ENG 342: Advanced Engineering Math II

Quiz #1

September 13, 2016

Problem 1 [4 pts]

Let $f(x) = \cos(m\pi x)$ and $g(x) = \sin(n\pi x)$ for integers $m > 0$, $n > 0$.

(a) Show that f and g are orthogonal to each other on the interval $[0, 2]$ for all possible values of m and n. [3 pts]

We must show that $(f, g) = 0$ for all possible values of m and n:

$$\begin{aligned}
(f,g) &= \int_0^2 \cos(m\pi x)\sin(n\pi x)\, dx \\
&= \int_0^2 \frac{1}{2}\left(\sin(m\pi x + n\pi x) - \sin(m\pi x - n\pi x)\right) dx \\
&= \frac{1}{2}\int_0^2 \sin((m+n)\pi x)dx - \frac{1}{2}\int_0^2 \sin((m-n)\pi x)dx \\
&= -\frac{\cos((m+n)\pi x)}{2(m+n)\pi}\Bigg|_0^2 + \frac{\cos((m-n)\pi x)}{2(m-n)\pi}\Bigg|_0^2
\end{aligned}$$

The right side is defined only for $m \neq n$. In this case, we have:

$$\begin{aligned}
(f,g) &= -\frac{\cos(2(m+n)\pi) - \cos(0)}{2(m+n)\pi} + \frac{\cos(2(m-n)\pi) - \cos(0)}{2(m-n)\pi} \\
&= -\frac{1-1}{2(m+n)\pi} + \frac{1-1}{2(m-n)\pi} = 0 + 0 = 0
\end{aligned}$$

since $\cos(2r\pi) = \cos(0) = 1$ for any integer r. In the case of $m = n$:

$$\begin{aligned}
(f,g) &= \int_0^2 \cos(n\pi x)\sin(n\pi x)\, dx \\
&= \int_0^2 \frac{1}{2}\left(\sin(2n\pi x) - \sin(0)\right) dx \\
&= \frac{1}{2}\int_0^2 \sin(2n\pi x)dx = -\frac{\cos(2n\pi x)}{4n\pi}\Bigg|_0^2 = -\frac{1-1}{4n\pi} = 0
\end{aligned}$$

(b) What is the norm of f on $[0, 1]$? [1 pt]
The squared norm is:

$$\begin{aligned}
\|f(x)\|^2 &= \int_0^1 \cos^2(m\pi x)dx \\
&= \int_0^1 \frac{1}{2}\left(1 + \cos(2m\pi x)\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{x}{2}\right|_0^1 + \left.\frac{\sin(2m\pi x)}{4m\pi}\right|_0^1 \\
&= \frac{1}{2} + 0 = \frac{1}{2}
\end{aligned}
$$

The norm is the square root of this:

$$\|f(x)\| = \frac{1}{\sqrt{2}}$$

Problem 2 [6 pts]

Let $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 2x & 0 \leq x < \pi \end{cases}$.

(a) Expand $f(x)$ in a Fourier series. (Write it as a summation.) [4 pts]

Recognizing that $p = \pi$ and that the integral over $(-\pi, 0)$ is always 0, the Fourier coefficients are calculated as follows:

$$a_0 = \frac{1}{\pi}\int_0^\pi 2x\ dx = \left.\frac{1}{\pi}x^2\right|_0^\pi = \pi$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^\pi 2x\cos(nx)\ dx \\
&= \frac{2}{\pi}\left(\left.\frac{x}{n}\sin(nx)\right|_0^\pi - \int_0^\pi \frac{1}{n}\sin(nx)\ dx\right) \\
&= \frac{2}{\pi}\left(0 + \left.\frac{1}{n^2}\cos(nx)\right|_0^\pi\right) \\
&= \frac{2}{\pi n^2}\left((-1)^n - 1\right)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^\pi 2x\sin(nx)\ dx \\
&= \frac{2}{\pi}\left(\left.-\frac{x}{n}\cos(nx)\right|_0^\pi + \int_0^\pi \frac{1}{n}\cos(nx)\ dx\right) \\
&= \frac{2}{\pi}\left(-\frac{\pi}{n}(-1)^n + \left.\frac{1}{n^2}\sin(nx)\right|_0^\pi\right)
\end{aligned}
$$

$$= \frac{2}{n}(-1)^{n+1}$$

since $-(-1)^n = (-1)^{n+1}$.

Therefore,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n^2}((-1)^n - 1)\cos(nx) + \frac{2}{n}(-1)^{n+1}\sin(nx) \right)$$

(b) Plot what the Fourier series from (a) will converge to (*i.e.*, with an infinite number of terms) over the interval $(-3\pi, 3\pi)$. [2 pts]

The Fourier series will converge to the periodic extension of f, with a fundamental period of 2π, and to $(2\pi - 0)/2$ at the points of discontinuity -3π, $-\pi$, π, 3π: