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Group Theory – Lecture 7

Summary of last week's lecture

	Lie group		Lie algebra
i)	G	$\longmapsto$	$\underline{g} \cong T_e G$
ii)	$\phi: G \to H$	$\longmapsto$	$\varphi := d\phi_e : \underline{g} \to \underline{h}$
iii)	$H \subset G$	$\longmapsto$	$\underline{h} \subset \underline{g}$
iv)	$\exists!$ simply connected Lie group $\tilde{G}$; other: $G = \tilde{G}/\Gamma$, $\Gamma \subset Z(\tilde{G})$	$\longleftarrow$	$\underline{g}$
v)	unique connected $H \subset G$	$\longleftrightarrow$	$\underline{h} \subset \underline{g}$
vi)	$\exists!\ \phi: G \to H$, G simply connected, H connected		$\varphi: \underline{g} \to \underline{h}$

Representations

$\phi: G \to GL(V) \quad \longmapsto \quad \varphi = (d\phi)_e : \underline{g} \to gl(V)$

$\exists!\ \phi: \tilde{G} \to GL(V) \quad \longleftarrow \quad \varphi: \underline{g} \to gl(V)$

simply conn. Lie group with Lie algebra $\underline{g}$. $G = \tilde{G}/\Gamma$, $\ker(\phi) \supset \Gamma$

Exponential map

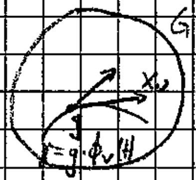
G: Lie group, $\mathrm{Lie}(G) = \underline{g}$. For every $v \in \underline{g}$ can define subalgebra $\underline{h} = \mathrm{span}(v) \subset \underline{g}$. Using v) from above table $\longrightarrow$ 1-parameter subgroup $H \subset G$ (unique connected)

$\phi_v : (\mathbb{R}, +) \longrightarrow G \quad \Big| \quad \varphi_v : \underline{h} \longrightarrow \underline{g}, \quad v \longmapsto v$

$\phi_v(s)\, \phi_v(t) = \phi_v(s+t)$

$(d\phi_v)_e = v$

i) $\phi_v(t): \mathbb{R} \longrightarrow G$ (integral curve of X_v)



$\dot{\gamma}(t) = X_v(\gamma(t))$

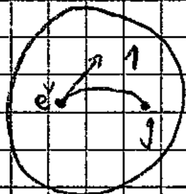
$\gamma = g\cdot\phi_v(t)$ integral curve of X_v through $g\in G$

$$\dot{\gamma}(t) = \frac{d}{ds}\Big|_{s=0} \gamma(t+s) = \frac{d}{ds}\Big|_{s=0} g\,\phi_v(t)\,\phi_v(s)$$

$$= d(L_{g\cdot\phi_v(t)})_e \underbrace{\frac{d}{ds}\Big|_{s=0}\phi_v(s)}_{v} = d(L_{g\cdot\phi_v(t)})_e\, v = X_v(\gamma(t))$$

ii) $\phi_v(t) = \phi_{vt}(1)$ (both integral curves of same vf)

Define: $\exp: \mathfrak{g} \longrightarrow G$, $v \longmapsto \phi_v(1)$



i) $\exp(tv)\exp(sv) = \exp((s+t)v)$

ii) $\underbrace{d(\exp)_0}_{id_{\mathfrak{g}}} : \mathfrak{g} \longrightarrow \mathfrak{g}$, $v \longmapsto v$

iii) $\phi: G \longrightarrow H \implies \phi(\exp(v)) = \exp(d\phi_e v)$

Matrix groups

$\exp = \mathrm{Exp}: Mat(n,\mathbb{R}) \longrightarrow GL(n)$, $v \longmapsto \sum_{i\geq 0} \frac{v^i}{i!}$ ↖ matrix multiplication

Baker-Campbell-Hausdorff formula

$$\exp(v)\exp(w) = \exp\left(v + w + \frac{1}{2}[v,w] + \frac{1}{12}\left([v,[v,w]] - [w,[w,v]]\right) + \dots\right)$$

The Lorentz group

Minkowski space: $\left(\mathbb{R}^{1,3}, \eta = \begin{pmatrix}1 &&& \\ & -1 && \\ && \ddots & \\ &&& -1\end{pmatrix}\right)$

$$\mathcal{L} = O(1,3) = \{\Lambda \in Mat(4,4) \mid \Lambda^t \eta \Lambda = \eta\}$$

$$\Lambda = \left(\begin{array}{c|c}\lambda_{00} & \lambda_2^t \\ \hline \lambda_1 & \hat{\Lambda}\end{array}\right), \quad \lambda_{00}\in\mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R}^3;\ \hat{\Lambda}\in Mat(3,3)$$

$$\Lambda^t \eta \Lambda = \left(\begin{array}{c|c}\lambda_{00}^2 - \lambda_1^t\lambda_1 & \lambda_{00}\lambda_2^t - \lambda_1^t\hat{\Lambda} \\ \hline \lambda_2\lambda_{00} - \hat{\Lambda}^t\lambda_1 & -\hat{\Lambda}^t\hat{\Lambda} + \lambda_2\lambda_2^t\end{array}\right) = \left(\begin{array}{c|c}1 & \\ \hline & \mathbb{1}_3\end{array}\right) \qquad (*)$$

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Polar decomposition of $O(1,3)$

Note that $\eta^t = \eta$, $\eta^2 = \mathbb{1}_4 \implies \eta \in O(1,3)$, $\Lambda^t = \eta \Lambda^{-1} \eta \in O(1,3)$

$\implies \Lambda^t \Lambda \in O(1,3)$, symmetric, >0

$L := \sqrt{\Lambda^t \Lambda}$

$L^t = L$, $L > 0$, $L \in O(1,3)$

$R := \Lambda L^{-1} \in O(1,3)$

$R^t R = (L^{-1})^t \Lambda^t \Lambda L^{-1} = \mathbb{1}_4 \implies R \in O(1,3) \cap O(4)$

$\implies R \in O(1) \times O(3)$, $R = \left(\begin{array}{c|ccc} \varepsilon & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & Q & \\ 0 & & & \end{array}\right)$, $\varepsilon \in O(1) = \{\pm 1\}$, $Q \in O(3)$

$\Lambda = \left(\begin{array}{c|c} \varepsilon & 0 \\ \hline 0 & Q \end{array}\right) L$, $L = \left(\begin{array}{c|c} \lambda_{00} & \lambda_2^t \\ \hline \lambda_1 & \hat{\Lambda} \end{array}\right)$, $\hat{\Lambda}^t = \hat{\Lambda}$

$(*) \implies \lambda_{00} = \sqrt{1 + \lambda^t \lambda}$, $\hat{\Lambda} = \sqrt{\mathbb{1} + \lambda \lambda^t}$

$$\Lambda = \left(\begin{array}{c|c} \overset{\text{time-reversal}}{\varepsilon} & 0 \\ \hline 0 & Q \end{array}\right)_{\leftarrow \text{space transf.}} \cdot \overset{\text{boosts}}{\left(\begin{array}{c|c} \sqrt{1+\lambda^t \lambda} & \lambda^t \\ \hline \lambda & \sqrt{\mathbb{1} + \lambda \lambda^t} \end{array}\right)}$$

$O(1,3) \overset{\text{homeo}}{\cong} O(1) \times O(3) \times \mathbb{R}^3$

$\longrightarrow$ 4 components distinguished by

$\det(\Lambda) = \varepsilon \det(Q) = \pm 1$ (+1: proper, preserve orientation)

$\operatorname{sign}(\lambda_{00}) = \varepsilon = \pm 1$ (+1: orthochronous, time-direction preserved)

Connected component of $e \in O(1,3)$ is called $O(1,3)_0 \trianglelefteq O(1,3)$

$O(1,3)/O(1,3)_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, P, T, PT\}$, $P = -\mathbb{1}_4$ parity-reversal

$T = \begin{pmatrix} -1 & \\ & \mathbb{1}_3 \end{pmatrix}$ time-reversal

Lie algebra $o(1,3)$

$O(1,3)$ is level set: $f(\Lambda) = \Lambda^t \eta \Lambda - \eta$

$df_e V = V^t \eta + \eta V$

$\ker(df_e) = \{V \mid V^t = -\eta V \eta\}$

$\overset{\shortparallel}{T_e O(1,3)} = o(1,3)$, $V = \left(\begin{array}{c|c} V_0 & c^t \\ \hline d & \hat{V} \end{array}\right) = \left(\begin{array}{c|c} -V_0 & +d^t \\ \hline +c & -\hat{V}^t \end{array}\right)$

$\Longrightarrow v_0 = 0, \quad c = d, \quad \hat{v}^T = -\hat{v}$

$$v = \left(\begin{array}{c|c} 0 & c^t \\ \hline c & \hat{v} \end{array}\right)$$

basis:

$$\omega_1 = \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & 0 & 1 & 0 \end{array}\right), \quad \omega_2 = \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ & -1 & 0 & 0 \end{array}\right), \quad \omega_3 = \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \end{array}\right)$$

$$K_1 = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & & & \\ 0 & & 0 & \\ 0 & & & \end{array}\right), \quad K_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & & & \\ 1 & & 0 & \\ 0 & & & \end{array}\right), \quad K_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & & & \\ 0 & & 0 & \\ 1 & & & \end{array}\right)$$

$o(1,3) = \text{span}\{\omega_i, k_i\}$ $\longleftarrow$ 6 dimensional real Lie algebra

$$\boxed{\begin{aligned} [\omega_i, \omega_j] &= \sum_k \varepsilon_{ijk}\, \omega_k \\ [K_i, K_j] &= -\sum_k \varepsilon_{ijk}\, \omega_k \\ [\omega_i, K_j] &= \sum_k \varepsilon_{ijk}\, K_k \end{aligned}}$$

$SL_2(\mathbb{C}) = \{A \in Mat(2,2,\mathbb{C}) \mid \det(A) = 1\}$

$sl_2(\mathbb{C}) = \{a \in Mat(2,2,\mathbb{C}) \mid tr(a) = 0\} = \text{span}_{\mathbb{C}}\{j_1 = \frac{i}{2}\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$

$j_2 = \frac{i}{2}\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j_3 = \frac{i}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\}$ $\quad$ σ_i Pauli matrices

$[j_a, j_b] = \sum_c \varepsilon_{abc}\, j_c$

$\longrightarrow$ complex 3-dimensional Lie algebra

Choose a real structure

$sl_2(\mathbb{C}) = \text{span}_{\mathbb{R}}\{j_1, j_2, j_3\} \otimes_{\mathbb{R}} \mathbb{C}$

$\varphi : (sl_2(\mathbb{C}))_{\mathbb{R}} \longrightarrow o(1,3), \quad j_a \longmapsto \omega_a, \quad i j_a \longmapsto K_a$

isomorphism of real Lie algebras

$sl_2(\mathbb{C})$ is in fact simply connected!

$\Longrightarrow \varphi$ integrates to a unique

$\phi : SL(2,\mathbb{C}) \longrightarrow O(1,3)_0$ $\quad$ which is possibly a covering

$d\phi_e = \varphi$

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identity: $\mathbb{R}^{1,3} \xrightarrow{\cong} i\,\overset{\text{hermitian } 2\times2\text{-matrices}}{U(2)}$, $x \longmapsto \tau(x) = \sum_i x_i \tau_i = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}$

$x^t \eta x = \det(\tau(x))$

Now $SL(2,\mathbb{C}) \curvearrowright iU(2)$, $\tau(x) \longmapsto A\tau(x)A^\dagger$, $A \in SL(2,\mathbb{C})$

But this action preserves η:

$\det(A\,\tau(x)\,A^\dagger) = \det(\tau(x))$ since $\det(A) = 1$

$\Longrightarrow A\tau(x)A^\dagger = \tau(\Psi(A)\cdot x)$, where $\Psi: SL(2,\mathbb{C}) \to O(1,3)$

check: $d\Psi_e = \varphi \Longrightarrow \Psi = \varphi$

$A(t)$ path in $SL(2,\mathbb{C})$: $A(0) = \mathbb{1}$, $\dot{A}(0) = a$

$\frac{d}{dt}\big|_{t=0} A(t)\tau(x)A^\dagger(t) = a\tau(x) + \tau(x)a^\dagger = \tau((d\Psi_e a)x)$

$a = j_i = \frac{i}{2}\tau_i \Longrightarrow \frac{d}{dt}\big|_{t=0} A(t)\tau(x)A^\dagger(t) = \frac{i}{2}[\tau_i, \tau(x)] = \sum \tau_c (\omega_i x)_c = \tau(\omega_i x)$

$a = -ij_i \Longrightarrow \frac{d}{dt}\big|_{t=0} A(t)\tau(x)A^\dagger(t) = \tau(k_i x)$

$\Longrightarrow \varphi = d\Psi_e : j_i \longmapsto \omega_i$, $-ij_i \longmapsto k_i$

$\Longrightarrow \Psi = \varphi$

$\ker(\varphi) = \{A \in SL(2,\mathbb{C}) \mid A\tau A^\dagger = \tau \;\forall \tau\} = \{\pm \mathbb{1}_2\}$

Remark: complexification

$o(1,3)_\mathbb{C} = o(1,3) \otimes_\mathbb{R} \mathbb{C} \longrightarrow$ 6-dimensional, complex Lie algebra

basis: $J_i^\pm := \frac{1}{2}(\omega_i \pm i k_i)$

complex basis of $o(1,3)_\mathbb{C}$

$[J_i^+, J_j^-] = 0 \quad \forall i,j$

$[J_i^\pm, J_j^\pm] = \sum_k \varepsilon_{ijk} J_k^\pm$

$$o(1,3)_{\mathbb{C}} \xrightarrow{\cong} sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$$

$$J_i^+ \longmapsto (j_i, 0)$$

$$J_i^- \longmapsto (0, j_i)$$

isomorphism of complex Lie algebras

$$\boxed{o(1,3)_{\mathbb{C}} \cong sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})}$$