Solutions Test 2

In problems 1. through 4. determine whether the **sequence {aₙ}** converges or diverges, where:

1. $a_n = \dfrac{5^{n-1}}{7^{n+1}} = \left(\dfrac{5}{7}\right)^{n-1} \cdot \dfrac{1}{49}$ $\quad \left|\dfrac{5}{7}\right| < 1 \quad \Rightarrow$ converges

2. $a_n = \dfrac{(\ln n)^2}{n}$ $\quad \lim\limits_{n\to\infty} \dfrac{(\ln n)^2}{n} = \lim\limits_{n\to\infty} \dfrac{2\ln n}{n \cdot 1} = \lim\limits_{n\to\infty} \dfrac{2}{n \cdot 1} = 0$

$\Rightarrow$ converges

3. $a_n = \ln(3n^2+5) - \ln(3n^2+1) = \ln\left(\dfrac{3n^2+5}{3n^2+1}\right)$

$\Rightarrow \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \ln\left(\dfrac{3+5/n^2}{3+1/n^2}\right) = \ln(1) = 0$

converges

4. $a_n = \dfrac{(3n+1)!}{(3n-1)!} = (3n).(3n+1)$

$\Rightarrow \lim\limits_{n\to\infty} a_n = \infty$ $\quad$ diverges

In problems 5. through 10. determine whether the **series** converge or diverge.

5. $\sum_{n=1}^{\infty} \frac{2n}{3n+5}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+5} = 2/3 \neq 0$

$\Rightarrow$ diverges

6. $\sum_{n=1}^{\infty} \frac{2+3^n}{5^n} = \sum \frac{2}{5^n} + \sum \left(\frac{3}{5}\right)^n$

$\Downarrow$ $\quad$ $\Downarrow$

$r = \frac{1}{5}$ $\quad$ $r = \frac{3}{5}$

$|r| < 1$ $\quad$ $|r| < 1$ $\Longrightarrow$ converges

7. $\sum_{n=1}^{\infty} \arctan(n)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \pi/2 \neq 0$

$\Rightarrow$ diverges.

8. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ $\quad f(x) = \frac{x^2}{x^3+1} \Rightarrow f' = \frac{x(2-x^3)}{(x^3+1)^2} < 0, \; x \geq 2$

$\Rightarrow \int_1^{\infty} \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int_1^{\infty} \frac{1}{u}\,du = \frac{1}{3}\lim_{b\to\infty} \left[\ln(b^3+1) - \ln 2\right]$

$= \infty \quad \Rightarrow$ series diverges.

9. $\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$ $\quad , \frac{2}{n(n+1)} = 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$

$S_n = 2\left[\left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots \left(\cancel{\tfrac{1}{n}} - \frac{1}{n-1}\right)\right]$

$S_n = 2\left[1 - \frac{1}{n+1}\right] \Rightarrow \lim_{n\to\infty} S_n = 2$

$\Rightarrow$ series converge

10. $\sum_{n=e}^{\infty} \frac{1}{n(\ln n)^3}$ $\quad f(x) = \frac{1}{x(\ln x)^3}$

$f' = -\frac{\left[\ln x^3 + 3\ln x^2\right]}{(x\ln x^3)^2} < 0, \quad x > 1$

$\int_e^{\infty} \frac{1}{x\ln x^3}\,dx = \lim_{b\to\infty} -\frac{1}{2}\frac{1}{\ln x^2}\Big]_e^b =$

$= \lim_{b\to\infty} -\frac{1}{2}\left[\frac{1}{\ln b^2} - 1\right] = \frac{1}{2}$

$\Rightarrow$ series <u>converge</u>

11. Find the **arclength** of the curve $y = x^2 - \frac{1}{8}\ln x$ from $x=e$ to $x=e^2$

$y' = 2x - \frac{1}{8x}$, $y'^2 = 4x^2 - \frac{1}{2} + \frac{1}{64x^2})$

$1+y'^2 = \left(2x + \frac{1}{8x}\right)^2 \Rightarrow \sqrt{1+y'^2} = \left(2x + \frac{1}{8x}\right)$

$\Rightarrow \text{Arcl} = \int_e^{e^2} 2x + \frac{1}{8x}\, dx = x^2 + \frac{1}{8}\ln x \Big]_e^{e^2}$

$= \left(e^4 + \frac{1}{4}\right) - \left(e^2 + \frac{1}{8}\right) = e^4 - e^2 + \frac{1}{8}$

12. The curve $y = \sqrt{1+e^x}, 0 \le x \le 1$, is rotated around the **x-axis**. Find the **surface area** of the solid generated.

$y' = \frac{1}{2} \frac{e^x}{\sqrt{1+e^x}}$, $y'^2 = \frac{1}{4} \frac{e^{2x}}{(\sqrt{1+e^x})^2}$, $1+y'^2 = \frac{e^{2x} + 4 + 4e^x}{4(1+e^x)}$

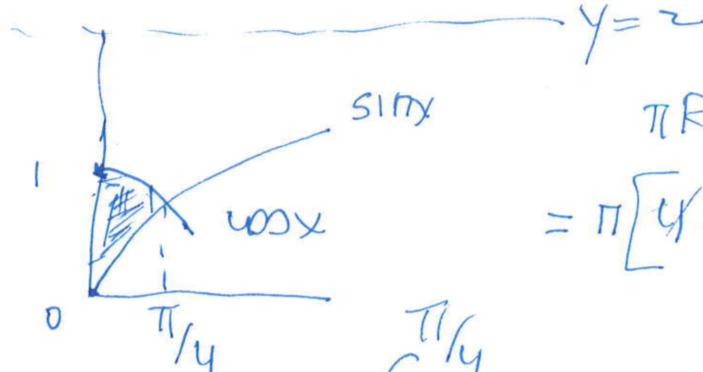
$\sqrt{1+y'^2} = \frac{e^x+2}{2\sqrt{1+e^x}} \Rightarrow \text{Area} = \int_0^1 \cancel{2}\pi \cancel{\sqrt{1+e^x}} \cdot \frac{e^x+2}{\cancel{2}\cancel{\sqrt{1+e^x}}}\, dx$

$= \pi \int_0^1 e^x + 2\, dx = \pi\left[e^x + 2x\right]_0^1 = \pi\left[e+1\right]$

13. The region bounded by the **three** curves $y = \sin x, y = \cos x$ and *x=0* is rotated around the line *y = 2* . Find the **volume** of the solid generated.

$R = 2 - \sin x$

$r = 2 - \cos x$



$\pi R^2 - \pi r^2 = \pi(2-\sin x)^2 - \pi(2-\cos x)^2$

$= \pi\left[\cancel{4} - 4\sin x + \sin^2 x - \cancel{4} + 4\cos x - \cos^2 x\right]$

$\Rightarrow Vol = \pi \int_0^{\pi/4} -4\sin x + \sin^2 x + 4\cos x - \cos^2 x \, dx =$

$= \pi \int_0^{\pi/4} -4\sin x + 4\cos x + \frac{1-\cos 2x}{2} - \frac{1+\cos 2x}{2} \, dx$

$= \pi\left[4\cos x + 4\sin x - \frac{\sin 2x}{2}\right]_0^{\pi/4}$

$= \pi\left[4\frac{\sqrt{2}}{2} + 4\frac{\sqrt{2}}{\sqrt{2}} - \frac{1}{2} - 4\right]$

$= \pi\left[4\sqrt{2} - \frac{9}{2}\right]$