(p. 111) <u>3.1.3. Nonhomogeneous Equations</u>

<u>General</u> <u>Solution</u> is :

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots c_n y_n(x) + y_p$$

where c_i, $i = 1, 2 \dots n$ are arbitrary constants

$\therefore$ $y = y_c(x) + y_p(x)$ where

$y_c(x)$ is the <u>complementary function</u>

and $y_p(x)$ is any <u>particular</u> <u>solution</u>

<u>Superposition</u> <u>Principle</u>:

Let $y_{p_1}, y_{p_2} \dots y_{p_k}$ be particular solutions of the general form of the a nonhomogeneous D.E. on the interval I, corresponding to k distinct functions $g_1, g_2 \dots g_k$ that is

$$a_n(x)y^n + a_{n-1}y^{n-1} \dots + a_1(x)y' + a_0(x)y = g_i(x)$$

where $i = 1, 2 \dots K$ then,

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \dots + y_{p_k}(x)$$

is a particular solution of;

$$a_n(x)y^n + a_{n-1}y^{n-1} + \dots + a_1(x)y' + a_0(x)y$$
$$= g_1(x) + g_2(x) + \dots g_k(x).$$

p113 <u>Example 11:</u> Superposition Principle.

Verify that y_{p_i} are particular solutions:

$y_{p_1} = -4x^2 \rightarrow y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2} = e^{2x} \rightarrow y'' - 3y' + 4y = 2e^{2x}$

$y_{p_3} = xe^x \rightarrow y'' - 3y' + 4y = 2xe^x - e^x$

① $y' = -8x \;,\; y'' = -8$

$-8 + 24x - 16x^2 = -16x^2 + 24x - 8$ ✓

② $y' = 2e^{2x} \,,\, y'' = 4e^{2x}$

$4e^{2x} - 6e^{2x} + 4e^{2x} = 2e^{2x}$ ✓

③ $y' = xe^x + e^x \,,\, y'' = xe^x + e^x + e^x$

$xe^x + 2e^x - 3xe^x - 3e^x + 4xe^x = 2xe^x - e^x$

$2xe^x - e^x = 2xe^x - e^x$ ✓

From the superposition principle the combination: $y = y_{p_1} + y_{p_2} + y_{p_3}$

OR $y = -4x^2 + e^{2x} + xe^x$ is a solution

of $y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{y_{p_1}} + \underbrace{2e^{2x}}_{y_{p_2}} + \underbrace{2xe^x - e^x}_{y_{p_3}}$

(7a)

p.115 34.) Verify that

$$y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15} x^2 - \frac{1}{6} x$$

$(0, \infty)$

is the general solution to:

$$2x^2 y'' + 5xy' + y = x^2 - x$$

$$y' = -\frac{1}{2} c_1 x^{-3/2} - c_2 x^{-2} + \frac{2}{15} x - \frac{1}{6}$$

$$y'' = +\frac{3}{4} c_1 x^{-5/2} + 2c_2 x^{-3} + \frac{2}{15}$$

$$2x^2 \left\{ \frac{3}{4} c_1 x^{-5/2} + 2c_2 x^{-3} + \frac{2}{15} \right)$$

$$+ 5x \left(-\frac{1}{2} c_1 x^{-3/2} - c_2 x^{-2} + \frac{2}{15} x - \frac{1}{6} \right)$$

$$+ c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15} x^2 - \frac{1}{6} x = x^2 - x$$

$$\frac{6}{4} c_1 x^{-5} + 4c_2 x^{-1} + \frac{4}{15} x^2$$

$$- \frac{5}{2} c_1 x^{-1/2} - 5c_2 x^{-1} + \frac{2}{3} x - \frac{5}{6} x$$

$$+ c_1 x^{-1/2} +$$

Messy ...

But if we use Superposition...

$$y_1(x) = C_1 x^{-1/2}, \quad y_1'(x) = -\tfrac{1}{2} C_1 x^{-3/2}$$

$$y_2(x) = C_2 x^{-1}, \quad y_2'(x) = -C_2 x^{-2}$$

homogeneous Equation is:

$$2x^2 y'' + 5xy' + y = 0$$

$$y_1(x): \; y_1''(x) = +\tfrac{3}{4} C_1 x^{-5/2} \quad \text{so}$$

$$2x^2\left(\tfrac{3}{4} C_1 x^{-5/2}\right) + 5x\left(-\tfrac{1}{2} C_1 x^{-3/2}\right) + C_1 x^{-1/2} = 0$$

$$\frac{3}{2} x^{-1/2} - \frac{5}{2} x^{-1/2} + \frac{2}{2} x^{-1/2} = 0 \checkmark$$

Now try $y_2(x)$ ...

and $y_p(x) = \frac{1}{15} x^2 - \frac{1}{6} x$ is a particular solution to the non homogeneous equation.

p.115 3.2 Reduction of Order

Consider the differential equation in standard form:

$$y'' + P(x)y' + Q(x)y = 0$$

Let $y_1(x)$ be a solution on the interval I and that $y_1(x) \neq 0$ for every x on the interval.

Product Rule

If we define $y = u(x)\,y_1(x)$, then

$$y' = uy_1' + y_1u', \quad y'' = uy_1'' + 2y_1'u' + y_1u''$$

$$\therefore y'' + Py' + Qy = u[y_1'' + Py_1' + Qy_1] + y_1u'' + (2y_1' + Py_1)u' = 0$$

but $y_1'' + Py_1' + Qy_1 = 0$ so

$y_1u'' + 2(y_1' + Py_1)u' = 0$; let $\boxed{w = u'}$

$\therefore y_1w' + (2y_1' + Py_1)w = 0$

Order has been reduced from second order to first order via the substitution to w.

The equation is now separable!!

$$y_1w' + 2y_1'w + Py_1w = 0$$

$$w' + \frac{2y_1'}{y_1}w + P(x)w = 0$$

Rewrite

$$\frac{dw}{dx} + \frac{2y_1'}{y_1} w + Pw = 0$$

$$\frac{dw}{w} + \frac{2y_1'}{y_1} dx + P dx = 0$$

Integrate

$$\ln|w| + \int \frac{2}{y_1}\frac{dy_1}{dx} dx + \int P dx = 0$$

$$\ln|w| + 2\ln|y_1| + \int P dx = C$$

$$\ln|w y_1^2| + \int P dx = C$$

$$\ln|w y_1^2| = -\int P dx + C$$

$$w y_1^2 = C e^{-\int P dx}$$

$$w = \frac{C e^{-\int P dx}}{y_1^2}, \text{ but } w = u'$$

$$u' = \frac{C e^{-\int P dx}}{y_1^2}$$

$$u = \int \frac{C e^{-\int P dx}}{y_1^2(x)} dx + C, \quad y = u(x) y_1(x)$$

$$\frac{y}{y_1} = \int \frac{C e^{-\int P dx}}{y_1^2(x)} dx$$

$$\boxed{y_2 = y_1(x) \int \frac{C e^{-\int P(x) dx}}{y_1^2(x)} dx} \text{ is a}$$

solution to $y'' + P(x) y' + Q(x) y = 0$
and y_1 and y_2 are independent for $y_1 \neq 0$

Example (#4, p.117)

Solve by reduction of order formula:

$$y'' + 9y = 0 \,, \quad \overset{\text{Given}}{y_1 = \sin 3x} \,, \quad \underline{\underline{\text{No } P(x)}}!$$

let $y = u(x)\, y_1(x) = u \sin 3x$

$$y' = 3u\cos 3x + u'\sin 3x$$

$$y'' = -9u\sin 3x + 3u'\cos 3x + u''\sin 3x + 3u'\cos 3x$$

$$= -9u\sin 3x + 6u'\cos 3x + u''\sin 3x$$

Substituting:

$$u''\sin 3x + 6\cos 3x\, u' - \cancel{9u\sin 3x} + \cancel{9u\sin 3x} = 0$$

$$\therefore u''\sin 3x + 6\cos 3x\, u' = 0$$

let $w = u'$, $w' = u''$

$$\therefore w'\sin 3x + 6\cos 3x\, w = 0$$

$$w' + \frac{6\cos 3x\, w}{\sin 3x} = 0$$

First order ODE!

$$\left\{ w' + 6(\cot 3x)\, w = 0 \right.$$

Integrating factor is:

$$e^{\int P\,dx} \quad \text{where } P = 6\cot 3x$$

$$\therefore e^{6\int \cot 3x\,dx} = e^{6\left[\frac{1}{3}\ln|\sin 3x|\right]}$$

$$= e^{2\ln|\sin 3x|} = \sin^2 3x$$

$$\therefore \frac{d}{dx}\left[(\sin^2 3x)\, w\right] = 0, \quad (\sin^2 3x)\, w = C$$

$$w = u' = C\csc^2 3x \,, \quad u = C\cot 3x$$

$$\therefore y_2 = \cot 3x \sin 3x = \cos 3x$$

Example (#10, p.117)

Solve by reduction of order formula or find a second solution to:

Given:

$$x^2y'' + 2xy' - 6y = 0, \quad y_1 = x^2$$

$$y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0$$

Since $P(x) = \frac{2}{x}$ we can use the integrating factor:

$$e^{-\int P(x)\,dx} = e^{-2\ln|x|} = x^{-2}$$

$$\therefore \quad y_2(x) = y_1(x)\int \frac{c\,e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

$$= x^2 \int c\,\frac{x^{-2}}{x^4}\,dx = x^2\left(-\frac{c}{5}\right)x^{-5}$$

$$y_2(x) = -\frac{c}{5}x^{-3} = \frac{c}{x^3}$$ (any constant works here)

A second solution is

$$\boxed{y_2(x) = \frac{c}{x^3}}$$

why? It satisfies the ODE!

We show that our formula works! (12)

Example (#10, p.117) (The Hard Way)

$$x^2y'' + 2xy' - 6y = 0, \; y_1 = x^2$$

$$y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0$$

$$p(x) = \frac{2}{x}, \; e^{\int p(x)dx} = e^{2\ln|x|} = x^2$$

let $y_2 = u(x)y_1 = ux^2$

Proof!

$$y' = 2ux + u'x^2$$

$$y'' = 2u + 2xu' + u''x^2 + 2u'x$$

$$= 2u + 4xu' + u''x^2$$

Substituting:

$$x^2(u''x^2 + 4xu' + 2u) + 2x(u'x^2 + 2ux) - 6ux^2 = 0$$

$$x^4u'' + 4x^3u' + 2\cancel{u}x^2 + 2x^3u' + 4\cancel{u}x^2 - 6\cancel{u}x^2 = 0$$

$$x^4u'' + 6x^3u' = 0$$

$$u'' + \frac{6}{x}u' = 0, \; \text{let } w = u'$$

$$w' + \frac{6}{x}w = 0 \leftarrow \text{Standard Form}$$

$$\frac{d}{dx}\left[x^6w\right] = 0, \; x^6w = c = x^6u'$$

$$u = -\frac{1}{5}cx^{-5}, \; y = cx^{-5}x^2, \; \boxed{y_2 = cx^{-3}}$$