Matrix Algebra:
Determinants

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Two-by-two determinant

Let matrix A be 2×2 matrix. Then

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

is called the determinant of the 2×2 matrix A.
A is invertible if and only if its determinant is nonzero.
If A is invertible, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.
If A is not invertible, then $Ax = b$ may have no solution or an infinite number of solutions.
When $\det A = 0$, we say that the matrix A is *singular*.

Three-by-three determinant

It is also straightforward to define the determinant for a 3×3 matrix.
Let matrix A be 3×3 matrix. Consider the system of equations $Ax = 0$ and determine the condition for which $x = 0$ is the only solution:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

We can do the messy algebra of elimination to solve for x_1, x_2, and x_3, then find that $x_1 = x_2 = x_3 = 0$ is the only solution when $\det A = 0$, where the definition, apart from a constant, is given by

$$\det A = aei + bfg + cdh - ceg - bdi - afh.$$

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$$\begin{pmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{pmatrix} - \begin{pmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{pmatrix}.$$

Write out the first two columns of the matrix to the right of the third column, giving five columns in a row. Then add the products of the diagonals going from top to bottom and subtract the products of the diagonals going from bottom to top.

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Answer: Unfortunately, this mnemonic only works for three-by-three matrices.

Problems

1. Find the determinant of the three-by-three identity matrix.
2. Show that the three-by-three determinant changes sign when the first two rows are interchanged.
3. Let A and B be two-by-two matrices. Prove by direct computation that $\det AB = \det A \det B$.

Laplace expansion

There is a way to write the three-by-three determinant that generalizes. It is called a Laplace expansion (also called a cofactor expansion or expansion by minors):

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a \det A_{11} - b \det A_{12} + c \det A_{13},$$

where A_{11}, A_{12} and A_{13} are obtained from A by deleting the first row and one of the three columns.

Evidently, the 3×3 determinant can be computed from lower-order 2×2 determinants, called minors. The rule here for a general $n \times n$ matrix is that one goes across the first row of the matrix, multiplying each element in the row by the determinant of the matrix obtained by crossing out that element's row and column, and adding the results with alternating signs.

In fact, this expansion in minors can be done across any row or down any column. When the minor A_{ij} is obtained by deleting the ith row and jth column, then the sign of the term is given by $(-1)^{i+j}$.
An easy way to remember the signs is to form a checkerboard pattern, exhibited here for the 3×3 and 4×4 matrices:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}, \quad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}.$$

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For example: Compute the determinant of matrix

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

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SOLUTION Compute $\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$:

$$\det A = 1 \cdot \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0-2) - 5(0-0) + 0(-4-0) = -2$$

Definition of a determinant

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For $n \geqslant 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{ij} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n}.$$

Cofactor expansion

Given $A = [a_{ij}]$, the (i,j)-cofactor of A is the number C_{ij} given by

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We omit the proof of the following fundamental theorem:

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Theorem

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

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cofactor expansion across the first row of A.
Note: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

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Let A be a square matrix.

(a) *If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.*

(b) *If two rows of A are interchanged to produce B, then $\det B = -\det A$.*

(b) *If one row of A is multiplied by k to produce B, then $\det B = k \det A$*

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Theorem

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Example:

Compute $\det A$, where

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}.$$

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SOLUTION The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

An interchange of rows 2 and 3 reverses the sign of the determinant, so

$$\det A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

A common use of Theorem 3(c) in hand calculations is to *factor out a common multiple of one row* of a matrix. For instance,

$$\begin{vmatrix} * & * & * \\ 5k & -2k & 3k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ 5 & -2 & 3 \\ * & * & * \end{vmatrix}$$

where the starred entries are unchanged. We use this step in the next example.

The main theorem

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\det U \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det U = 0$

FIGURE 1
Typical echelon forms of square matrices.

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges. (This is always possible. See the row reduction algorithm in Section 1.2.) If there are r interchanges, then Theorem 3 shows that

$$\det A = (-1)^r \det U$$

Since U is in echelon form, it is triangular, and so $\det U$ is the product of the diagonal entries $u_{11}, \ldots, u_{nn}$. If A is invertible, the entries u_{ii} are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1's). Otherwise, at least u_{nn} is zero, and the product $u_{11} \cdots u_{nn}$ is zero. See Figure 1. Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases} \qquad (1)$$

It is interesting to note that although the echelon form U described above is not unique (because it is not completely row reduced), and the pivots are not unique, the *product* of the pivots *is* unique, except for a possible minus sign.

Formula (1) not only gives a concrete interpretation of $\det A$ but also proves the main theorem of this section:

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

This theorem adds the statement $\det A \neq 0$ to the Invertible Matrix Theorem.

Determinants and Matrix Products

Theorem (Multiplicative property)

If A and B are $n \times n$ matrices, then $\det(AB) = \det A \cdot \det B$.

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SOLUTION

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

and

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

Since $\det A = 9$ and $\det B = 5$,

$$(\det A)(\det B) = 9 \cdot 5 = 45 = \det AB$$

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For example:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

This linearity property of the determinant turns out to have many useful consequences that are studied in more advanced courses.

Useful properties

- The determinant of the identity matrix is one;
- The determinant changes sign under row interchange;
- The determinant is a linear function of any row, holding all other rows fixed;
- If a matrix has two equal rows, then the determinant is zero;
- If we add k times row-i to row-j, the determinant doesn't change;
- The determinant of a matrix with a row of zeros is zero;
- A matrix with a zero determinant is not invertible;
- The determinant of a diagonal matrix is the product of the diagonal elements;
- The determinant of an upper or lower triangular matrix is the product of the diagonal elements;
- The determinant of the product of two matrices is equal to the product of the determinants;
- The determinant of the inverse matrix is equal to the reciprical of the determinant;
- The determinant of the transpose of a matrix is equal to the determinant of the matrix.

Cramer's Rule

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For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

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Theorem (Cramer's Rule)

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}.$$

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Use Cramer's rule to solve the system

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SOLUTION View the system as $A\mathbf{x} = \mathbf{b}$. Using the notation introduced above,

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

Since $\det A = 2$, the system has a unique solution. By Cramer's rule,

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{24 + 16}{2} = 20$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{24 + 30}{2} = 27 \qquad \blacksquare$$

Formula for the inverse matrix

Cramer's rule leads easily to a general formula for the inverse of an $n \times n$ matrix A.
The jth column of A^{-1} is $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{e}_j$, where $\mathbf{e}_j$ is the jth column of the identity matrix, and the ith entry of $\mathbf{x}$ is the (i, j)-entry of A^{-1}.
Let A_{ji} be the submatrix of A formed by deleting row j and column i. Then

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$$\{(i,j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A},$$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

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Theorem (An inverse formula)

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

THANK YOU FOR YOUR ATTENTION!