SIGNALS AND SYSTEMS.
MIDTERM EXAM I: MARKING GUIDE

Solution 1 (25 Marks).

$$y[n] = A\,x[\alpha n + \beta] + B$$

a) Time Transformations
- Time scaling through α
- Time shifting through β
- Time reversal for $\alpha < 0$

5 points (for any 5)

Amplitude Transformations
- Amplitude scaling through A
- Amplitude shifting through B
- Amplitude reversal for $A < 0$

b)

$$y[n] = A\,x[\alpha n + \beta] + B \quad \text{———} (1)$$

Re-writing (1) gives

$$A\,x[\alpha n + \beta] = y[n] - B$$

$$x[\alpha n + \beta] = \frac{1}{A}\left(y[n] - B\right) \quad \text{———} (2)$$

Define $m = \alpha n + \beta \Rightarrow n = \frac{1}{\alpha}(m - \beta)$ and substituting in (2) gives

$$x[m] = \frac{1}{A}\left(y\left[\tfrac{1}{\alpha}(m-\beta)\right] - B\right)$$

$$= \frac{1}{A}\,y\left[\tfrac{1}{\alpha}(m-\beta)\right] - \frac{B}{A}$$

10 points

Hence we may write

$$x[n] = \frac{1}{A}\,y\left[\frac{n-\beta}{\alpha}\right] - \frac{B}{A} \quad \text{———} (3)$$

(c) Comparing $y[n] = 0.5\,x[-n+1] + 2$ with (1) gives $A = 0.5$, $\alpha = -1$, $\beta = 1$ and $B = 2$.

From (3), we have

$$x[n] = \frac{1}{0.5}\, y\left[\frac{n-1}{-1}\right] - \frac{2}{0.5} = 2y[-n+1] - 4.$$

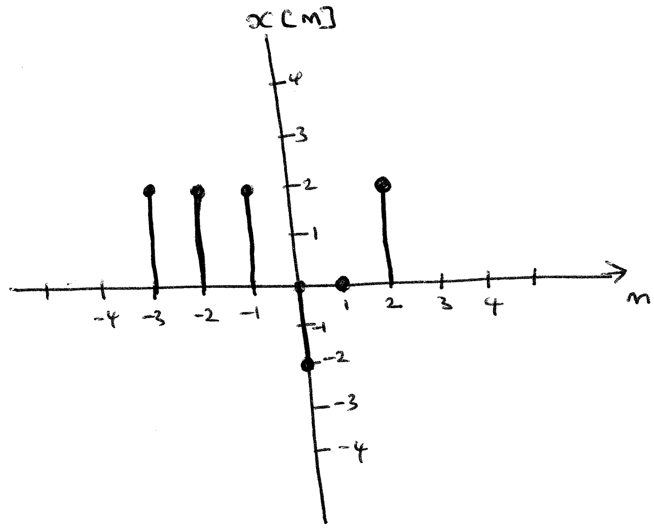
First stage: Amplitude transformation

Let $m = -n+1$, we have

$$x[m] = 2y[m] - 4$$

We then generate the following table and figure

m	$y[m]$	$x[m]$
-3	3	2
-2	3	2
-1	3	2
0	1	-2
1	2	0
2	3	2

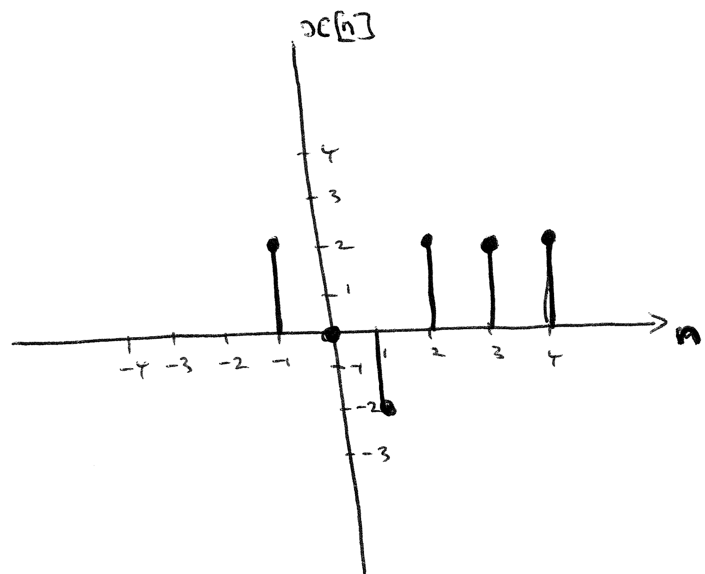


Second stage: Time transformation

using the relationship $m = -n+1 \Rightarrow n = -m+1$

We can generate the following table and graph.

m	n	$x[n]$
-3	4	2
-2	3	2
-1	2	2
0	1	-2
1	0	0
2	-1	2



10 points

Solution 2 (25 Marks).

$$x[n] = \sin(n\pi T)$$

(a) $x[n] = \sin(n\pi T)$ 5 points

$$x[-n] = \sin(-n\pi T) = -\sin(n\pi T)$$

Since $x[n] = -x[n]$, the sequence is "ODD".

(b) Sampling period $T = 0.5s$

The period of the continuous-time signal $x(t)$ is obtained as follows:

$$x(t) = \sin \pi t \equiv \sin \omega t$$

Comparing gives

$$\pi = \omega \quad \text{or} \quad \pi = 2\pi f_0 \quad \text{or} \quad \pi = \frac{2\pi}{T_0}$$

It follows that $T_0 = 2s$.

For periodicity, the ratio $\frac{T}{T_0}$ must be ratio of integers.

$$\frac{T}{T_0} = \frac{0.5s}{2s} = \frac{5}{20} = \frac{1}{4} \rightarrow$$ This is a ratio of integers.

We conclude that $x[n]$ is <u>periodic</u>.

From $\frac{T}{T_0} = \frac{1}{4}$ 10 points

we have $T_0 = 4T$

This implies there are <u>4 samples</u> in one period of $x(t)$.

c)

$$y(t) = \frac{1}{2}\int_{-\infty}^{\infty} x(\tau)\left[\delta(\tau-t) - \delta(\tau+t)\right] d\tau$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x(\tau)\,\delta(\tau-t)\,d\tau - \frac{1}{2}\int_{-\infty}^{\infty} x(\tau)\,\delta(\tau+t)\,d\tau$$

$$= \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$

Since $x(t) = \sin(\pi t)$, we have

10 points

$$y(t) = \frac{1}{2}\sin(\pi t) - \frac{1}{2}\sin(-\pi t)$$

$$= \frac{1}{2}\sin(\pi t) + \frac{1}{2}\sin(\pi t) = \sin \pi t = x(t).$$

Solution 3 (25 Marks).

$$y(t) = x(t+1)$$

a) 1. Linearity check:

Let $y_1(t) = x_1(t+1)$, $y_2(t) = x_2(t+1)$ with

$x(t) = \alpha x_1(t) + \beta x_2(t)$

We have

$$y(t) = x(t+1)$$
$$= \alpha x_1(t+1) + \beta x_2(t+1)$$
$$= \alpha y_1(t) + \beta y_2(t).$$

5 points

Hence the system is Linear.

(2) Casuality check:

Since the current output $y(t)$ depends on future values of the input $x(t+1)$, the system is non-causal. 5 point

3). Time-invariance check:

Let the input be delayed by "t_0" to obtain

$$x(t-t_0+1) \text{ ——— } (1)$$

Delaying the output $y(t)$ by same amount "t_0" gives

$$y(t-t_0) = x(t-t_0+1) \text{ ——— } (2)$$

5 points

Since (1) is exactly (2), the system is time-invariant.

(b) The impulse response is obtained by setting the input as $\delta(t)$.

Setting $x(t) = \delta(t) \implies x(t+1) = \delta(t+1)$

Hence the impulse response $\boxed{h(t) = \delta(t+1)}$

Alternatively, $y(t)$ can be expressed as

$$y(t) = \int_{\theta=-\infty}^{\infty} x(\theta+1)\,\delta(t-\theta)\,d\theta = \int_{\tau=-\infty}^{\infty} x(\tau)\,\delta(t-\tau+1)\,\delta\tau$$

Comparing the above with the convolution integral definition

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \equiv \int_{\tau=-\infty}^{\infty} x(\tau)\,\delta(t-\tau+1)\,\delta\tau$$ 5 points

It follows that $h(t-\tau) = \delta(t-\tau+1)$ or $\boxed{h(t) = \delta(t+1)}$

c) Given that $x(t) = e^{jt}$

and $h(t) = \delta(t+1)$

using the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} e^{j\tau}\,\delta(t-\tau+1)\,\delta\tau = e^{j(t+1)}$$

$$= e^{j}e^{jt}.$$

Alternatively,

$$y(t) = x(t+1)$$

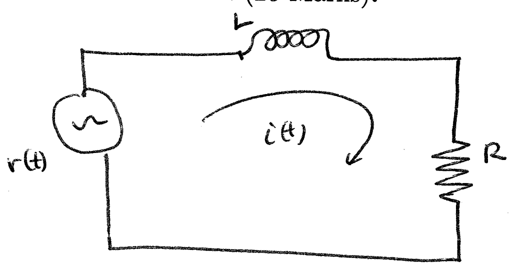
If $x(t) = e^{jt}$ then $x(t+1) = e^{j(t+1)}$.

It follows that

$$y(t) = x(t+1) = e^{j(t+1)}$$
$$= e^{j}e^{jt}.$$ 5 points

Solution 4 (25 Marks).

(a)



Applying KVL gives

7 points

$$\boxed{v(t) = L\frac{d\,i(t)}{dt} + i(t)R} \quad (1)$$

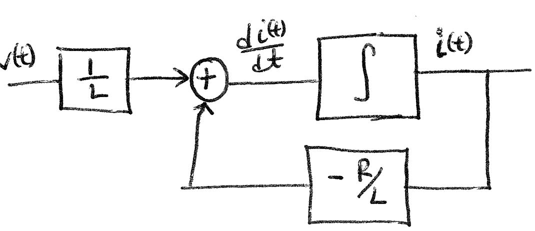
This is the differential equation description of the system.

(b) To draw the simulation diagram, we re-arrange (1) as follows:

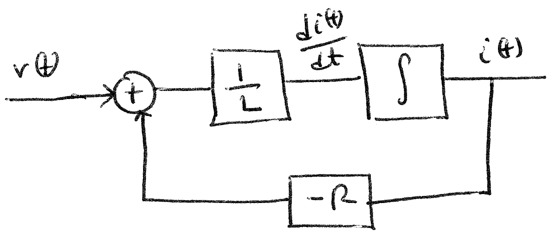
$$L\frac{d\,i(t)}{dt} = -i(t)R + v(t)$$

$$\frac{d\,i(t)}{dt} = -\frac{R}{L}i(t) + \frac{1}{L}v(t)$$

This can be represented as follows:



or



8 points

(c) Using the method of undetermined coefficients

Stage 1: complementary solution

Set $i_c(t) = Ae^{-st}$ and substituting into the homogenous equation $L\frac{d\,i(t)}{dt} + R\,i(t) = 0$ gives

$$-ALse^{-st} + ARe^{-st} = 0 \Rightarrow (-Ls + R)Ae^{-st} = 0$$

Solving for the non-trivial solution gives $-Ls + R = 0$ and

$$s = R/L.$$

Hence the complementary solution is

$$i_c(t) = A e^{-R/L\, t}.$$

<u>Stage 2</u>: Particular solution

Set $i_p(t) = B$ [Since $v(t) = u(t) = 1$ $(t>0)$] and substituting into $L \frac{d\, i(t)}{dt} + R\, i(t) = v(t)$ gives

$$RB = 1 \ (t>0) \quad \Rightarrow \quad B = 1/R$$

<u>Stage 3</u>: <u>General solution</u>

Set $i(t) = i_c(t) + i_p(t)$

$$= A e^{-R/L\, t} + \frac{1}{R}$$

Applying the initial condition ($i(0) = 0$) gives

$$0 = A e^{0} + \frac{1}{R}$$

$$A = -1/R$$

Hence the general solution becomes

10 points

$$i(t) = -\frac{1}{R} e^{-R/L\, t} + \frac{1}{R}$$

or

$$i(t) = \frac{1}{R}\left(1 - e^{-R/L\, t}\right), \quad t > 0.$$