Test 1 Solutions

Evaluate the indefinite integrals 1. through 10.

1. $\int (\sin x)^3(\cos x)^4\,dx = \int \sin x\,(1-\cos^2 x)\cos^4 x\,dx =$

$= \int \sin x \cos^4 x\,dx - \int \sin x \cos^6 x\,dx =$

$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

2. $\int \frac{\ln x}{x}dx$ let $u = \ln x$ $du = \frac{1}{x}dx$

$= \int u\,du = \frac{u^2}{2} + C = \frac{\ln^2 x}{2} + C$

3. $\int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+1}dx = \arctan(x+1) + C$

4. $\int \sqrt{1-x^2}\,dx$ let $x = \sin\theta$, $dx = \cos\theta$, $\sqrt{1-x^2} = \cos\theta$

$= \int \cos^2\theta\,d\theta = \frac{1}{2}\int 1+\cos 2\theta\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$

$= \frac{1}{2}\left[\arcsin x + x\sqrt{1-x^2}\right] + C$

5. $\int \frac{\cos x}{(\sin x)^2 - \sin x} dx$

Let $u = \sin x$, $du = \cos x\, dx$

$$\int \frac{du}{u^2 - u} = \int \frac{1}{u(u-1)} du,$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \Rightarrow A(u-1) + Bu = 1$$

$$\Rightarrow A = -1 \text{ and } B = 1 \Rightarrow \int \frac{-1}{u} du + \int \frac{1}{u-1} du$$

$$= \ln\left|\frac{u-1}{u}\right| = \ln\left|\frac{\sin x - 1}{\sin x}\right| + C$$

6. $\int (\sec x)^3 dx$

let $u = \sec x$ $\quad du = \sec x \tan x\, dx$

$du = \sec^2 x\, dx \quad v = \tan x$

$$\Rightarrow \sec x \tan x - \int \sec x \tan^2 x\, dx =$$

$$= \sec x \tan x - \int \sec^3 x - \sec x\, dx \Rightarrow$$

$$\int \sec^3 x\, dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

7. $\int \frac{x^2 + 2x - 1}{x^3 - x} dx = \int \frac{x^2 + 2x - 1}{x(x-1)(x+1)} dx = \int \frac{A\,dx}{x} + \int \frac{B\,dx}{x-1} + \int \frac{C}{x+1} dx$

$$x^2 + 2x - 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$A + B + C = 1$, $B - C = 2$, $\boxed{A = 1} \Rightarrow B + C = 0$

$$\Rightarrow \boxed{B = 1}, \boxed{C = -1}$$

$$\Rightarrow \int \frac{1}{x} dx + \int \frac{1}{x-1} dx - \int \frac{1}{x+1} dx = \ln\left|\frac{x(x-1)}{x+1}\right| + C$$

Let $u = \ln^2 x$ $\quad du = \frac{2\ln x}{x}$

$dv = x\,dx$ $\quad v = \frac{x^2}{2}$

8. $\int x(\ln x)^2\, dx$

$\frac{x^2}{2}\ln^2 x - \int x \ln x\, dx = \frac{x^2}{2}\frac{\ln^2 x}{2} - \frac{x^2}{2}\ln x + \frac{1}{4}x^2 + C$

9. $\int (\tan x)^2 (\sec x)^4\, dx$ $= \int \tan^2 x\,(\sec^2 x)^2\, dx$

$= \int \tan^2 x\,(1+\tan^2 x)\sec^2 x\, dx$ $\quad$ let $u = \tan x$

$du = \sec^2 x\, dx$

$= \int u^2 + u^4\, du = \frac{u^3}{3} + \frac{u^5}{5} + C$

$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$

10. $\int \frac{1}{\sqrt{1+x^2}}dx$

let $x = \tan\theta$ $\quad dx = \sec^2\theta\, d\theta$

$\sqrt{1+x^2} = \sec\theta$

$= \int \frac{\sec^2\theta}{\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\tan\theta + \sec\theta|$

$= \ln|x + \sqrt{1+x^2}| + C$

For problems 11. through 13., determine whether the integral converges or diverges

11. $\int_1^{\infty} \frac{x}{x^3+x+1} dx$, $\frac{x}{x^3+x+1} < \frac{x}{x^3} = \frac{1}{x^2}$ and

$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} -\frac{1}{x}\Big]_1^b = \lim_{b\to\infty} -\frac{1}{b} + 1 = 1$ conv.

$\Rightarrow$ by Comp TR. $\int_1^{\infty} \frac{x}{x^3+x+1}$ also conv.

12. $\int_1^{\infty} \frac{1}{\sqrt{x+5}} dx = \lim_{b\to\infty} \sqrt{x+5}\Big]_1^b = \lim_{b\to\infty} \sqrt{b+5} - \sqrt{6} = \infty$

Div.

13. $\int_0^1 \ln x \, dx = \lim_{a\to 0^+} x\ln x - x\Big]_a^1 = \lim_{a\to 0^+} \left[-1 - a\ln a + a\right]$

and $\lim_{a\to 0} a\ln a = \lim_{a\to 0} \frac{\ln a}{1/a} = \lim_{a\to 0} \frac{\frac{1}{a}}{-1/a^2} = 0$

$\Rightarrow \int_0^1 \ln x \, dx = -1$ conv.

14. Use the Midpoint Rule with n=6 to approximate arctan(3)

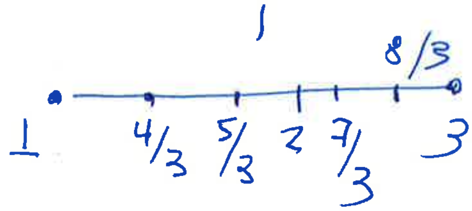
$$\arctan 3 = \int_0^3 \frac{1}{1+x^2}\,dx \quad , \quad \Delta x = \frac{3-0}{6} = 0{,}5$$

$$\bar{x}_1 = 0{,}25, \quad \bar{x}_2 = 0{,}75, \quad \bar{x}_3 = 1{,}25$$
$$\bar{x}_4 = 1{,}75, \quad \bar{x}_5 = 2{,}25, \quad \bar{x}_6 = 2{,}75$$

$$\arctan 3 \sim 0{,}5\left[\frac{1}{1+(,25)^2} + \frac{1}{1+(,75)^2} + \frac{1}{1+(1,25)^2} + \frac{1}{1+(1,75)^2} + \frac{1}{1+(2,25)^2} + \frac{1}{1+(2,75)^2}\right]$$

15. Use the Trapezoidal Rule with n=6 to approximate ln(3)

$$\ln 3 = \int_1^3 \frac{1}{x}\,dx \qquad \Delta x = \frac{3-1}{6} = \frac{1}{3}$$



$$\ln 3 \sim \frac{1}{3}\cdot\frac{1}{2}\left[1 + 2\cdot\frac{3}{4} + 2\cdot\frac{3}{5} + 2\cdot\frac{1}{2} + 2\cdot\frac{3}{7} + 3\cdot\frac{3}{8} + \frac{1}{3}\right]$$