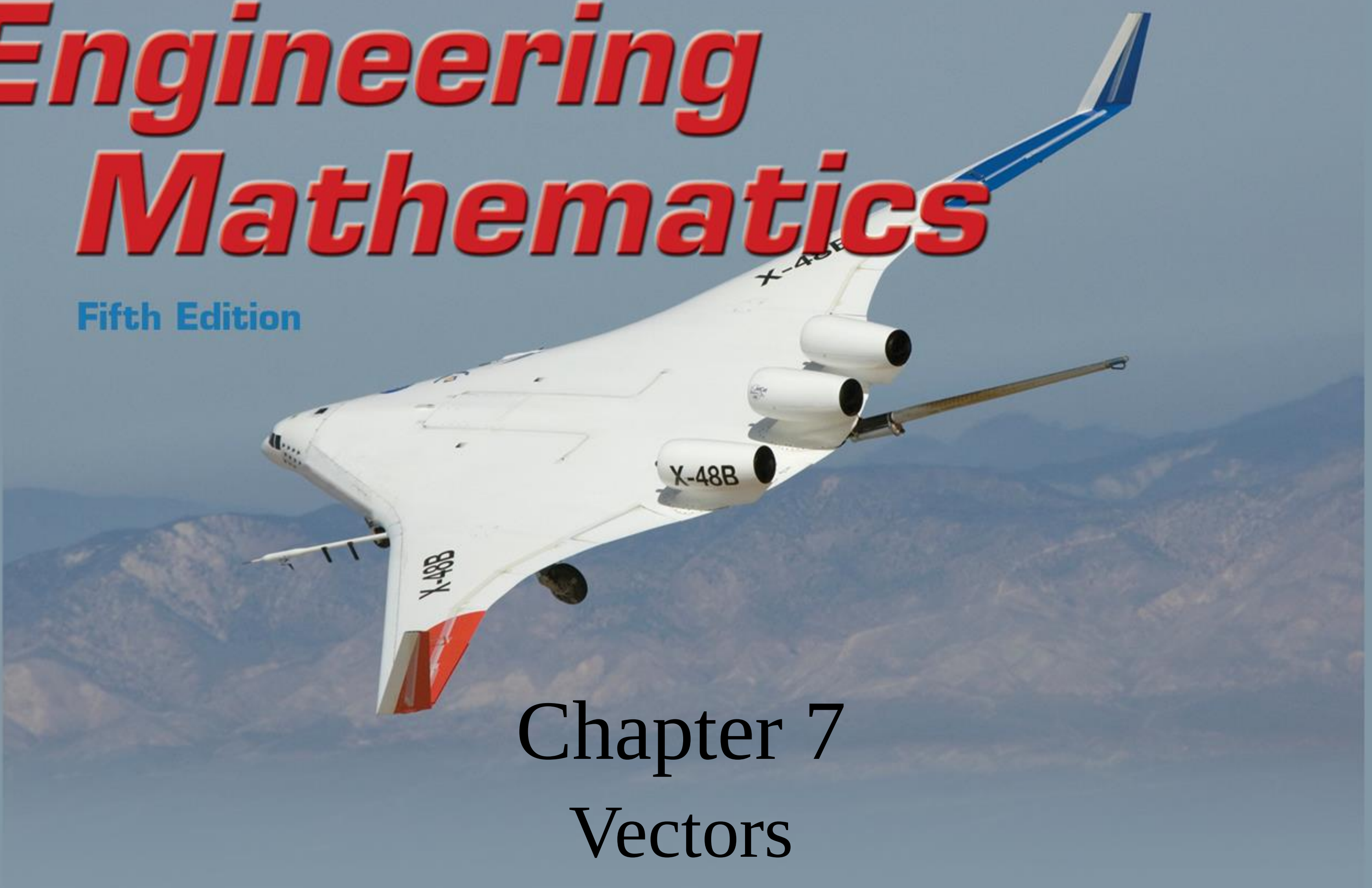
Dennis G. Zill Warren S. Wright

Advanced Engineering Mathematics

Fifth Edition



Chapter 7

Vectors

Outline

Vectors in 2-Space

- A **scalar** is a real number or quantity that has a magnitude, such as length and temperature
- A **vector** has both magnitude and direction and it can be represented by a boldface symbol or a symbol under an arrow, **v** or $\overrightarrow{AB}$

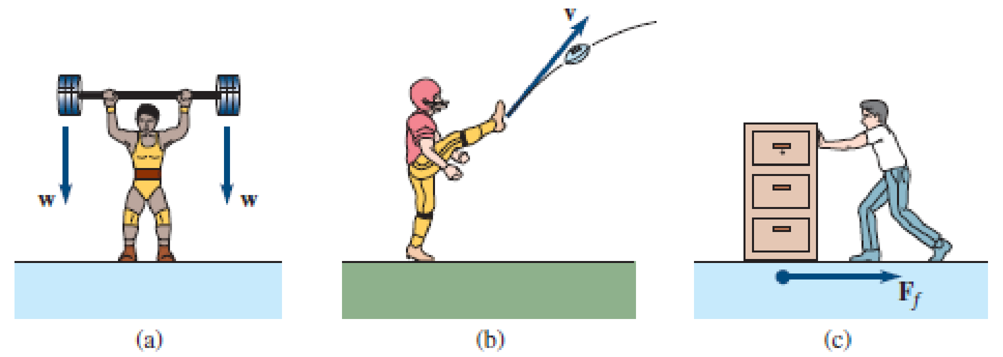


FIGURE 7.1.1 Examples of vector quantities

Vectors in 2-Space (cont'd.)

- Characteristics of vectors
  - The vector $\overrightarrow{AB}$ has an initial point at *A* and a terminal point at *B*
  - Equal vectors have the same magnitude and direction
  - Vectors are **free**, meaning they can be moved from one position to another provided magnitude and direction do not change
  - The **negative** of a vector has the same magnitude and opposite direction

Vectors in 2-Space (cont'd.)

- Characteristics of vectors (cont'd.)
  - If $k \neq 0$ is a scalar, the **scalar multiple** of a vector $k\overrightarrow{AB}$ is a vector that is $|k|$ times as long as $\overrightarrow{AB}$
  - Two vectors are **parallel** if they are nonzero scalar multiples of each other

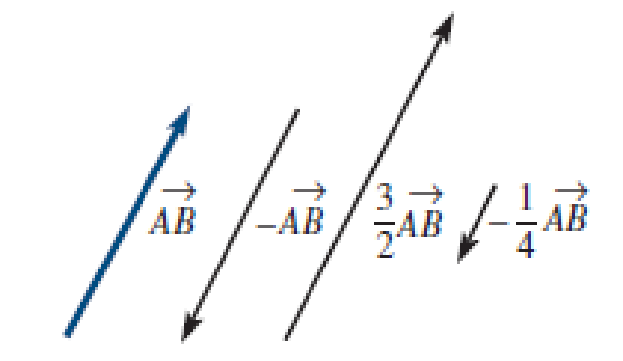


FIGURE 7.1.3 Parallel vectors

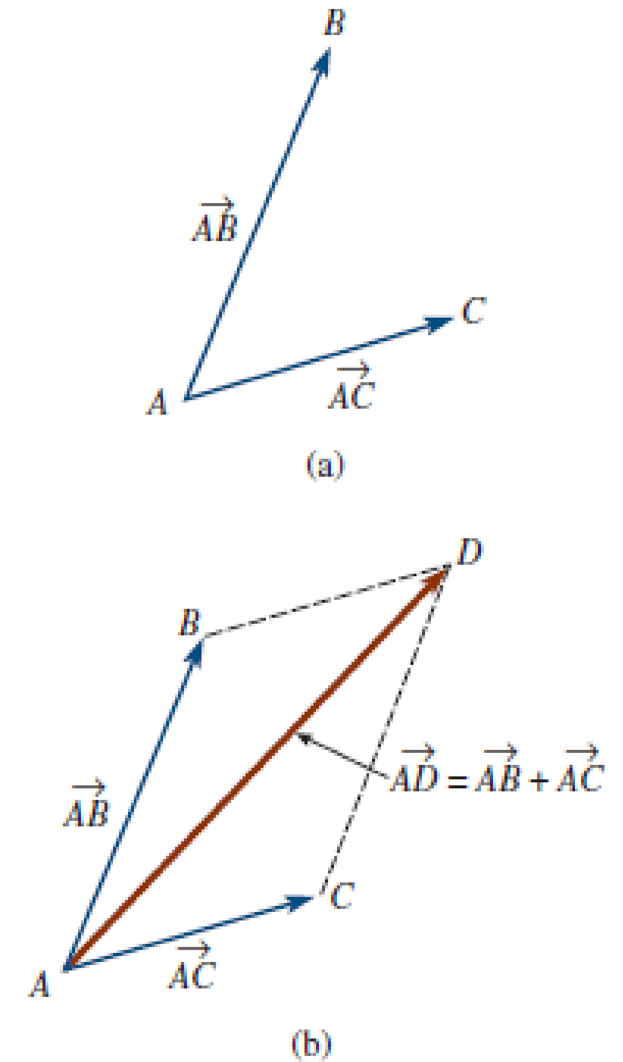


FIGURE 7.1.4 Vector $\overline{AD}$ is the sum of $\overline{AB}$ and $\overline{AC}$

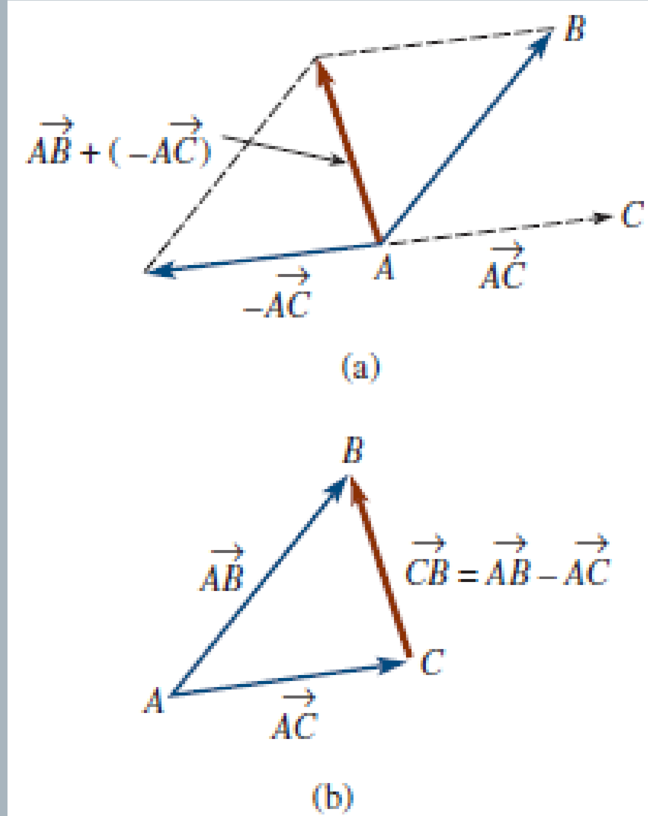


FIGURE 7.1.5 Vector $\overline{CB}$ is the difference of $\overline{AB}$ and $\overline{AC}$

Addition and Subtraction of 2-space vectors

Vectors in 2-Space (cont'd.)

- A vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is an ordered pair of real numbers where a_1 and a_2 are the **components** of the vector
  - Addition and subtraction of vectors, multiplication of vectors by scalars, and so on, are defined in terms of components

Definition 7.1.1 Addition, Scalar Multiplication, Equality

Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be vectors in R^2.

(*i*) Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$ (1)

(*ii*) Scalar multiplication: $k\mathbf{a} = \langle ka_1, ka_2 \rangle$ (2)

(*iii*) Equality: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1, a_2 = b_2$ (3)

Vectors in 2-Space (cont'd.)

- The component definition of a vector can be used to verify the following properties of vectors

Theorem 7.1.1 Properties of Vectors

(*i*)	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	← commutative law
(*ii*)	$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$	← associative law
(*iii*)	$\mathbf{a} + \mathbf{0} = \mathbf{a}$	← additive identity
(*iv*)	$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$	← additive inverse
(*v*)	$k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$, k a scalar	
(*vi*)	$(k_1 + k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}$, k_1 and k_2 scalars	
(*vii*)	$k_1(k_2\mathbf{a}) = (k_1k_2)\mathbf{a}$, k_1 and k_2 scalars	
(*viii*)	$1\mathbf{a} = \mathbf{a}$	
(*ix*)	$0\mathbf{a} = \mathbf{0}$	← zero vector

Vectors in 2-Space (cont'd.)

- The **magnitude**, **length**, or **norm** of a vector **a** is denoted by $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$
- A vector **u** with magnitude 1 is a **unit vector**
  - $\mathbf{u} = \left(1/\|\mathbf{a}\|\right)\mathbf{a}$ is the normalization of **a**
  - The unit vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ are the standard basis for two-dimensional vectors

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

where a_1 and a_2 are **horizontal** and **vertical components** of **a**, respectively

Vectors in 3-Space

- In three dimensions, or **3-space**, a rectangular coordinate system is constructed with three mutually orthogonal axes

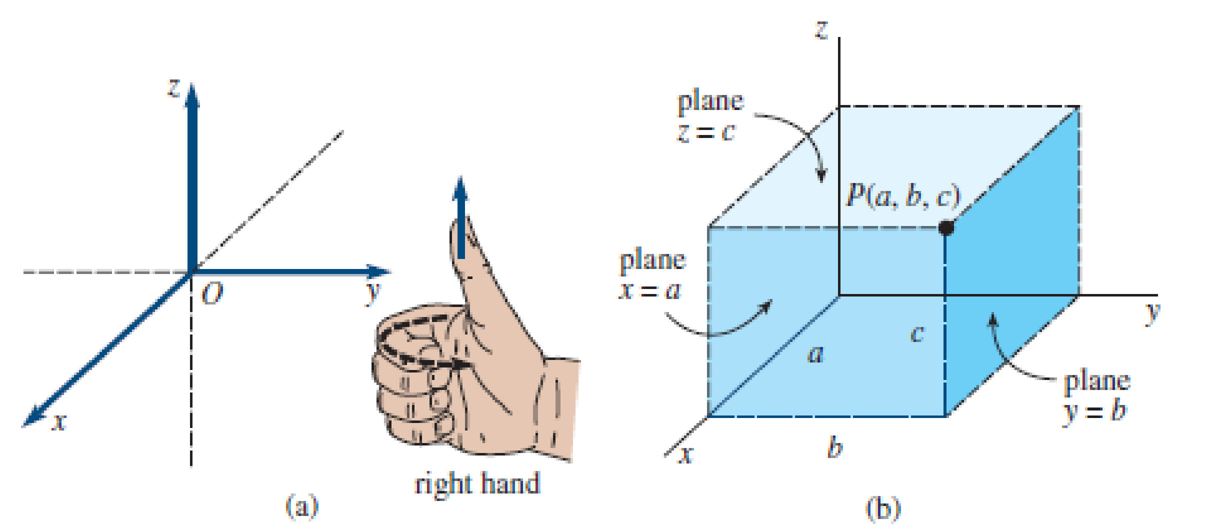
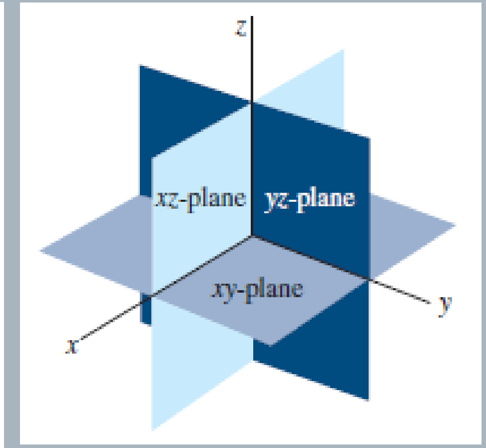


FIGURE 7.2.2 Rectangular coordinates in 3-space



Vectors in 3-Space (cont'd.)

- The **distance** between two points

$$d\left(P_1, P_2\right) = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

- The coordinates of the **midpoint** of a line segment between two points

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Vectors in 3-Space (cont'd.)

- A vector **a** in 3-space is any ordered triple of real numbers

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Definition 7.2.1 Component Definitions in 3-Space

Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors in R^3.

(*i*) Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

(*ii*) Scalar multiplication: $k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$

(*iii*) Equality: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1, a_2 = b_2, a_3 = b_3$

(*iv*) Negative: $-\mathbf{b} = (-1)\mathbf{b} = \langle -b_1, -b_2, -b_3 \rangle$

(*v*) Subtraction: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

(*vi*) Zero vector: $\mathbf{0} = \langle 0, 0, 0 \rangle$

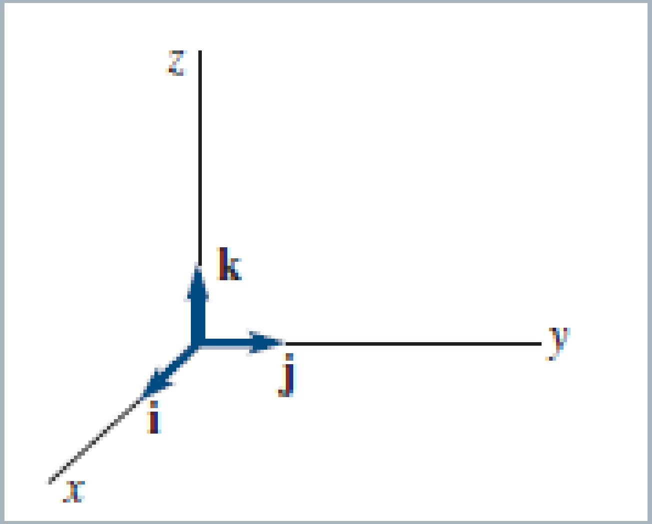
(*vii*) Magnitude: $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Vectors in 3-Space (cont'd.)

- Any vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ can be expressed as a linear combination of the unit vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$



Dot Product

- In 2-space, $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$
- In 3-space, $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Theorem 7.3.1 Properties of the Dot Product

(*i*) $\mathbf{a}\cdot\mathbf{b} = 0$ if $\mathbf{a} = 0$ or $\mathbf{b} = 0$
(*ii*) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ ← commutative law
(*iii*) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ ← distributive law
(*iv*) $\mathbf{a}\cdot(k\mathbf{b}) = (k\mathbf{a})\cdot\mathbf{b} = k(\mathbf{a}\cdot\mathbf{b})$, k a scalar
(*v*) $\mathbf{a}\cdot\mathbf{a} \geq 0$
(*vi*) $\mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2$

- Alternative form of the dot product

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

Dot Product (cont'd.)

- Two vectors are orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$
- The angle between two vectors is given by

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

- The component of **a** on **b**

$$\text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$

- The projection of **a** on **b**

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b}$$

Cross Product

- Recall determinants

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

- The **cross product** of two vectors is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \left(a_2 b_3 - a_3 b_2\right)\mathbf{i} - \left(a_1 b_3 - a_3 b_1\right)\mathbf{j} + \left(a_1 b_2 - a_2 b_1\right)\mathbf{k}$$

Cross Product (cont'd.)

- $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane containing **a** and **b**
- The magnitude of the cross product is given by

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$$

- Two nonzero vectors are parallel if $\mathbf{a} \times \mathbf{b} = 0$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ if **a**, **b**, and **c** are coplanar

Cross Product (cont'd.)

Theorem 7.4.1 Properties of the Cross Product

(*i*) $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$

(*ii*) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

(*iii*) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ ← distributive law

(*iv*) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$ ← distributive law

(*v*) $\mathbf{a} \times (k\mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$, k a scalar

(*vi*) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

(*vii*) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

(*viii*) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

Lines and Planes in 3-Space

- The **vector equation** for a line is $\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$
  - $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$ are vectors from the origin to points on the line
  - Scalar t is a **parameter**, nonzero vector **a** is a **direction vector**, components of **a** are direction numbers

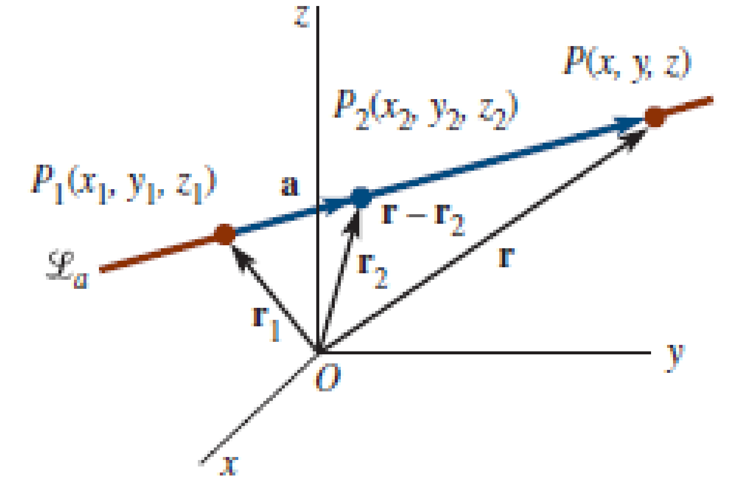


FIGURE 7.5.1 Line through distinct points in 3-space

Lines and Planes in 3-Space (cont'd.)

- The **vector equation** for a plane is $\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_1)=0$
  - Plane passes through a given point and has a specified normal vector **n**
  - **r** and $\mathbf{r}_1$ are vectors from the origin to points on the plane

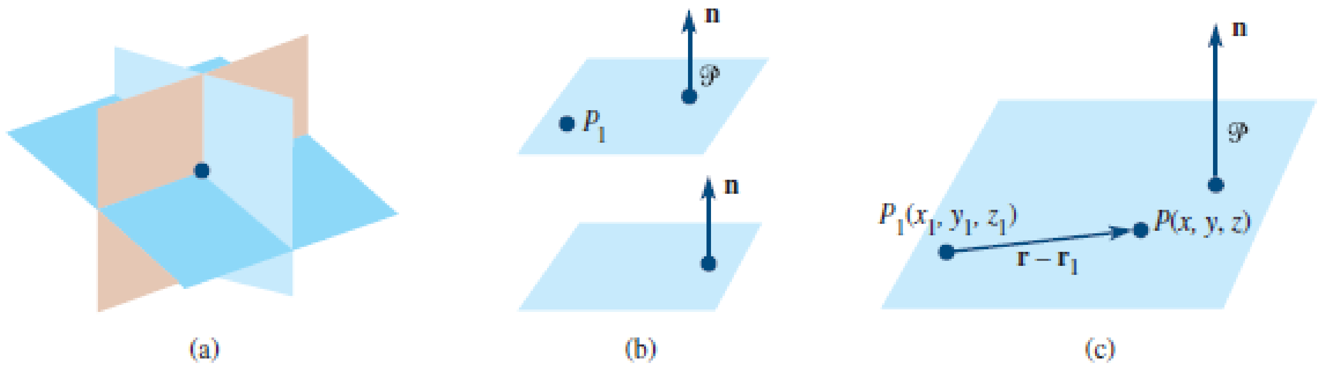


FIGURE 7.5.3 Vector n is perpendicular to a plane

Vector Spaces

- The set of all vectors $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ in ***n*-space** is R^n, and the set is a **vector space** V if certain axioms are met

Definition 7.6.1 Vector Space

Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following 10 properties are satisfied.

Axioms for Vector Addition:

(*i*) If $\mathbf{x}$ and $\mathbf{y}$ are in V, then $\mathbf{x} + \mathbf{y}$ is in V.

(*ii*) For all $\mathbf{x}$, $\mathbf{y}$ in V, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. ← commutative law

(*iii*) For all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in V, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$. ← associative law

(*iv*) There is a unique vector $\mathbf{0}$ in V such that

$$\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}.$$ ← zero vector

(*v*) For each $\mathbf{x}$ in V, there exists a vector $-\mathbf{x}$ such that

$$\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}.$$ ← negative of a vector

Axioms for Scalar Multiplication:

(*vi*) If k is any scalar and $\mathbf{x}$ is in V, then $k\mathbf{x}$ is in V.

(*vii*) $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$ ← distributive law

(*viii*) $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$ ← distributive law

(*ix*) $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$

(*x*) $1\mathbf{x} = \mathbf{x}$

Vector Spaces (cont'd.)

- A set of vectors $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ in a vector space *V* is a **basis** for *V* if
  - *B* is linearly independent
  - Every vector in *V* can be expressed as a linear combination of these vectors
- A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is linearly independent if the only constants satisfying $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n = 0$ are $k_1 = k_2 = \cdots = k_n = 0$

Gram–Schmidt Orthogonalization Process

- If $B=\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_n\}$ is an orthonormal basis for R^n and $\mathbf{u}$ is any vector in R^n , then
$$\mathbf{u}=(\mathbf{u}\cdot\mathbf{w}_1)\mathbf{w}_1+(\mathbf{u}\cdot\mathbf{w}_2)\mathbf{w}_2+\cdots+(\mathbf{u}\cdot\mathbf{w}_n)\mathbf{w}_n$$
- A basis B of R^n can be converted into an orthonormal basis $B'=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ then into $B''=\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_n\}$ by normalizing the vectors in B'
- The vectors $\mathbf{v}_n$ and $\mathbf{w}_n$ are mutually orthogonal and are unit vectors

Gram–Schmidt Orthogonalization Process (cont'd.)

- Example: Transform the set $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ into an orthonormal basis $B'' = \{\mathbf{w}_1, \mathbf{w}_2\}$ (where $\mathbf{u}_1 = \langle 3,1 \rangle$ and $\mathbf{u}_2 = \langle 1,1 \rangle$)
  - Choose $\mathbf{v}_1 = \mathbf{u}_1 = \langle 3,1 \rangle$ and

$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \langle 1,1 \rangle - \frac{4}{10} \langle 3,1 \rangle = \left\langle -\frac{1}{5}, \frac{3}{5} \right\rangle$$

  - The set $B' = \left\{ \langle 3,1 \rangle, \left\langle -\frac{1}{5}, \frac{3}{5} \right\rangle \right\}$ is an orthogonal basis for R^2

Gram–Schmidt Orthogonalization Process (cont'd.)

- Example (cont'd.):
  - Finish by normalizing the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle \text{ and } \mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

  - The new orthonormal basis is

$$B'' = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle, \left\langle -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle \right\}$$