4. REPRESENTATIONS of FINITE GROUPS

$|G| < \infty$, complex representations.

Fact: Any representation of G is fully decomposable.

$\ulcorner$ $W \subset V \leftarrow$ repr. is invariant subspace.

choose a projector $\pi_0 : V \longrightarrow V$ onto W, i.e. $\mathrm{Im}(\pi_0) = W$

Define:
$$\pi := \frac{1}{|G|} \sum_{g \in G} \rho(g)\, \pi_0\, \rho(g^{-1}) \quad V \longrightarrow V$$

i) π is still a projector on W

ii) π is an intertwiner

From this it follows $\quad V \overset{\swarrow \pi \text{ is a projector}}{=} \underbrace{\mathrm{Im}(\pi)}_{W} \oplus \mathrm{Ker}(\pi)$ $\nwarrow$ is invariant subspace, because π is intertwiner.

i) $$\pi^2 = \frac{1}{|G|^2} \sum_{g,k \in G} \rho(g)\, \pi_0\, \rho(g^{-1})\, \rho(k)\, \pi_0\, \rho(k^{-1}) =$$

$$= \frac{1}{|G|} \sum_{k \in G} \rho(k)\, \pi_0\, \rho(k^{-1}) = \pi$$

$\pi|_W = id_W$

ii) $$\pi \circ \rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\, \pi_0\, \overbrace{\rho(g^{-1})\rho(h)}^{\rho(g^{-1}h)} = \quad (g \longmapsto hg)$$

$$= \frac{1}{|G|} \sum_{g \in G} \underbrace{\rho(hg)}_{\rho(h)\rho(g)}\, \pi_0\, \rho(g) =$$

$$= \rho(h)\, \pi$$

$\lrcorner$

Any representation ρ of a finite group decomposes into a sum of irreducible representation:

$$\rho = \rho_1^{\oplus n_1} \oplus \dots \rho_k^{\oplus n_k}, \quad n_i \in \mathbb{N}_0$$

Remark: Proof by averaging over group $\rightarrow$ can be repeated for non-finite but compact groups, where instead of suming, one can integrate

Decomposition

start with representation $\rho: G \longrightarrow GL(V)$

$$\Pi := \frac{1}{|G|} \sum_{g \in G} \rho(g) \quad V \longrightarrow V$$

- it is a projector on the invariant subspace.

$$V^G := \{ v \in V \mid \rho(g) v = v \quad \forall g \in G \}$$

(i) $\Pi^2 = \frac{1}{|G|^2} \sum_{g,h \in G} \underbrace{\rho(g)\rho(h)}_{\rho(gh)} \qquad (g \longmapsto gh^{-1})$

$$= \frac{1}{|G|} \sum_{g \in G}' \rho(g) = \Pi$$

(ii) $v \in V^G : \Pi v = v \Rightarrow \Pi|_{V^G} = id_{V^G} \Rightarrow V^G \subset \mathrm{Im}(\Pi)$

$$\rho(g)\Pi w = \rho(g) \frac{1}{|G|} \sum_{h \in G} \rho(h) \, w = \Pi w$$

$\Rightarrow \mathrm{Im}(\Pi) \subset V^G \Rightarrow V^G = \mathrm{Im}(\Pi$

$\longrightarrow \Pi$ projects on $V^G = \mathrm{Im}(\Pi)$

We want to know the dim of V^G.

$$\dim(V^G) = \mathrm{Tr}_V(\Pi) = \frac{1}{|G|} \sum_{g \in G} \underbrace{\mathrm{Tr}_V(\rho(g))}$$

$\chi_\rho(g) = \chi_V(g)$ Character of the representation ρ, V

$$= \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$$

Apply this

$\mathrm{Hom}(V_1, V_2) \cong V_1^* \otimes V_2$ carries reprs. $\bar{\rho}_1 \otimes \rho_2$

$$\bar{\rho}_1 \otimes \rho_2(g) : \quad f \longmapsto \rho_2(g) \circ f \circ \rho_1(g^{-1})$$

$$\mathrm{Hom}(V_1, V_2)^G = \{ f \in \mathrm{Hom}(V_1, V_2) \mid \rho_2(g) \circ f \circ \rho_1(g^{-1}) = f \} =$$

$=$ space of intertwiners

If ρ_1 and ρ_2 are irreducible, by Schur's lema

$$\dim(\mathrm{Hom}(V_1,V_2)^G) = \begin{cases} 1, & \rho_1 \cong \rho_2 \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$~~\dim~~ \frac{1}{|G|} \sum_{g\in G} \chi_{\bar{\rho}_1\otimes\rho_2}(g) = \begin{cases} 1, & \rho_1 \cong \rho_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\|$$

$$\frac{1}{|G|} \sum_{g\in G} \mathrm{Tr}_{V_1^*\otimes V_2}(\bar{\rho}_1(g)\otimes\rho_2(g))$$

$$\| \qquad \bar{\rho}_1(g) = \rho_1(g^{-1}) \text{ on } V_1$$

$$\frac{1}{|G|} \sum_{g\in G} \mathrm{Tr}_{V_1^*}(\bar{\rho}_1(g)) \cdot \underbrace{\mathrm{Tr}_{V_2}(\rho_2(g))}_{\chi_2(g)}$$

$$\|$$

$$\frac{1}{|G|} \sum_{g\in G} \chi_1(g^{-1})\, \chi_2(g)$$

$$\frac{1}{|G|} \sum_{g\in G} \chi_i(g^{-1}) \cdot \chi_j(g) = \delta_{ij}$$

sum over irreducible representations

$$V = V_1^{\oplus n_1} \oplus \dots \oplus V_k^{\oplus n_k}$$

$$\mathrm{Hom}(V_i, V) = \oplus\, \mathrm{Hom}(V_i, V_a)^{\oplus n_a}$$

$$\boxed{n_i = \mathrm{Tr}_{\mathrm{Hom}(V_j,V)}(\Pi_i) = \frac{1}{|G|} \sum_{g\in G} \chi_{\rho_i}(g^{-1})\, \chi_\rho(g)}$$

From characters one can obtain the decomposition of V.

Can do more: identify the ~~representatio repr~~ respective subrepresentation.

Next step: how to localize a subgroup V_i in ~~among~~ V

Projector on $V_1^{\oplus n_1}$

$$\Pi_{\rho_1} := \frac{\overset{=d_1}{\dim(V_1)}}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1})\, \rho(g) : V \to V$$

i) Π_{ρ_1} is intertwiner

$$\Pi_{\rho_1} \circ \rho(h) = \frac{d_1}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1})\, \rho(gh) =$$

$$= \frac{d_1}{|G|} \sum_{g\in G} \underbrace{\chi_{\rho_1}(hg^{-1}h^{-1})}_{=\chi_{\rho_1}(g^{-1})} \rho(hg) =$$

cyclicity of trace

$$\chi_{\rho_1}(hg^{-1}h^{-1}) = \mathrm{Tr}(\rho(hg^{-1}h^{-1})) =$$

$$= \mathrm{Tr}(\rho(h)\rho(g^{-1})\rho(h^{-1})) =$$

$$= \mathrm{Tr}(\rho(h^{-1})\rho(h)\rho(g^{-1})) = \mathrm{Tr}(\rho(g^{-1}))$$

$$= \rho(h)\Pi_{\rho_1}$$

ii) Π_ρ is a projector:

$$\Pi_\rho^2 = \frac{d_1^2}{|G|^2} \sum_{g,h\in G} \chi_{\rho_1}(g^{-1})\, \chi_{\rho_1}(h^{-1})\, \rho(gh) \qquad g \mapsto gh^{-1}$$

$$= \frac{d_1^2}{|G|^2} \sum_{g,h} \chi_{\rho_1}(hg^{-1})\, \chi_{\rho_1}(h^{-1})\, \rho(g) =$$

[Use Ex. 3c) from PS 2

$$\sum_h \chi_{\rho_i}(hg^{-1})\, \chi_{\rho_j}(h^{-1}) = \chi_{\rho_1}(g^{-1}) \frac{|G|}{d_1} \delta_{ij}$$

$$= \frac{d_1}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1})\, \rho(g) = \Pi_{\rho_1}$$

iii) Π_{ρ_1} projects onto $V_1^{\oplus n_1} \subset V$

Look at representation $\mathrm{Im}(\Pi_{\rho_1}) = V_1^{\oplus n_1'} \oplus \dots \oplus V_k^{\oplus n_k'}$

Want to show that $n_1' = n_1$

$$n_i' = \frac{1}{|G|} \sum_{g\in G} \chi_{\rho_i}(g^{-1})\, \chi_{\mathrm{Im}(\Pi_{\rho_1})}(g) =$$

$$\chi_{\mathrm{Im}(\Pi_{\rho_1})}(g) = \mathrm{Tr}_{\mathrm{Im}(\Pi_{\rho_1})}\, \rho(g) = \mathrm{Tr}_V(\Pi_{\rho_1} \rho(g))$$

$$= \frac{d_1}{|G|} \sum_{g\in G} \chi_{\rho_1}(h^{-1})\, \underbrace{\mathrm{Tr}_V(\underbrace{\rho(h)\rho(g)}_{\rho(hg)})}_{\chi_V(hg)} = \frac{d_1}{|G|} \sum_{h\in G} \chi_{\rho_1}(h^{-1})\, \chi_V(hg)$$

$$n_i' = \frac{d_1}{|G|^2} \sum_{g,h} \chi_{\rho_1^*}(g^{-1})\, \chi_{\rho_1}(h^{-1})\, \chi_V(hg) \qquad g \longmapsto h^{-1}g$$

$$= \frac{d_1}{|G|^2} \sum_{g,h} \chi_{\rho_1^*}(g^{-1}h)\, \chi_{\rho_1}(h^{-1})\, \chi_V(g) =$$

$$= \delta_{i,1} \underbrace{\frac{1}{|G|} \sum_{g\in G} \chi_{\rho_i}(g^{-1})\, \chi_V(g)}_{n_i} =$$

Ex 3.

$$= \delta_{i,1}\, n_i$$

$\to \Pi_{\rho_1}$ projects onto $V_1^{\oplus n_1} \subset V$

- Characters of irreducible representations.

$$\chi_\rho(g) := \mathrm{Tr}(\rho(g))$$

<u>Properties</u> i) $\chi_{\rho_1 \oplus \rho_2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$

ii) $\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g) \cdot \chi_{\rho_2}(g)$

iii) $\chi_\rho(e) = \dim(\rho)$ ← dimension of the representation space

iv) $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$

v) $\chi_\rho(h^{-1}gh) = \chi_\rho(g)$ ← by cyclicity of trace

χ_S only depends on conjugacy class of g !

$\to \chi_\rho$ is a <u>class function</u>.

Space of class functions:

$$\mathbb{C}^{class}(G) := \{ f : G \to \mathbb{C} \,/\, f(h^{-1}gh) = f(g) \}$$

is a vector space.

$\dim(\mathbb{C}^{class}(G)) = \#$ conjugacy classes in G

On this space define a Hermitian form

$$(\alpha, \beta) := \frac{1}{|G|} \sum_{g\in G} \overline{\alpha(g)}\, \beta(g)$$

$\chi_i := \chi_{\rho_i} \in \mathbb{C}^{class}(G)$ ← irreducible

$(\chi_i, \chi_j) = \delta_{i,j}$ orthonormal set

$\to \#$ irred repres $\leq \#$ conjugacy classes

Want to show: χ_i an ON basis of $\mathbb{C}^{class}(G)$

χ_i are linearly independent

$\rightarrow$ caracterize representation completely.

$\ulcorner$ We have already seen that χ_i are an ON set in $\mathbb{C}^{class}(G)$

If it is a basis: if $(\alpha, \chi_j) = 0 \ \forall j \Rightarrow \alpha = 0$

Assume $(\alpha, \chi_i) = 0 \ \forall i$

$$\varphi_\alpha^{\rho_i} := \sum' \overline{\alpha(g)}\, \rho_i(g) : V_i \rightarrow V_i .$$

This is an intertwiner:

$$\varphi_\alpha^{\rho_i} \rho_i(h) = \sum_g' \overline{\alpha(g)}\, \rho_i(gh) \qquad g \mapsto hgh^{-1}$$

$$= \sum_g \overline{\alpha(hgh^{-1})}\, \rho_i(hg)$$

$$= \rho_i(h) \sum' \underbrace{\overline{\alpha(hgh^{-1})}}_{\overline{\alpha(g)}}\, \rho_i(g) =$$

$$= \rho_i(h)\, \varphi_\alpha^{\rho_i} \longrightarrow \text{intertwiner}$$

V_i irreducible: by Schur's lema: $\varphi_\alpha^{\rho_i} = \lambda_i \, id_{V_i}$

$$\lambda_i = \frac{Tr(\varphi_\alpha^{\rho_i})}{dim(V_i)} = \frac{1}{d_i} \sum_g \overline{\alpha(g)}\, \chi_{\rho_i}(g)$$

$$= \frac{|G|}{d_i} (\alpha, \chi_{\rho_i}) = 0 \Rightarrow \boxed{\varphi_\alpha^{\rho_i} \equiv 0} \quad \forall i$$

$$\Rightarrow \sum_g' \overline{\alpha(g)}\, \rho_i(g) = 0 \quad \forall i$$

$$\Rightarrow \sum_g \overline{\alpha(g)}\, \rho(g) = 0 \quad \forall \text{ repres. } \rho$$

Find a representation ρ such that all $\rho(g_i)$ are linearly independent.

$$\Rightarrow \overline{\alpha(g)} = 0 \quad \forall g$$

A representation for which this holds is the so-called regular represent.

$$V = \mathbb{C}[G] = \{ \sum_i \alpha_i g_i \mid \alpha_i \in \mathbb{C} \}$$

group algebra of G.

$$\rho(g)\left(\sum \alpha_i g_i\right) = \sum \alpha_i (g g_i)$$

$\sum \alpha_i \rho(g_i) g = \sum \alpha_i (g_i g) = 0$ only if $\alpha_i = 0 \; \forall i$

→ $\rho(g_i)$ are linearly independent.

→ χ_i is an orthonormal basis of $\mathbb{C}^{\text{class}}[G]$, $\dim \mathbb{C}^{\text{class}}[G] = \#$ conj. classes of G

→ $\boxed{\text{as many irreducible representations as there are conjugacy classes}}$

→ Characters determine repr.

* Characters table

conjugacy class of the identity

G	C_1	C_2	...	C_k
ρ_1	$\chi_1(C_1) = d_1$	$\chi_2(C_2)$		
⋮				
ρ_k	$\chi_k(C_1) = d_k$	....		$\chi_k(C_k)$

$k = \#$ conj.-classes of G

$\boxed{\chi_{ia} := \chi_i(C_a)}$

$$\delta_{ij} = \frac{1}{|G|} \sum_g \overline{\chi_i(g)}\, \chi_j(g) = \sum_a \overline{\chi_i(C_a)}\, \chi_j(C_a) \frac{|C_a|}{|G|}$$

orthogonality of rows

$$D_{a,b} := \frac{|C_a|}{|G|} \delta_{a,b}$$

$$X D \bar{X}^T = \mathbb{1}_k \Rightarrow (\bar{X}^T) X = D^{-1}$$

$$\Rightarrow \boxed{\sum_i \bar{\chi}_i(C_a) \cdot \chi_i(C_b) = \delta_{a,b} \frac{|G|}{|C_a|}}$$

orthogonality of colums

Constrains character table

Another constraint from regular representation.

$$\rho^{reg}(g)\left(\sum \alpha_i g_i\right) = \sum \alpha_i (g g_i)$$

$$\chi_{irreg}(g) = Tr_{\mathbb{C}[G]}(\rho^{rep}(g)) = \#\{ i \mid g\cdot g_i = g_i \} = \begin{cases} |G|, & g = e \\ 0 & \text{otherwise} \end{cases}$$

basis g_i

$\rho(g) g_i = (g\cdot g_i)$

$$\rho^{reg} = \rho_1^{\oplus n_1} \oplus \dots \oplus \rho_k^{\oplus n_k}$$

$$n_i = (\chi_{\rho_i}, \chi_{reg}) = \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_i}(g^{-1})\, \chi_{\rho_{reg}}(g) =$$

$$= \frac{1}{|G|} \dim(\rho_i)\cdot |G| = \dim(\rho_i)$$

$$\Rightarrow \rho^{reg} = \rho_1^{\oplus d_1} \oplus \dots \oplus \rho_k^{\oplus d_k} \quad , d_i = \dim(\rho_i)$$

$$\dim(\rho^{reg}) = \dim(\mathbb{C}[G]) = |G| = \sum_i d_i^2 \quad \Rightarrow \boxed{|G| = \sum d_i^2}$$

$6 = 1 + 1 + d_2^2 \Rightarrow d_2 = 2$

<u>Example</u> $D_3 \cong S_3$ $\quad |G|$

$|D_3| = 6$

Conjugacy classes

$(\cdot)\ (\cdot)\ (\cdot)$  $C_0 = C(R_0) = \{R_0, e\}$

$(\cdot\ \cdot)(\cdot)$  $C_1 = C(S_0) = \{S_0, S_1, S_2\}$

$(\cdot\ \cdot\ \cdot)$  $C_2 = C(R_1) = \{R_1, R_3\}$

3 irreducible representations

D_3/S_3	C_0 1	C_1 3	C_2 2
ρ^{triv}	1	1	1
ρ^{sign}	1	-1	1
ρ_2 ⊛	2	$\alpha = 0$	$\beta = -1$
⊛⊛ ρ^P	3	1	0

⊛ $|G| = \sum d_i^2$

$6 = 1 + 1 + d_2^2 \rightarrow d_2 = 2$

Obtain α and β using orthogonality:

$0 = (\chi_{triv}, \chi_2) = \frac{1}{6}(2 \cdot 1 + 3 \cdot \alpha + 2 \cdot \beta)$

$0 = (\chi_{sign}, \chi_2) = \frac{1}{6}(2 - 3 \cdot \alpha + 2\beta)$

$\Rightarrow \alpha = 0, \ \beta = -1$

$n_1(\chi_{triv}, \chi_3) = \frac{1}{6}(3 \cdot 1 + 3 \cdot 1) = 1$

$n_2(\chi_{sign}, \chi_3) = \frac{1}{6}(3 \cdot 1 - 3 \cdot 1) = 0$

$n_3(\chi_2, \chi_3) = \frac{1}{6}(3 \cdot 2) = 1$

$\boxed{\rho^P \cong \rho_{triv} \oplus \rho_2}$

Projectors

$\Pi_{triv} = \frac{1}{6} \sum_g \overline{\chi_{triv}(g)} \rho^P(g) =$

$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ projector on $\mathbb{C}(e_1+e_2+e_3)$

$\Pi_2 = \underbrace{\frac{2}{6}}_{\text{dimensionality}}\left(2\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right) =$

$= \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$\rightarrow$ projector on the space span$\{e_1 - e_2, e_2 - e_3\}$

Look at

$\rho(S_0) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $\rho(R_i) = \begin{pmatrix} \cos(\frac{2\pi}{3} i) & -\sin(\frac{2\pi i}{3}) \\ \sin(\frac{2\pi}{3} i) & \cos(\frac{2\pi i}{3}) \end{pmatrix}$

$\chi_\rho(C_0) = 2 \qquad \chi_\rho(C_1) = 0 \qquad \chi(C_2) = 2\cos(\frac{2\pi}{3}) = -1$

$\rho \cong \rho_2$ because the characters agree.

⊛⊛ Permutation representation.

$\rho^P(e) = \mathbb{1}_3 \qquad \chi_P(C_0) = 3$

$\rho^P(R_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \chi_P(C_1) = 1$

$\rho^P(S_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \chi_P(C_2) = 0$

$\rho^P = \rho_{triv}^{\oplus n_1} \oplus \rho_{sign}^{\oplus n_2} \oplus \rho_2^{\oplus n_3}$

27/05/15

Sumary from last lecture:

- complex representation of finite groups are fully decomposable.

$$\rho = \underbrace{\bigoplus_i \rho_i^{\oplus n_i}}_{\text{irreducible repres.}} = \underbrace{\rho_1 \oplus \dots \oplus \rho_1}_{n_1} \oplus \underbrace{\rho_2 \oplus \dots \oplus \rho_2}_{n_2} \oplus \dots$$

- as many irred. reprs. as there are conjugacy classes in G

- repres. determined by <u>characters</u>: $\chi_\rho(g) = \mathrm{Tr}_{V_\rho}(\rho(g))$

  $(\chi_i = \chi_{\rho_i})$, irr. reps. are ON basis of class-fct

- $n_i = (\chi_i, \chi_\rho) = \frac{1}{|G|} \sum_{g\in G} \underbrace{\chi_i(g^{-1})}_{\overline{\chi_i(g)}} \chi_\rho(g)$

- projector $\pi_i = \frac{d_i}{|G|} \sum_{g\in G} \chi_i(g^{-1}) \rho(g) : V \longrightarrow V$ projects onto $V_i^{\oplus n_i} \subset V$

  $d_i \equiv \dim(\rho_i)$

- Characters - table:

G	$C_1 = [e]$, $\lvert C_1\rvert = 1$	C_2, $\lvert C_2\rvert$	$\dots$	C_k, $\lvert C_k\rvert$
ρ_1	$d_1 = \chi_1(C_1)$		$\dots$	χ_1
$\vdots$	$\vdots$			
ρ_k	$d_k = \chi_k(C_1)$		$\dots$	$\chi_k(C_k$

$$\sum_a \frac{|C_a|}{|G|} \bar{\chi}_i(C_a)\chi_i(C_a) = \delta_{i,j}$$

$$\sum_i \frac{|C_a|}{|G|} \bar{\chi}_i(C_a)\chi_j(C_b) = \delta_{ab}$$

$$\sum_i d_i^2 = |G|$$

▸ <u>Example</u>: $D_3 = \{ \underset{\overset{\shortparallel}{e}}{R_0}, R_1, R_2, S_0, S_1, S_2 \}$

$C_1 = C(R_0) = \{R_0\}$

$C_2 = C(S_0) = \{S_0, S_1, S_2\}$

$C_3 = C(R_1) = \{R_1, R_2\}$

D_3	C_1 1	C_2 3	C_3 2
ρ_0	1	1	1
ρ_1	1	-1	1
ρ_2	2	0	-1
ρ_3	3	1	0
	$\mathrm{tr}\,\rho_3(R_0)$	$\mathrm{Tr}\,\rho_3(S_0)$	$\mathrm{Tr}\,\rho_3(R_1)$

ρ_0: trivial repr. $V = \mathbb{C}$, $\rho_0(g) = 1$

$\chi_0(C_1) = 1$

ρ_1: sign-repr. $V = \mathbb{C}$

ρ_1 $\quad R_i \longmapsto 1$
$\quad\quad S_i \longmapsto -1$

$\chi_1(C_1) = 1$
$\chi_1(C_3) = 1$
$\chi_1(C_2) = -1$

ρ_2: 2d-rep.

$\rho_2(R_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \rho_2(R_1) = \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$

$\rho_2(S_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \rho(R_2) = \dots$

$\rho_2(S_1) = \dots$

<u>Permutation representation</u>

$V = \mathrm{span}\{e_1, e_2, e_3\}$

R_0: $e_1 \longmapsto e_1$, $e_2 \longmapsto e_2$, $e_3 \longmapsto e_3$

R_1: $e_1 \longmapsto e_2$, $e_2 \longmapsto e_3$, $e_3 \longmapsto e_1$

R_2: $e_1 \longmapsto e_3$, $e_2 \longmapsto e_1$, $e_3 \longmapsto e_2$

S_0: $e_1 \longmapsto e_1$, $e_2 \longmapsto e_3$, $e_3 \longmapsto e_2$

...

$\rho_3(R_0) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\quad \rho_3(S_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$\rho_3(R_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad \rho_3(S_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$\rho_3(R_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad \rho_3(S_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\rho_3 = \rho_0^{\oplus n_0} \oplus \rho_1^{\oplus n_1} \oplus \rho_2^{\oplus n_2}$

$n_0 = (\chi_0, \chi_3) = \frac{1}{6}(3 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 3) = 1$

$$\begin{array}{r} 8 \\ \times 4 \\ \hline 32 \end{array} - 10 = 22$$

$15 \to 7$

$n_1 = (\chi_1, \chi_3) = \frac{1}{6}(3 - 3) = 0$

$n_2 = (\chi_2, \chi_3) = \frac{1}{6} \cdot 6 = 1$

$$\boxed{\rho_3 \cong \rho_1 \oplus \rho_2}$$

Projector:

$$\Pi_0 = \frac{1}{6} \sum_{g \in G} \bar{\chi}_0(g)\, \rho_3(g) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \text{Projection on span}\{e_1 + e_2 + e_3\}$$

$V_0 \subset V_3 = \text{span}(e_1, e_2, e_3)$

$\overset{''}{\mathbb{C}}(e_1 + e_2 + e_3)$

$$\Pi_2 = \frac{2}{6} \sum_{g \in G} \bar{\chi}_2(g)\, \rho_3(g) =$$

$$= \frac{1}{3}\left(2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right) = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \rightsquigarrow \text{Projection on } V_2 = \text{span}\{e_1 - e_2,\ e_2 - e_3\}$$

$$\rho_4 = \rho_2 \otimes \rho_2 = \rho_0^{\oplus n_0} \oplus \rho_1^{\oplus n_1} \oplus \rho_2^{\oplus n_2}$$

$$\chi_4(g) = (\chi_2(g))^2$$

$$\left.\begin{aligned} n_0 &= (\chi_0, \chi_4) = \frac{1}{6}(4+2) = 1 \\ n_1 &= (\chi_1, \chi_4) = \frac{1}{6}(4+2) = 1 \\ n_2 &= (\chi_2, \chi_4) = \frac{1}{6}(8-2) = 1 \end{aligned}\right\} \Rightarrow \rho_4 \cong \rho_0 \oplus \rho_1 \oplus \rho_2$$