solutions

Problems 1. through 4. determine whether the series converge or diverge

1. $\sum_{n=0}^{\infty} \frac{3+\cos 7n}{n^3}$ $\quad \left| \frac{3+\cos 7n}{n^3} \right| \le \frac{4}{n^3}$

and $4 \sum \frac{1}{n^3}$ converges, p-series $p=3>1$

$\Rightarrow$ converges

2. $\sum_{n=0}^{\infty} \frac{n!}{5^n}$ $\quad \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!} \right| = \left| \frac{n+1}{5} \right| \to \infty$

diverges

3. $\sum_{n=0}^{\infty} \frac{n^2+3}{n^5+1}$ $\quad \dfrac{\frac{n^2+3}{n^5+1}}{\frac{1}{n^3}} = \frac{n^5+3n^3}{n^5+1} = \frac{1+3/n^2}{1+1/n^5} \to 1$

by limit comp since $\sum \frac{1}{n^3}$ conv.

$\sum \frac{n^2+3}{n^5+1}$ also converges

4. $\sum_{n=0}^{\infty} \frac{3n^2+1}{\sqrt{n^5+1}}$ $\quad \dfrac{\frac{3n^2+1}{\sqrt{n^5+1}}}{\frac{1}{\sqrt{n}}} = \frac{3n^2\sqrt{n}+\sqrt{n}}{n^{5/2}\sqrt{1+\frac{1}{n^5}}} =$

$= \frac{3n^{5/2}\left(1+\frac{1}{n^2}\right)}{n^{5/2}\left(1+\frac{1}{n^5}\right)^{1/2}} \longrightarrow 1$

by lmt comp since $\sum \frac{1}{\sqrt{n}}$ div

this series also diverges

Problems 5. and 6. determine whether the series converge absolutely, converge conditionally or diverge.

5. $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+2}$

(a) Alt series $f(x) = \frac{1}{3x+2}$

$f' = \frac{-3}{(3x+2)^2} < 0$ dec.

and $\lim_{n\to\infty} \frac{1}{3n+2} = 0$

$\Rightarrow$ conv.

(b) $\frac{\frac{1}{3n+2}}{1/n} = \frac{n}{3n+2} = \frac{n}{n} \cdot \frac{1}{(3+2/n)} \to \frac{1}{3}$

$\Rightarrow$ by limit comp $\sum \frac{1}{3n+2}$ div.

$\Rightarrow$ ser. converge conditionally

6. $\sum_{n=1}^{\infty} \frac{(-1)^n(\arctan n + \cos n)}{n^3+1}$

$\left| \frac{\arctan + \cos n}{n^3+1} \right| < \frac{\pi/2 + 1}{n^3}$

and $\sum \frac{1}{n^3}$ conv.

$\Rightarrow$ ser. converge absolutely

Problems 7. and 8. find the Radius of convergence and the interval of convergence of the following power series

7. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$

$$\left| \frac{(x-5)^{n+1}}{n+1} \cdot \frac{n}{(x-5)^n} \right| = |x-5| \left| \frac{n}{n+1} \right|$$

$$\lim_{n \to \infty} |x-5| \left| \frac{n}{n+1} \right| = |x-5| < \boxed{1 = R}$$

$$-1 < x-5 < 1 \Rightarrow 4 < x < 6$$

check $x = 4 \Rightarrow \sum \frac{(-1)^n}{n}$ conv. by Altser test

$x = 6$ $\sum \frac{1}{n}$ div, harmonic ser.

$\Rightarrow$ interval of convergence

$[4, 6)$

8. $\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+1}}$

$$\left| \frac{2^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{2^n x^n} \right| =$$

$$= \left| 2x \sqrt{\frac{n+1}{n+2}} \right| \longrightarrow |2x| < 1, \ \boxed{|x| = \frac{1}{2} = R}$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

check $x = -\frac{1}{2}$ $\sum \frac{1}{\sqrt{n+1}}$ div limit comp with $\frac{1}{\sqrt{n}}$

$x = \frac{1}{2}$ $\sum \frac{(-1)^n}{\sqrt{n+1}}$ conv. by alt ser. test

$\Rightarrow$ interval $(-\frac{1}{2}, \frac{1}{2}]$

9. (a) Derive the Maclaurin series of *sinx*

$\sin 0 = 0$

$\cos 0 = 1$

$-\sin 0 = 0$

$-\cos 0 = -1$

$\sin 0 = 0$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - - -$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(b) Using power series evaluate

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

$$\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - - -$$

$$\frac{\sin x - x}{x^3} = -\frac{1}{6} + \frac{x^2}{5!} - - - -$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x - x}{x^3} = \boxed{-\frac{1}{6}}$$

10. (a) Derive the Maclaurin series of

$$\frac{x}{2+x} = x \cdot \frac{1}{2} \cdot \frac{1}{1-\left(\frac{-x}{2}\right)} = \frac{x}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{2^{n+1}}$$

(b) Using power series evaluate

$$\int \frac{x}{2+x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{(n+2)2^{n+1}}$$