ENG 342: Advanced Engineering Math II

Quiz #3

October 13, 2016

Problem 1 [7 pts]

Consider a thin rod that coincides with the x-axis on the interval $[0, 1]$. The left side is held at temperature 0, the right side is insulated, and the initial temperature is $f(x)$ throughout.

(a) Set up the boundary-value problem for the temperature $u(x, t)$. [2 pts]

The boundary conditions on the left and right sides mean $u(0, t) = 0$ and $u_x(1, t) = 0$. Therefore, the BVP is

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad 0 \le x \le 1, t > 0$$
$$u(0, t) = 0, u_x(1, t) = 0 \quad t > 0$$
$$u(x, 0) = f(x) \quad 0 \le x \le 1$$

(b) Solve for $u(x, t)$ using separation of variables. Be sure to consider all possible cases. Your final answer should be in terms of $f(x)$ and k. [3 pts]

Assuming $u = XT$, we have

$$kX''T = XT'$$
$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

This gives two differential equations:

$$X'' + \lambda X = 0 \qquad T' + \lambda kT = 0$$

Starting with $X'' + \lambda X = 0$, there are three cases:

Case I: $\lambda = -\alpha^2 < 0, \alpha > 0$
The roots are real: $m = \pm\alpha$, so $X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. Since $X'(x) = c_1\alpha \sinh \alpha x + c_2\alpha \cosh \alpha x$, $X(0) = 0$ implies $c_1 = 0$ and $X'(1) = 0$ implies $c_2\alpha \sinh \alpha = 0$. But $\sinh b$ is always positive for $b > 0$, so this means $c_2 = 0$ also. Therefore this case is trivial.

Case II: $\lambda = 0$
This is a repeated root $m = 0, 0$. So, $X(x) = c_1 + c_2x$, and $X'(x) = c_2$. $X(0) = 0$ means $c_1 = 0$, and $X'(1) = 0$ means $c_2 = 0$. Again, this case is trivial.

Case III: $\lambda = +\alpha^2, \alpha > 0$
The roots are imaginary: $m = \pm\alpha i$. So, $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$. $X(0) = 0$ implies $c_1 = 0$. $X'(x) = c_2\alpha \cos \alpha x$, so $X'(1) = c_2\alpha \cos \alpha = 0$. This is true for $\alpha = \pi/2, 3\pi/2, ...$, so

$$\alpha_n = -\frac{\pi}{2} + \pi n = \frac{\pi}{2}(2n-1) \quad n = 1, 2, ...$$

And

$$X(x) = c_2 \sin \frac{\pi}{2}(2n-1)x$$

Knowing $\lambda_n = [\frac{\pi}{2}(2n-1)]^2$, we can solve the T component: $m = -k\lambda$, so

$$T(t) = c_3 e^{-k[\frac{\pi}{2}(2n-1)]^2}$$

By superposition of the product solutions $u_n(x,t)$ $n = 1, 2, ...$, we have

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left[\frac{\pi}{2}(2n-1)x\right] e^{-k[\frac{\pi}{2}(2n-1)]^2 t}$$

Then applying the initial condition,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{\pi}{2}(2n-1)\, x = f(x)$$

This is a Fourier sine series, so the formula for B_n is

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x) \sin \frac{\pi}{2}(2n-1)x \, dx \\ &= 2\int_0^1 f(x) \sin \frac{\pi}{2}(2n-1)x \, dx \end{aligned}$$

(c) Letting $f(x) = 10$ and $k = 1$, find $u(x,t)$. [2 pts]

To find B_n we integrate:

$$\begin{aligned} B_n &= 2\int_0^1 10 \sin \frac{\pi}{2}(2n-1)\, x \, dx = -\frac{40}{(2n-1)\pi} \cos \frac{\pi}{2}(2n-1)\, x \Big|_0^1 \\ &= \frac{40}{(2n-1)\pi}\left(1 - \cos \frac{\pi}{2}(2n-1)\right) = \frac{40}{(2n-1)\pi} \end{aligned}$$

So,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{(2n-1)\pi} \sin\left[\frac{\pi}{4}\,(2n-1)\,x\right] e^{-[\frac{\pi}{2}(2n-1)]^2 t}$$

Problem 2 [3 pts]

Consider a string stretched taught on the x-axis over $[0, \pi]$, freely vibrating in the vertical plane. The vertical displacement $u(x, t)$ of the string takes the following form:

$$u(x, t) = 0.5 \cos(at) \sin(x) + \sin(3at) \sin(3x)$$

(a) What is the boundary-value problem for $u(x, t)$ in terms of a? [2 pts]

Since we are given $u(x, t)$, we just have to reverse-engineer the boundary conditions. For the Wave Equation, we need $u(0, t)$, $u(\pi, t)$, $u(x, 0)$, and $u_t(x, 0)$:

$u(0, t) = u(\pi, t) = 0$, which is always the case.

$u(x, 0) = 0.5 \cdot 1 \cdot \sin(x) + 0 \cdot \sin(3x) = 0.5 \sin(x)$.

$u_t(x, t) = -0.5a \sin(at) \sin(x) + 3a \cos(3at) \sin(3x)$. So $u_t(x, 0) = -0.5a \cdot 0 \cdot \sin(x) + 3a \cdot 1 \cdot \sin(3x) = 3a \sin(3x)$.

Therefore, the BVP is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad 0 \leq x \leq \pi, t > 0$$
$$u(0, t) = 0, u_x(\pi, t) = 0 \quad t > 0$$
$$u(x, 0) = 0.5 \sin(x), u_t(x, 0) = 3a \sin(3x) \quad 0 \leq x \leq \pi$$

(b) Which of the standing waves, if any, are non-zero? [1 pt]

The standing waves correspond to the different values of n making up the solution. In this case, $u(x, t) = u_1(x, t) + u_3(x, t)$, where $u_1(x, t) = 0.5 \cos(at) \sin(x)$ and $u_3(x, t) = \sin(3at) \sin(3x)$. So, the first and third standing waves are non-zero.