2} [1 - (1 - 2\epsilon)^n]$$

and therefore,

$$C = 1 - h(p)$$

If  $n \rightarrow \infty$ , then  $(1 - 2\epsilon)^n \rightarrow 0$  and  $p \rightarrow \frac{1}{2}$ . In this case

$$C = \lim_{n \rightarrow \infty} C(n) = 1 - h\left(\frac{1}{2}\right) = 0$$

122. In the transmission and reception of signals to and from moving vehicles, the transmitted signal frequency is shifted in direct proportion to the speed of the vehicle. The so-called *Doppler frequency shift* imposed on a signal that is received in a vehicle traveling at a velocity  $v$  relative to a (fixed) transmitter is given by the formula

$$f_D = \pm \frac{v}{\lambda},$$

where  $\lambda$  is the wavelength and the sign depends on the direction (moving toward or moving away) that the vehicle is traveling relative to the transmitter. Suppose that a vehicle is traveling at a speed of 100 km/hr relative to a base station in a mobile cellular communication system. The signal is narrowband and transmitted at a carrier frequency of 1 GHz.

- Determine the Doppler frequency shift.
- Suppose that the transmitted signal bandwidth is 1 MHz centered at 1 GHz. Determine the Doppler frequency spread between the upper and lower frequencies in the signal.

**Solution:**

- (a) The wavelength
- $\lambda$
- is

$$\lambda = \frac{3 \times 10^8}{10^9} \text{ m} = \frac{3}{10} \text{ m}$$

Hence, the Doppler frequency shift is

$$f_D = \pm \frac{u}{\lambda} = \pm \frac{100 \text{ Km/hr}}{\frac{3}{10} \text{ m}} = \pm \frac{100 \times 10^3 \times 10}{3 \times 3600} \text{ Hz} = \pm 92.5926 \text{ Hz}$$

The plus sign holds when the vehicle travels towards the transmitter whereas the minus sign holds when the vehicle moves away from the transmitter.

- (b) The maximum Doppler frequency shift is obtain when
- $f = 1 \text{ GHz} + 1 \text{ MHz}$
- and the vehicle moves towards the transmitter. In this case

$$\lambda_{\min} = \frac{3 \times 10^8}{10^9 + 10^6} \text{ m} = 0.2997 \text{ m}$$

and therefore

$$f_{D_{\max}} = \frac{100 \times 10^3}{0.2997 \times 3600} = 92.6853 \text{ Hz}$$

Thus, the Doppler frequency spread is  $B_d = 2f_{D_{\max}} = 185.3706 \text{ Hz}$ .

123. A multipath fading channel has a multipath spread of  $T_m = 1 \text{ s}$  and a Doppler spread  $B_d = 0.01 \text{ Hz}$ . The total channel bandwidth at bandpass available for signal transmission is  $W = 5 \text{ Hz}$ . To reduce the effect of intersymbol interference, the signal designer selects a pulse duration of  $T = 10 \text{ s}$ .

- (a) Determine the coherence bandwidth and the coherence time.
- (b) Is the channel frequency selective? Justify your answer.
- (c) Is the channel fading slowly or rapidly? Justify your answer.

**Solution:**

- (a) Since
- $T_m = 1 \text{ second}$
- , the coherence bandwidth

$$B_{cb} = \frac{1}{2T_m} = 0.5 \text{ Hz}$$

and with  $B_d = 0.01 \text{ Hz}$ , the coherence time is

$$T_{ct} = \frac{1}{2B_d} = 100/2 = 50 \text{ seconds}$$

- (b) Since the channel bandwidth  $W \gg B_{cb}$ , the channel is frequency selective.
- (c) Since the signal duration  $T \ll T_{ct}$ , the channel is slowly fading.

124. Demonstrate that a DS spread-spectrum signal (without coding) provides no improvement in performance against AWGN.

**Solution:** The probability of error for DS spread spectrum with binary PSK may be expressed as

$$P_2 = Q \left( \sqrt{\frac{2W/R_b}{P_J/P_S}} \right)$$

where  $W/R$  is the processing gain and  $P_J/P_S$  is the jamming margin. If the jammer is a broadband, WGN jammer, then

$$\begin{aligned} P_J &= W J_0 \\ P_S &= \mathcal{E}_b/T_b = \mathcal{E}_b R_b \end{aligned}$$

Therefore,

$$P_2 = Q \left( \sqrt{\frac{2\mathcal{E}_b}{J_0}} \right)$$

which is identical to the performance obtained with a non-spread signal.

125. A DS spread-spectrum system is used to resolve the multipath signal component in a two-path radio signal propagation scenario. If the path length of the secondary path is 300 m longer than that of the direct path, determine the minimum chip rate necessary to resolve the multipath components.

**Solution:** The radio signal propagates at the speed of light,  $c = 3 \times 10^8 m/sec$ . The difference in propagation delay for a distance of 300 meters is

$$T_d = \frac{300}{3 \times 10^8} = 1 \mu sec$$

The minimum bandwidth of a DS spread spectrum signal required to resolve the propagation paths is  $W = 1 MHz$ . Hence, the minimum chip rate is  $10^6$  chips per second.

## **SAMPLE EXAMS**

## Solution Midterm Exam # 1-Type B

**Problem 1 (25 points)** Consider the bandpass signal

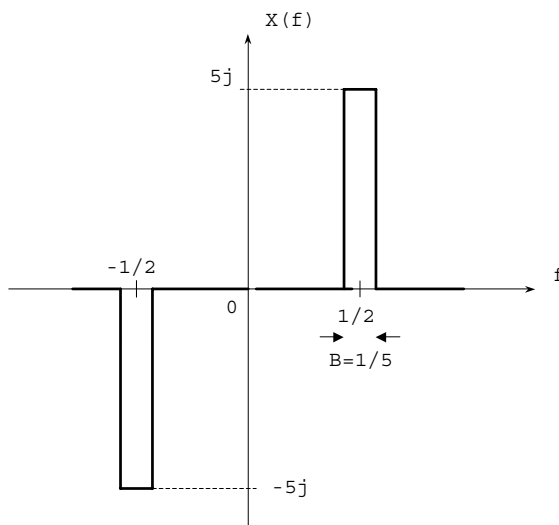
$$x(t) = 2 \operatorname{sinc}(t/5) \sin(\pi t).$$

- (a) Sketch carefully the spectrum of this signal. What is its bandwidth?

The center frequency is  $f_0 = 1/2$ . The spectrum is

$$X(f) = -5j \Pi(5(f + 1/2)) + 5j \Pi(5(f - 1/2)),$$

and shown in the figure below. The bandwidth is  $B = 1/5$ .



Spectrum of a bandpass signal.

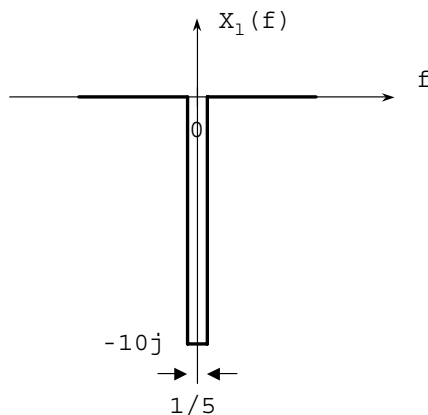
- (b) Give an expression of the complex baseband equivalent  $x_\ell(t)$ .

By inspection, we have  $x_s(t) = -2 \operatorname{sinc}(t/5)$ ,  $x_c(t) = 0$ . Therefore  $x_\ell(t) = j x_s(t) = -2j \operatorname{sinc}(t/5)$ .

- (c) Sketch carefully the complex baseband equivalent spectrum  $X_\ell(f)$ .

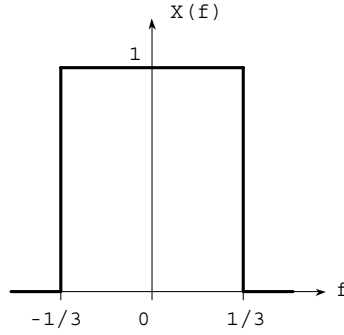
$$X_\ell(f) = j X_s(f) = -2j \cdot 5 \Pi(5f),$$

which is sketched below.



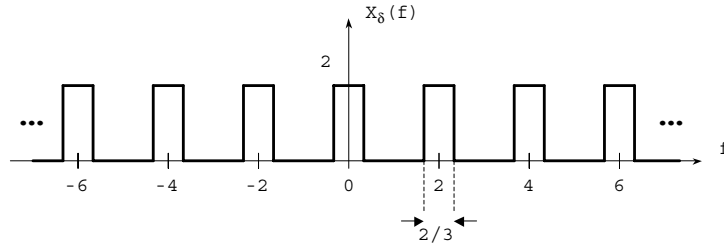
Spectrum of the complex baseband signal.

**Problem 2 (30 points)** The spectrum of a lowpass signal is shown below:



Spectrum of a lowpass signal.

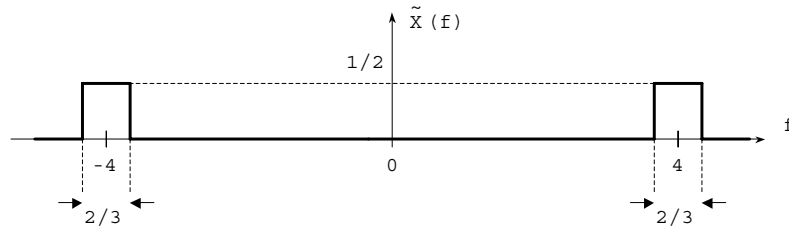
- (a) Determine the value of the Nyquist sampling rate,  $f_{s,\min}$ .  
 $W = \frac{1}{3}$ . Therefore the Nyquist rate is  $f_{s,\min} = \frac{2}{3}$ .
- (b) Sketch the ideal sampled spectrum  $X_\delta(f)$ , for the sampling rate  $f_s = 3f_{s,\min}$ .  
 The sampled spectrum  $X_\delta(f)$  for  $f_s = 2$  is shown below.



Sampled spectrum  $X_\delta(f)$ .

- (c) Sketch the output  $\tilde{x}(t)$  of an ideal (rectangular) *bandpass* filter with  $f_0 = 4$ ,  $B = 0.7$ , and gain  $1/4$ .

The spectrum  $\tilde{X}(f)$  at the output of the filter is:



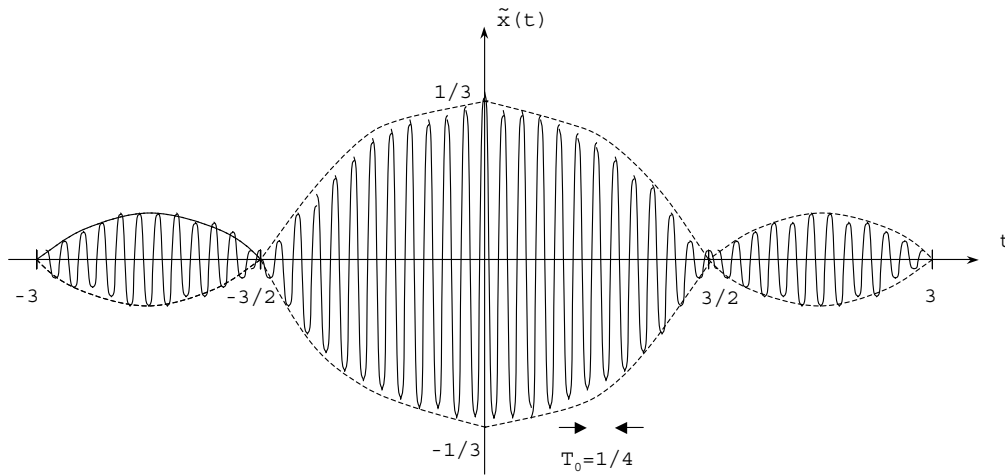
Output spectrum  $\tilde{X}(f)$ .

We have

$$\begin{aligned}\tilde{X}(f) &= \frac{1}{2} \left\{ \Pi \left[ \frac{3}{2}(f+4) \right] + \Pi \left[ \frac{3}{2}(f-4) \right] \right\} \\ &= \frac{1}{3} \left\{ \frac{1}{2} \cdot 3 \Pi \left[ \frac{3}{2}(f+4) \right] + \frac{1}{2} \cdot 3 \Pi \left[ \frac{3}{2}(f-4) \right] \right\} \iff \tilde{x}(t) = \frac{1}{3} \operatorname{sinc} \left( \frac{2t}{3} \right) \cos(8\pi t)\end{aligned}$$



The amplitude envelope is a sinc function with zero crossings at multiples of  $3/2$ . Also, the sinusoidal waveform has period  $T_0 = 1/4$ . A sketch is shown below:



Sketch of output signal  $\tilde{x}(t)$ .

**Problem 3 (15 points)** Using the properties of the Fourier transform, find the value of the integral

$$I = \int_{-\infty}^{\infty} \text{sinc}^2(t) \cos(6\pi t) dt.$$

Use the modulation property and

$$\int_{-\infty}^{\infty} x(t) dt = [X(f)]_{f=0}.$$

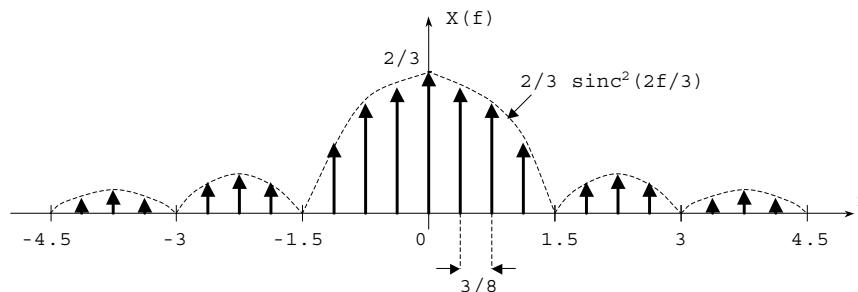
This gives

$$I = \frac{1}{2} [\Lambda(f+3) + \Lambda(f-3)]_{f=0} = 0.$$

Note (based on an idea of C. Igwebuikwe): The same result is obtained using Parseval's theorem. This follows from the evaluation property of the impulse function:

$$\int_{-\infty}^{\infty} X(f) \delta(f - f_0) df = X(f_0).$$

**Problem 4 (30 points)** The spectrum  $X(f)$  of a signal is depicted below.



# Spectrum of a signal.

- (a) Is this signal energy-type or power-type? Justify your answer. (Hint: You do not need to compute the power or the energy.)

The signal spectrum,

$$X(f) = \sum_{n=-\infty}^{\infty} \frac{2}{3} \operatorname{sinc}^2 \left( \frac{2}{3} \frac{3n}{8} \right) \delta \left( f - \frac{3n}{8} \right),$$

is that of a periodic signal. Consequently, the signal is of the power type.

- (b) Sketch carefully the time-domain signal  $x(t)$ .

Compare the given spectrum with that of a periodic signal:

$$X(f) = \sum_{n=-\infty}^{\infty} x_n \delta \left( f - \frac{n}{T_0} \right).$$

Then  $T_0 = 8/3$ , and

$$x_n = \frac{2}{3} \operatorname{sinc}^2 \left( \frac{2}{3} \frac{3n}{8} \right) = \frac{1}{T_0} X_{T_0} \left( \frac{n}{T_0} \right).$$

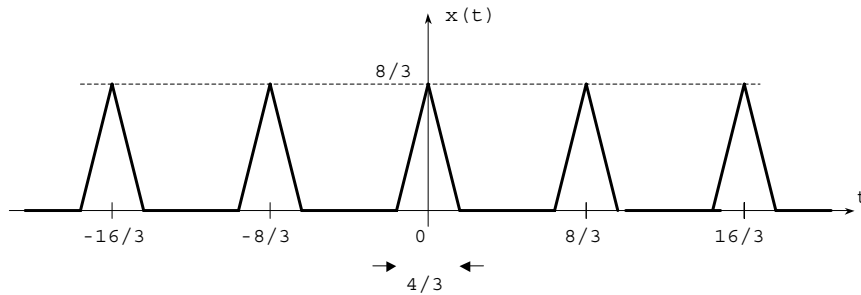
From this it follows that

$$X_{T_0}(f) = \frac{16}{9} \operatorname{sinc}^2 \left( \frac{2}{3} f \right) \iff x_{T_0}(t) = \frac{8}{3} \Lambda \left( \frac{3}{2} t \right),$$

and

$$x(t) = \sum_{n=-\infty}^{\infty} x_{T_0}(t - nT_0) = \frac{8}{3} \sum_{n=-\infty}^{\infty} \Lambda \left( \frac{3}{2} \left( t - \frac{8n}{3} \right) \right),$$

which is sketched below:



Sketch of the time signal  $x(t)$ .

- (c) Find the average value of  $x(t)$ . (Hint: No integrals required.)

The average of the signal is

$$\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) dt = x_0 = \frac{2}{3},$$

the area of the impulse at the origin in  $X(f)$ .

## Solution Midterm Exam # 1 - B

1. (30 points) A periodic signal  $s(t)$  is depicted in Fig. 1.

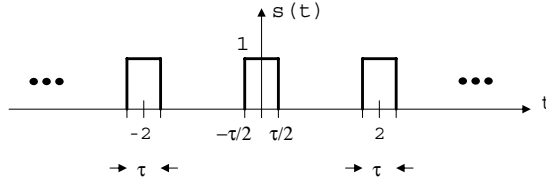


Figure 1: Periodic signal  $s(t)$ .

- (a) Find the Fourier series expansion of  $s(t)$ .  
 (b) Sketch carefully the spectrum,  $S(f)$ , of signal  $s(t)$ .

**Solution:**

- (a) The Fourier transform of the signal  $s_T(t)$ , over one period  $T = 2$  seconds, is

$$S_T(f) = \mathcal{F}(\Pi(t)) = \tau \text{sinc}(\tau f).$$

The Fourier series coefficients  $s_n$  can be computed from  $S_T(f)$  as

$$s_n = \frac{1}{T} S_T\left(\frac{n}{T}\right) = \frac{1}{2} \tau \text{sinc}\left(\frac{n\tau}{2}\right).$$

The Fourier series expansion of  $s(t)$  is therefore

$$\begin{aligned} s(t) &= \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n t/T} = \sum_{n=-\infty}^{\infty} \frac{1}{2} \tau \text{sinc}\left(\frac{n\tau}{2}\right) e^{j\pi n t} \\ &= \frac{\tau}{2} + \tau \sum_{n=1}^{\infty} \text{sinc}\left(\frac{n\tau}{2}\right) \cos(\pi n t) \end{aligned}$$

- (b) The signal is periodic with period  $T = 2$ . As a result, its spectrum is

$$\begin{aligned} S(f) &= \sum_{n=-\infty}^{\infty} s_n \delta\left(f - \frac{n}{2}\right) \\ &= \frac{\tau}{2} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\tau}{2}\right) \delta\left(f - \frac{n}{2}\right) \end{aligned}$$

and shown in the figure below:

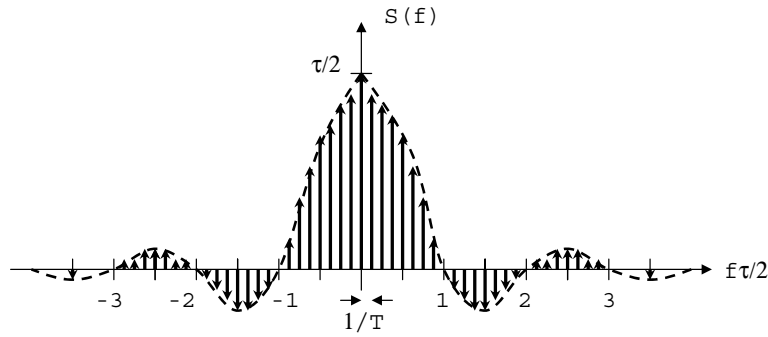
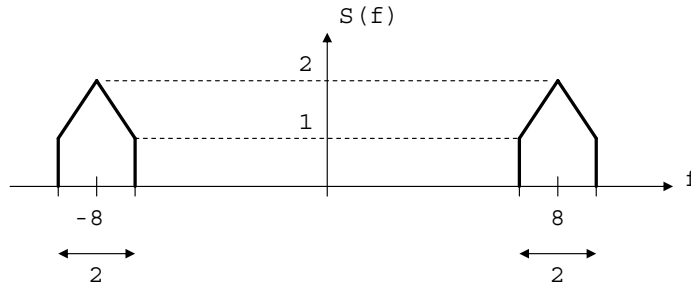


Figure 2: Spectrum of the periodic signal of problem 1.

2. (50 points) The spectrum of a bandlimited signal  $s(t)$  is shown below.



Spectrum of a bandlimited signal.

- Find an expression of signal  $s(t)$ . (Hint: The spectrum is a linear combination of a rectangular pulse and a triangular pulse.)
- Sketch carefully the spectrum of the sampled signal using a sampling rate equal to  $R_s = 1.5R_{\text{nyq}}$ , where  $R_{\text{nyq}}$  denotes the Nyquist rate.
- Specify a lowpass filter characteristic that allows reconstruction of  $s(t)$  from its samples.

**Solution:**

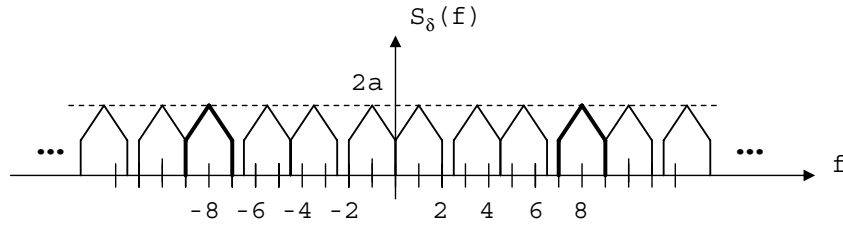
- (a) The spectrum can be written as

$$S(f) = \frac{1}{2} \left\{ 2 [\Lambda(f-8) + \Lambda(f+8)] + 2 \left[ \Pi\left(\frac{f-8}{2}\right) + \Pi\left(\frac{f+8}{2}\right) \right] \right\}$$

The inverse Fourier transform yields

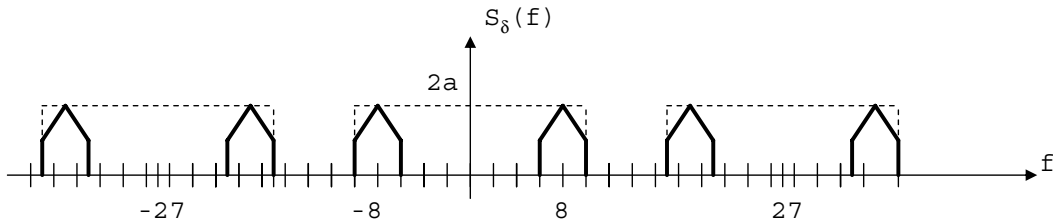
$$s(t) = 2 \{ \text{sinc}^2(t) + 2 \text{sinc}(2t) \} \cos(16\pi t).$$

- (b) This is a bandpass signal. Nyquist rate is  $R_s = 2f_u/[f_u/2] = 4.5$ . With this rate, the spectrum is shown in the figure below, where  $a = 4.5$



Spectrum of a sampled bandpass signal.

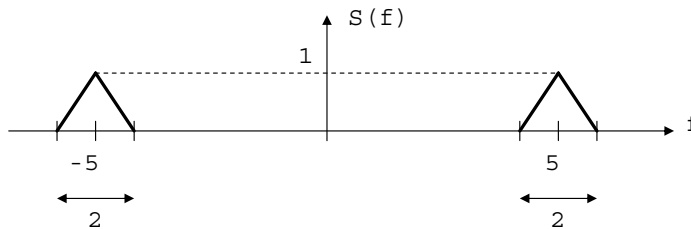
However, sampling at  $1.5R_s$  results in overlapping of spectra. Therefore, we regard the signal as a lowpass signal and sample at  $R_s = 1.5(2)(9) = 27$ .



Spectrum of a sampled bandpass signal regarded as a lowpass signal.

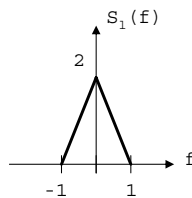
- (c) The signal can be recovered by a reconstruction filter that has constant response  $1/27$  in the frequency band of the signal  $7 < |f| \leq 9$  and zero for  $|f| \geq 18$ .

3. (20 points) Sketch the spectrum of the complex baseband equivalent of the bandpass signal whose spectrum is depicted in the following figure.



Spectrum of a bandpass signal.

**Solution:** The spectrum is below:



Spectrum of complex baseband signal.

## Solution Midterm Exam # 2

1. (35 points) A binary baseband communication system transmits one bit every  $T$  seconds, with  $T = 4$ . The pulse used is shown in Fig. 3 below.

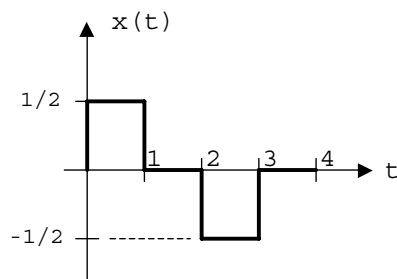


Figure 3: Pulse shape used in a binary baseband communication system.

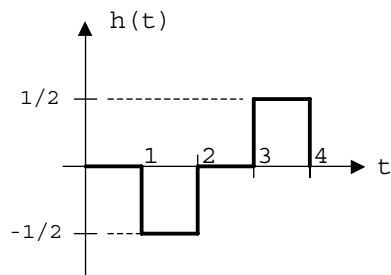
- Find the impulse response  $h(t)$  of the matched filter for  $x(t)$ .
- Determine the output of the matched filter at  $t = 4$  when the input is  $x(t)$ .
- For transmission over an AWGN channel, polar mapping is employed, i.e., a bit  $b_k$  is mapped onto a voltage level  $a_k$  as follows:

$b_k$	$\mapsto$	$a_k$
0		-1
1		+1

After polar mapping the levels are input to a transmit filter with impulse response  $x(t)$  every 4 seconds. The AWGN process has power spectral density equal to  $1/2$ . Determine the probability of a bit error of the optimal receiver. Express your answer in terms of the Gaussian  $Q$ -function.

**Solution:**

- (a) Matched filter:



- (b) Sampled output of matched filter at  $t = T = 4$ :

$$y(4) = E_x = \int_0^4 |x(\tau)|^2 d\tau = 2 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

(c) Polar mapping:

$$P[\epsilon] = Q\left(\sqrt{\frac{2E_x}{N_0}}\right) = Q\left(\sqrt{\frac{2(\frac{1}{2})}{(1)}}\right) = Q(1)$$

2. (30 points) Binary data at a rate of  $4 \times 10^6$  bits/s are transmitted using  $M$ -PAM over a bandlimited channel using a raised cosine roll-off characteristic as shown in Fig. 4 below.

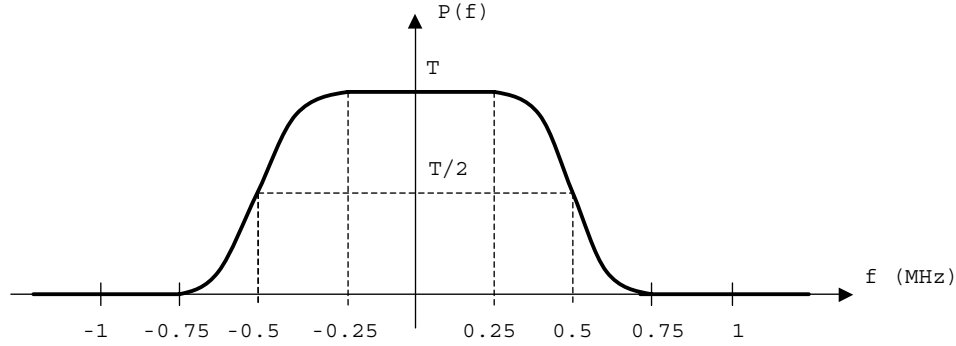


Figure 4: Raised cosine roll-off characteristic.

- (a) Determine the roll-off factor  $\alpha$  and the value of  $M$  of this system.
- (b) Signals are transmitted over a bandlimited AWGN channel. Determine the minimum required bit energy-to-noise ratio,  $E_b/N_0$ , to achieve a probability of a bit error less than  $10^{-3}$ . Express your result in decibels (dB).

**Solution:**

- (a)  $R_b = 4$  Mbits/s. Note that  $1/2T = 0.5$  MHz, the point of symmetry of the raised cosine. As a result,  $1/T = 1$  MHz. Since  $R_b = 1/T_b = \log_2 M (1/T)$ , it follows that  $\log_2 M = 4$  and therefore  $M = 16$ .

To find the roll-off factor, note that  $2\alpha(1/2T) = 0.75 - 0.25 = 0.5$  MHz (the region of the cosine characteristic) and therefore, with  $1/2T = 0.5$  MHz, we obtain  $\alpha = 0.5$ .

- (b) The probability of a bit error for  $M$ -PAM is

$$P[\epsilon] = \frac{M-1}{M} Q\left(\sqrt{\frac{6 \log_2 M}{M^2-1} \frac{E_b}{N_0}}\right).$$

For  $M = 16$ ,

$$P[\epsilon] = \frac{15}{16} Q\left(\sqrt{\frac{24}{255} \left(\frac{E_b}{N_0}\right)_{\min}}\right) < 1 \times 10^{-3}$$

From the table of the  $Q$ -function, we have  $Q(3.07) = 0.0011$ . Therefore,

$$\sqrt{\frac{8}{85} \left(\frac{E_b}{N_0}\right)_{\min}} = 3.07 \longrightarrow \left(\frac{E_b}{N_0}\right)_{\min} = \frac{85}{8} (3.07)^2 = 100.14 = 20 \text{ dB}$$

3. (35 points) A communication system is designed for binary data transmission over an ideal AWGN channel, with RZ unipolar pulses of average energy equal to  $E_s = A^2T/4$ . This system is then applied to an optical link that has the feature that the variance of the AWGN process varies with the signal level.

Specifically, it is found that the sampled output of the matched filter (with  $k_{MF} = 1$ )  $Y$  is a Gaussian random variable with conditional PDF given by

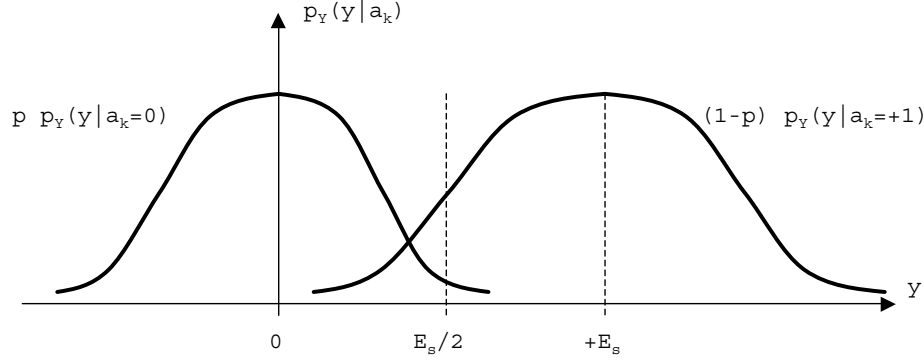
$$p_Y(y|a_k) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2 E_s}} \exp \left[ -\frac{y^2}{2\sigma^2 E_s} \right], & a_k = 0 \quad (b_k = 0); \\ \frac{1}{\sqrt{4\pi\sigma^2 E_s}} \exp \left[ -\frac{(y-E_s)^2}{4\sigma^2 E_s} \right], & a_k = +1 \quad (b_k = 1). \end{cases}$$

In other words, the variance of  $Y$  equals  $\sigma^2 E_s$  if the transmitted level is 0 and  $2\sigma^2 E_s$  if the transmitted level is +1.

- Find the probability of a bit error for the optical link in terms of the energy-to-noise power ratio,  $E_s/\sigma^2$ . Express your result in terms of the Gaussian  $Q$ -function.
- Give an expression to determine the threshold of the decision device used in an optimal receiver for the optical link.

**Solution:**

- The conditional PDF's of the matched filter output are sketched below:



Probability of error:

$$\begin{aligned} P[\epsilon] &= p Q \left( \frac{E_s/2 - 0}{\sqrt{\sigma^2 E_s}} \right) + (1-p) \left[ 1 - Q \left( \frac{E_s/2 - E_s}{\sqrt{2\sigma^2 E_s}} \right) \right] \\ &= p Q \left( \sqrt{\frac{E_s}{4\sigma^2}} \right) + (1-p) Q \left( \sqrt{\frac{E_s}{8\sigma^2}} \right) \end{aligned}$$

- The optimum threshold  $y = \lambda$  is obtained by equating the a-posteriori PDF's:

$$\begin{aligned} p p_Y(\lambda|a_k = 0) &= (1-p) p_Y(\lambda|a_k = +1) \\ p \frac{1}{\sqrt{2\pi\sigma^2 E_s}} \exp \left[ -\frac{\lambda^2}{2\sigma^2 E_s} \right] &= (1-p) \frac{1}{\sqrt{4\pi\sigma^2 E_s}} \exp \left[ -\frac{(\lambda - E_s)^2}{4\sigma^2 E_s} \right] \end{aligned}$$

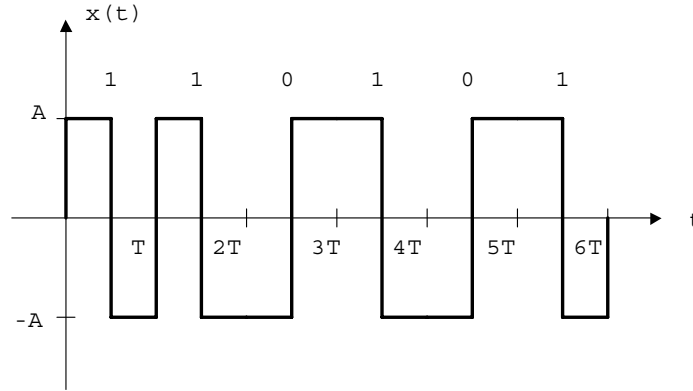


$$\begin{aligned}
-\frac{\lambda^2}{2\sigma^2 E_s} &= -\frac{(\lambda - E_s)^2}{4\sigma^2 E_s} - \frac{1}{2} \ln 2 + \ln \frac{1-p}{p} \\
\longrightarrow \lambda^2 + 2\lambda E_s - E_s^2 + \sigma^2 E_s \left[ 2 \ln \frac{1-p}{p} - \ln 2 \right] &= 0.
\end{aligned}$$

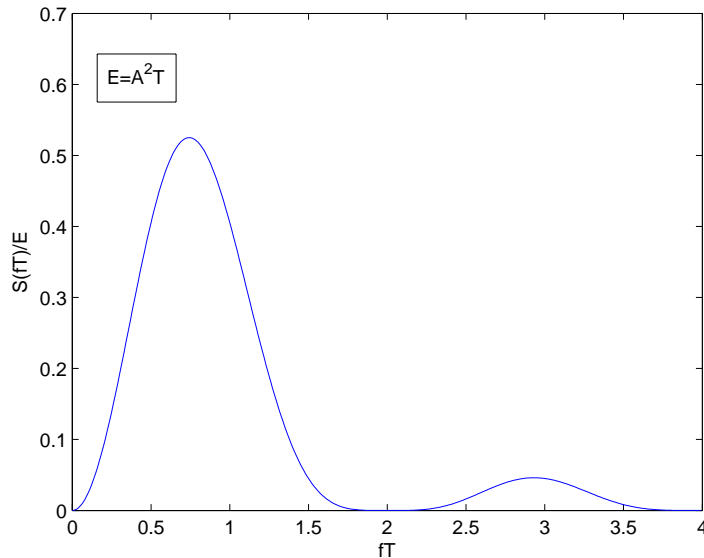
## Solution Midterm Exam # 2-B

**Problem 1 (25 points)** A line coding scheme uses Manchester encoding with rectangular pulses.

- (a) Sketch the signal corresponding to the bit sequence “110101”



- (b) Sketch carefully, showing all relevant labels, the power spectral density of this scheme.



- (c) The energy per bit is  $E_b = 1 \times 10^{-9}$  Joules and the bit rate is  $R_b = 1 \times 10^6$  bits/sec. What is the amplitude of the pulses?

The bit rate is  $R_b = 1/T = 10^6$ . The energy per bit is  $E_b = A^2 T$  from which it follows that  $A = \sqrt{E_b/T} = \sqrt{E_b R_b} = \sqrt{(10^{-9})(10^6)} = \sqrt{10^{-3}} = 31.62$  mV.

- (d) The noise at the receiver is AWGN with  $\sigma_n^2 = 1.25 \times 10^{-10}$  W/Hz. Determine the probability of a bit error.

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2 \cdot 10^{-9}}{2 \cdot 1.25 \times 10^{-10}}}\right) = Q(\sqrt{8}) = 2.34 \times 10^{-3}.$$

**Problem 2 (40 points)** A binary communication system uses NRZ pulses and polar mapping:

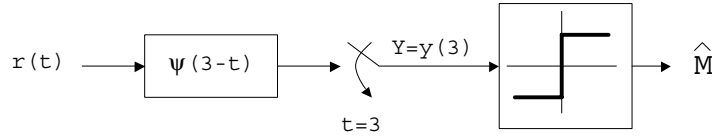
$$s(t) = \begin{cases} 2\psi(t), & \text{bit} = \text{"0"} \\ -2\psi(t), & \text{bit} = \text{"1"}, \end{cases}$$

where  $\psi(t)$  is a unit-energy pulse. The bit duration is 3 seconds.

- (a) Determine the energy per bit,  $E_b$ .

We have  $s(t) = \pm\sqrt{E_b}\psi(t)$ , and therefore  $\sqrt{E_b} = 2$  or  $E_b = 4$ .

- (b) Sketch carefully a block diagram of the demodulator. In particular, you must give a detailed specification of the impulse response of the matched filter, the sampling time, and the decision rule.



Decision rule:

$$\hat{M} = \begin{cases} 0, & Y > 0; \\ 1, & Y \leq 0, \end{cases}$$

- (c) Noise at the receiver is AWGN with  $\sigma_n^2 = 3$ . Determine the probability of a bit error,  $P_e$ .

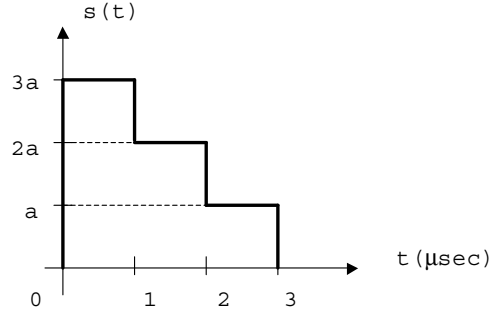
$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2 \cdot 4}{2 \cdot 3}}\right) = Q\left(\sqrt{\frac{4}{3}}\right) = 1.24 \times 10^{-1}.$$

- (d) By mistake, the transmitter sends RZ pulses with polar mapping. Determine the resulting probability of a bit error,  $P_{e,\text{mistake}}$ .

If RZ pulses of amplitude  $A$  are sent and the receiver has a filter (or correlator) matched to NRZ pulses, then the energy is scaled by one half. Consequently,

$$P_{e,\text{mistake}} = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{4}{2 \cdot 3}}\right) = Q\left(\sqrt{\frac{2}{3}}\right) = 2.07 \times 10^{-1}.$$

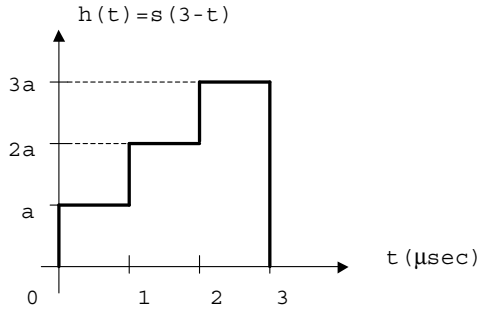
**Problem 3 (35 points)** The pulse waveform shown below is used for binary communication with unipolar mapping (also known as on-off keying).



- (a) Determine the value of  $a$  in terms of the average energy per bit  $E_b$ .

The energy of the pulse is  $E_s = (3a)^2(1) + (2a)^2(1) + a^2(1) = 14a^2$ , and therefore  $a = \sqrt{E_s/14}$ . On the other hand, the average energy per bit with unipolar mapping is  $E_b = E_s/2$ . Therefore,  $a = \sqrt{E_b/7}$ .

- (b) Sketch the impulse response of the filter matched to  $s(t)$ .



- (c) It is desired that the bit rate  $R_b$  (bits/sec) be at least  $1 \times 10^6$ . Using  $M$ -PAM modulation, you are asked to determine the minimum value of  $M$ .

The pulse duration is  $T = 4 \times 10^{-6}$  sec. The bit rate of  $M$ -PAM, with  $M = 2^m$ , is  $R_b = m/T$ . The requirement is

$$\frac{m}{T} \geq 10^6 \rightarrow m \geq (3 \times 10^{-6}) (10^6) \rightarrow m \geq 3.$$

Therefore, the minimum value is  $M = 2^3 = 8$ .

- (d) With the value of  $M$  found in part (c), determine the new value of  $a$  in terms of the average energy per symbol  $E_{\text{ave}}$ .

For 8-PAM, the average energy is:  $E_{\text{ave}} = (M^2 - 1)E_s/3 = 21E_s$ . As a result, the new value is  $a = \sqrt{E_s/7} = \sqrt{E_{\text{ave}}/147}$ .

## Solution Final Exam

**Problem 1 (35 points)** The raised-cosine spectrum  $X(f)$  of a digital communication system is shown in Fig. 5 below. The excess bandwidth is 25%.

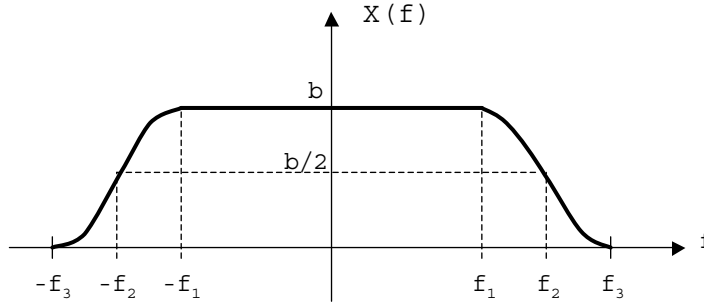


Figure 5: Raised-cosine spectrum of a digital communication system.

- (a) The system transmits information at a rate of 10 Mbps and utilizes 4-PAM modulation. Determine the values of  $b$ ,  $f_1$ ,  $f_2$  and  $f_3$ .

We have  $m = 2$  and  $R_b = 10 \times 10^6$  bps. Since  $R_b = m/T$ , it follows that  $1/T = R_b/2 = 5 \times 10^6$  baud. Therefore,

$$b = T = 2 \times 10^{-7} = 200 \text{ ns}$$

$$f_2 = \frac{1}{2T} = 2.5 \times 10^6 = 2.5 \text{ MHz}$$

$$f_3 = \frac{(1 + \alpha)}{2T} = (1.25) (2.5 \times 10^6) = 3.125 \times 10^6 = 3.125 \text{ MHz}$$

$$f_1 = \frac{(1 - \alpha)}{2T} = (0.75) (2.5 \times 10^6) = 1.875 \times 10^6 = 1.875 \text{ MHz}$$

(Type B:  $b = 400$  ns,  $f_2 = 1.25$  MHz,  $f_3 = 1.6625$  MHz, and  $f_1 = 837.5$  KHz.)

- (b) RZ rectangular pulses are utilized. Determine the amplitude of the pulses if  $N_0 = 1 \times 10^{-8}$  and the probability of a bit error  $P_b$  is desired to be lower than  $10^{-2}$ .

With RZ pulses,  $E = a^2 T/2$ . For 4-PAM modulation, the requirement that  $P_b < 10^{-2}$  translates into

$$P_b \approx \frac{3}{4} Q \left( \sqrt{\frac{4E_b}{5N_0}} \right) < 10^{-2} \quad \longrightarrow \quad Q \left( \sqrt{\frac{4E_b}{5N_0}} \right) < 1.33 \times 10^{-2}.$$

With the help of Table 1, we find that

$$\sqrt{\frac{4E_b}{5N_0}} > 2.3 \quad \longrightarrow \quad \frac{E_b}{N_0} > \frac{5}{4} (2.3)^2 = 6.6125 = 8.2 \text{ dB}.$$

With  $N_0 = 1 \times 10^{-8}$ ,

$$E_b = 6.6125 \times 10^{-8} = \frac{a^2 T}{2} \quad \longrightarrow \quad a = \sqrt{\frac{2(6.6125 \times 10^{-8})}{2 \times 10^{-7}}} = 0.813 \text{ V.}$$

(Type B:  $E_b/N_0 = 6.8$  dB, and  $a = 1.095$  V.)

- (c) The reliability is improved by requiring that  $P_b < 10^{-3}$ ? What amount of additional power will be needed? Express your answer in dB.

With the aid of Table 1, the requirement  $P_b < 10^{-3}$  translates into  $\sqrt{\frac{4E_b}{5N_0}} > 3.1$ , from which it follows that

$$\left(\frac{E_b}{N_0}\right)_{\text{impr}} = \frac{5}{4} (3.1)^2 = 12.0125 = 10.8 \text{ dB.}$$

Consequently, the amount of additional power is  $10.8 - 8.2 = 2.6$  dB.

(Type B:  $(E_b/N_0)_{\text{impr}} = 9.6$  dB, additional power is 2.8 dB.)

**Problem 2 (35 points)** A binary communication system transmits information at a rate of 1 Mbps. The mapping is polar and a correlation-type receiver is employed. The waveform used is shown in Fig. 6 below.

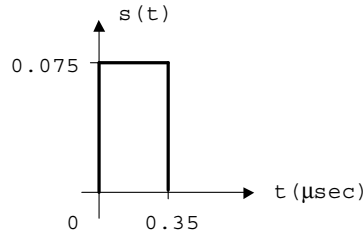


Figure 6: Waveform used in a binary communication system.

- (a) The noise at the receiver is AWGN with  $\sigma_n^2 = 5 \times 10^{-10}$ . Determine the probability of a bit error.

$T = 1 \mu\text{s}$  and mapping is polar. The energy per bit is

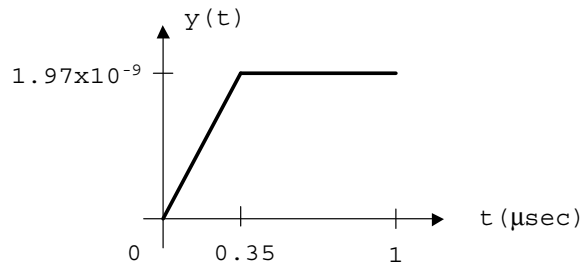
$$E_b = (0.075)^2 (0.35 \times 10^{-6}) = 1.96875 \times 10^{-9}.$$

Also,  $\sigma_n^2 = N_0/2 = 5 \times 10^{-10}$  or  $N_0 = 1 \times 10^{-9}$ . Therefore,

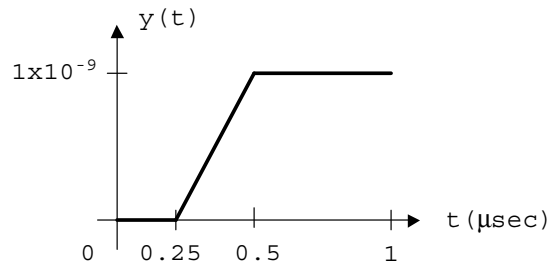
$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2(1.96875 \times 10^{-9})}{1 \times 10^{-9}}}\right) = Q(1.984) \approx 2 \times 10^{-2}.$$

(Type B:  $a = 64.6$  mV.)

- (b) Sketch the output of the correlator in the range 0 to 1  $\mu\text{sec}$ . (Hint: No integrals.)

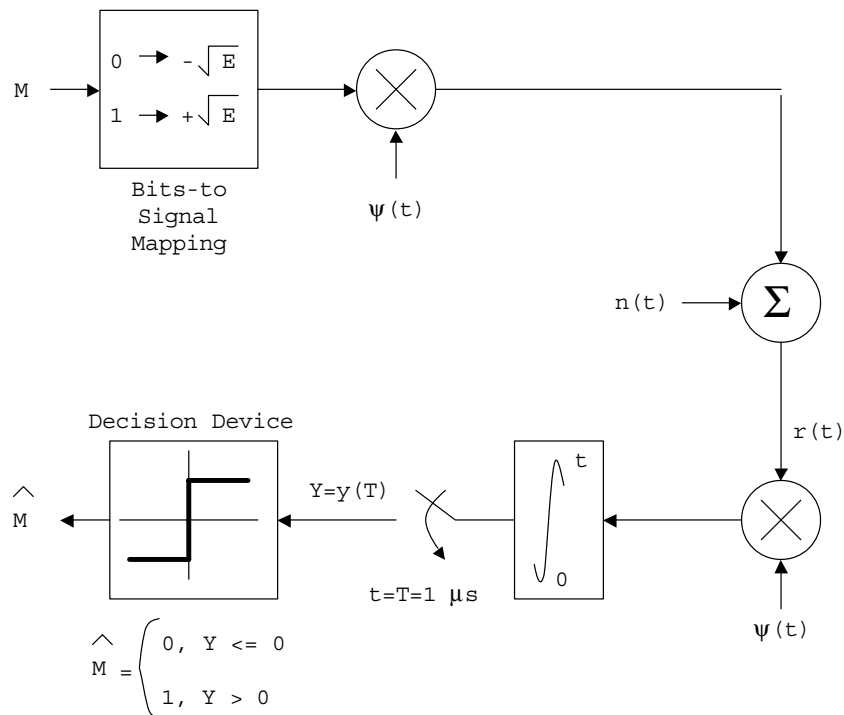


(Type B:



)

- (c) Sketch carefully, with as much detail as possible, a block diagram of this system, including both transmitter and receiver. (The more detail is shown the higher the grade of this part.)



The waveform  $\psi(t)$  is a rectangular pulse of duration  $0.35 \mu\text{s}$  and amplitude  $\sqrt{1/(0.35 \times 10^{-6})}$ .

- (d) If the transmitter sends NRZ pulses at the same rate and with the same energy as the pulse of Fig. 6, and the receiver is the same as that of part (c), determine the resulting probability of a bit error.

An NRZ pulse with the same energy and duration is a rectangular pulse of width  $1 \times 10^{-6}$  s and amplitude  $\sqrt{1.96875 \times 10^{-3}}$  V. The correlator output is a waveform similar to that of the previous figure with maximum amplitude (energy)  $a = (1.96875 \times 10^{-3})(0.35 \times 10^{-6}) = 6.89 \times 10^{-10}$ . Therefore,

$$P_b = Q\left(\sqrt{\frac{2(6.89 \times 10^{-10})}{1 \times 10^{-9}}}\right) = Q(1.174) \approx 4 \times 10^{-2}.$$

(Type B:  $P_b \approx 3.59 \times 10^{-2}$ .)

**Problem 3 (35 points)** A bandpass QPSK modulation system uses the following orthonormal signals,

$$\psi_1(t) = 110 \cos(10^4 \pi t), \quad \psi_2(t) = 110 \sin(10^4 \pi t).$$

Noise is AWGN with  $N_0 = 1$  and the target probability of a bit error is  $P_b = 10^{-5}$ .

- (a) Determine the maximum bit rate (bps) of this system.

For QPSK modulation,  $m = 2$  and  $P_b = 10^{-5}$ , we have that (Homework 9!)  $E_b/N_0 \geq 9.12$ . Therefore, with  $N_0 = 1$ ,  $E = 2E_b \geq 18.24$ . Also, the amplitude of the signal is

$$\sqrt{\frac{2E}{T}} = 110 \quad \longrightarrow \quad \frac{1}{T} \leq \frac{(110)^2}{2 \times 18.24} = 331.7 \text{ baud.}$$

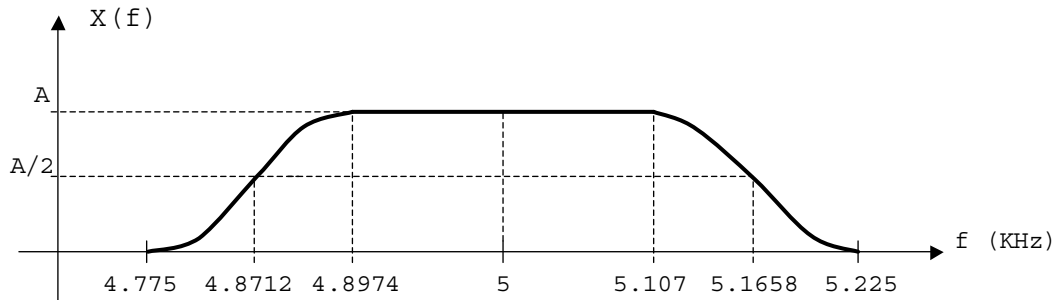
As a result,

$$R_{b,\max} = \frac{2}{T} = 663.38 \text{ bps.}$$

(Type B:  $R_{b,\max} = \frac{4}{T} = 4002 \text{ bps.}$ )

- (b) The channel bandwidth is 450 Hz and a raised-cosine spectrum is employed to eliminate ISI. Determine the excess bandwidth and sketch carefully the spectrum, paying particular attention to the frequency values.

We have  $B = \frac{1}{T}(1 + \alpha) = 450$ . Therefore,  $\alpha = \frac{450}{331.7} - 1 = 0.3567$ , or 35.7%

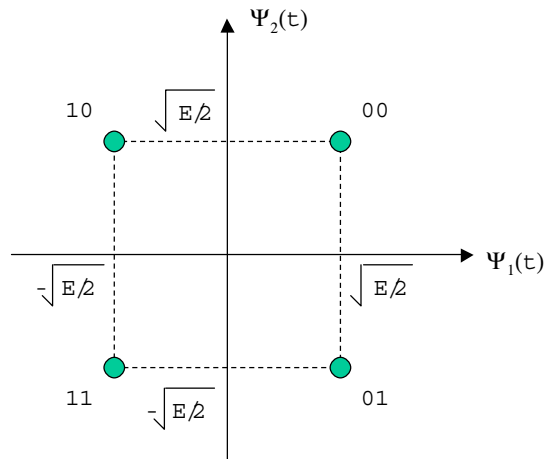




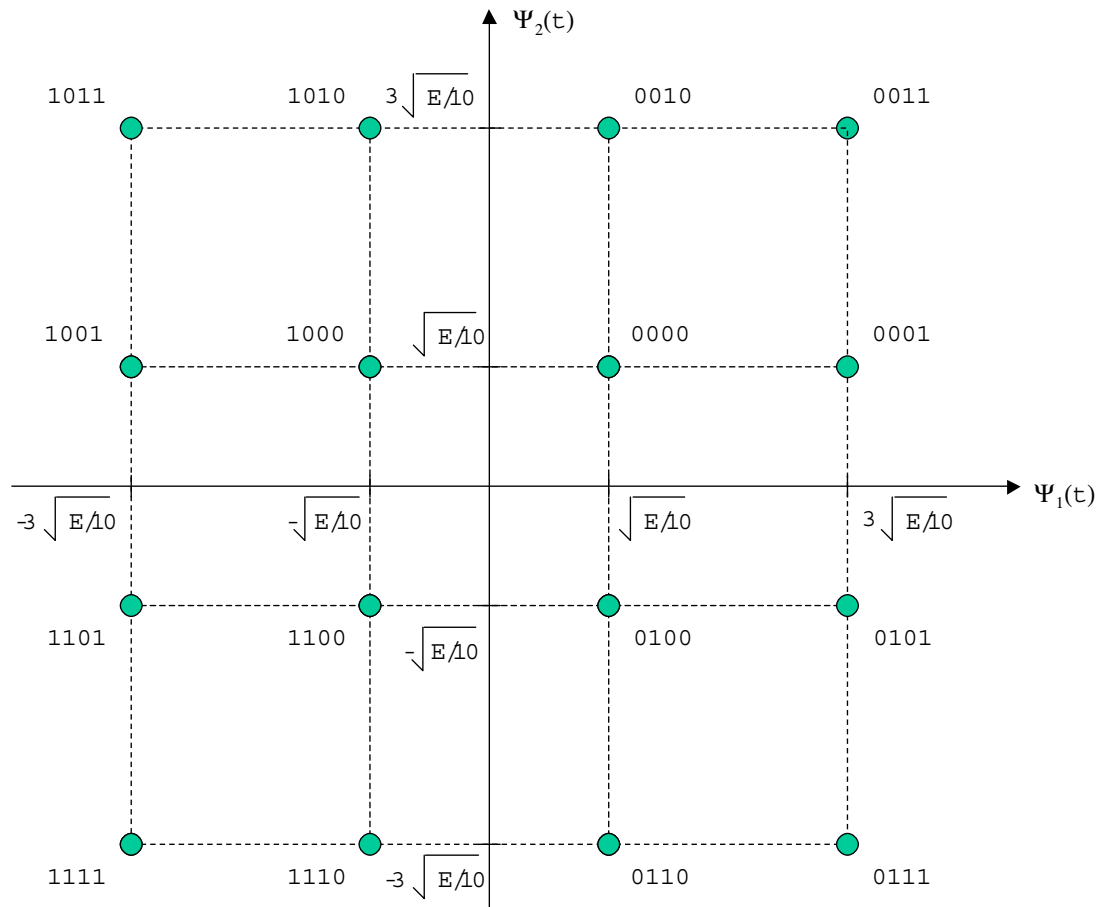
In the figure above,  $A = 2.12 \times 10^{-3}$ .

(Type B:  $B = 1220$ ,  $\alpha = 0.22$  or 21.94%, and  $R_b/B = 3.28$  bps/Hz. Also,  $A = 7.07 \times 10^{-4}$ .)

(c) Sketch the constellation points used in the bits-to-signal mapper.



(Type B:



)

- (d) A new RF technology produces a 5 dB improvement in SNR at the receiver. Discuss how this can be used to enhance performance.

A larger SNR at the receiver can be used in three alternative ways:

- (1) To achieve a reduced probability of a bit error or bit error rate (BER), for the same transmitted power
- (2) A lower transmitted power with the same target BER
- (3) An increased transmission rate for the same BER. For example, the bit rate could be doubled to  $R_{b,\text{new}} = 2 \times 663.38 = 1266.8$  bps by using 16-QAM modulation, which requires 14.6 dB (5 dB more than QPSK modulation). See also the solution of Homework 9. (Type B: Can use 32-QAM and increase the rate to  $R_{b,\text{new}} = \frac{5}{4} \times 4002 = 5002.5$  bps.)

## Solutions to final exams

1A. The orthonormal pulses shown in Fig. 7 are used in a transmission system using quadrature modulation.

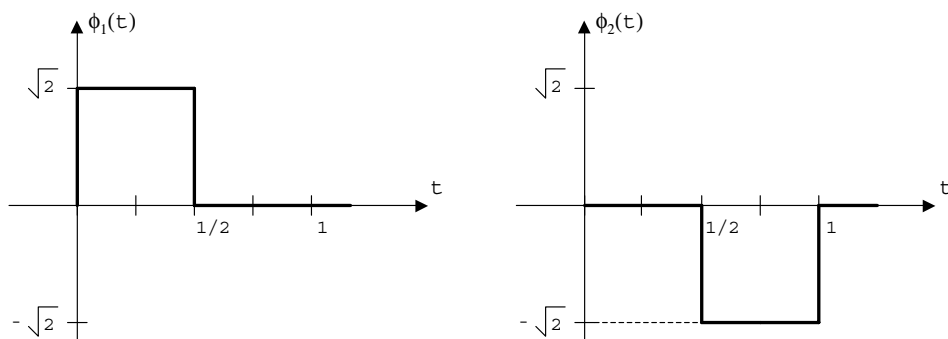


Figure 7: Two orthonormal signals.

- (a) Determine the pulses associated with the points in the  $\phi_1\phi_2$ -plane shown in Fig. 8 (a).
- (b) Determine the signal point  $s$  associated with the pulse  $s(t)$  shown in Fig. 8 (b).

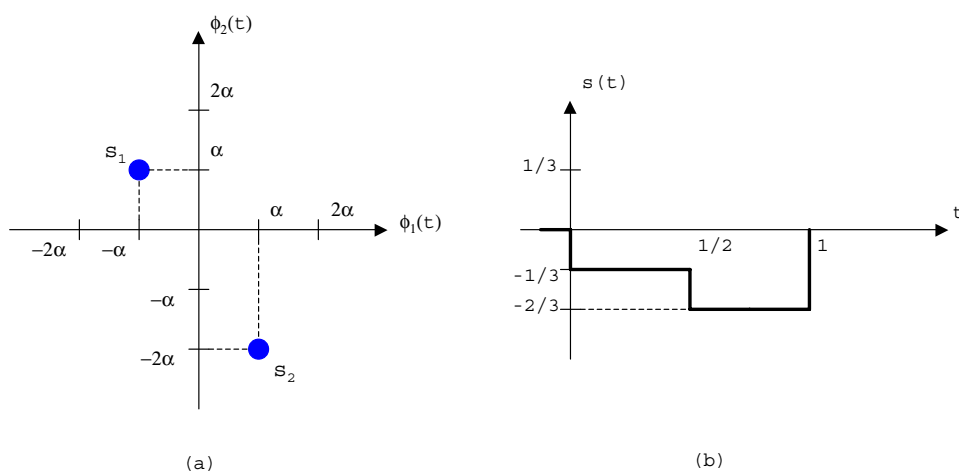


Figure 8: (a) Two signal points; (b) a pulse.

**Solution:**

(a)

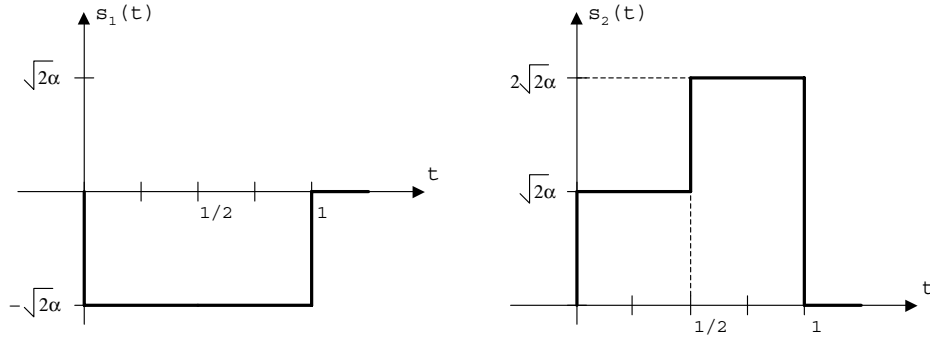


Figure 9: Pulse signals associated with points  $s_1$  and  $s_2$ .

(b) The coordinates of the point are equal to the outputs of the filters matched to  $\phi_1(t)$  and  $\phi_2(t)$ , respectively, and given by

$$s_1 = \int_0^{1/2} s(t)\phi_1(t) dt = \frac{1}{2} \left( -\frac{1}{3} \right) (\sqrt{2}) = -\frac{\sqrt{2}}{6},$$

$$s_2 = \int_{1/2}^1 s(t)\phi_2(t) dt = \frac{1}{2} \left( -\frac{2}{3} \right) (-\sqrt{2}) = \frac{\sqrt{2}}{3},$$

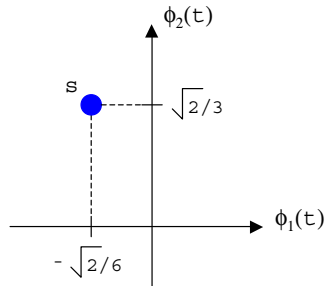


Figure 10: Point in the  $\phi_1\phi_2$ -plane associated with pulse  $s(t)$ .

**1B.** The orthonormal pulses shown in Fig. 11 are used in a transmission system using quadrature modulation.

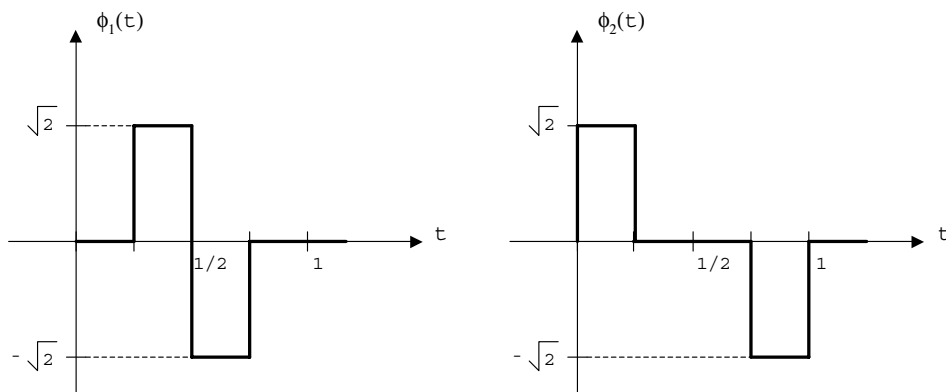


Figure 11: Two orthonormal signals.

- (a) Determine the pulses associated with the points in the  $\phi_1\phi_2$ -plane shown in Fig. 12 (a).
- (b) Determine the signal point  $s$  associated with the pulse  $s(t)$  shown in Fig. 12 (b).

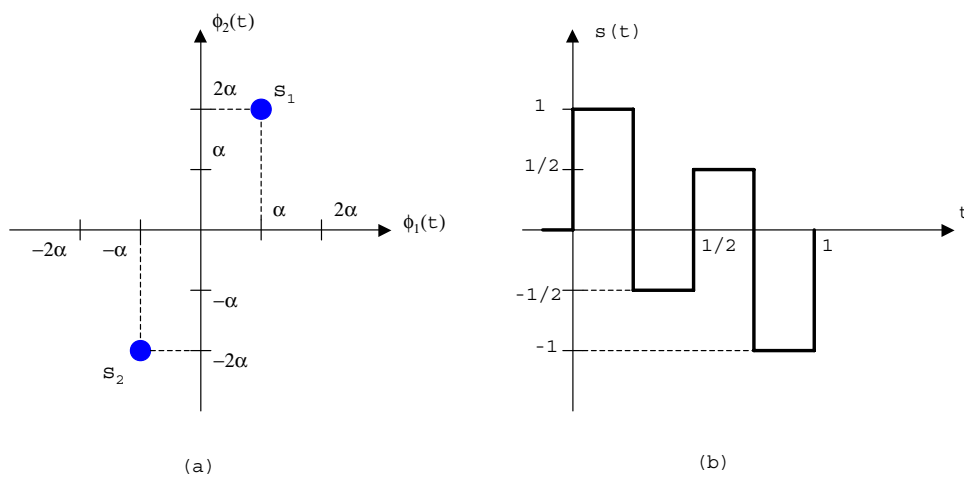


Figure 12: (a) Two signal points; (b) a pulse.

**Solution:**

(a) The signals are shown in Fig. 13.

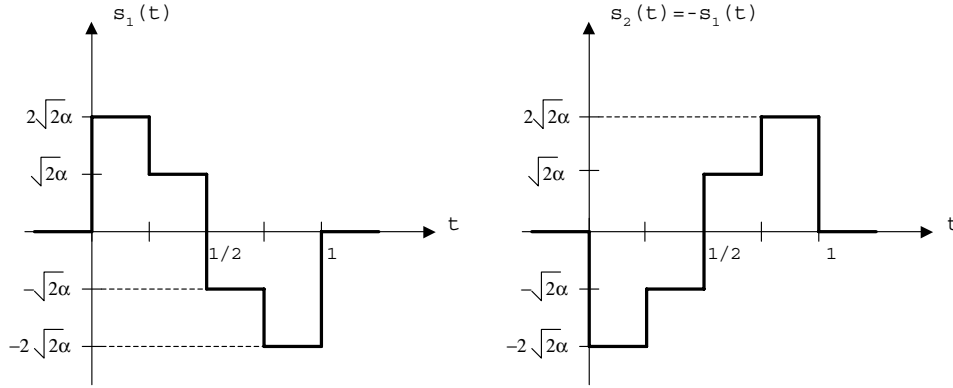


Figure 13: Pulse signals associated with points  $s_1$  and  $s_2$ .

(b) The coordinates of the point are equal to the outputs of the filters matched to  $\phi_1(t)$  and  $\phi_2(t)$ , respectively, and given by

$$s_1 = \int_{1/4}^{3/4} s(t)\phi_1(t) dt = \frac{1}{4} \left( -\frac{1}{2} \right) (\sqrt{2}) + \frac{1}{4} \left( \frac{1}{2} \right) (-\sqrt{2}) = -\frac{\sqrt{2}}{4},$$

$$s_2 = \int_0^{1/4} s(t)\phi_2(t) dt + \int_{3/4}^1 s(t)\phi_2(t) dt = \frac{1}{4}(1)(\sqrt{2}) + \frac{1}{4}(-1)(-\sqrt{2}) = \frac{\sqrt{2}}{2}.$$

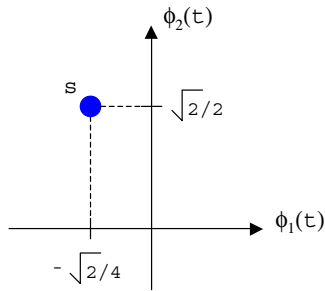


Figure 14: Point in the  $\phi_1\phi_2$ -plane associated with pulse  $s(t)$ .

2. A binary modulation system uses quadrature modulation with the signal points shown in Fig. 15. Transmission of equally likely bits takes place over an AWGN channel with power spectral density  $N_0/2$ .

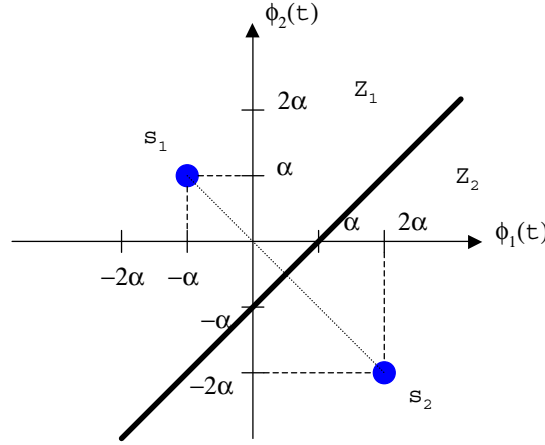


Figure 15: Binary signal constellation.

- Find the value of the constant  $\alpha$  as a function of the average signal energy  $E$ .
- Sketch carefully the decision regions.
- Find the probability of a bit error  $P[\epsilon]$ , in terms of  $E/N_0$  and the  $Q$ -function, and compare it with binary transmission using square pulses and polar mapping.

**Solution:**

- $\alpha = \sqrt{\frac{E}{5}}$ .
- Shown in Fig. 15.
- The probability of a bit error is given by

$$P[\epsilon] = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{9E}{5N_0}}\right).$$

For polar mapping (regardless of the pulse shape),

$$P[\epsilon] = Q\left(\sqrt{\frac{2E}{N_0}}\right).$$

As a result, the constellation in question has a gain of

$$10 \log_{10} \left( \frac{9E/5N_0}{2E/N_0} \right) = 10 \log_{10} \left( \frac{9}{10} \right) = -0.46 \text{ dB}$$

with respect to polar mapping. That is, the constellation in Fig. 15 needs 0.46 dB more power to achieve the same error performance.

3. This problem deals with a Hamming code with  $m = 2$  redundant bits. This  $(3, 1)$  code is also known as a *repetition code* of length  $n = 3$ .

- (a) Specify a parity-check matrix  $H$  for this code.
- (b) Construct a look-up table with syndrome  $s$  as input and error vector  $e$  as output.
- (c) If the vector  $r = (1 \ 0 \ 0)$  is received, determine the estimated code vector  $v$ . (Type B:  $r = (1 \ 0 \ 1)$ .)

**Solution:**

- (a) A parity-check matrix is:

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

- (b) Look-up table:

$s^T$	$e$
0 0	0 0 0
1 1	1 0 0
1 0	0 1 0
0 1	0 0 1

- (c) The syndrome corresponding to the received vector is

$$s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore,  $e = (1 \ 0 \ 0)$  and

$$v = r \oplus e = (1 \ 0 \ 0) \oplus (1 \ 0 \ 0) = (0 \ 0 \ 0).$$

For final exam type B,

$$s = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore,  $e = (0 \ 1 \ 0)$  and

$$v = r \oplus e = (1 \ 0 \ 1) \oplus (0 \ 1 \ 0) = (1 \ 1 \ 1).$$



**Table 1: The Gaussian  $Q$ -function**

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	5.00e-01	4.60e-01	4.21e-01	3.82e-01	3.45e-01	3.09e-01	2.74e-01	2.42e-01	2.12e-01	1.84e-01
1.0	1.59e-01	1.36e-01	1.15e-01	9.68e-02	8.08e-02	6.68e-02	5.48e-02	4.46e-02	3.59e-02	2.87e-02
2.0	2.27e-02	1.79e-02	1.39e-02	1.07e-02	8.20e-03	6.21e-03	4.66e-03	3.47e-03	2.56e-03	1.87e-03
3.0	1.35e-03	9.68e-04	6.87e-04	4.83e-04	3.37e-04	2.33e-04	1.59e-04	1.08e-04	7.23e-05	4.81e-05
4.0	3.17e-05	2.07e-05	1.33e-05	8.54e-06	5.41e-06	3.40e-06	2.11e-06	1.30e-06	7.93e-07	4.79e-07
5.0	2.87e-07	1.70e-07	9.96e-08	5.79e-08	3.33e-08	1.90e-08	1.07e-08	5.99e-09	3.32e-09	1.82e-09

Example:  $Q(2.5) = 6.21\text{e-}03 = 6.21 \times 10^{-3}$ .

**Table 2: Selected values of the inverse Gaussian  $Q$ -function**

$Q(x)$	$x$
$10^{-1}$	1.28
$10^{-2}$	2.33
$10^{-3}$	3.10
$10^{-4}$	3.73
$10^{-5}$	4.27
$10^{-6}$	4.76
$10^{-7}$	5.20
$10^{-8}$	5.61
$10^{-9}$	6.00
$10^{-10}$	6.63
$10^{-11}$	6.71
$10^{-12}$	7.03
$10^{-13}$	7.35
$10^{-14}$	7.65