p.56 <u>Section 2.4</u> <u>Exact Equations</u>

Given a function $z = f(x,y)$; its differential is:

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

If $f(x,y) = c$, it follows that

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

(That is, given a one-parameter family of curves $f(x,y)=c$, we can generate a first-order DE by computing the differential

$\therefore$ $M(x,y)\,dx + N(x,y)\,dy = 0$ is an <u>exact</u> equation if $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential; that is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

<u>Example (p.65 #2)</u>

Is equation exact?

$$(2x+y)\,dx - (x+6y)\,dy = 0$$

$$\left.\frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = -1\right\}\ \text{NOT EXACT}$$

<u>Example (p.65 #10)</u>

$$(x^3+y^3)\,dx + 3xy^2\,dy = 0$$

$$\left.\frac{\partial M}{\partial y} = 3y^2,\ \frac{\partial N}{\partial x} = 3y^2\right\}\ \text{Equation is exact}\ \checkmark$$

Method of Solving Exact Equations

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact

Since the equation is exact, there exists a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)$$

$$\therefore f(x,y) = \int M(x,y)\,dx + g(y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx + g(y)\right] = N(x,y)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

Using the last example (p. 65 #10)

$$f(x,y) = \int M(x,y)\,dx = \int x^3 + y^3\,dx = \tfrac{1}{4}x^4 + xy^3 + g(y)$$

$$\frac{\partial}{\partial y}\left[\tfrac{1}{4}x^4 + xy^3 + g(y)\right] = 3xy^2 + g'(y)$$

$$3xy^2 + g'(y) = 3xy^2$$

$$g'(y) = 0, \quad g(y) = c$$

Implicit Solution is:

$$\tfrac{1}{4}x^4 + xy^3 = c$$

Explicit solution is: $y = \left[\frac{c - x^4/4}{x}\right]^{1/3}$

Defined for $x \neq 0$

pp. 60-61 Example 3: Initial Value Problem

Solve: $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$

Rearrange:

(1) $y(1-x^2)\,dy - (xy^2 - \cos x \sin x)\,dx = 0$

Is it Exact?

(2) $N(x,y) = y(1-x^2)$, $M(x,y) = \cos x \sin x - xy^2$

$\dfrac{\partial M}{\partial y} = -2xy$, $\dfrac{\partial N}{\partial x} = -2xy$ ✓ exact

(3) $\therefore \dfrac{\partial f}{\partial y} = y(1-x^2)$

(4) $f(x,y) = \dfrac{y^2}{2}(1-x^2) + h(x)$

(5) $\dfrac{\partial f}{\partial x} = -xy^2 + h'(x) = M(x,y)$

$$-xy^2 + h'(x) = \cos x \sin x - xy^2$$

$$h'(x) = \cos x \sin x$$

$$h(x) = \int \cos x \sin x \, dx \quad \begin{matrix} u = \cos x \\ du = -\sin x\,dx \end{matrix}$$

$$= -\frac{1}{2}\cos^2 x + c_1$$

$\therefore$ A family of solutions is:

$$\frac{y^2}{2}(1-x^2) - \frac{1}{2}\cos^2 x = c_1$$

OR $y^2(1-x^2) - \cos^2 x = c$

Particular Solution: $y(0)=2$, $(0,2)$

$4(1) - (1) = c$, $c = 3$

$\therefore \boxed{y^2(1-x^2) - \cos^2 x = 3}$

Example: (p. 62 #2)

$(2x+y)dx + (x+6y)dy = 0$

Exact? $M(x,y) = 2x+y$, $N(x,y) = x+6y$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 1$ ✓ exact

$\frac{\partial f}{\partial y} = N(x,y) = x+6y$

$f(x,y) = \int (x+6y)\,dy = xy + 3y^2 + h(x)$

$\frac{\partial f}{\partial x} = y + h'(x) = M(x,y) = 2x+y$

$h'(x) = 2x$, $h(x) = x^2 + c$

$\therefore$ Solution is $\boxed{xy + 3y^2 + x^2 = c}$

Now is $(2x+y)dx - (x+6y)dy = 0$ exact

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$. Not exact

(See p. 65 #2)

<u>Example (p.62 #6):</u>

$$\left(2y - \frac{1}{x} + \cos 3x\right)\frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$$

$$\left(2y - \frac{1}{x} + \cos 3x\right)dy = 4x^3 - \frac{y}{x^2} - 3y\sin 3x\,dx$$

$$\left(4x^3 - \frac{y}{x^2} - 3y\sin 3x\right)dx - \left(2y - \frac{1}{x} + \cos 3x\right)dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -\frac{1}{x^2} - 3\sin 3x \\ \frac{\partial N}{\partial x} &= -\frac{1}{x^2} + 3\sin 3x \end{aligned}\right\} \therefore \text{ Not exact}$$

<u>Example (p.63 #18):</u>

$$\left(2y\sin x\cos x - y + 2y^2 e^{xy^2}\right)dx = \left(x - \sin^2 x - 4xye^{xy^2}\right)dy$$

$$M(x,y) = 2y\sin x\cos x - y + 2y^2 e^{xy^2}$$
$$N(x,y) = -x + \sin^2 x + 4xy\,e^{xy^2}$$

$$\frac{\partial M}{\partial y} = 2\sin x\cos x - 1 + 4ye^{xy^2} + 2y^2 \cdot 2xy\,e^{xy^2}$$

$$\frac{\partial N}{\partial x} = -1 + 2\cos x\sin x + 4ye^{xy^2} + 4xy \cdot y^2 e^{xy^2}$$

They are exact

$\therefore M_y = N_x = 2\sin x\cos x - 1 + 4ye^{xy^2} + 2xy^3e^{xy^2}$

$$\frac{\partial f}{\partial x} = M(x,y) = 2y\sin x\cos x - y + 2y^2e^{xy^2}$$

$$f(x,y) = \int 2y\sin x\cos x\,dx - \int y\,dx + \int 2y^2e^{xy^2}\,dx$$

$$= y\sin^2 x - xy + 2e^{xy^2} + h(y)$$

$$\frac{\partial f}{\partial y} = \sin^2 x - x + 4xye^{xy^2} + h'(y)$$

But $\frac{\partial f}{\partial y} = N(x,y) = -x + \sin^2 x + 4xye^{xy^2}$

$$= \sin^2 x - x + 4xye^{xy^2} + h'(y)$$

$\therefore h'(y) = 0, \quad h(y) = c$

A solution is:

$$\boxed{y\sin^2 x - xy + 2e^{xy^2} = c}$$

Example (p.63 # 26)

Solve the initial value problem:

$$\left(\frac{1}{(1+y^2)} + \cos x - 2xy\right)\frac{dy}{dx} = y(y+\sin x),$$
$$y(0) = 1$$

Rearranging:

$$y(y+\sin x)\,dx + \left(2xy - \cos x - \frac{1}{(1+y^2)}\right)dy = 0$$

$$M(x,y) = y^2 + y\sin x$$
$$N(x,y) = 2xy - \cos x - \frac{1}{(1+y^2)}$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2y + \sin x\\ \frac{\partial N}{\partial x} &= 2y + \sin x\end{aligned}\right\} \checkmark \text{ exact}$$

$$M_y = N_x = 2y + \sin x$$

$$\frac{\partial f}{\partial x} = M(x,y) = y^2 + y\sin x$$

$$f(x,y) = \int y^2\,dx + \int y\sin x\,dx$$
$$= xy^2 - y\cos x + h(y)$$

$$\frac{\partial f}{\partial y} = 2xy - \cos x + h'(y) = N(x,y)$$
$$= 2xy - \cos x - \frac{1}{(1+y^2)}$$

$$h'(y) = \frac{-1}{(1+y^2)}$$

$$h(y) = \int \frac{-dy}{(1+y^2)} = -\tan^{-1} y + c$$

$\therefore$ A family of solutions is:

$$xy^2 - y\cos x - \tan^{-1} y = c$$

Initial value problem

$$y(0) = 1 \quad , \quad (0,1)$$

$$(0)(1) - (1)(1) - \tan^{-1}(1) = c$$

$$c = -\pi/4$$

$\therefore$ A particular solution is:

$$\boxed{xy^2 - y\cos x - \tan^{-1} y = -\frac{\pi}{4}}$$

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Integrating Factor for Making a Non-Exact Differential Equation Exact

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, then it is sometimes possible to find an integrating factor $\mu(x,y)$, so that after multiplying, the equation:

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$$

Now, the above equation is exact if and only if $(\mu M)_y = (\mu N)_x$

Partial derivative

Using the product rule:

$$\mu M_y + \mu_y M = \mu N_x + \mu_x N \quad \text{or} \ldots$$

$$\mu_x N - \mu_y M = (M_y - N_x)\mu$$

Simplifying Assumption:

μ is a function of one variable

$\therefore \mu_x = \dfrac{d\mu}{dx}$ and

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

$$\mu(x) = e^{\int (M_y - N_x)/N \, dx}$$

Now, if μ depends only on y, then

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$$

and $\mu(y) = e^{\int (N_x - M_y)/M \, dy}$

Results for $M(x,y)\,dx + N(x,y)\,dy = 0$ are as follows:

① If $\left(\frac{M_y - N_x}{N}\right)$ is a function of x alone

then $\mu(x) = e^{\int (M_y - N_x)/N \, dx}$

② If $\frac{(N_x - M_y)}{M}$ is a function of y alone

then $\mu(y) = e^{\int (N_x - M_y)/M \, dy}$

where $\mu(x)$ and $\mu(y)$ are the integrating factor options for the non-exact differential Equation

Choose one of them for:

$$\mu M(x,y)\,dx + \mu N(x,y) = 0$$

p.62 Example 4: A non exact D.E. made exact:

$$xy\,dx + (2x^2+3y^2-20)dy = 0$$

is not exact. Proof:

$$M(x,y) = xy, \quad N(x,y) = 2x^2+3y^2-20$$

$$\frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = 4x$$

Can this be made exact using the integrating factors?

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2+3y^2-20} = \frac{-3x}{2x^2+3y^2-20}$$

depends on x and y (doesnt work)

Try $\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y}$

depends only on y. Therefore

$$\mu(x,y) = e^{\int \frac{3\,dy}{y}} = y^3$$

$$\therefore (y^3)xy\,dx + y^3(2x^2+3y^2-20)dy = 0$$

$$xy^4\,dx + (2x^2y^3+3y^5-20y^3)dy = 0$$

Is this exact?

$$\frac{\partial M}{\partial y} = 4xy^3, \quad \frac{\partial N}{\partial x} = 4xy^3 \quad \checkmark$$

$$\frac{\partial f}{\partial x} = M(x,y) = xy^4$$

$$f(x,y) = \frac{1}{2}x^2y^4 + h(y)$$

$$\frac{\partial f}{\partial y} = 2x^2y^3 + h'(y) = N(x,y)$$

$$2x^2y^3 + h'(y) = 2x^2y^3 + 3y^5 - 20y^3$$

$$h'(y) = 3y^5 - 20y^3$$

$$h(y) = 3\int y^5 dy - 20\int y^3 dy$$

$$= \frac{1}{2}y^6 - 5y^4 + c$$

A Family of solutions for this D.E. is therefore:

$$\boxed{\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c}$$