Lecture notes:
Matrix Algebra

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Definition of a matrix

An $m \times n$ matrix is a rectangular array of numbers (or other mathematical objects) with m rows and n columns.
For example, a 2×2 matrix A, with two rows and two columns, looks like

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A useful notation for writing a general $m \times n$ matrix A is:

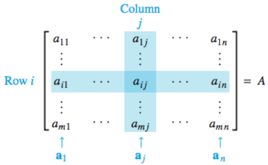

FIGURE 1 Matrix notation.

The matrix element of A in the ith row and the jth column is denoted as a_{ij}.
Each column of A is a list of m real numbers, which identifies a vector in $\mathbf{R}^m$.
Often, these columns are denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ and the matrix A is written as $A = (\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n)$.

Column matrices, row matrices

Column matrices and row matrices are also called vectors.
The *column vector* is in general $n \times 1$.
The *row vector* is $1 \times n$.
For example, when $n = 3$:
A column vector:

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

A row vector:

$$y = \begin{pmatrix} a & b & c \end{pmatrix}.$$

Practice Problems

1. The diagonal of a matrix A are the entries a_{ij} where $i = j$.

a) Write down the three-by-three matrix with ones on the diagonal and zeros elsewhere.

b) Write down the three-by-four matrix with ones on the diagonal and zeros elsewhere.

c) Write down the four-by-three matrix with ones on the diagonal and zeros elsewhere.

Addition and multiplication of matrices

Matrices can be added only if they are the same size. *Addition* proceeds element by element.

For example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

Matrices can also be *multiplied by a scalar*. The rule is to just multiply every element of the matrix.

For example:

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}.$$

Matrix multiplication: Matrices (other than the scalar) can be multiplied only if the number of columns of the left matrix equals the number of rows of the right matrix. In other words, an $m \times n$ matrix on the left can only be multiplied by an $n \times k$ matrix on the right. The resulting matrix will be $m \times k$.

For example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix}.$$

First, the first row of the left matrix is multiplied against and summed with the first column of the right matrix to obtain the element in the first row and first column of the product matrix. Second, the first row is multiplied against and summed with the second column. Third, the second row is multiplied against and summed with the first column. And fourth, the second row is multiplied against and summed with the second column.

In general: Let A be an $m \times n$ matrix with matrix elements a_{ij} and B be an $n \times p$ matrix with matrix elements b_{ij}.

Then $C = AB$ is an $m \times p$ matrix, and its ij matrix element can be written as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Notice that if B is an $n \times p$ matrix with columns $\mathbf{b}_1$, ..., $\mathbf{b}_p$, then the product

$$AB = A \left[\mathbf{b}_1 \ \dots \ \mathbf{b}_p \right] = \left[A\mathbf{b}_1 \ \dots \ A\mathbf{b}_p \right]$$

.

Theorem

Let A, B, and C be matrices of the same size, and let r and s be scalars.

- $A + B = B + A$;
- $(A + B) + C = A + (B + C)$;
- $A + 0 = A$;
- $r(A + B) = rA + rB$;
- $(r + s)A = rA + sA$;
- $r(sA) = (rs)A$.

The following theorem lists the standard properties of matrix multiplication.

Theorem

Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined.

- $(AB)C = A(BC)$ *(associative law of multiplication);*
- $A(B + C) = AB + AC$ *(left distributive law);*
- $(B + C)A = BA + CA$ *(right distributive law);*
- $r(AB) = (rA)B = A(rB)$ *for any scalar r;*
- $I_m A = A = AI_n$ *(identityformatrixmultiplication).*

Note that if $AB = BA$, we say that A and B commute with one another.

Warnings:

- In general, $AB \neq BA$;
- The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$;
- If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$.

1. Define the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 1 \\ 2 & -4 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$
$$D = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Compute if defined: $B - 2A$, $3C - E$, AC, CD, CB.

2. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Compute AD and DA.

4. Prove the associative law for matrix multiplication. That is, let A be an m-by-n matrix, B an n-by-p matrix, and C a p-by-q matrix. Then prove that $A(BC) = (AB)C$.

Special matrices

The zero matrix: denoted by 0, can be any size and is a matrix consisting of all zero elements. Multiplication by a zero matrix results in a zero matrix.

For example:

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The identity matrix denoted by I, is a square matrix (number of rows equals number of columns) with ones down the main diagonal. If A and I are the same sized square matrices, then

$$AI = IA = A.$$

Multiplication by the identity matrix leaves the matrix unchanged.

For example:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The zero and identity matrices play the role of the *numbers zero* and *one* in matrix multiplication.

A diagonal matrix has its only nonzero elements on the diagonal.
For example, a 2×2 diagonal matrix is given by

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Diagonal matrices refer to square matrices, but they can also be rectangular.
An upper or lower triangular matrix is a square matrix that has zero elements below or above the diagonal.
For example, a 3×3 upper and lower triangular matrices:

$$U = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \quad L = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}.$$

Practice Problems

1. Let

$$A = \begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix}.$$

Construct a two-by-two matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

2. Verify that

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{pmatrix}.$$

Prove in general that the product of two diagonal matrices is a diagonal matrix, with elements given by the product of the diagonal elements.

3. Verify that

$$\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 + a_2 b_3 \\ 0 & a_3 b_3 \end{pmatrix}.$$

Prove in general that the product of two upper triangular matrices is an upper triangular matrix, with the diagonal elements of the product given by the product of the diagonal elements.

The transpose of a $m \times n$ matrix A denoted by $n \times m$ matrix A^T and spoken as A-transpose, switches the rows and columns of A.

$$\text{if } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \text{then } A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

In other word: $a^T_{ij} = a_{ji}$.

For example,

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}^T = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

Transpose matrix

If A is a square matrix with $A^T = A$, then A is *symmetric*.
If $A^T = -A$ then A is *skew symmetric*.
For example,

$$\text{symmetric: } \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \quad \text{skew symmetric: } \begin{pmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{pmatrix}.$$

The following are useful and easy to prove facts:

Theorem

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

- $(A^T)^T = A$;
- $(A + B)^T = A^T + B^T$;
- *for any scalar* k, $(kA)^A = kA^T$;
- $(AB)^T = B^T.A^T$

Practice Problems

1. Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?

2. Let A be a 4×4 matrix and let $\mathbf{x}$ be a vector in $\mathbb{R}^4$. What is the fastest way to compute $A^2\mathbf{x}$? Count the multiplications.

3. Suppose A is an $m \times n$ matrix, all of whose rows are identical. Suppose B is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in AB?

Inverse matrix

The inverse matrix of a $n \times n$ matrix A denoted by $n \times n$ matrix A^{-1}. The inverse matrix satisfies:

$$AA^{-1} = A^{-1}A = I,$$

where $I = I_n$, the $n \times n$ identity matrix.
Note: Square matrices may have inverses. When a matrix A has an *inverse*, we say it is *invertible*.

- If A is an invertible matrix, then A^{-1} is invertible and: $(A^{-1})^{-1} = A$;
- If A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$;
- If A is invertible then so is A^T, and $(AT)^{-1} = (A^{-1})T$.

For example:

Solve the problem

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where x_1, x_2,y_1, y_2 are unknown.

The corresponding linear system of equations:

$$ax_1 + by_1 = 1, \qquad cx_1 + dy_1 = 0,$$

$$ax_2 + by_2 = 1, \qquad cx_2 + dy_2 = 0.$$

The solution for the inverse matrix is found to be:

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The term $ad - bc$ is just the definition of the determinant of the two-by-two matrix:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Theorem

Let matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $\det A = ad - bc \neq 0$, *then* A *is invertible and:*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If $\det A = ad - bc = 0$, *then* A *is not invertible.*

Theorem says that 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Notice that the inverse of a two-by-two matrix, in words, is found by switching the diagonal elements of the matrix, negating the off-diagonal elements, and dividing by the determinant.

Theorem

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ *in R^n, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution*

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Theorem

If A is an invertible $n \times n$ matrix, then for each **b** *in R^n, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution*

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

For example:

Use the inverse of the matrix to solve the system:

$$3x_1 + 4x_2 = 3; \qquad 5x_1 + 6x_2 = 7.$$

Theorem

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For example:

Use the inverse of the matrix to solve the system:

$$3x_1 + 4x_2 = 3; \qquad 5x_1 + 6x_2 = 7.$$

This system is equivalent to $A\mathbf{x} = \mathbf{b}$, so:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -3 & 2 \\ 5/2 & -3/2 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

Practice Problems

1. Use determinants to determine which of the following matrices are invertible.

a. $\begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix}$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

3. If A is an invertible matrix, prove that $5A$ is an invertible matrix.

THANK YOU FOR YOUR ATTENTION!