Computer Graphics and Animation

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Putting your Mobile away and paying attention to those talking to you? There's **App** for that, it's called **RESPECT**!

Computer Graphics and Animation

Unit 2 : TWO DIMENSIONAL AND THREE DIMENSIONAL TRANSFORMATIONS

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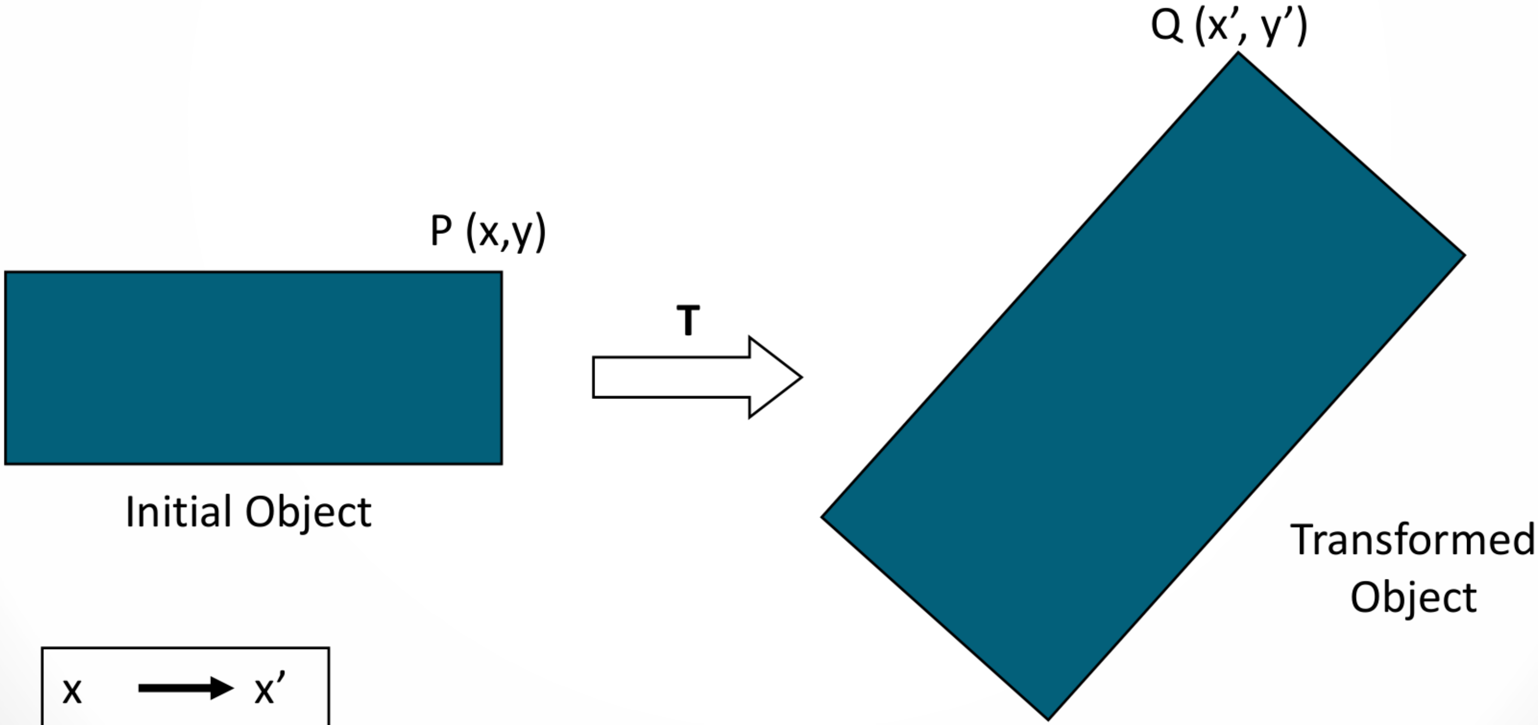
2D Transformation

Introduction to 2D Transformation

Geometric Transformations

- The orientation, size, and shape of the output primitives are accomplished with geometric transformations that alter the coordinate descriptions of objects.



$x \rightarrow x'$

$y \rightarrow y'$

The point Q is the image of P under the transformation T.

Introduction to 2D Transformation

Geometric Transformations

- The basic geometric transformations are **translation, rotation, and scaling**.
- Other transformations that are often applied to objects include **reflection and shear**.
- In these all cases we consider the reference point is origin.
- If we have to do these transformations about any point then we have to shift these point to the origin first.
- After this, perform required operation and then again shift to that position.

Introduction to 2D Transformation

Types of Transformations

1. Translation
2. Rotation
3. Reflection

Rigid Body Transformation

(Transformation without deformation in shape.)

4. Scaling
5. Shearing

Non Rigid Body Transformation

(Transformation with change in shape.)

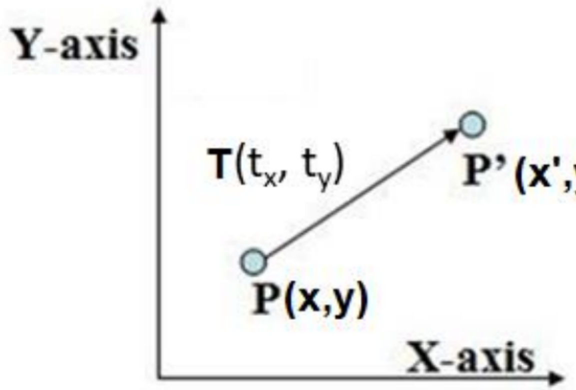
2D Translation

- A translation is applied to an object by repositioning it along a straight-line path from one coordinate location to another.
- We translate a two-dimensional point by adding translation distances, tx, and ty, to the original coordinate position (x, y) to move the point to a new position (x ' , y').

x’ = x + tx

y’ = y + ty

The vector (t_x, t_y) is called the *offset vector* or *translation vector* or *shift vector*.



Translating a point from position P to a position P’ using translation vector T

2D Translation

We can write equation as a single matrix equation by using column vectors to represent coordinate points and translation vectors. Thus,

$$P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$P' = P + T$$

So we can write

In homogeneous representation if position P = (x, y) is translated to new position P'= (x', y') then:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = T(t_x, t_y) \cdot P$$

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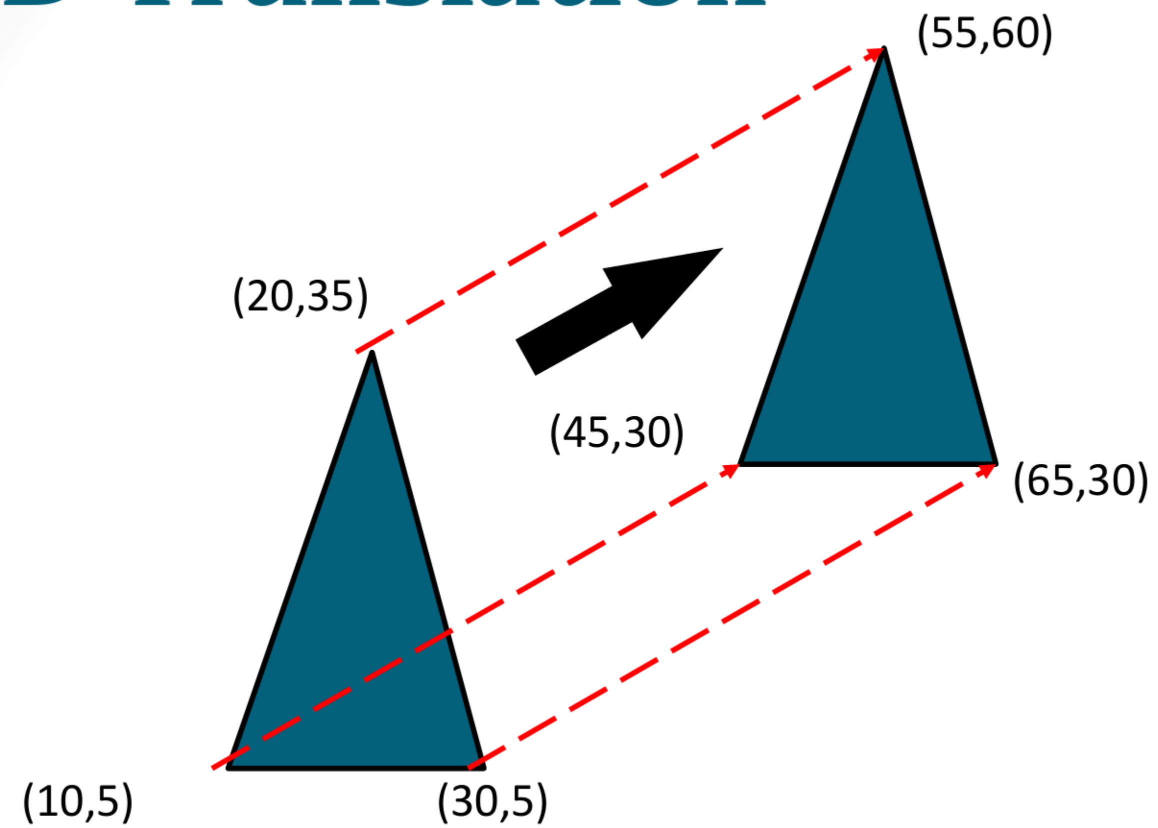
2D Translation



Specifying a 2D-Translation:

```
glTranslatef(tx, ty, 0.0);
```

(The z component is set to 0 for 2D translation).

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2D Translation

- **Given:**

$$P = (x, y)$$
$$T = (t_x, t_y)$$

$$P = (-3.7, -4.1)$$
$$T = (7.1, 8.2)$$

- **We want:**

$$x' = x + t_x$$
$$y' = y + t_y$$

$$x' = -3.7 + 7.1$$
$$y' = -4.1 + 8.2$$

- **Matrix form:**

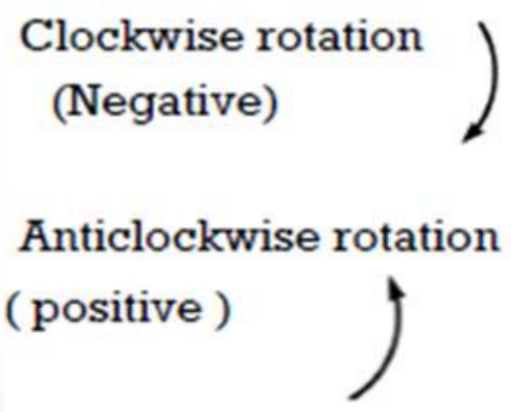
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$
$$P' = P + T$$

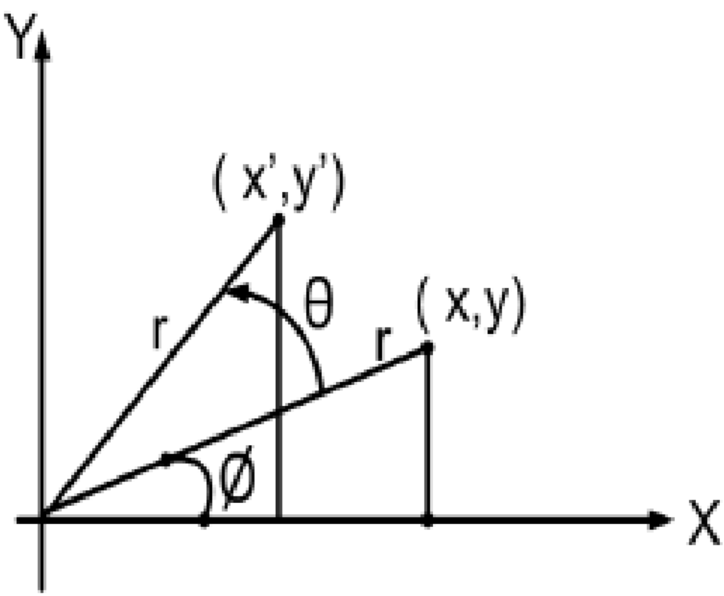
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3.7 \\ -4.1 \end{bmatrix} + \begin{bmatrix} 7.1 \\ 8.2 \end{bmatrix}$$
$$x' = 3.4$$
$$y' = 4.1$$

2D Rotation

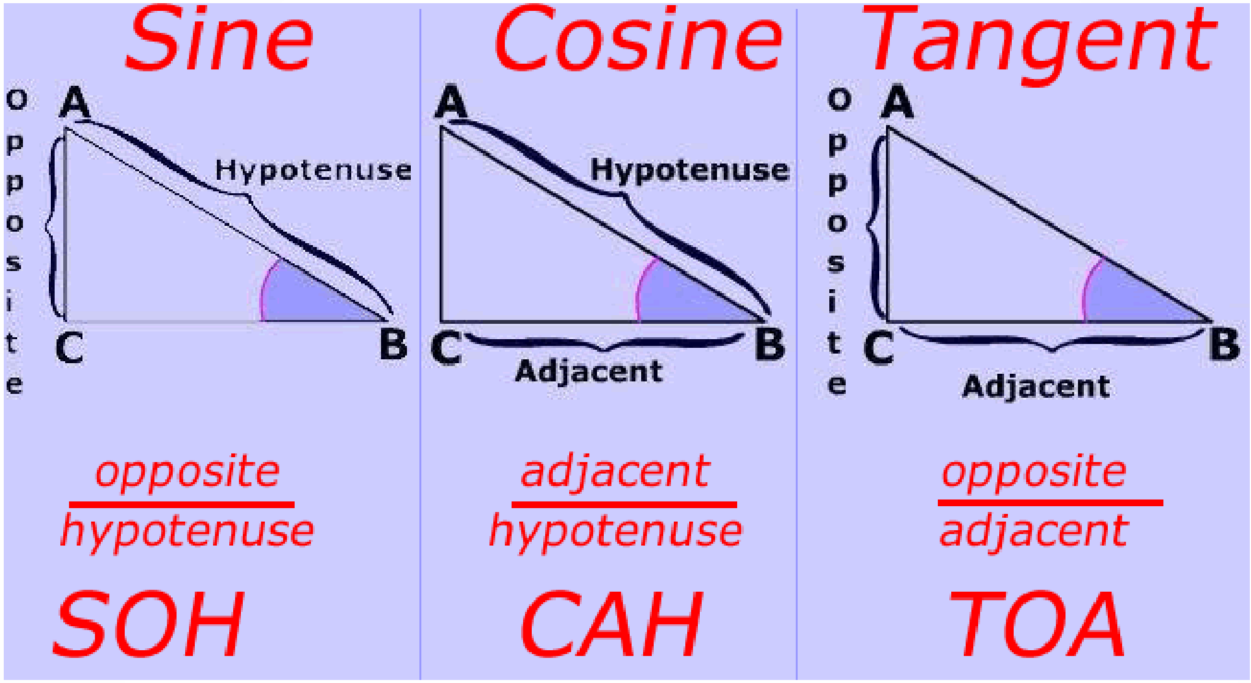
- A two-dimensional rotation is applied to an object by repositioning it along a circular path in the **xy** plane.
- To generate a rotation, we specify a rotation angle θ and the position (x_r , y_r) of the rotation point (or pivot point) about which the object is to be rotated.

+ Value for ‘θ’ define *counter-clockwise* rotation about a point
- Value for ‘θ’ defines *clockwise* rotation about a point





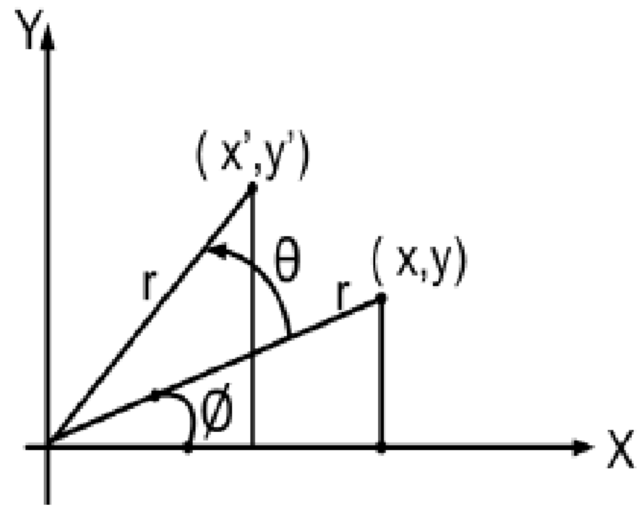
2D Rotation



$$\sin(\alpha+\beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta$$

2D Rotation

Anticlockwise Direction



At Origin

Coordinates of point (x,y) in polar form

$$x = r\cos\phi, \qquad y = r\sin\phi$$

$$x' = r\cos(\phi+\theta) = r\cos\phi.\cos\theta - r\sin\phi.\sin\theta$$
$$y' = r\sin(\phi+\theta) = r\cos\phi.\sin\theta + r\sin\phi.\cos\theta$$
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$P' = R\,.\,P$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & \text{Cos}\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

ROTATION MATRX

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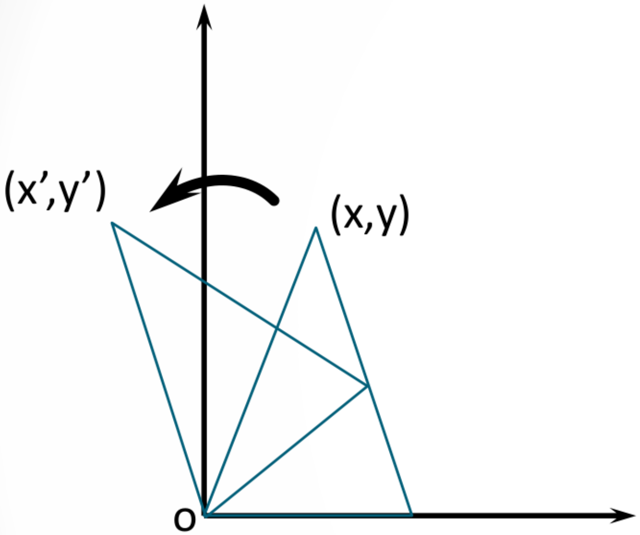
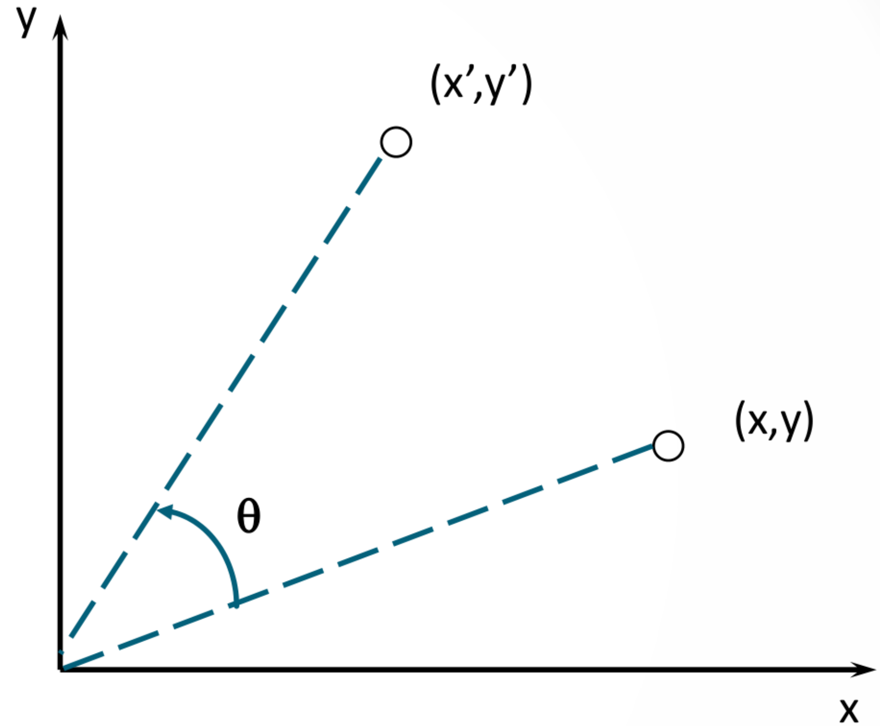
2D Rotation

Rotation About the Origin





$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

The above 2D rotation is actually a rotation about the z-axis (0,0,1) by an angle θ.

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2D Rotation

Rotation About the Origin

Specifying a 2D-Rotation about the origin:

```
glRotatef(theta, 0.0, 0.0, 1.0);
```

`theta`: Angle of rotation in degrees.
The above function defines a rotation about the z-axis (0,0,1).

2D Rotation

In homogeneous co-ordinate

Anticlockwise Direction

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = R(\theta).P$$

Clockwise Direction

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = R(\theta).P$$

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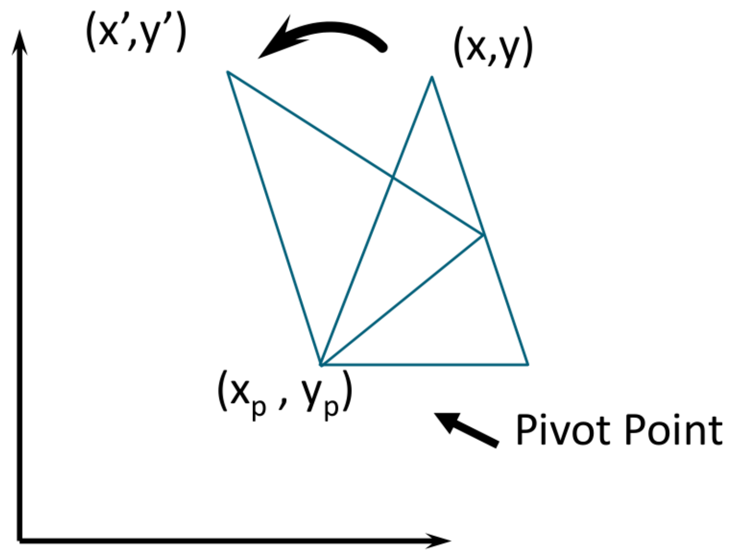
2D Rotation

Rotation About a Pivot Point

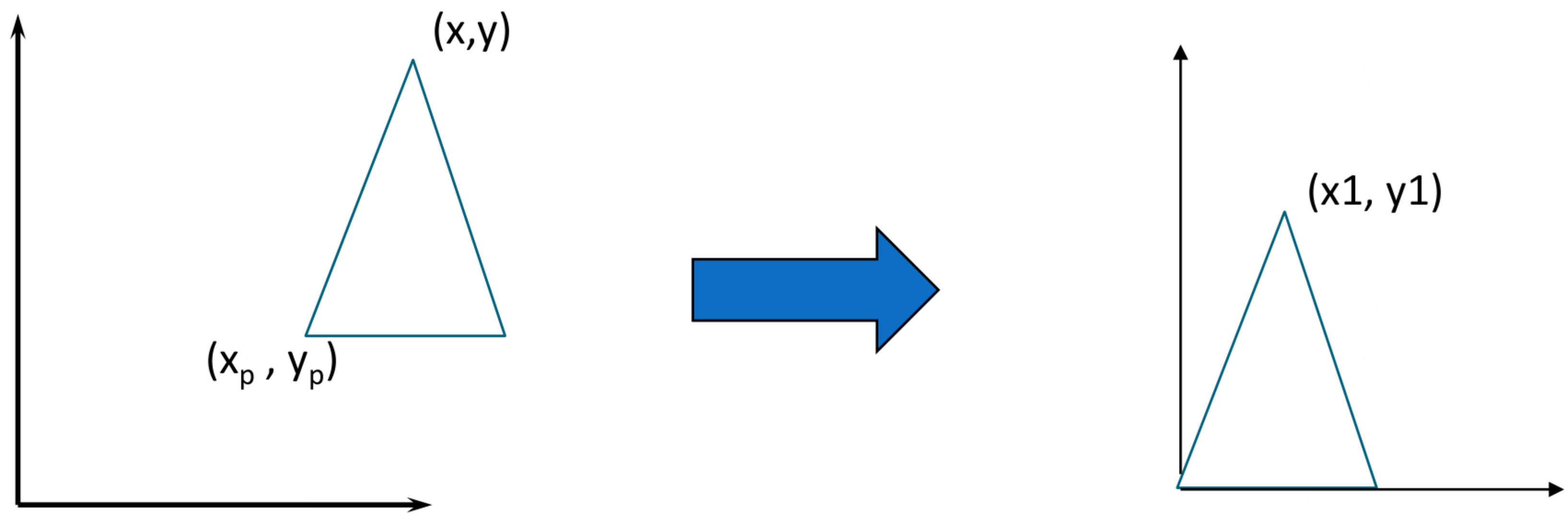


- ❑ Pivot point is the point of rotation
- ❑ Pivot point need not necessarily be on the object

2D Rotation

Rotation About a Pivot Point

STEP-1: Translate the pivot point to the origin



$$x1 = x - x_p$$

$$y1 = y - y_p$$

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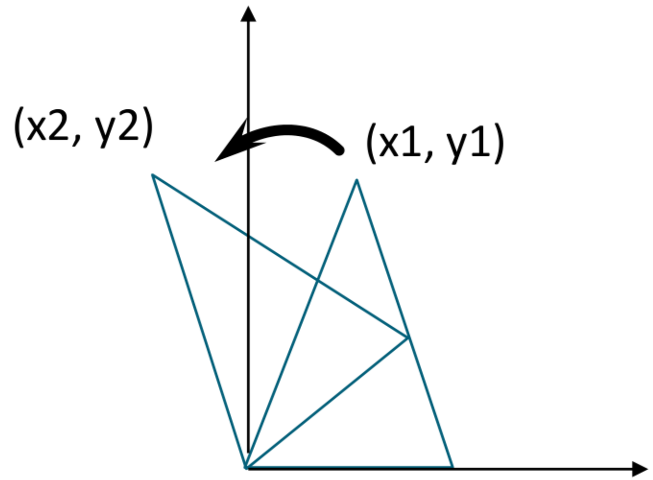
2D Rotation

Rotation About a Pivot Point

STEP-2: Rotate about the origin



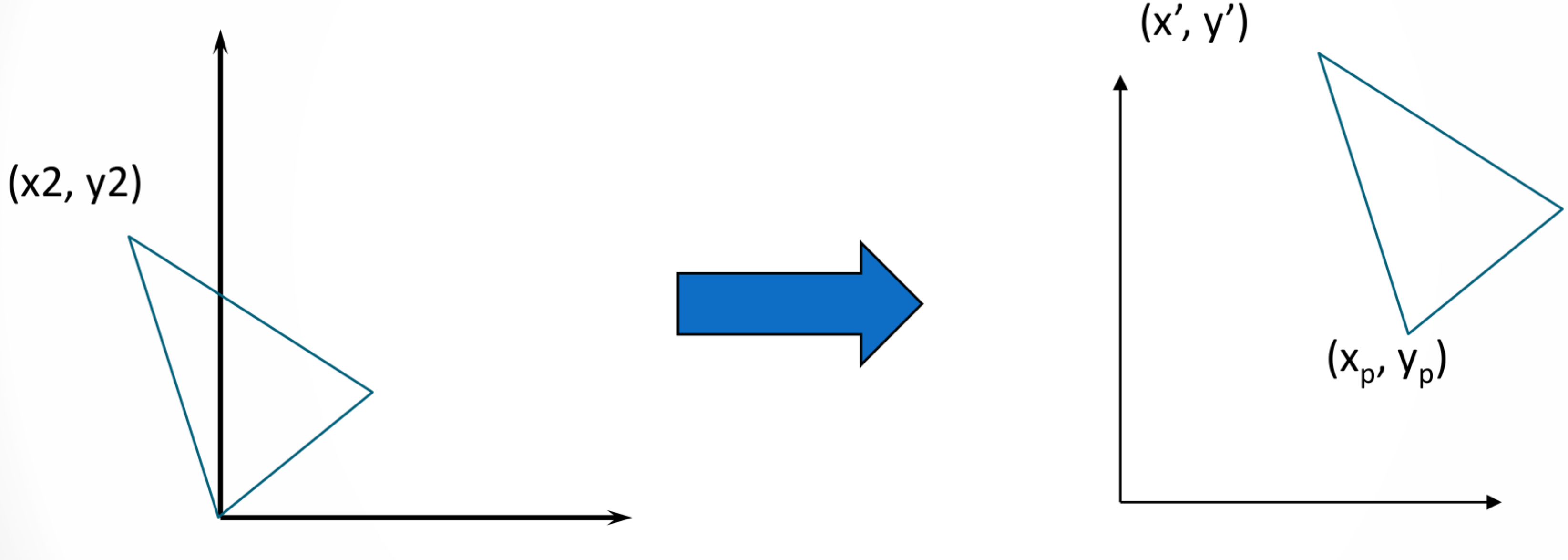
$$x2 = x1\cos\theta - y1\sin\theta$$

$$y2 = x1\sin\theta + y1\cos\theta$$

2D Rotation

Rotation About a Pivot Point

STEP-3: Translate the pivot point to original position



$$x' = x2 + x_p$$

$$y' = y2 + y_p$$

2D Rotation

Rotation About a Pivot Point

$$x' = (x - x_p)\cos\theta - (y - y_p)\sin\theta + x_p$$

$$y' = (x - x_p)\sin\theta + (y - y_p)\cos\theta + y_p$$

Specifying a 2D-Rotation about a pivot point (xp,yp):

```
glTranslatef(xp, yp, 0);
glRotatef(theta, 0, 0, 1.0);
glTranslatef(-xp, -yp, 0);
```

Note the OpenGL specification of the sequence of transformations in the reverse order !

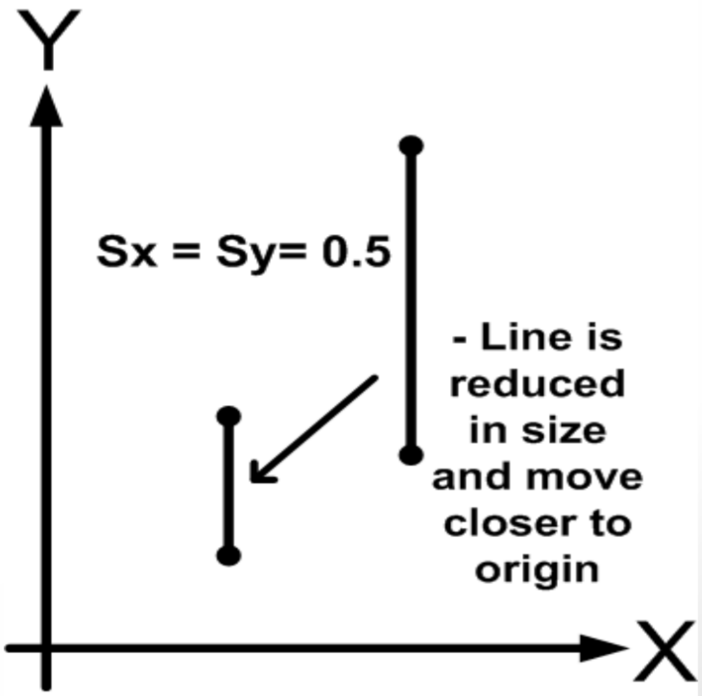
2D Scaling

- A scaling transformation alters the size of an object.
- This operation can be carried out for polygons by multiplying the coordinate values (x, y) of each vertex by scaling factors $\mathbf{S_x}$ and $\mathbf{S_y}$ to produce the transformed coordinates (x', y').

s_x scales object in ‘x’ direction

s_y scales object in ‘y’ direction

The parameters S_x, S_y are called *scale factors*.



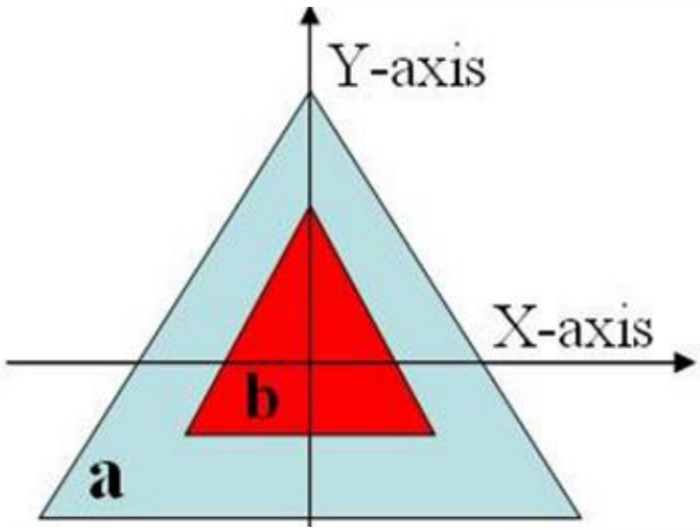
2D Scaling

Thus, for equation form,

$$x' = x.\ s_x$$
$$y' = y.\ s_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$(s_x, s_y > 0)$$



Turning triangle "a" into triangle "b" with scaling factors S_x = -2 and S_y = -2

- Values greater than 1 for s_x , s_y produce ***enlargement***
- Values less than 1 for s_x , s_y ***reduce*** size of object
- $s_x = s_y = 1$ leaves the size of the object ***unchanged***
- When s_x , s_y are assigned the same value $s_x = s_y = 3$ or 4 etc. then a ***Uniform Scaling*** is produced

2D Scaling

$P' = S.P$

In homogeneous co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = S(s_x, s_y).P$$

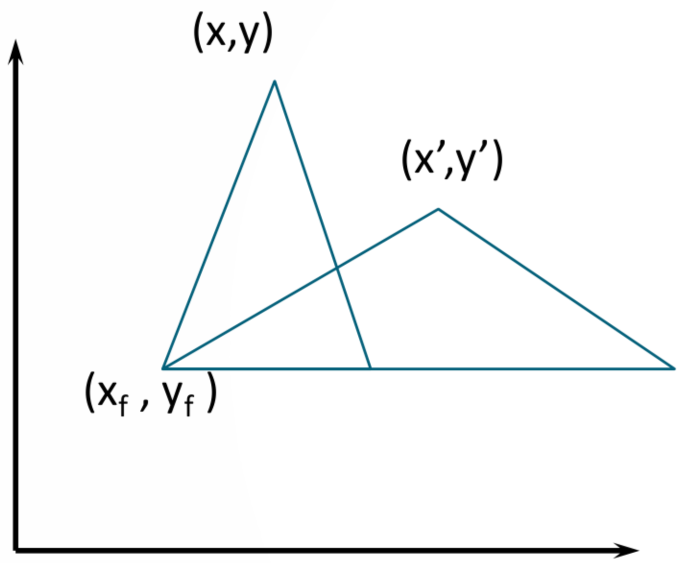
Specifying a 2D-Scaling with respect to the origin:

```
glScalef(sx, sy, 1.0);
```

`sx, sy`: Scale factors along x, y.
For proper scaling `sx, sy` must be positive.
For 2D scaling, the third scale factor must be set to 1.0.

2D Scaling

Scaling About a Fixed Point



- ❑ Translate the fixed point to origin
- ❑ Scale with respect to the origin
- ❑ Translate the fixed point to its original position.

$$x' = (x - x_f).s_x + x_f$$

$$y' = (y - y_f).s_y + y_f$$

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2D Scaling

- Given:

$$P = (x, y)$$
$$S = (s_x, s_y)$$

$$P = (1.4, 2.2)$$
$$S = (3, 3)$$

- We want:

$$x' = s_x x$$
$$y' = s_y y$$

$$x' = 3 * 1.4$$
$$y' = 3 * 2.2$$

- Matrix form:

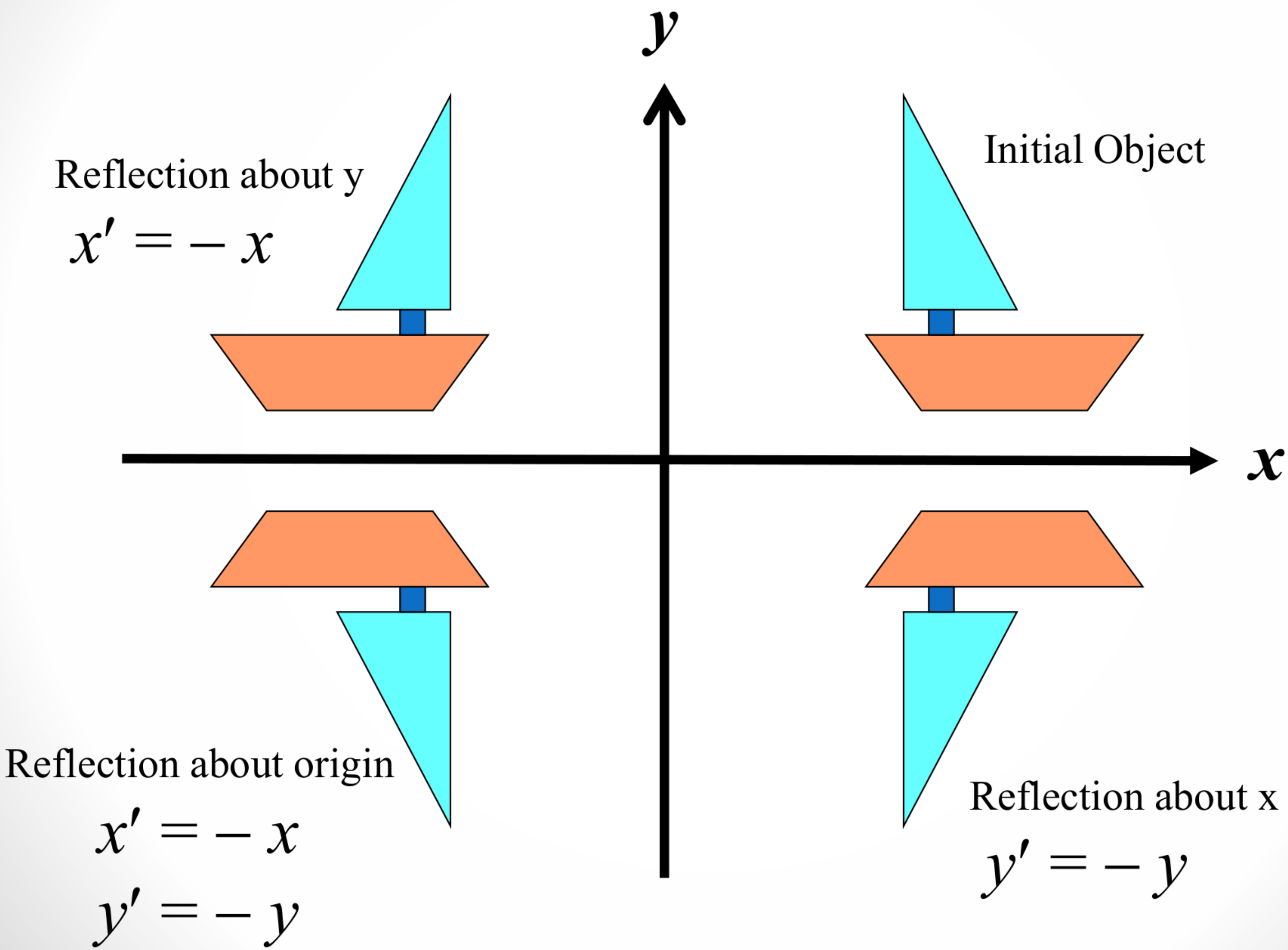
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$P' = S \cdot P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1.4 \\ 2.2 \end{bmatrix}$$
$$x' = 4.2$$
$$y' = 6.6$$

2D Reflection

- A reflection is a transformation that produces a mirror image of an object.
- The mirror image for a 2D reflection is generated relative to an axis of reflection by rotating the object 180" about the reflection axis.
- We can choose an axis of reflection in the **xy**-plane or perpendicular to the xy plane.
- When the reflection axis is a line in the xy plane, the rotation path about this axis is in a plane perpendicular to the **xy**-plane.
- For reflection axes that are perpendicular to the **xy**-plane, the rotation path is in the xy plane.

2D Reflection



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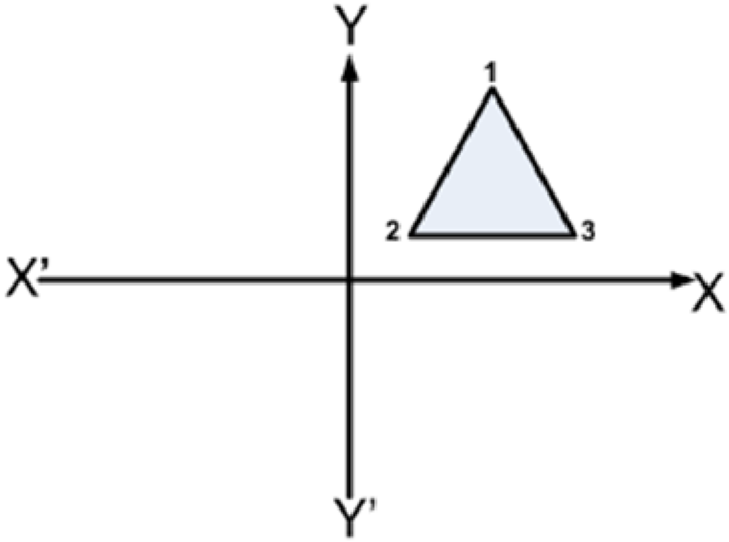
2D Reflection

(i) Reflection about x axis or about line y = 0

Keeps **X** value same but flips **Y** value of coordinate points



$x' = x$
$y' = -y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Homogeneous Co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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2D Reflection

(ii) Reflection about y axis or about line x = 0

Keeps 'y' value same but flips x value of coordinate points

$x' = -x$
$y' = y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Homogeneous Co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

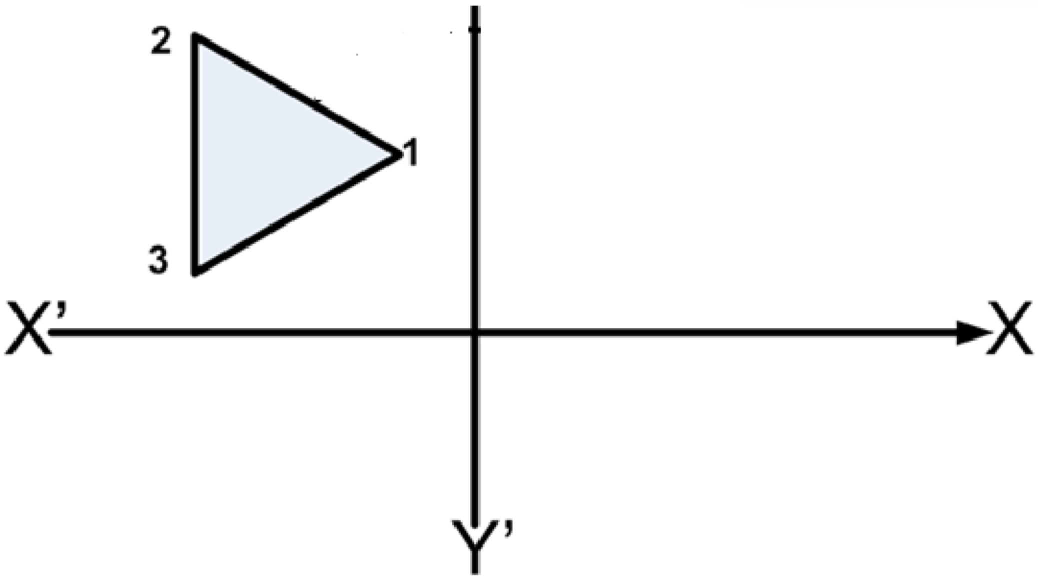


Fig: Reflection of object about y-axis (x=0)

2D Reflection

(iii) Reflection about origin

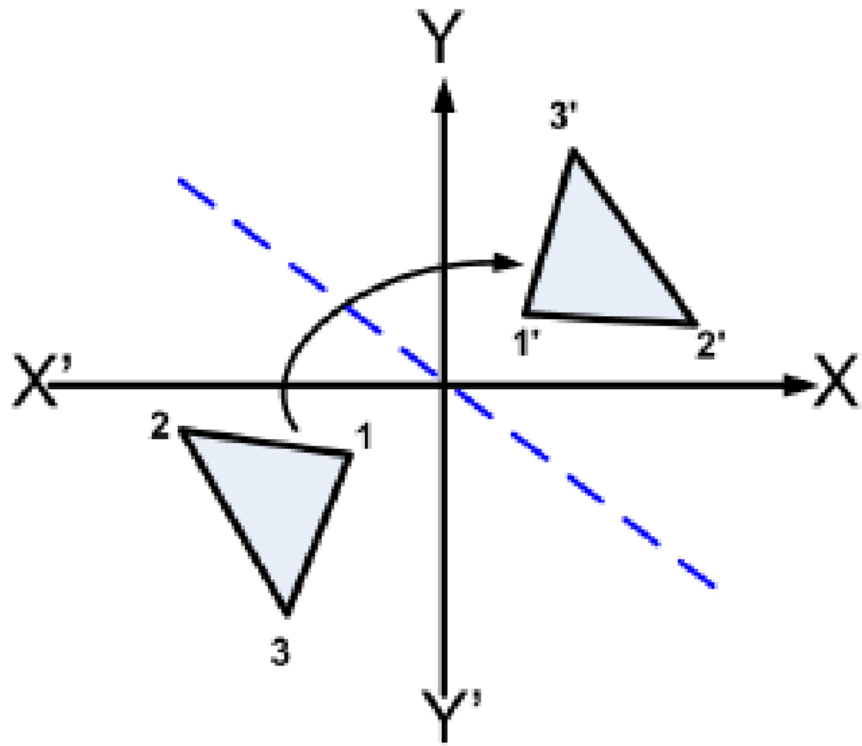
Flip both ‘x’ and ‘y’ coordinates of a point

x’ = -x
y’ = -y

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Homogeneous co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



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2D Reflection

(iv) Reflection about line y = x

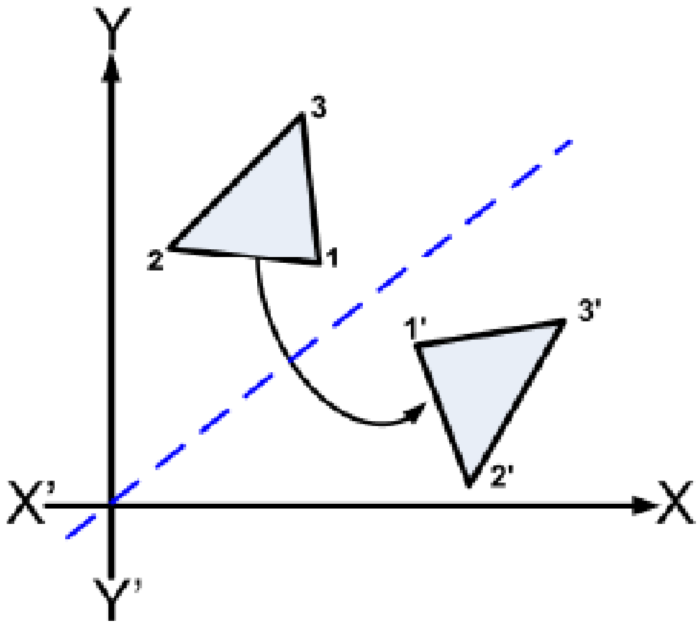
$x' = y$
$y' = x$

Thus, reflection against x=y-axis (i.e. θ =45)

$$R_{x=y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Homogeneous co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Equivalent to:

- ❑ Reflection about x-axis
- ❑ Rotate anticlockwise **90˚**

OR

- ❑ Clockwise rotation 45 ˚
- ❑ Reflection with x-axis
- ❑ anticlockwise rotation 45 ˚

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2D Reflection

(v) Reflection about line y = -x

$x' = -y$

$y' = -x$

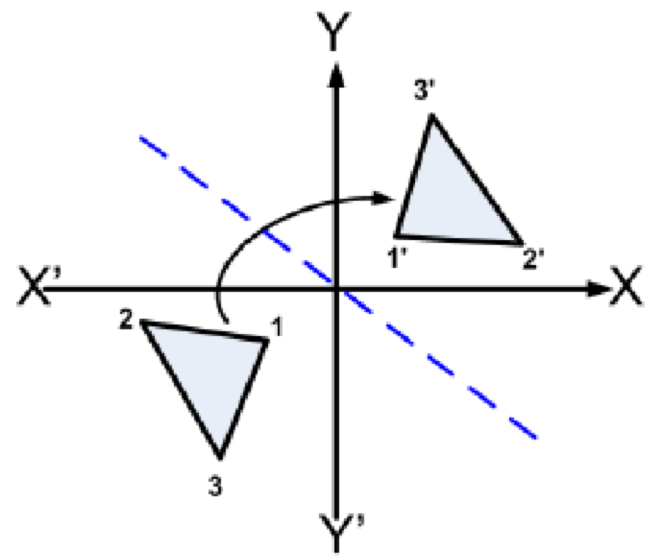
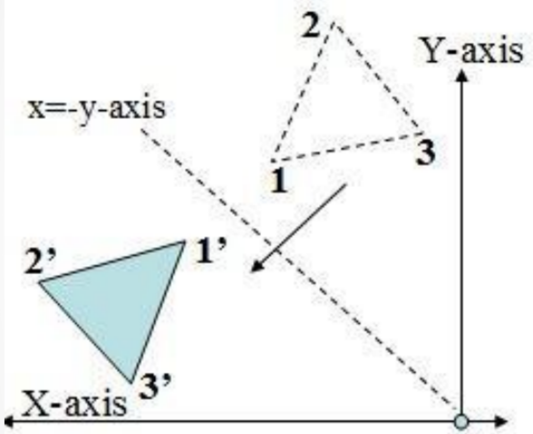
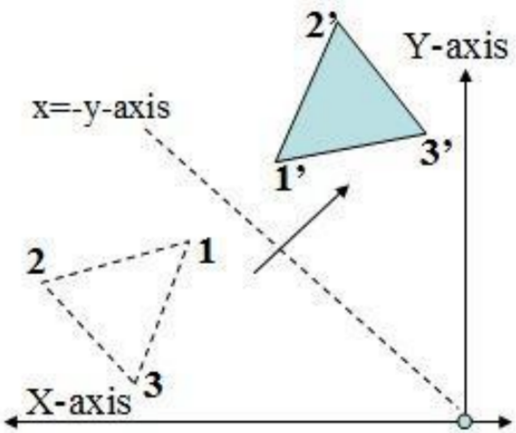


Fig: Reflection of object about y = - x



Reflection of an object about x= -y in anti-clockwise direction



Reflection of an object about x= -y in clockwise direction

Thus, reflection against x=y-axis in anti-clockwise direction (i.e. θ = 45)

Hence

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Where $R_{x=-y} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Thus, reflection against x=y-axis in clockwise direction (i.e. θ = - 45)

Hence

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Where $R_{x=-y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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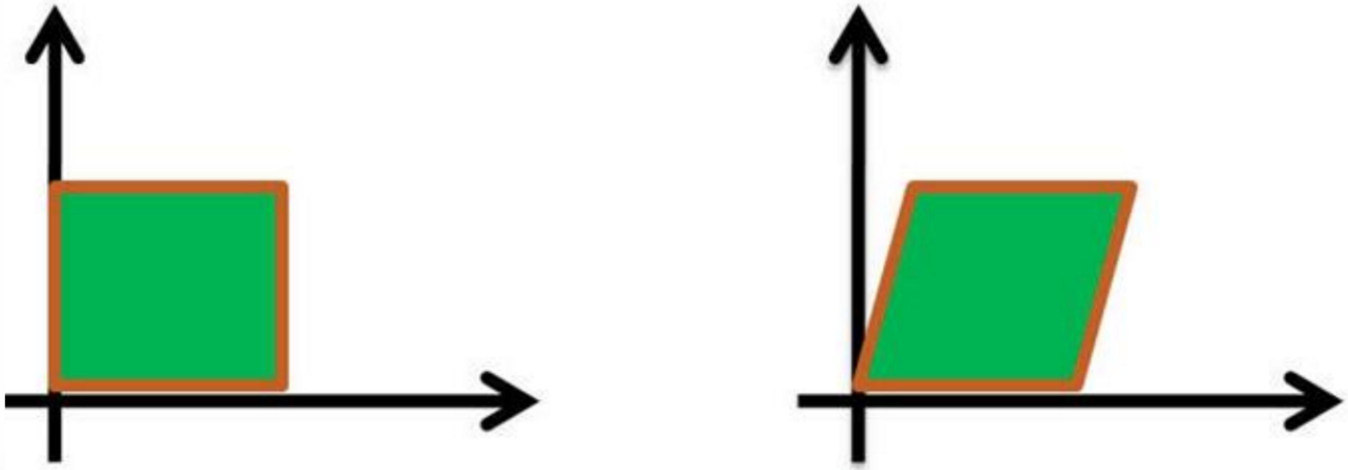
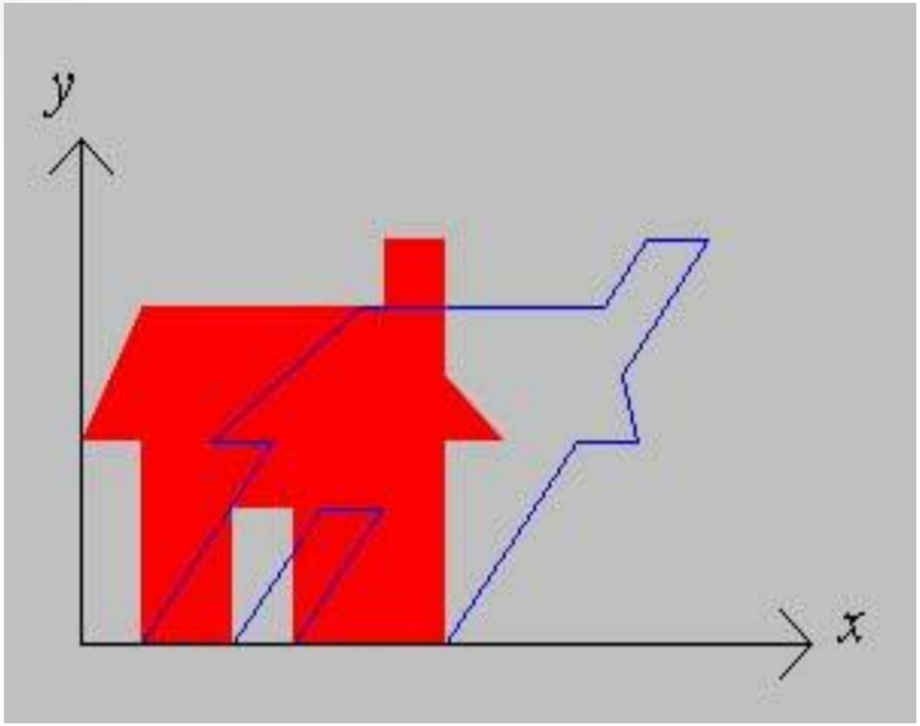
2D Shearing

- It distorts the shape of object in either 'x' or 'y' or both direction.
- In case of single directional shearing (e.g. in 'x' direction can be viewed as an object made up of very thin layer and slid over each other with the *base* remaining where it is).
- Shearing is a ***non-rigid-body transformation*** that moves objects with deformation.



2D Shearing



$$x' = x + h_x y$$

$$y' = y$$

- A shear transformation in the x-direction (along x) shifts the points in the x-direction proportional to the y-coordinate.
- The y-coordinate of each point is unaffected.

2D Shearing

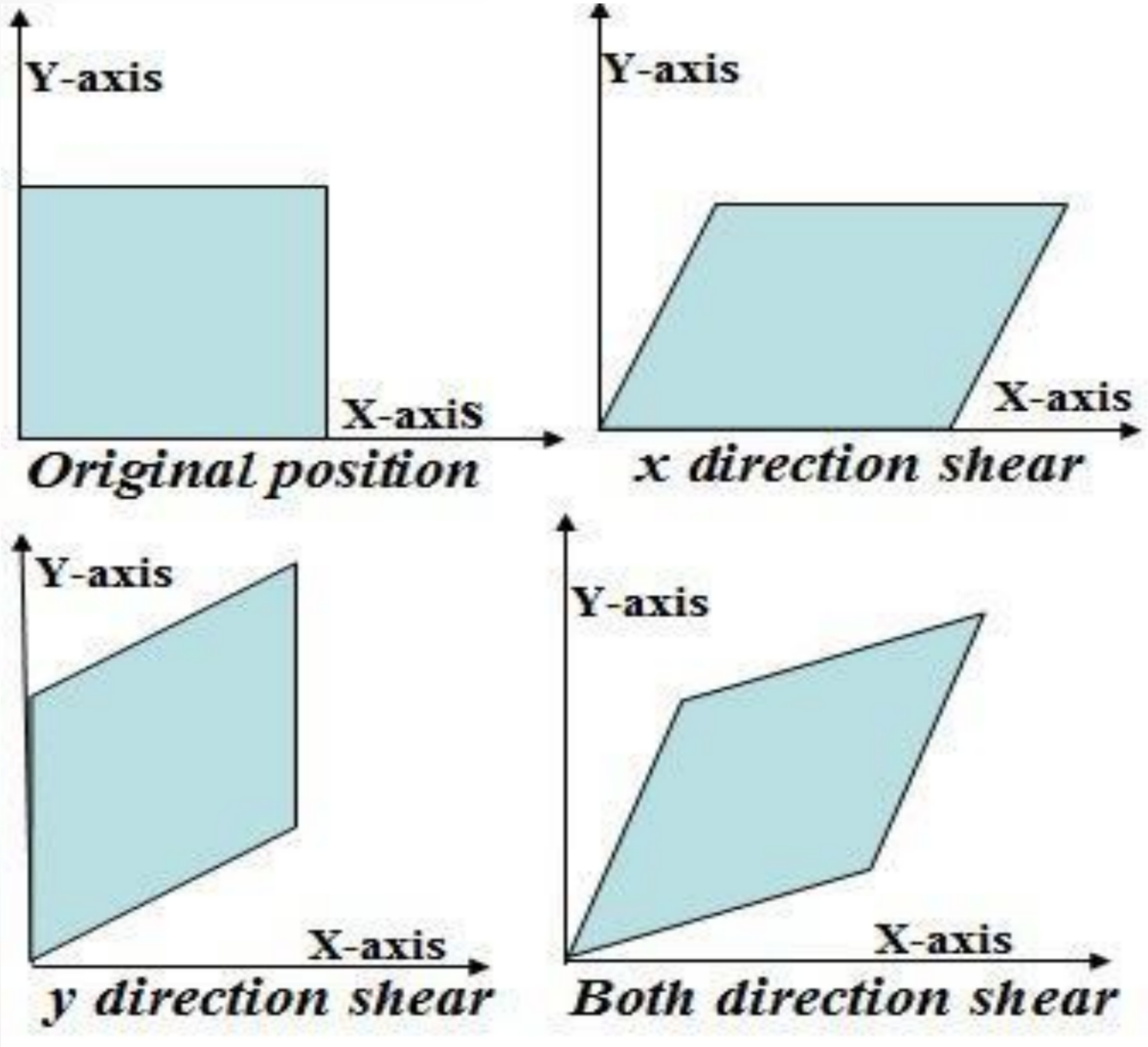


Fig. Two Dimensional shearing

2D Shearing

In 'x' direction,

$x' = x + s_{hx} \cdot y$

$y' = y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & S_{hx} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In 'y' direction,

$x' = x$

$y' = y + s_{hy} \cdot x$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ S_{hy} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In both directions,

$x' = x + s_{hx} \cdot y$

$y' = y + s_{hy} \cdot x$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & S_{hx} \\ S_{hy} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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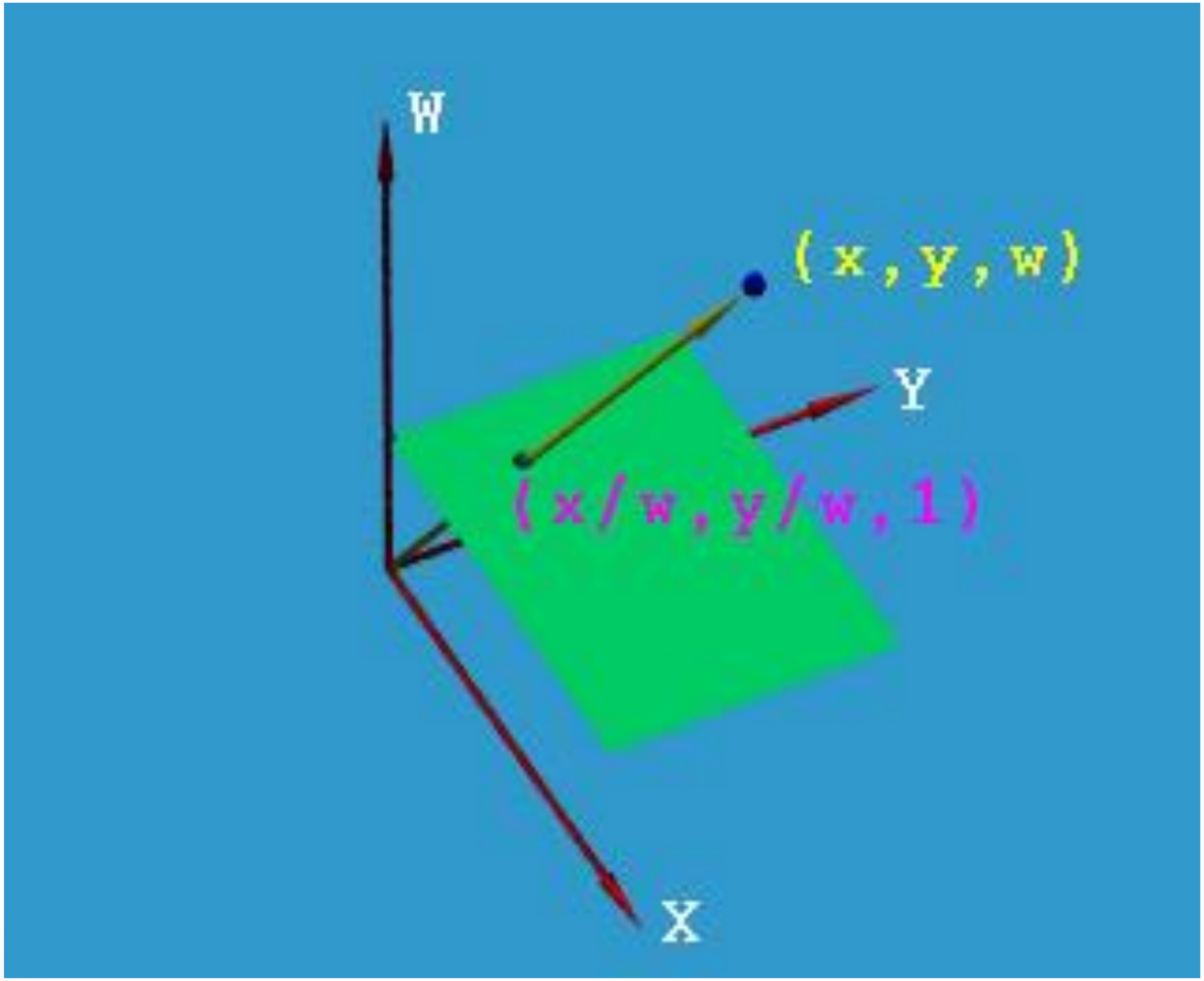
Homogenous Coordinates

- To obtain square matrices an additional row was added to the matrix and an additional coordinate, the *w*-coordinate, was added to the vector for a point.
- In this way a point in 2D space is expressed in three-dimensional homogeneous coordinates.
- This technique of representing a point in a space whose dimension is one greater than that of the point is called ***homogeneous representation***.
- Expressing position in homogeneous coordinates allows us to represent all geometric transformations equation as matrix multiplications.
- It provides a consistent, uniform way of handling affine transformations.

Homogenous Coordinates



Homogenous Coordinates

- If we use homogeneous coordinates, the geometric transformations given above can be represented using only a matrix pre-multiplication.
- A composite transformation can then be represented by a product of the corresponding matrices.

Cartesian — **Homogeneous**

$$(x, y) \longrightarrow (xh, yh, h), h \neq 0$$

$$\left(\frac{a}{c}, \frac{b}{c}\right) \longleftarrow (a, b, c), c \neq 0$$

Examples: (5, 8)	→	(15, 24, 3)
(x, y)	→	(x, y, 1)

Homogenous Coordinates

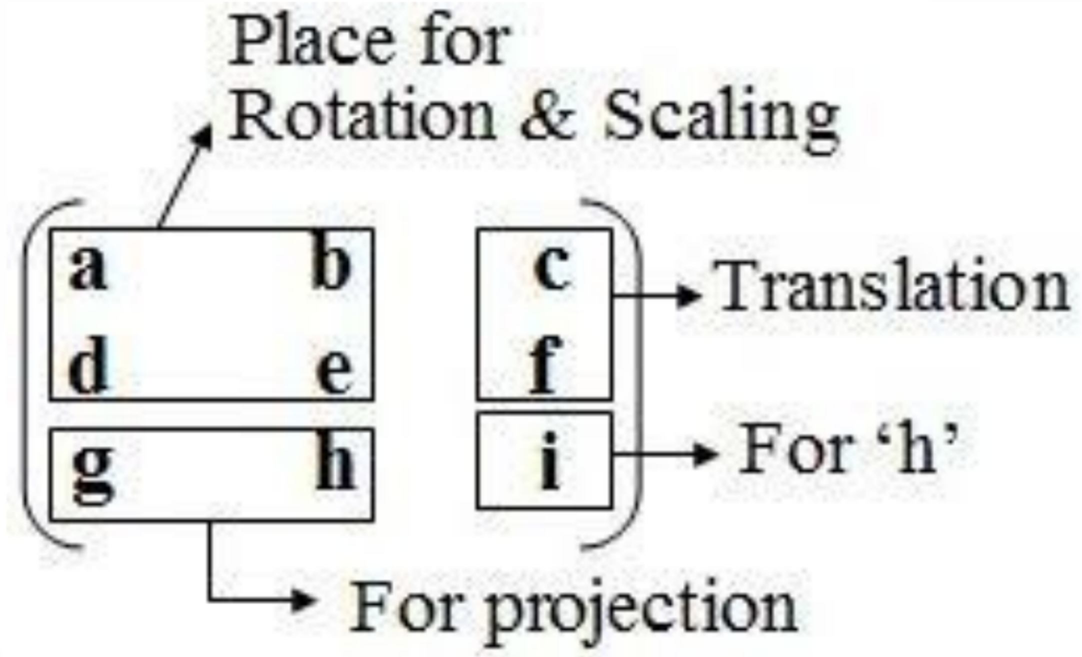
The matrix representations for translation, scaling and rotation are respectively:

- Translation: **P' = T + P** (*Addition*)
- Scaling: **P' = S . P** (*Multiplication*)
- Rotation: **P' = R . P** (*Multiplication*)

Since, the composite transformation such as include many sequence of translation, rotation etc. and hence the many naturally differ addition & multiplication sequence have to perform by the graphics allocation. Hence, the applications will take more time for rendering.

Thus, we need to treat all three transformations in a consistent way so they can be combined easily & compute with one mathematical operation. If points are expressed in homogenous coordinates, all geometrical transformation equations can be represented as matrix multiplications.

Homogenous Coordinates



Here, in case of homogenous coordinates we add a third coordinate '*h*' to a point (x, y) so that each point is represented by (*h*x, *h*y, *h*). The '*h*' is normally set to 1. If the value of 'h' is more the one value then all the co-ordinate values are scaled by this value.

Homogenous Coordinates

- **For Translation**

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

With T(tx , ty) as translation matrix, inverse of this translation matrix is obtained by representing tx, ty with -tx, - ty

- **For Rotation**

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}\theta & -\mathrm{Sin}\,\theta & 0 \\ \mathrm{Sin}\theta & \mathrm{Cos}\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

(a)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}\theta & \mathrm{Sin}\,\theta & 0 \\ -\mathrm{Sin}\theta & \mathrm{Cos}\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

(b)

Here, figure-a shows the Counter Clockwise (CCW) rotation & figure-b shows the Clockwise (CW) rotation.

Homogenous Coordinates

- **For Scaling**

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} S_{hx} & 0 & 0 \\ 0 & S_{hy} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- **For Reflection**
  - Reflection about x-axis

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

  - Reflection about y-axis

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Homogenous Coordinates

- **For Reflection**
  - Reflection about y=x-axis

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

  - Reflection about y=-x-axis

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

  - Reflection about any line y=mx+c

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-(m^2-1)}{(m^2+1)} & \frac{2m}{(m^2+1)} & \frac{2mc}{(m^2+1)} \\ \frac{2m}{(m^2+1)} & \frac{(m^2-1)}{(m^2+1)} & \frac{2c}{(m^2+1)} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Homogenous Coordinates

Examples

- Translate [1,3] by [7,9]

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 1 \end{bmatrix}$$

- Scale [2,3] by 5 in the X direction and 10 in the Y direction

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 1 \end{bmatrix}$$

- Rotate [2,2] by 90° (π/2)

$$\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) & 0 \\ \sin(\pi/2) & \cos(\pi/2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Composite Transformation

- With the matrix representation of transformation equations it is possible to setup a matrix for any sequence of transformations as a composite transformation matrix by calculating the matrix product of individual transformation.
- Forming products of transformation matrices is often referred to as a **concatenation,** or **composition,** of matrices.
- For column matrix representation of coordinate positions we form composite transformation by multiplying matrices in order from right to left.

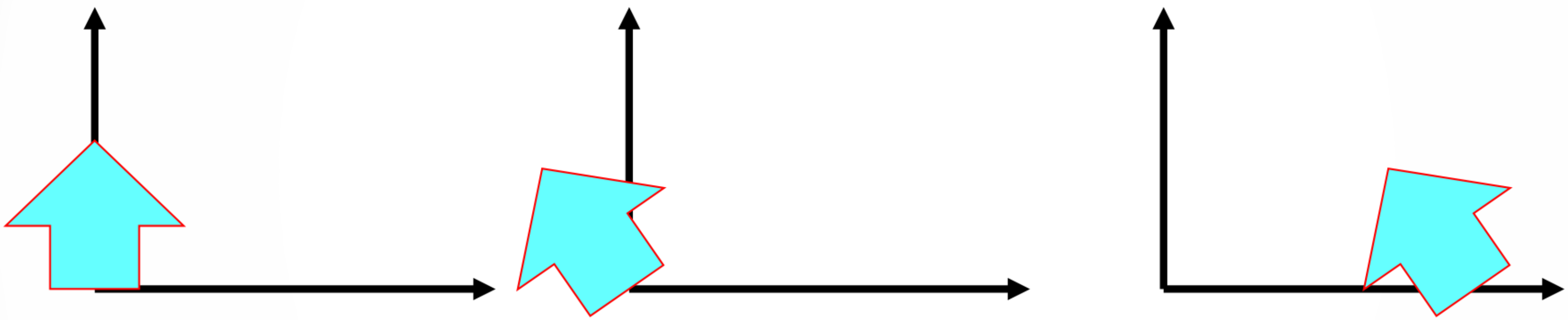
Right to Left

A.B != B.A

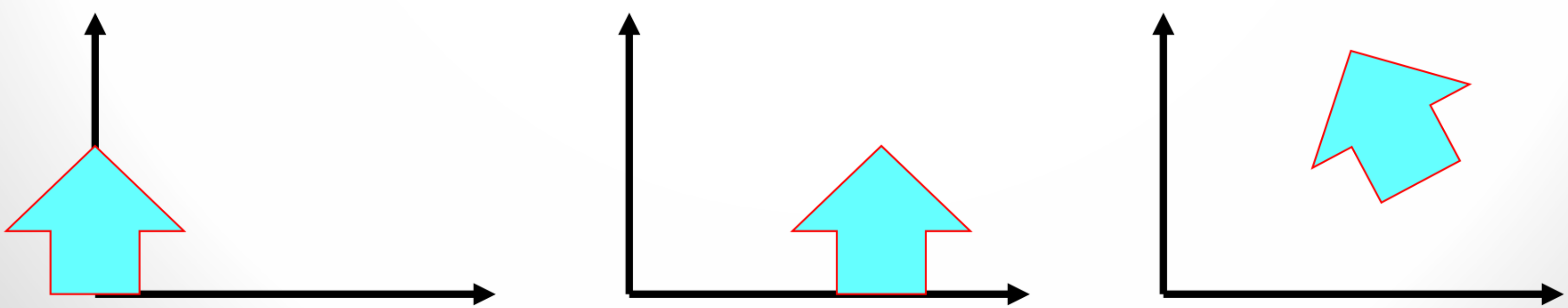
Composite Transformation

- In composite transformations, the order of transformations is very important.

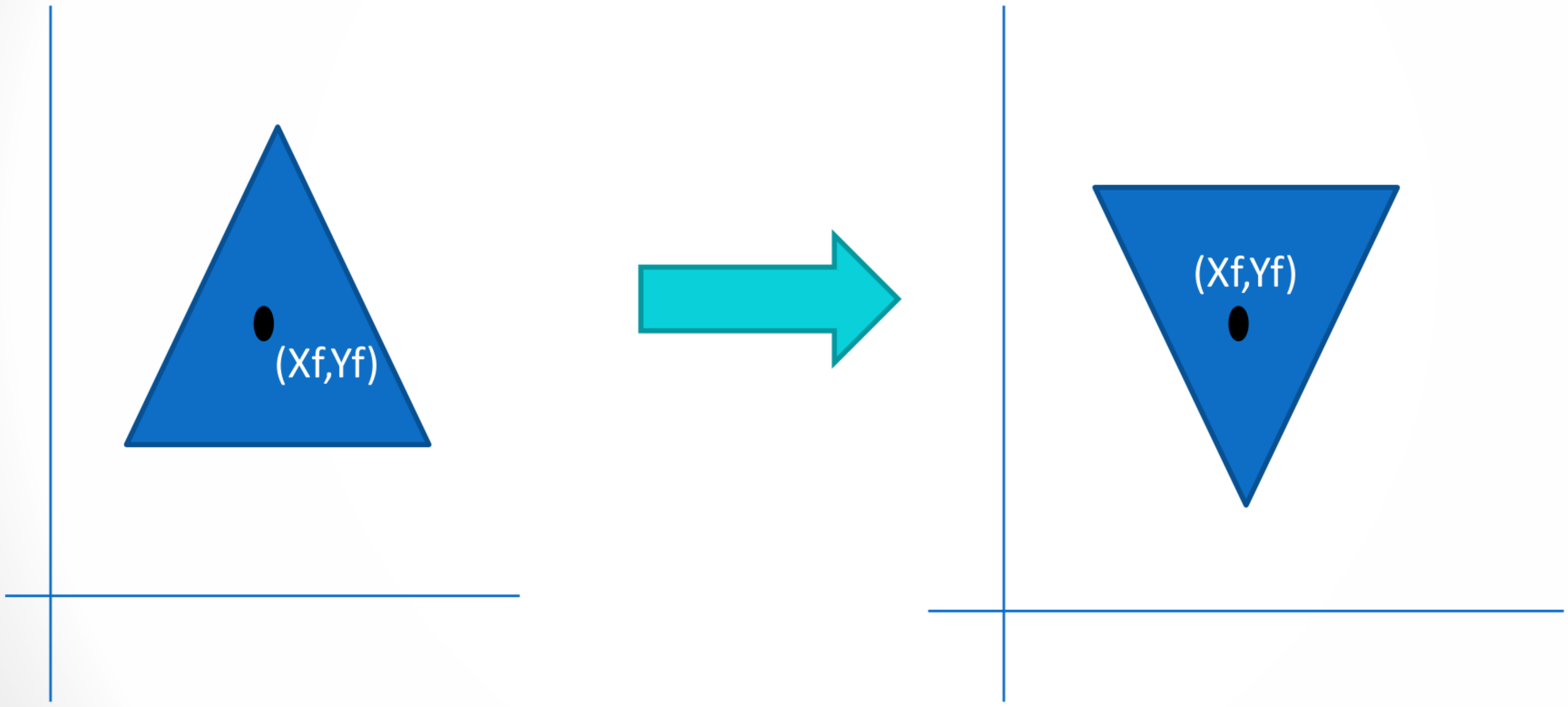
Rotation followed by Translation:



Translation followed by Rotation:

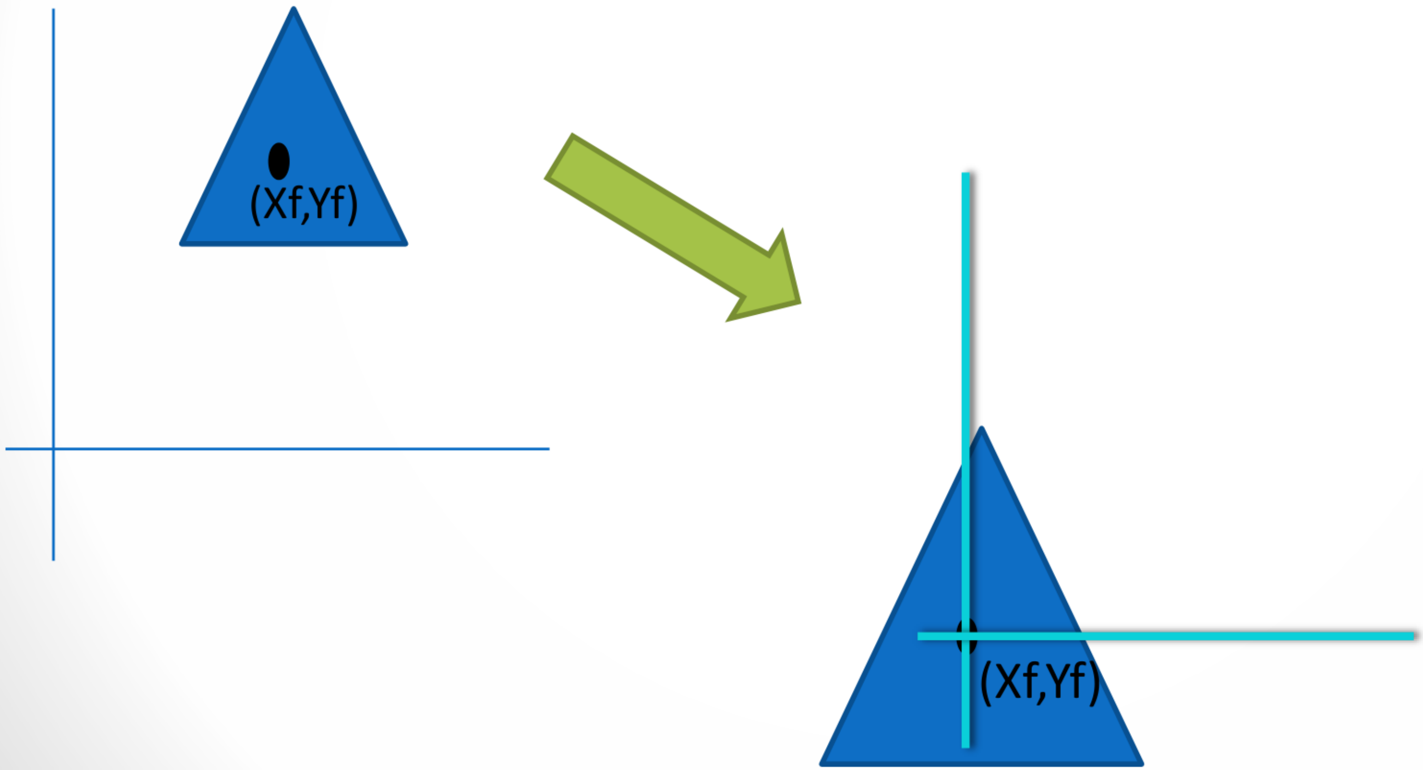


Fixed Point Rotation



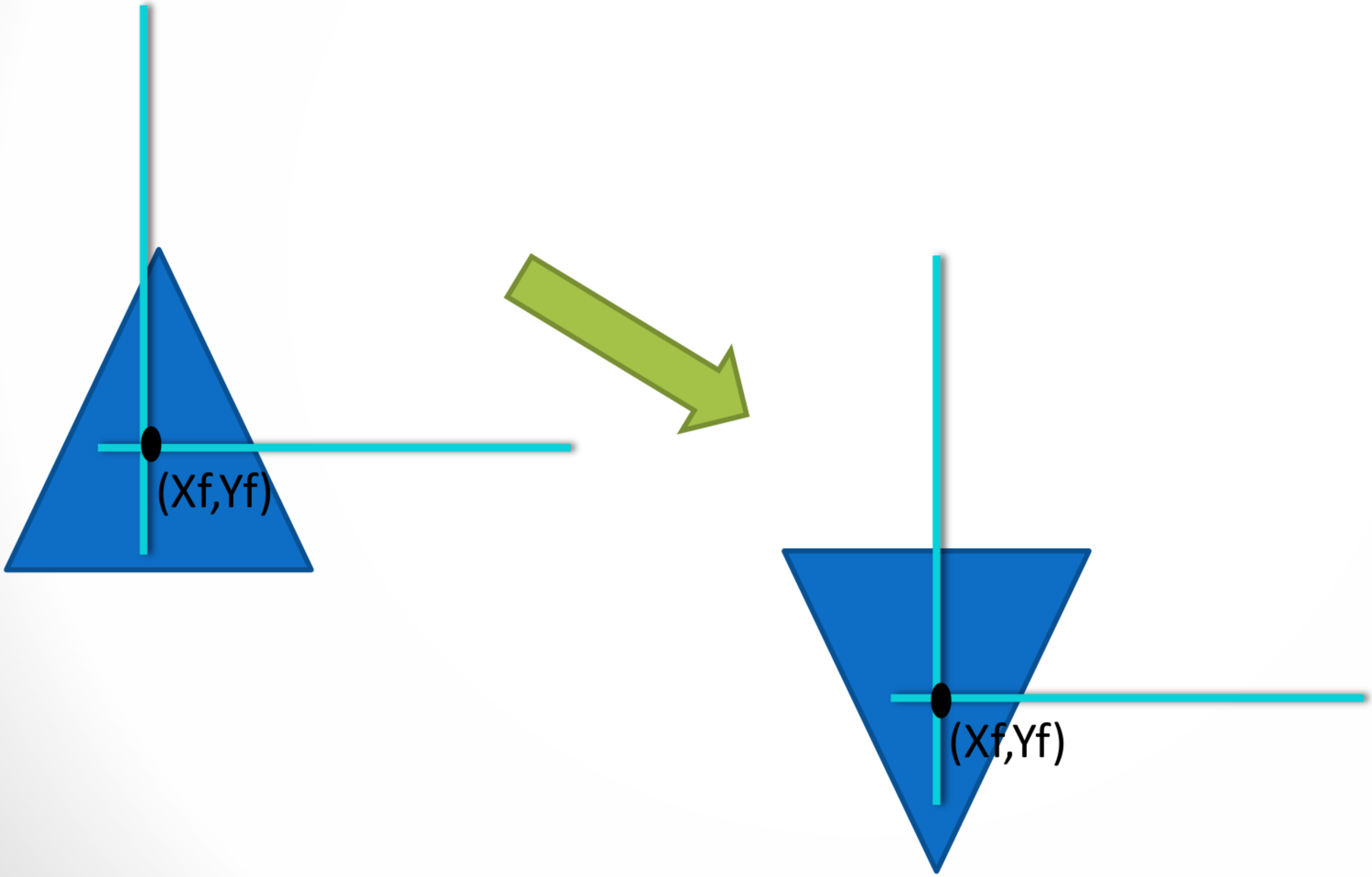
Fixed Point Rotation

Step 1: The fixed point along with the object is translated to coordinate origin.



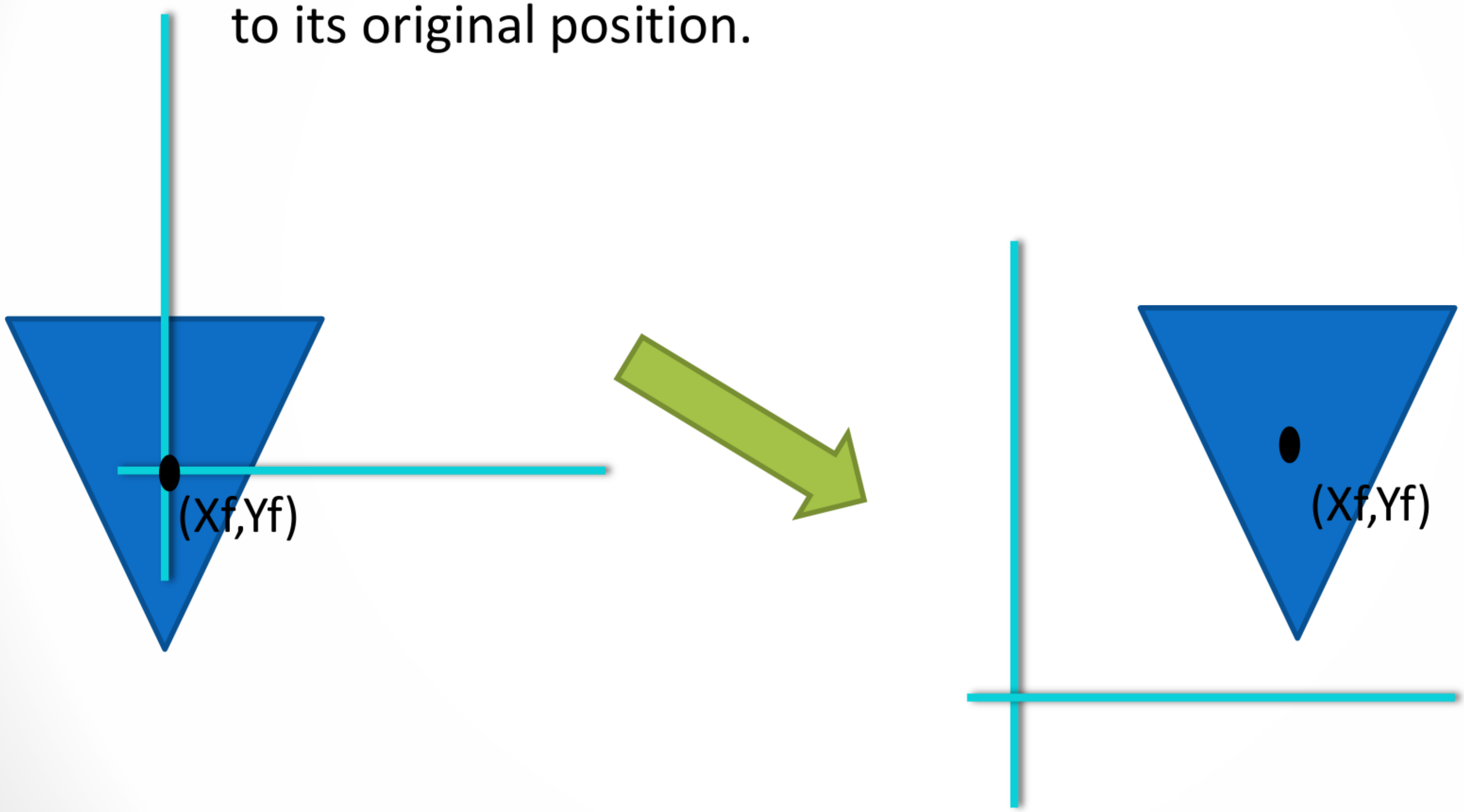
Fixed Point Rotation

Step 2: Rotate the object about origin



Fixed Point Rotation

Step 3: The fixed point along with the object is translated back to its original position.



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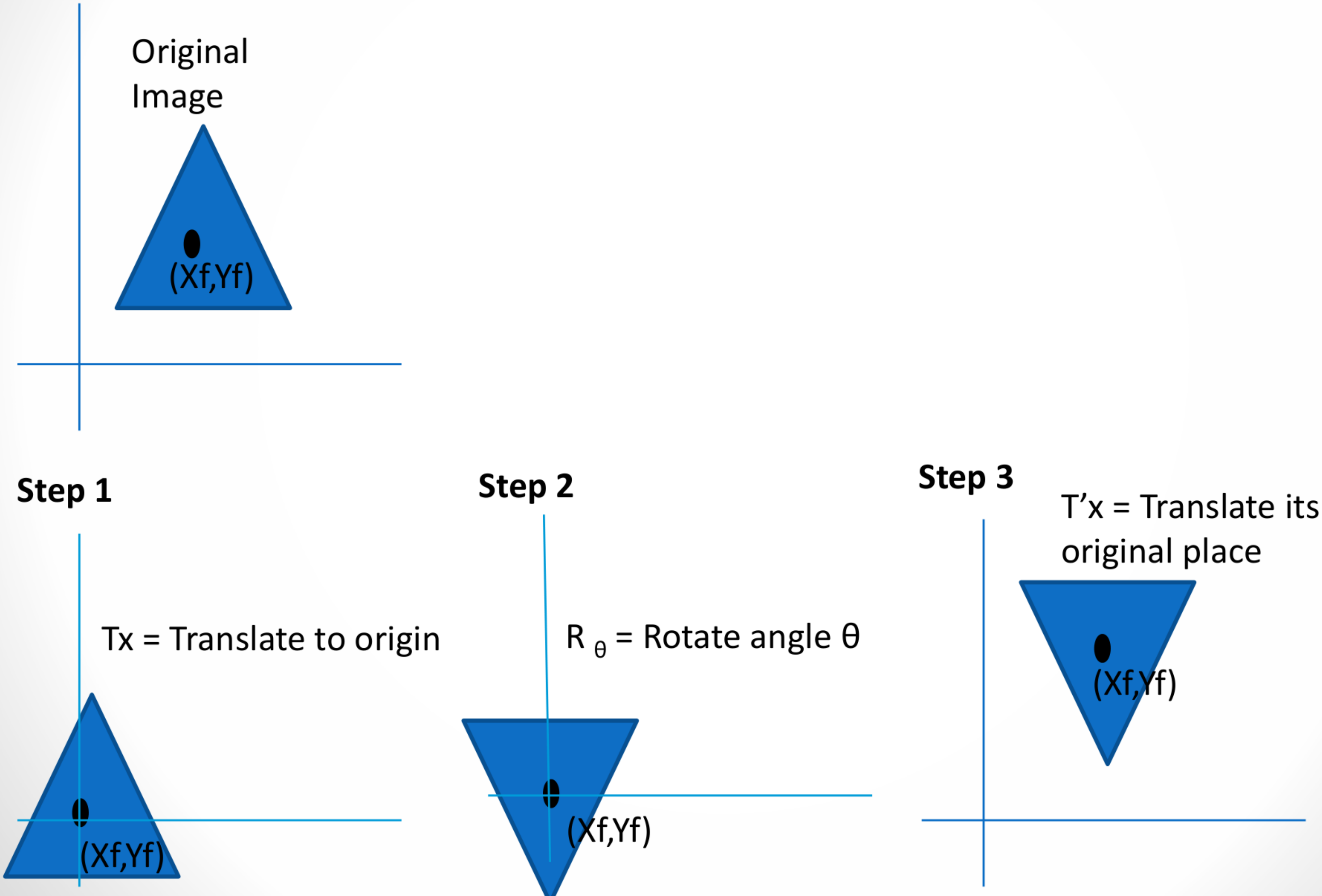
Fixed Point Rotation



Fixed Point Rotation

Composite Transformation

$$= T'_{(Xf\,,Yf)} \cdot R_{\theta} \cdot T_{(-Xf\,,-Yf)}$$

$$= \begin{bmatrix} 1 & 0 & Xf \\ 0 & 1 & Yf \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -Xf \\ 0 & 1 & -Yf \\ 0 & 0 & 1 \end{bmatrix}$$

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Exercise 1

Question: Rotate the triangle (5, 5), (7, 3), (3, 3) in counter clockwise (CCW) by 90 degree.

Answer: *P' = R. P*

$$= \begin{pmatrix} Cos90 & -Sin90 & 0 \\ Sin90 & Cos90 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 7 & 3 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 7 & 3 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} -5 & -3 & -3 \\ 5 & 7 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

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Exercise 2

Question: Rotate the triangle (5, 5), (7, 3), (3, 3) about fixed point (5, 4) in counter clockwise (CCW) by 90 degree.

Answer: Here, the required steps are:

1. Translate the fixed point to origin.
2. Rotate about the origin by specified angle θ .
3. Reverse the translation as performed earlier.

Thus, the composite matrix is given by

$$Com = T(xf, yf)\ .\ R_\theta\ .T(-xf, -yf)$$

$$= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Cos}90 & -\text{Sin}90 & 0 \\ \text{Sin}90 & \text{Cos}90 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

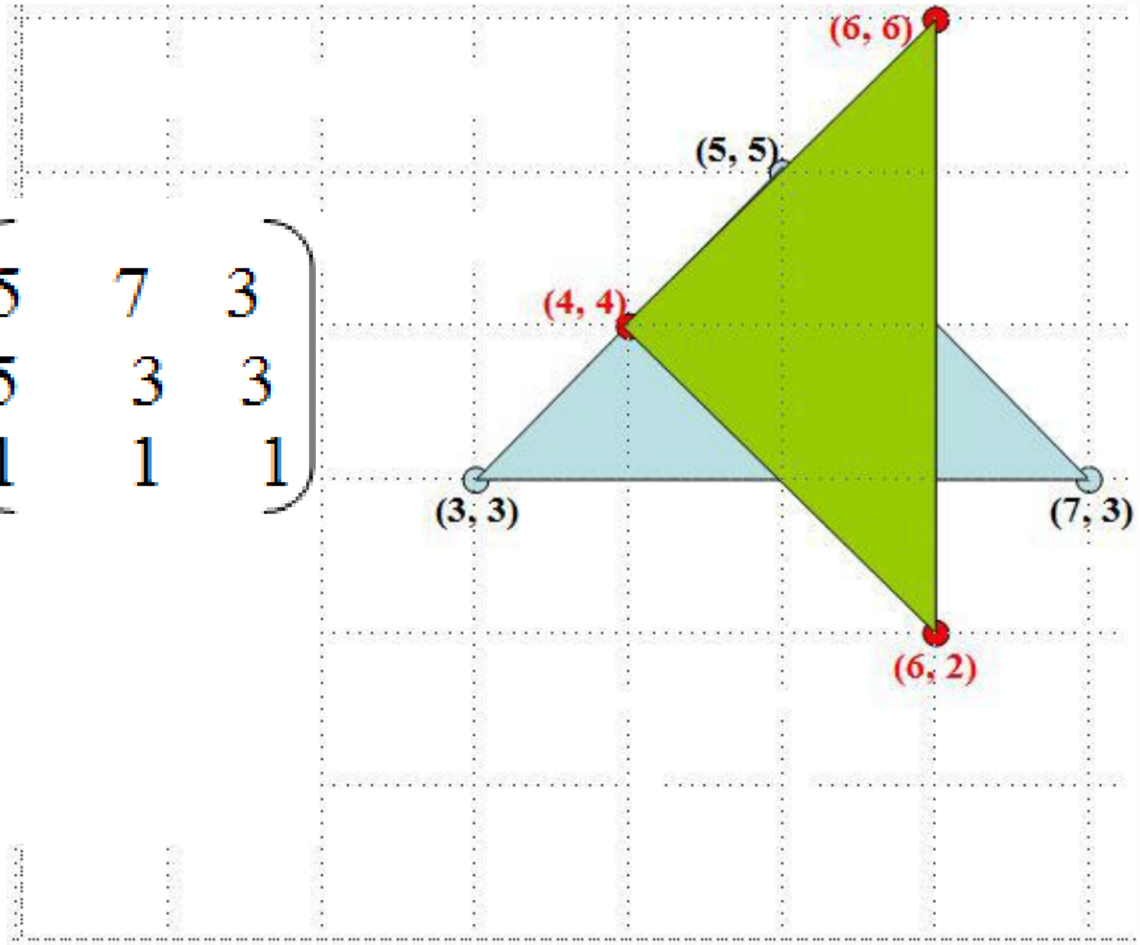
$$= \begin{pmatrix} 0 & -1 & 9 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, the required co-ordinate can be calculated as:

P' = Com . P

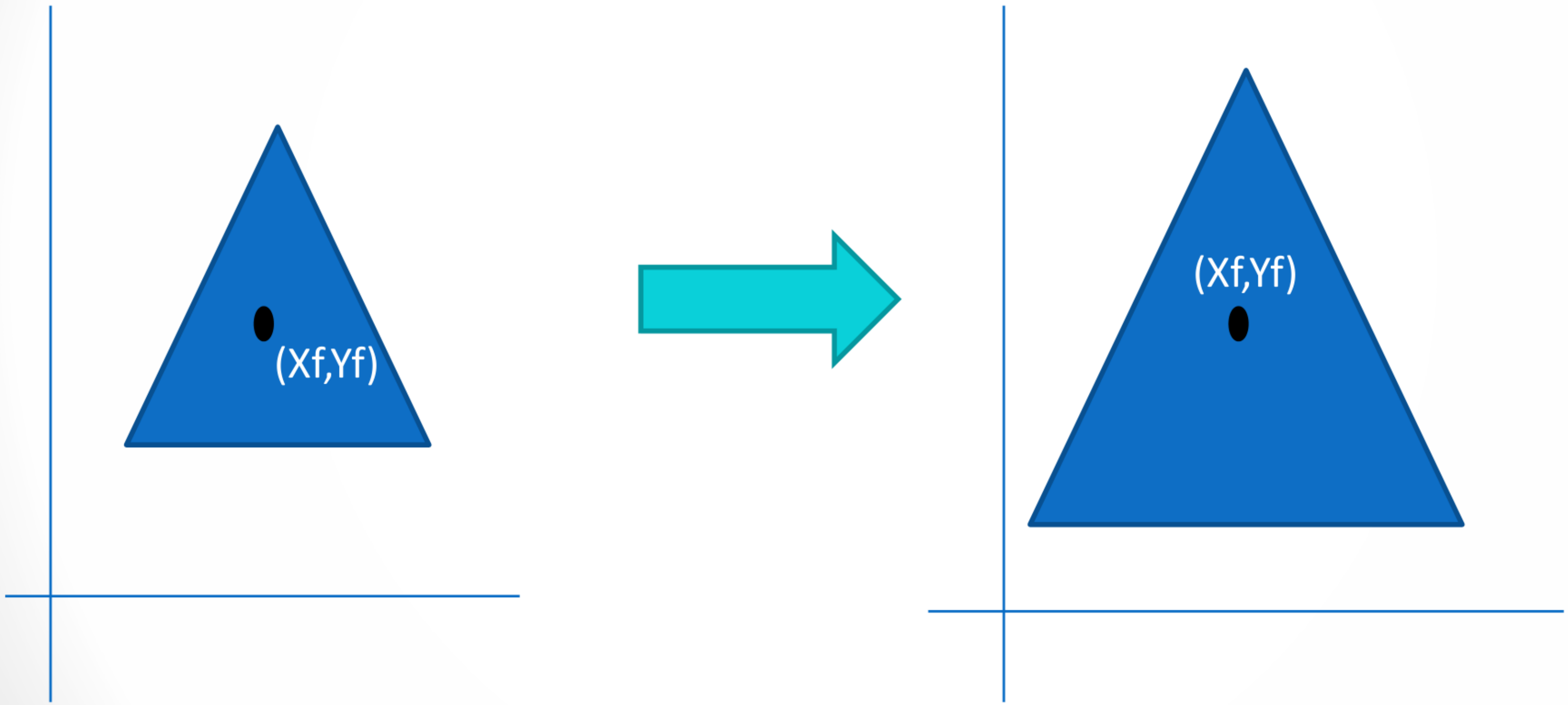
$$= \begin{pmatrix} 0 & -1 & 9 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 7 & 3 \\ 5 & 3 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 & 6 \\ 4 & 6 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$



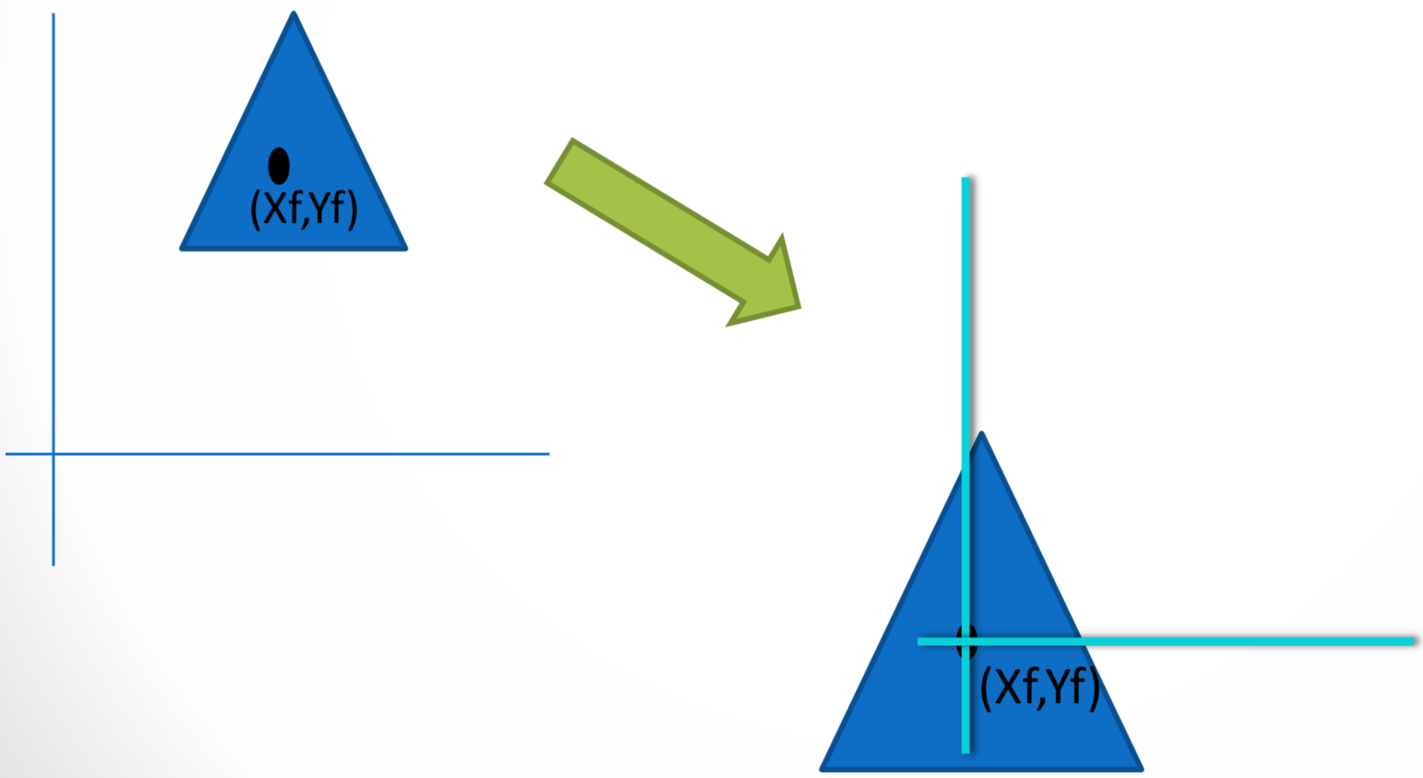
Hence, the new required coordinate points are **(4, 4), (6, 6) & (6, 2).**

Fixed Point Scaling



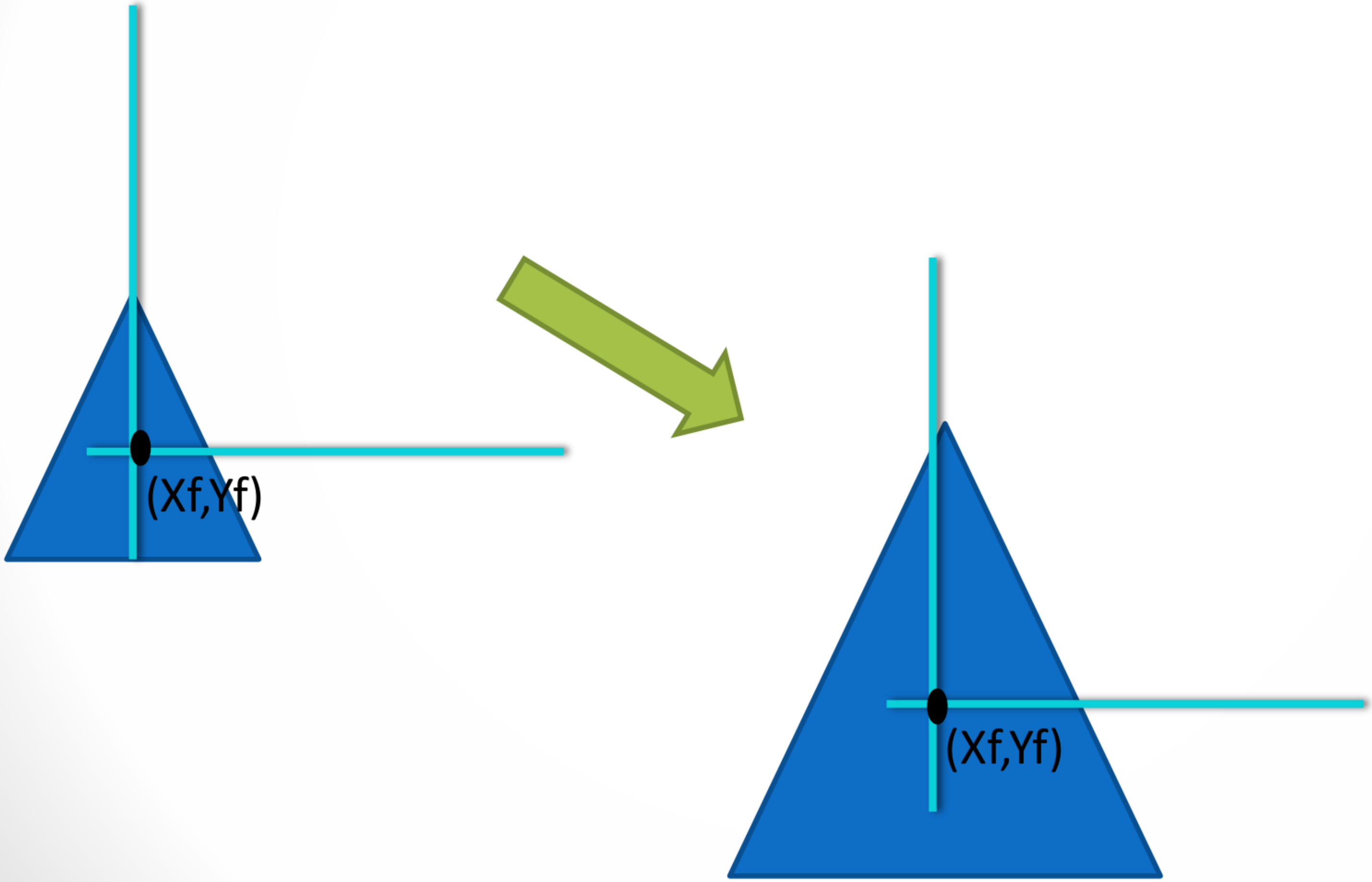
Fixed Point Scaling

Step 1: The fixed point along with the object is translated to coordinate origin.



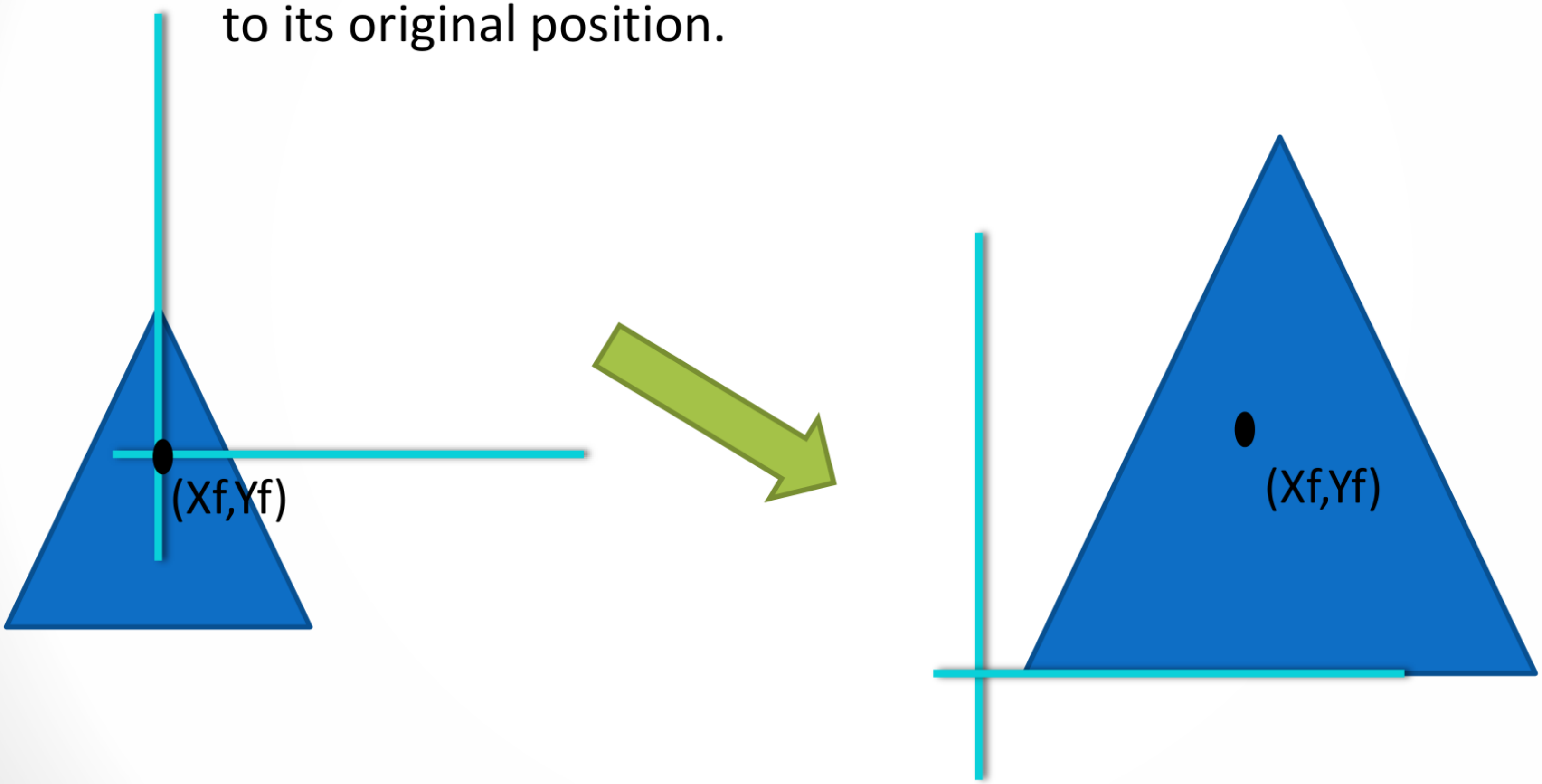
Fixed Point Scaling

Step 2: Scaling the objet about origin

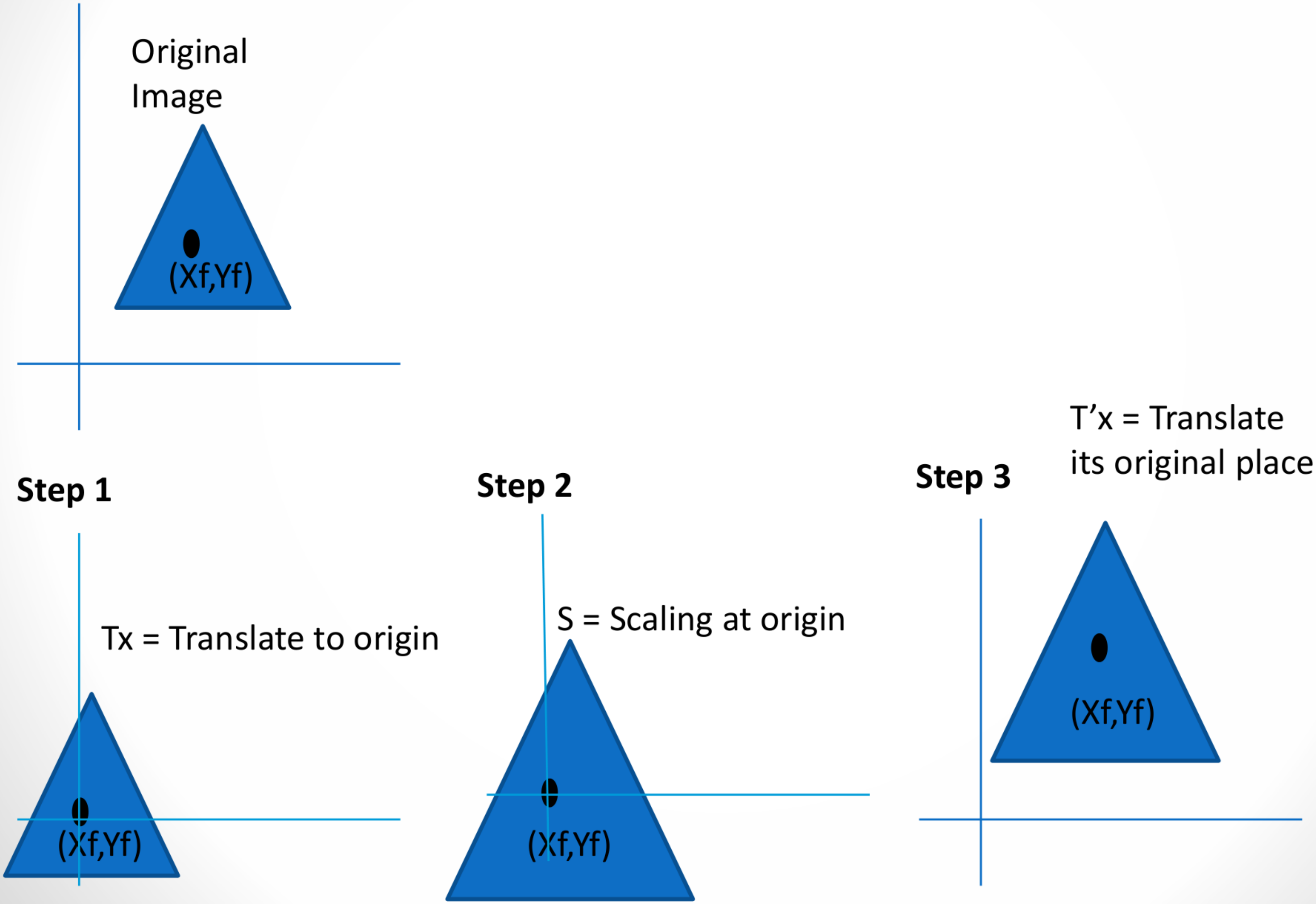


Fixed Point Scaling

Step 3: The fixed point along with the object is translated back to its original position.



Fixed Point Scaling



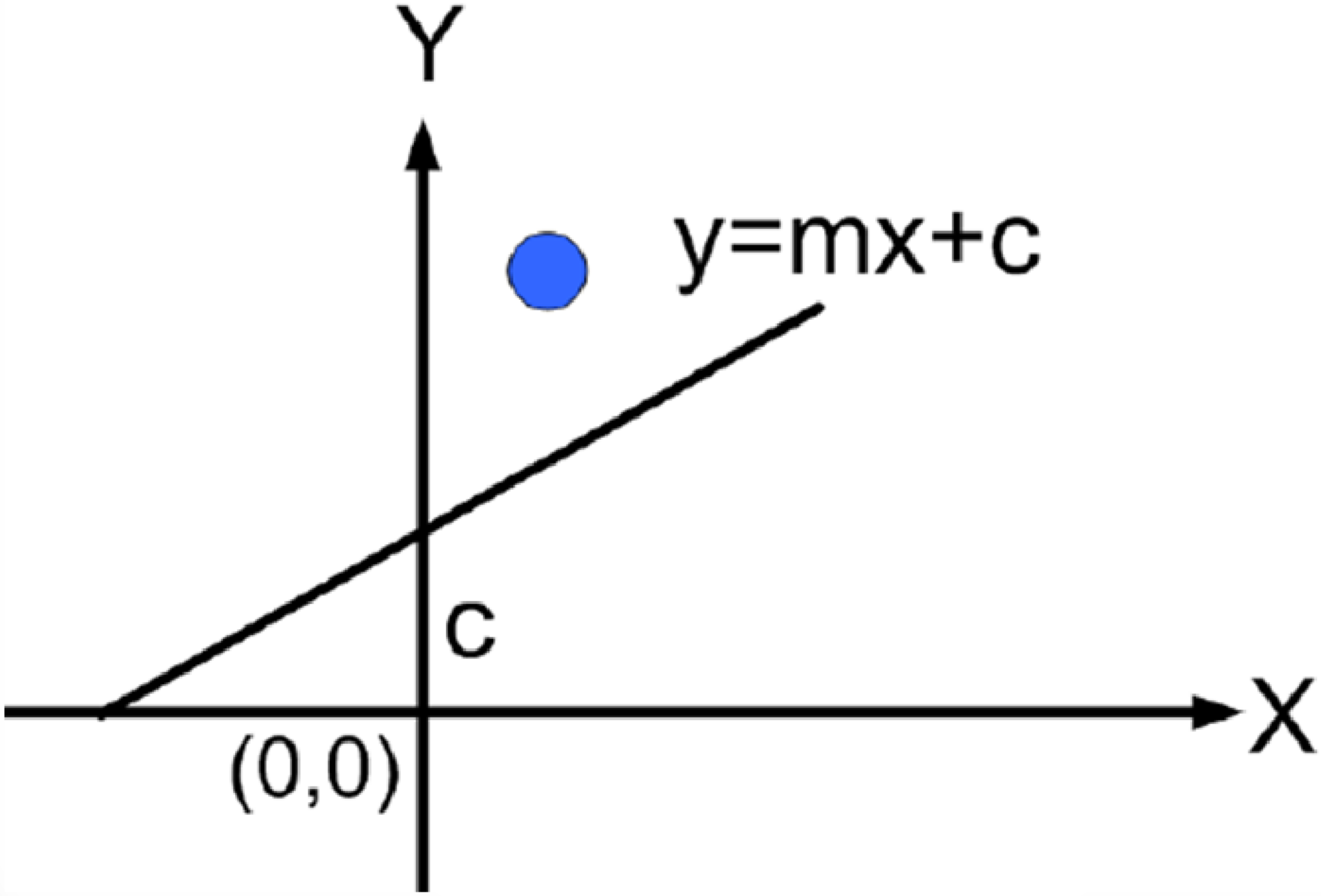
Fixed Point Rotation

Composite Transformation

$$= T'_{(Xf\ ,Yf)} \cdot S_{axis} \cdot T_{(-Xf\ ,-Yf)}$$

$$= \begin{bmatrix} 1 & 0 & Xf \\ 0 & 1 & Yf \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -Xf \\ 0 & 1 & -Yf \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about Line y=mx+c

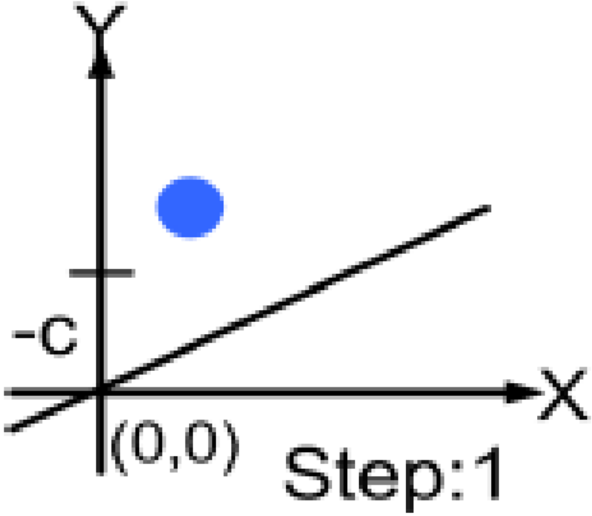


Reflection about Line y=mx+c

1. First translate the line so that it passes through the origin

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

translatio n



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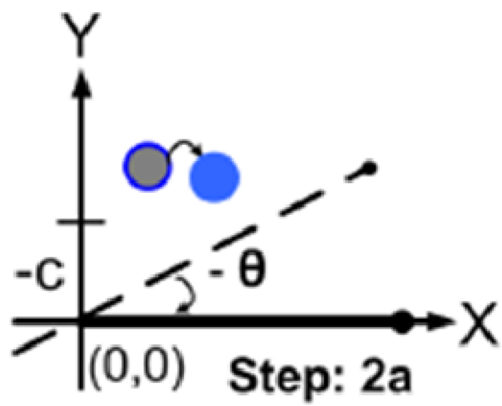
Reflection about Line y=mx+c

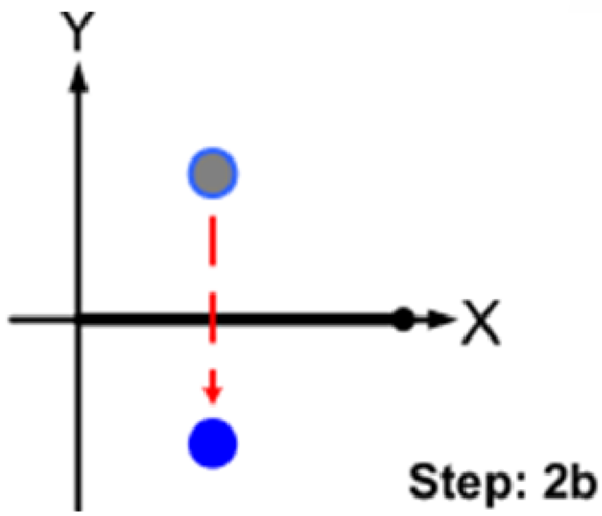
2. Rotate the line onto one of the coordinate axes(say x-axis) and reflect about that axis (x-axis)

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} --- \textit{rotation}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reflection



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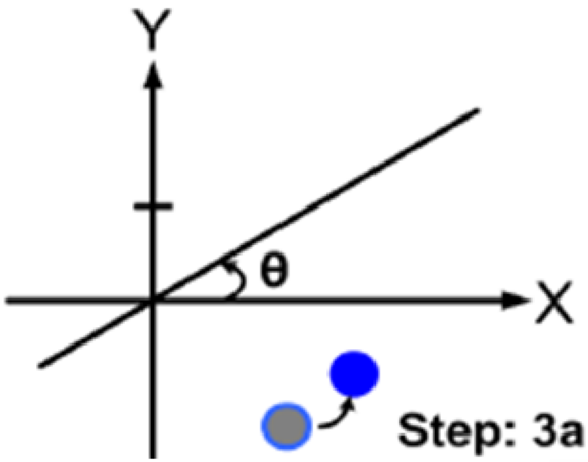
Reflection about Line y=mx+c

3. Finally, restore the line to its original position with the inverse rotation and translation transformation.

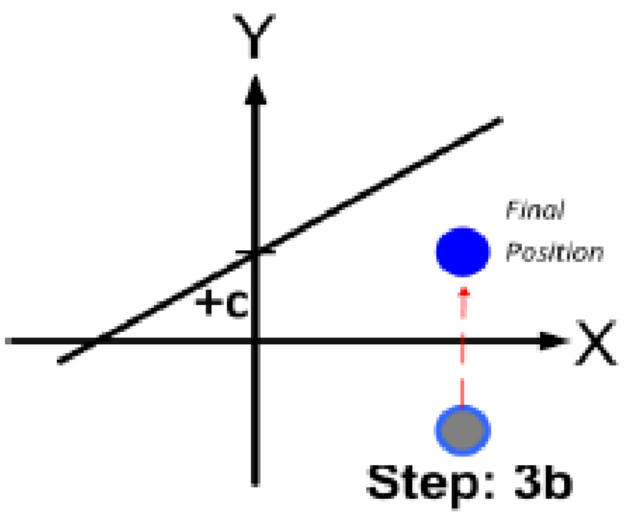
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

translation



Reflection about Line y=mx+c

$$CM = T'_{(0,C)} \cdot R'_{\theta} \cdot R_{refl} \cdot R_{\theta} \cdot T_{(0,-C)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise

Question: *Reflect an object (2, 3), (4, 3), (4, 5) about line y = x +1.*

Solution:

Here,

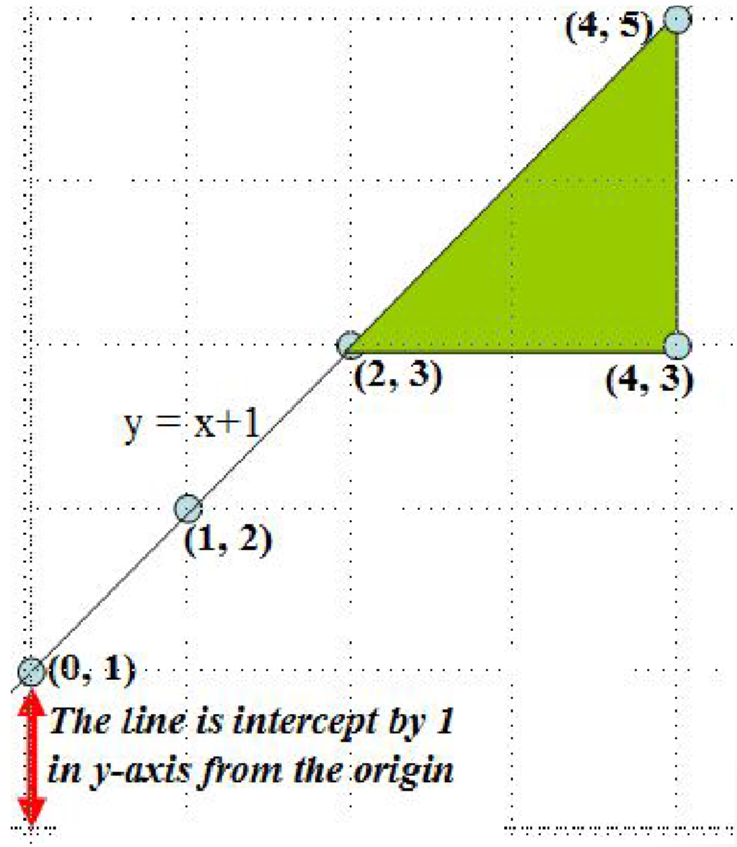
The given line is $y = x +1$.

Thus,

When x = 0, y=1

When x = 1, y=2

When x = 2, y=3



Also,

The slope of the line (m) = 1

Thus, the rotation angle (θ) = $Tan^{-1}(m) = Tan^{-1}(1) = 45^0$

Here, the required steps are:

- Translate the line to origin by decreasing the y-intercept with one.
- Rotate the line by angle 45^0 in clockwise direction so that the given line must overlap x-axis.
- Reflect the object about the x-axis.
- Reverse rotate the line by angle -45^0 in counter-clockwise direction.
- Reverse translate the line to original position by adding the y-intercept with one.

Thus, the composite matrix is given by:

$$\mathbf{CM = T'_{(0,C)} \cdot R'_{\theta} \cdot R_{refl} \cdot R_{\theta} \cdot T_{(0,-C)}}$$

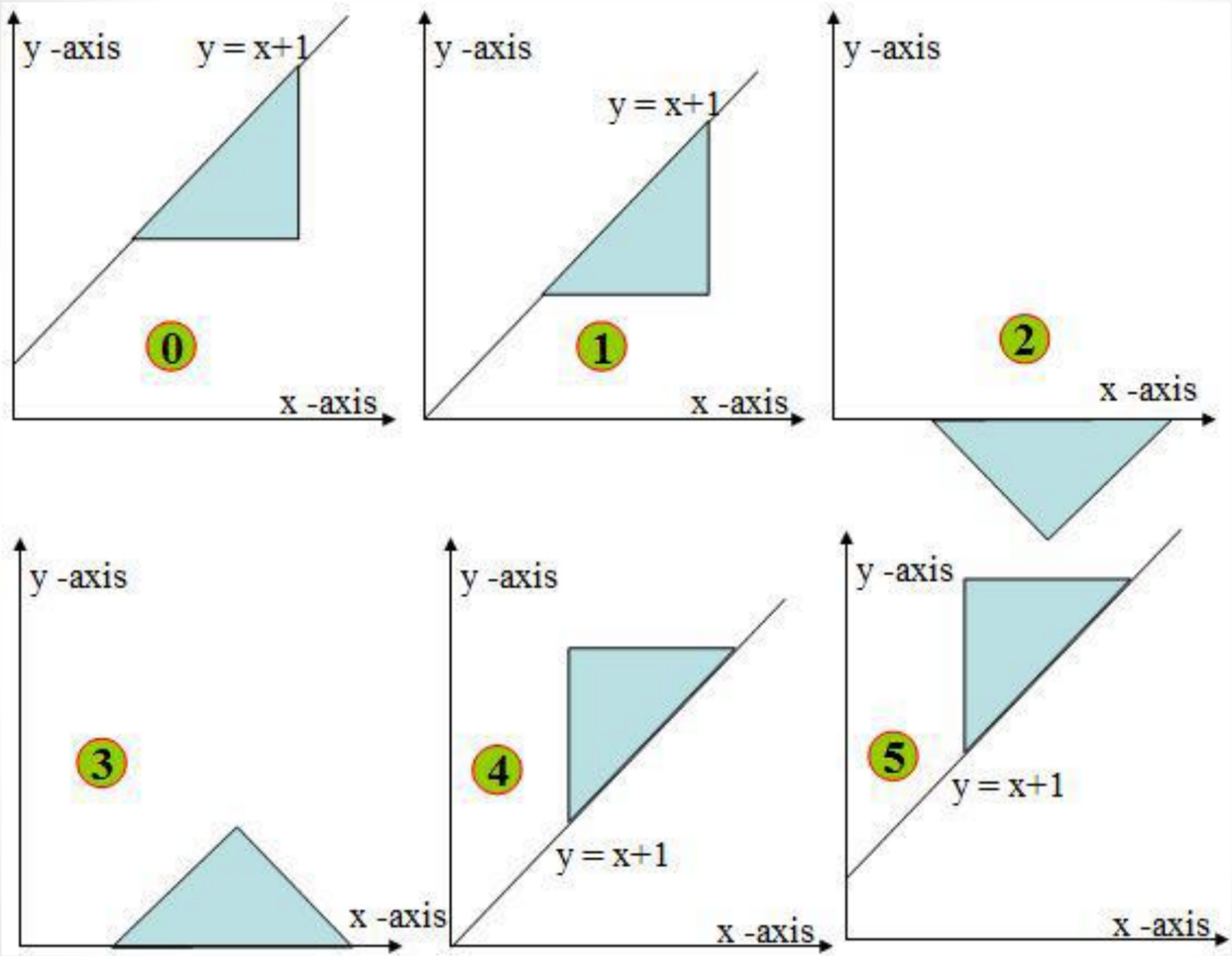
$$= \overset{\text{Addition y-intercept}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \overset{\text{CCW Rotation}}{\begin{pmatrix} \text{Cos}45 & -\text{Sin}45 & 0 \\ \text{Sin}45 & \text{Cos}45 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{\text{Reflection about x-axis}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{\text{CW Rotation}}{\begin{pmatrix} \text{Cos}45 & \text{Sin}45 & 0 \\ -\text{Sin}45 & \text{Cos}45 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{\text{Reduce y-intercept}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



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Now, the required co-ordinate can be calculated as:

P' = Com x P

$$= \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 4 \\ 3 & 3 & 5 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 4 \\ 3 & 5 & 5 \\ 1 & 1 & 1 \end{pmatrix}$$

Hence, the final coordinates are (2, 3), (2, 5) & (4, 5).

Exercise

Question: Reflect a triangle A(1,8) B(3,8) and C(1,6) about line $y= x+2$

X	Y
0	2
1	3
2	4
3	5

Exercise

Question: *A mirror is placed such that it passes through (0,10), (10, 0). Fin the mirror image of an object (6,7), (7, 6), (6, 9).*

Solution

Here,

The given mirror or line is passing through the points *(0, 10) & (10, 0).*

Now, the slope of the line

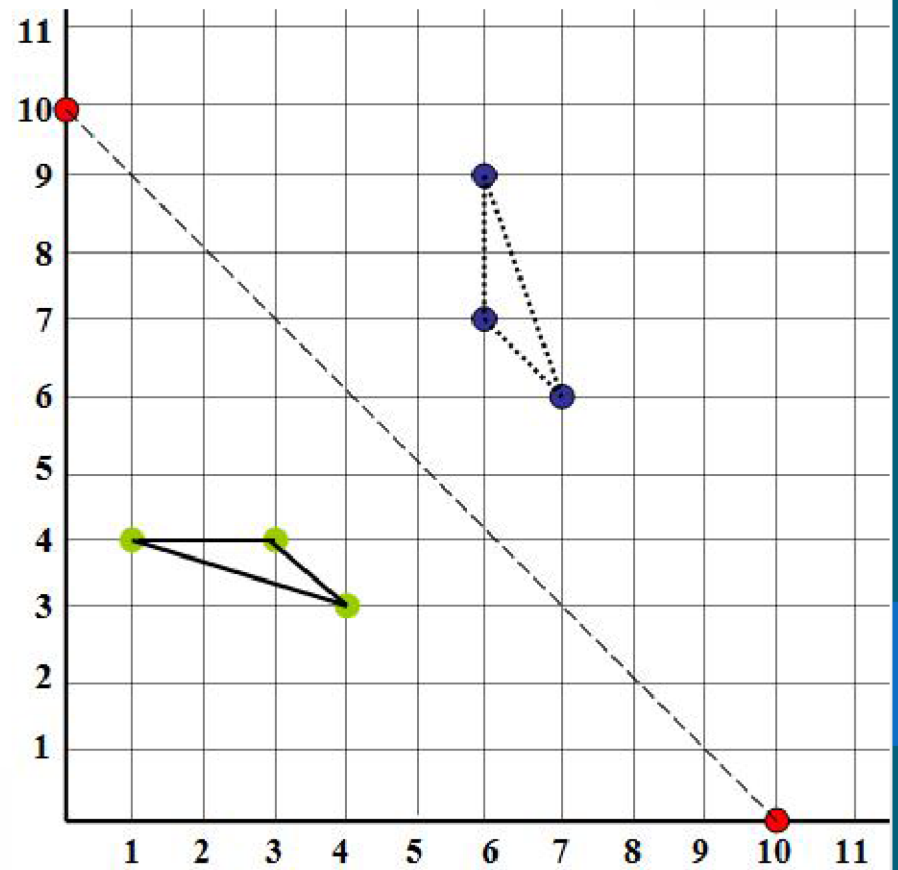
(m) = (y2-y1) / (x2-x1)

= (0 - 10) / (10 - 0) = -1

Thus, the rotation angle (**θ**)

= Tan^{-1}(m) = Tan^{-1}(-1)

= - 45^0



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The composite matrix is given by:

Com = $\boldsymbol{T}_{(0,\,10)\text{ or }(10,\,0)}.\,\boldsymbol{R}_{\theta\text{ in }CW}.\,\boldsymbol{R}_{fx}.\,\boldsymbol{R}_{\theta\text{ in }CCW}.\,\boldsymbol{T}_{(0,\,-10)\text{ or }(-10,\,0)}$

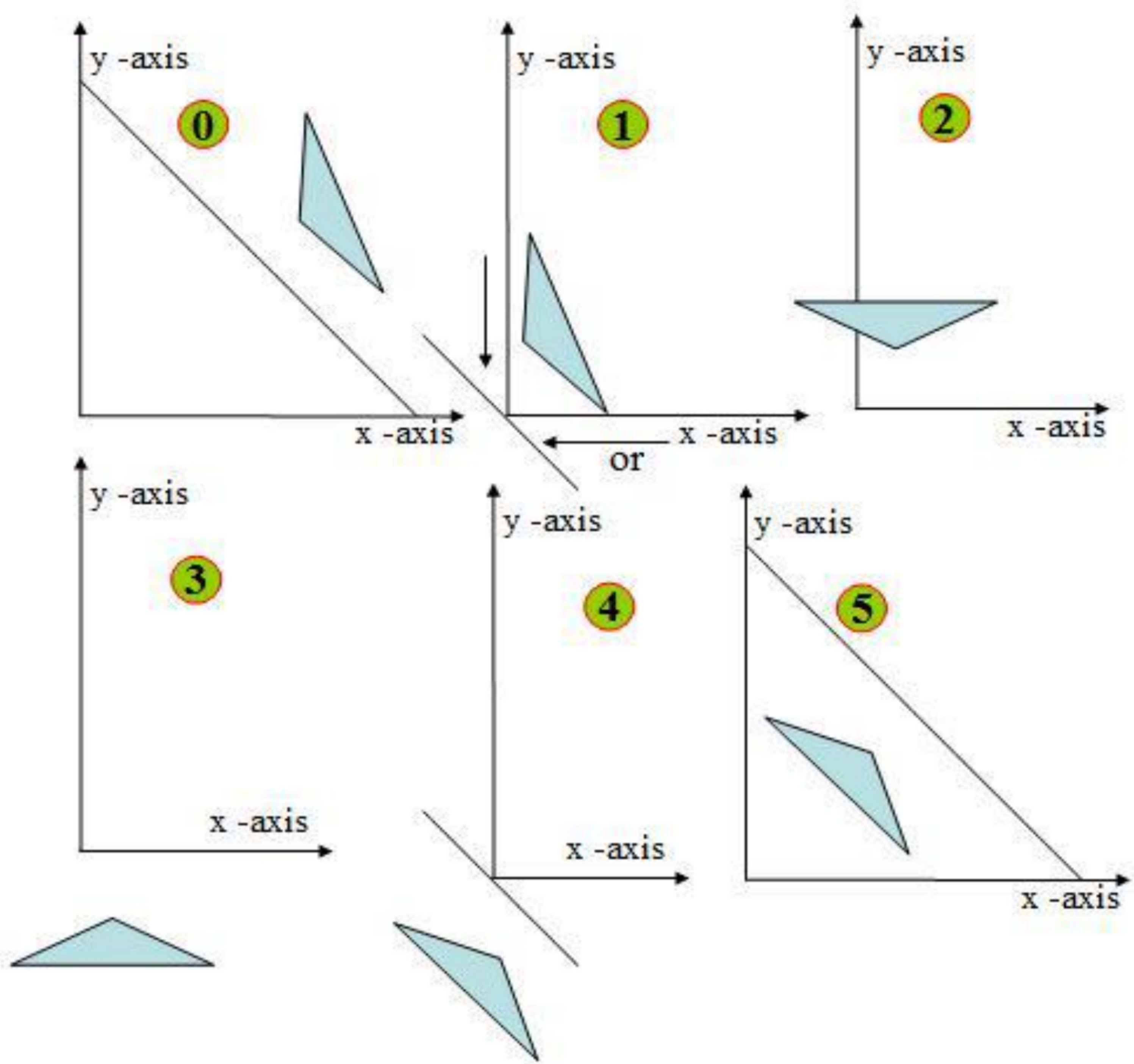
$$= \overset{\underline{\textit{Addition x-intercept}}}{\begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{\underline{\textit{CW Rotation}}}{\begin{pmatrix} \text{Cos45} & \text{Sin45} & 0 \\ -\text{Sin45} & \text{Cos45} & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{\underline{\textit{Reflection about x-axis}}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{\underline{\textit{CCW Rotation}}}{\begin{pmatrix} \text{Cos45} & -\text{Sin45} & 0 \\ \text{Sin45} & \text{Cos45} & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{\underline{\textit{Reduce x-intercept}}}{\begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & -10/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & -10/\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & -10/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 10/\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 10 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 10 \\ -1 & 0 & 10 \\ 0 & 0 & 1 \end{pmatrix}$$


y -axis
0
x -axis
y -axis
1
x -axis
or
y -axis
2
x -axis
y -axis
3
x -axis
y -axis
4
x -axis
y -axis
5
x -axis

Now, the required co-ordinate can be calculated as:

P' = Com . P

$$= \begin{pmatrix} 0 & -1 & 10 \\ -1 & 0 & 10 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 7 & 6 \\ 7 & 6 & 9 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 1 \\ 4 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

Hence, the final coordinates are (3, 4), (4, 3) & (1, 4).

General Properties

Preserved Attributes

	Line	Angle	Distance	Area
Translation	Yes	Yes	Yes	Yes
Rotation	Yes	Yes	Yes	Yes
Scaling	Yes	No	No	No
Reflection	Yes	Yes	Yes	Yes
Shear	Yes	No	No	Yes

Assignment

1. Find the coordinate of a triangle A(1,3) , B(2,5) and C(3,3) after being rotated about fixed point p(2,4) by 45 in clockwise direction and then translate by 3 unit along x-direction.

2. What do you mean by homogeneous coordinates? Rotate a triangle A(5,6), B(6,2) and C(4,1) by 45 degree about an arbitrary pivot point (3,3).

3. Find the coordinate of a *triangle (5, 5), (7, 3), (3, 3) after scaling (2,3) and then rotate 45 CCW then translate (1,1) and then reflect y=0 line.*

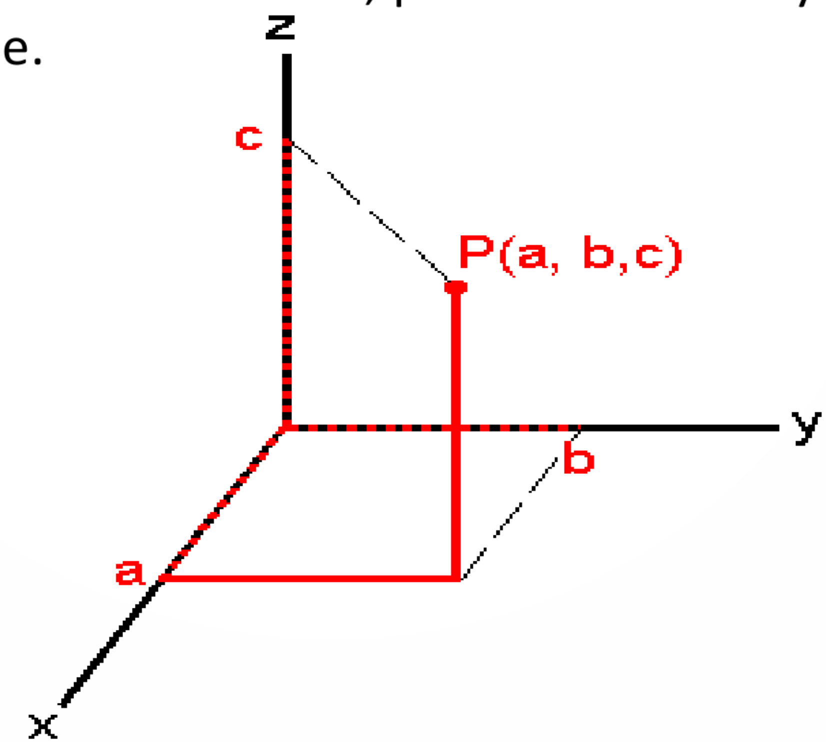
4. Find the coordinate of a triangle A(1,3) , B(2,5) and C(3,3) after being rotated about fixed point p(2,4) by 45 in clockwise direction and then translate (3,4) .

5. Find the coordinate of a triangle A(1,3) , B(2,5) and C(3,3) after twice its original size
   a) about origin
   b) about fixed point (2,3)

3D Transformation

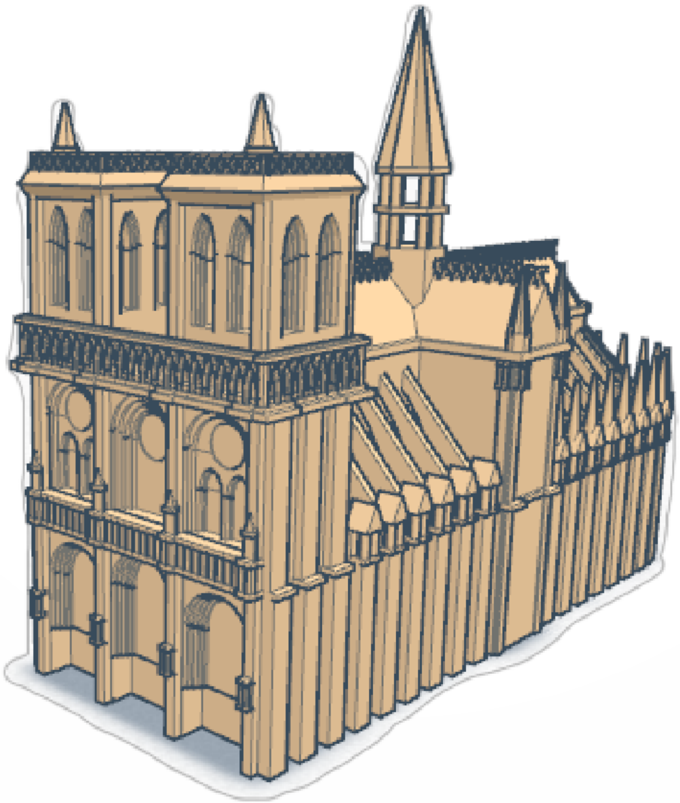
3-Dimension

- Three-dimensional space is a geometric 3-parameters model of the physical universe (without considering time) in which all known matter exists.
- These three dimensions can be labeled by a combination of length, breadth, and depth.
- Any three directions can be chosen, provided that they do not all lie in the same plane.



3-Dimensional Object

- An object that has height, width and depth, like any object in the real world is a 3 dimensional object.
- Types of objects: Geometrical shapes, trees, terrains, clouds, rocks, glass, hair, furniture, human body, etc.



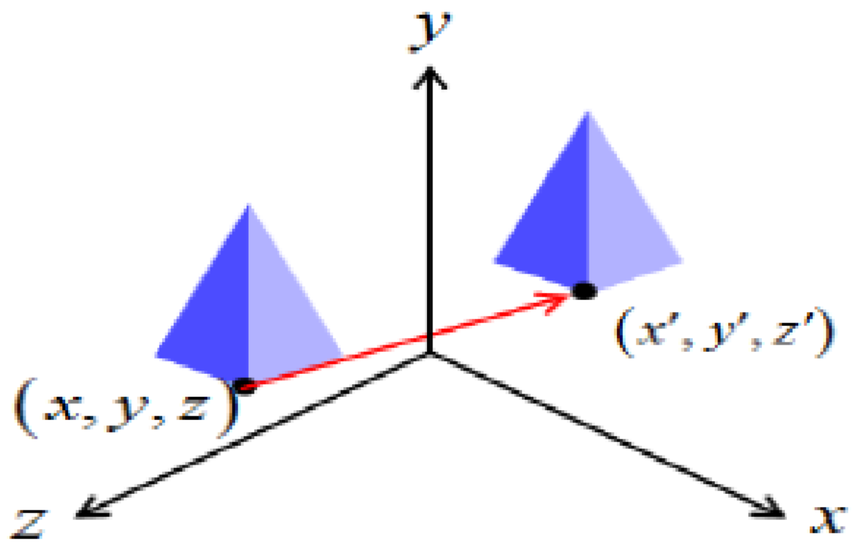
3D Transformations

- Just as 2D-transfromtion can be represented by 3x3 matrices using homogeneous co-ordinate can be represented by 4x4 matrices, provided we use homogenous co-ordinate representation of points in 3D space as well.
  1. Translation
  2. Rotation
  3. Scaling
  4. Reflection
  5. Shear

3D Translation

- Translation in 3D is similar to translation in the 2D except that there is one more direction parallel to the z-axis. If, tx, ty, and tz are used to represent the translation vectors. Then the translation of the position P(x, y, z) into the point P' (x', y', z') is done by
  - $x' = x + t_x$
  - $y' = y + t_y$
  - $z' = z + t_z$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = T.P$$

- In matrix notation using homogeneous coordinate this is performed by the matrix multiplication

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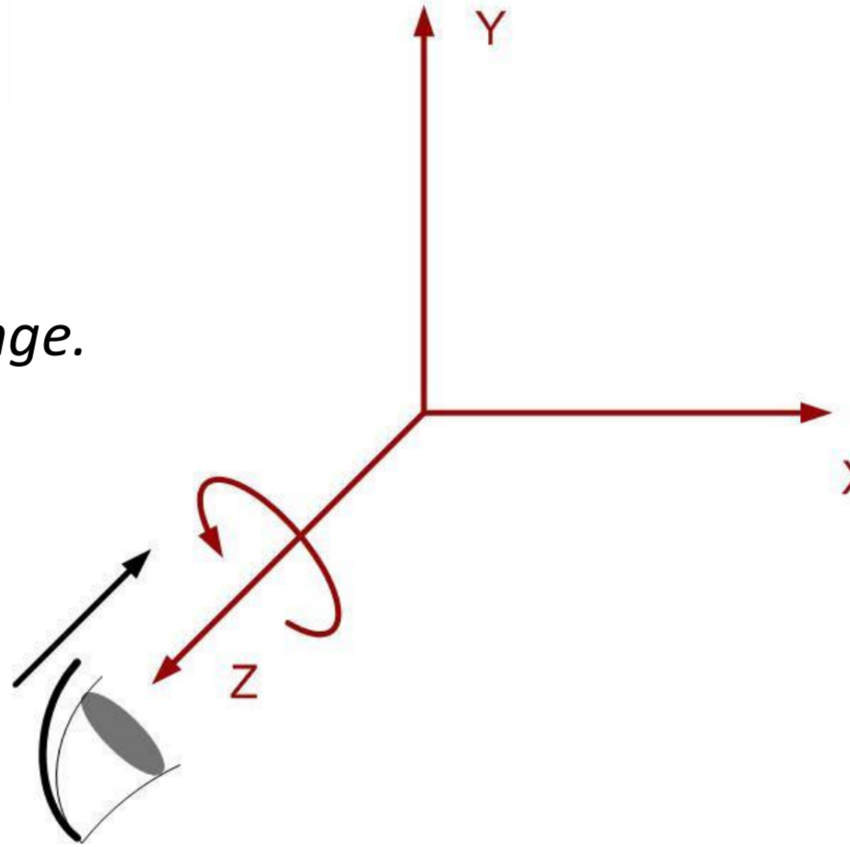
3D Rotation

(i) Rotation About z-axis:

Z-component does not change.

X' = X cosθ – Y sinθ
Y' = X sinθ + Y cosθ
Z' = Z



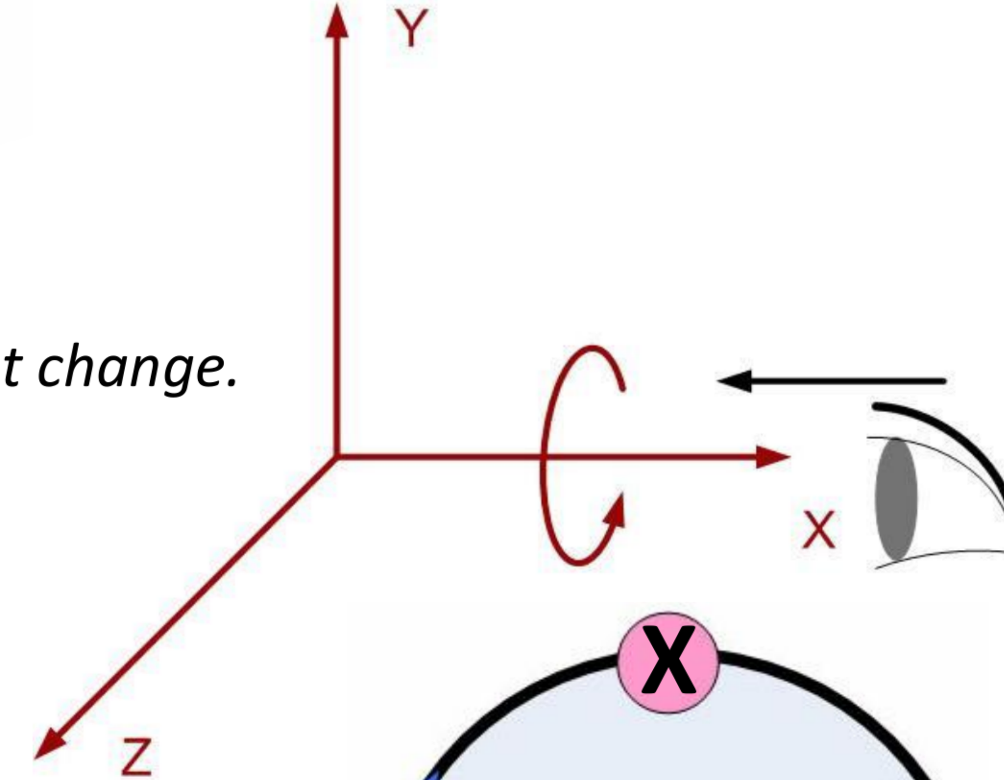
Matrix representation for rotation around z-axis,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

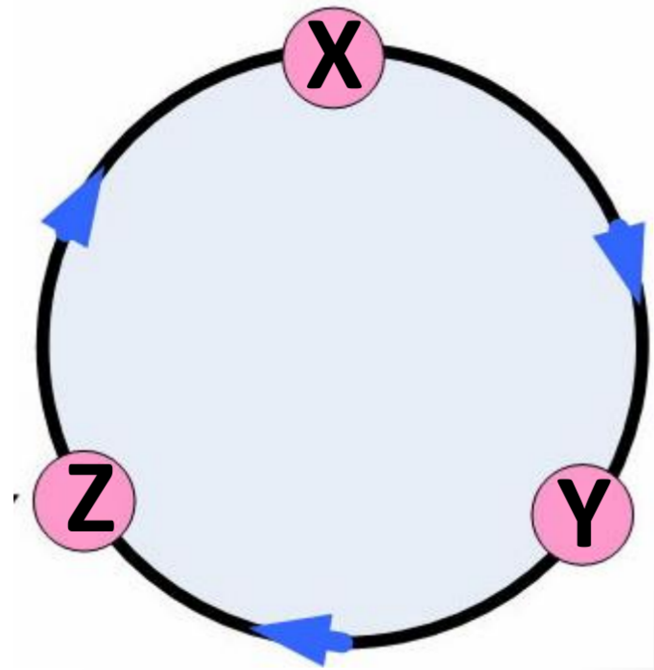
3D Rotation

(ii) Rotation About x-axis:

X-component does not change.



Y' = Y cosθ - Z sinθ
Z' = Y sinθ + Z cosθ
X' = X



Matrix representation for rotation around X-axis,

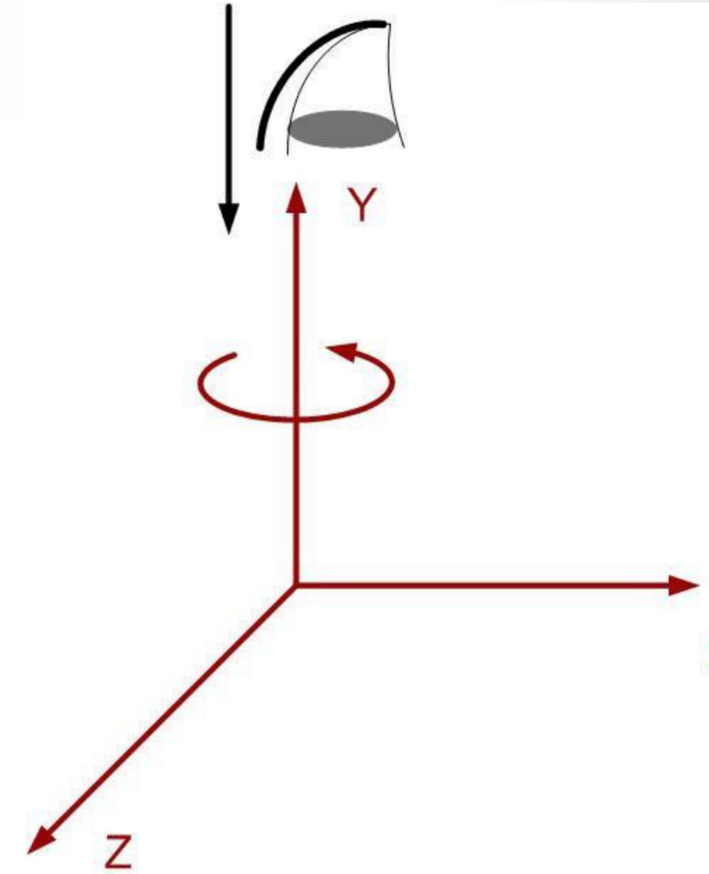
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Rotation

(iii) Rotation About Y-axis:

Y-component does not change.

Z' = Z cosθ – X sinθ
X' = Z sinθ + X cosθ
Y' = Y



Matrix representation for rotation around Y-axis,

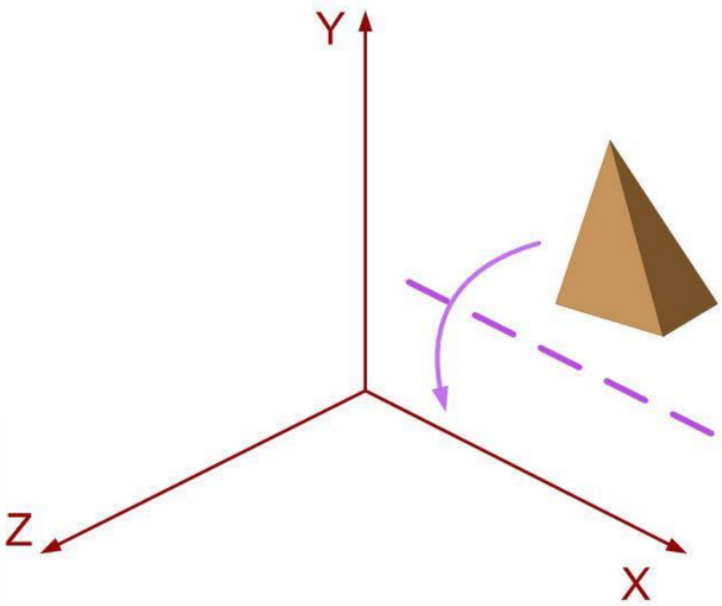
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Rotation

❑General 3D Rotations:

(a) Rotation about an axis parallel to any of the co-axis: *When an object is to be rotated about an axis that is parallel to one of the co-ordinate axis, we need to perform series of transformation.*

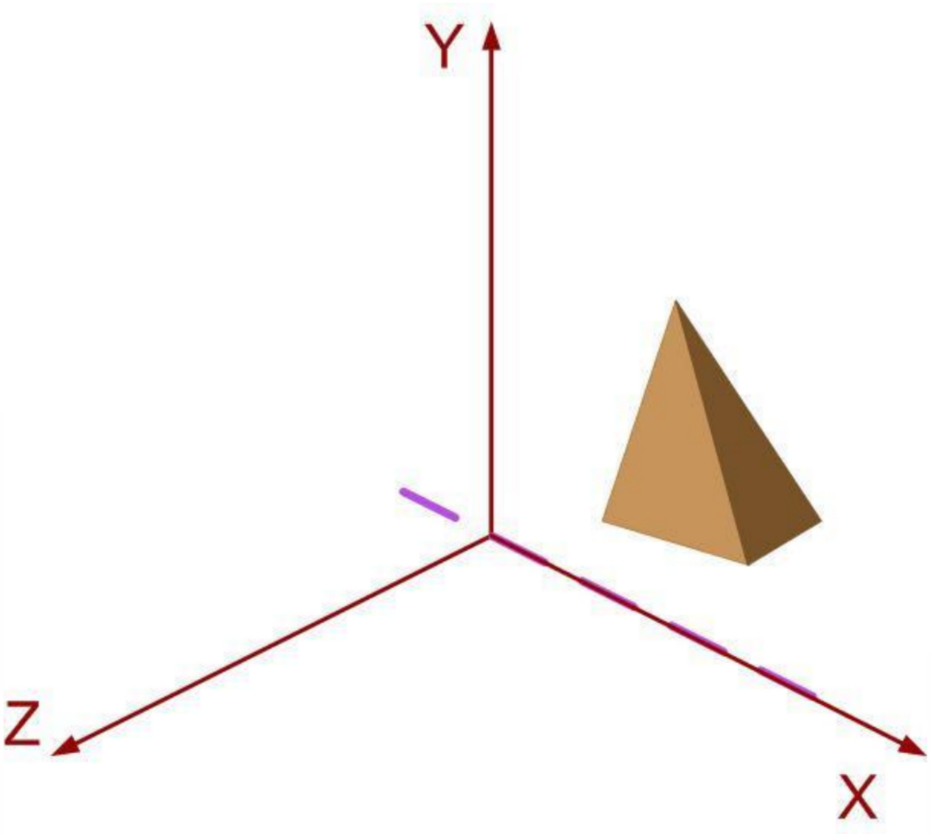
i. Translate the object so that the rotation axis coincides with the parallel co-ordinate axis.
ii. Perform the specified rotation about the axis.
iii. Translate the object so that the rotation axis is moved to its original position.



a) Rotation about an axis parallel to any of the co-axis:

Step 1

- Translate the object so that the rotation axis coincides with the parallel co-ordinate axis.



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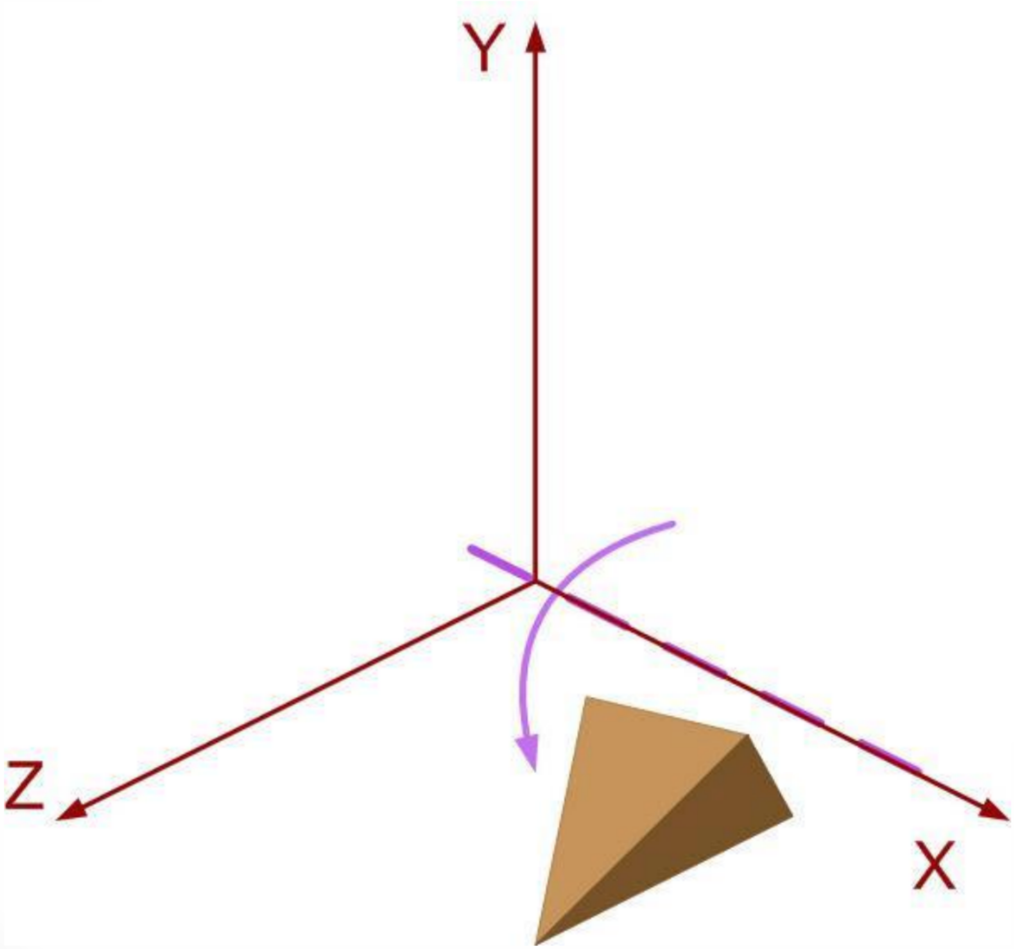
a) Rotation about an axis parallel to any of the co-axis:

Step 2

- Perform the specified rotation about the axis.



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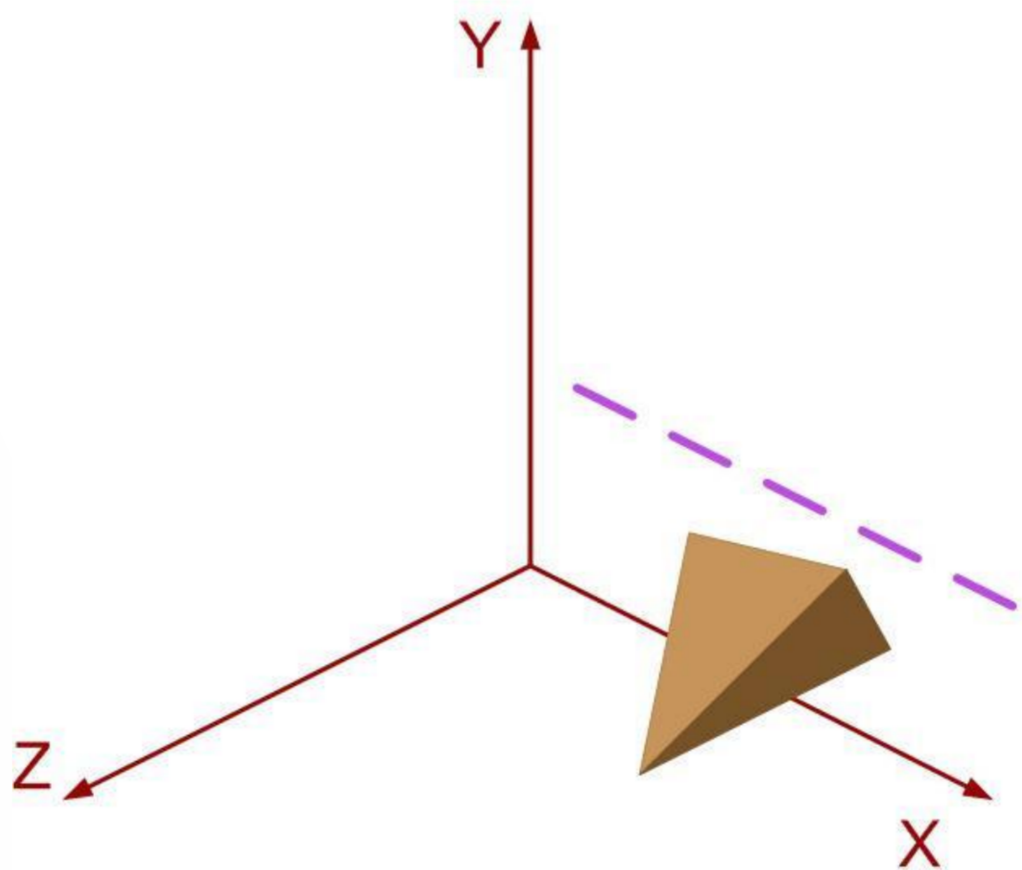
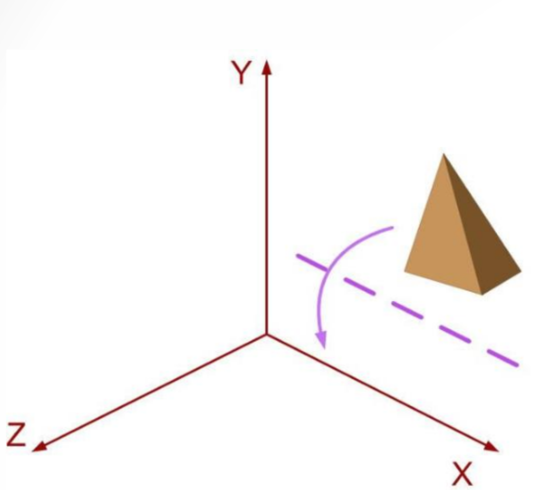
a) Rotation about an axis parallel to any of the co-axis:

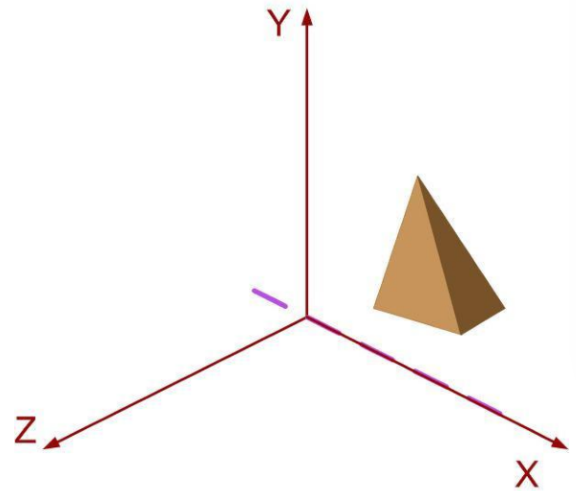
Step 3

- Translate the object so that the rotation axis is moved to its original position.

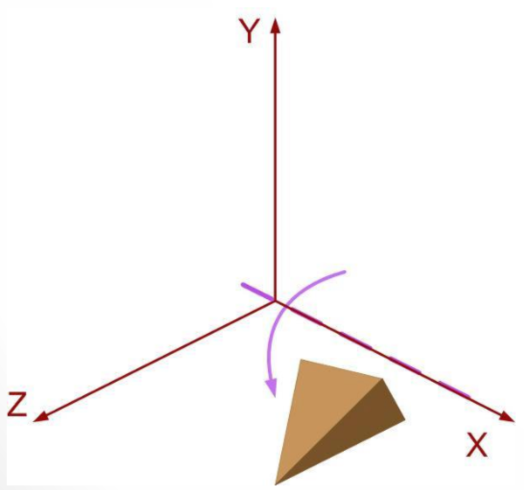


a) Rotation about an axis parallel to any of the co-axis:

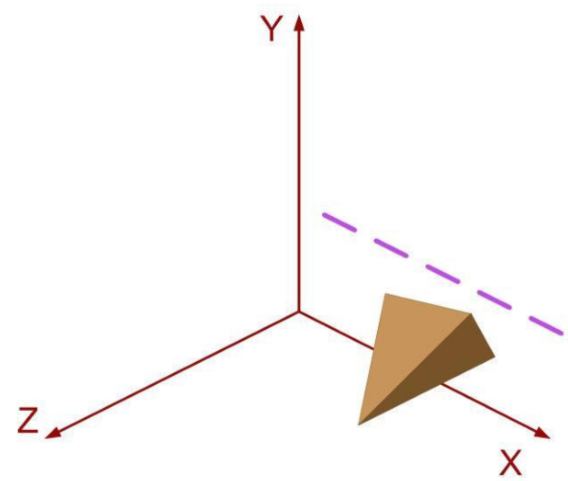




Step 1: Translate



Step 2: Rotation



Step 3: Translate to original place

3D Rotation

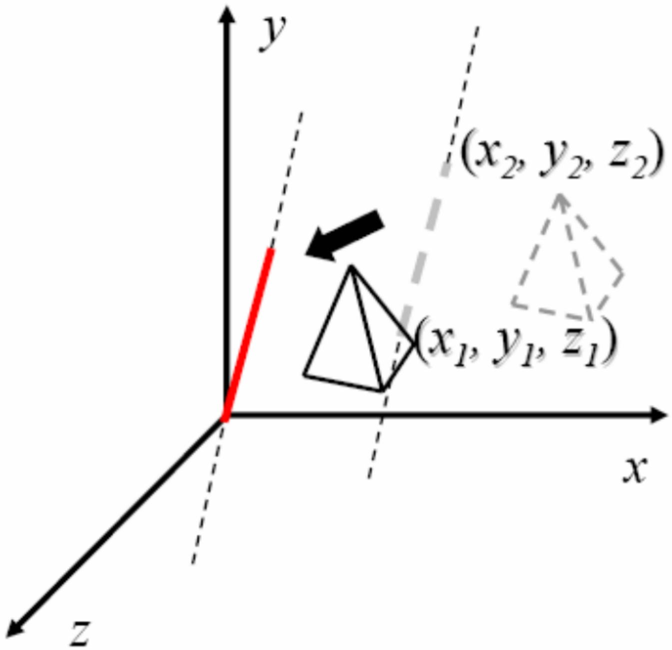
- **General 3D Rotations:**

  (b) Rotation about an axis not parallel to any of the co-axis:

  i. Translate the object such that rotation axis passes through co-ordinate origin.
  ii. Rotate the axis such that axis of rotation coincides with one of the co-ordinate axis.
  iii. Perform the specific rotation about the ordinate axis.
  iv. Apply inverse rotation to bring the rotation axis back to its original orientation.
  v. Apply inverse translation to bring the rotation axis back to its original position.

(b) Rotation about an axis not parallel to any of the co-axis:

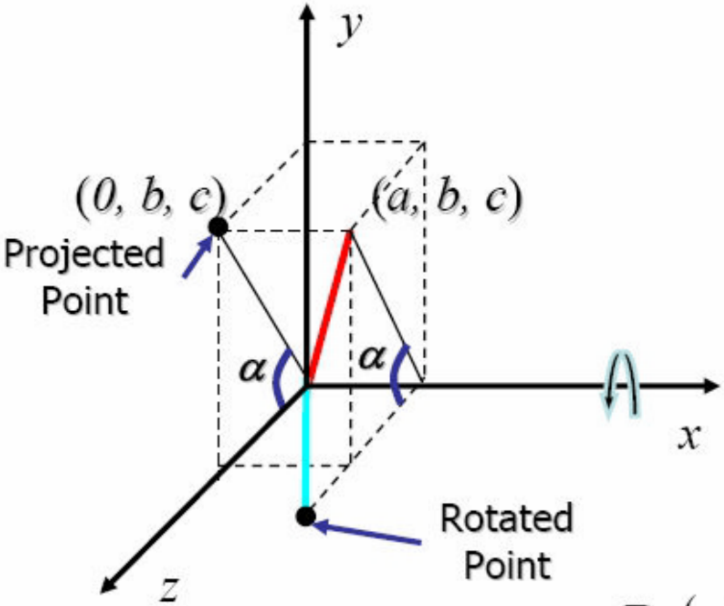
- Step 1. Translate



$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation about an axis not parallel to any of the co-axis:

- Step 2. Rotate about x axis by α



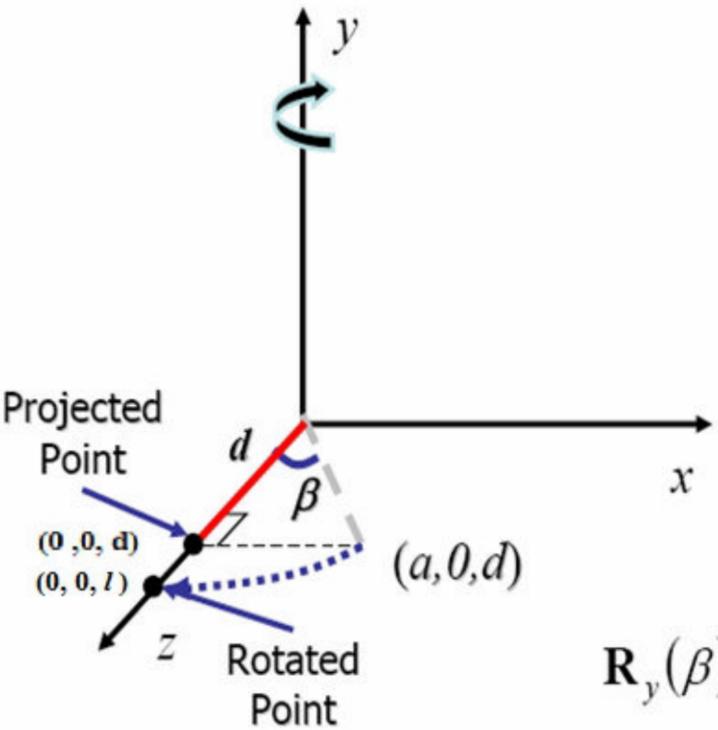
$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation about an axis not parallel to any of the co-axis:

- Step 3. Rotate about y axis by β (clockwise)



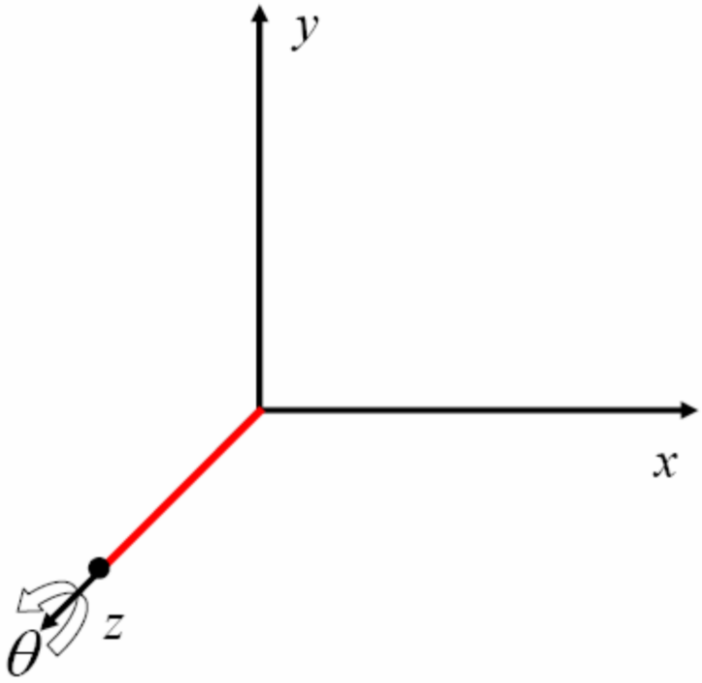
$$\sin\beta = \frac{a}{l}, \qquad \cos\beta = \frac{d}{l}$$

$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

$$\mathbf{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Rotation about an axis not parallel to any of the co-axis:

- Step 4. Rotate about *z* axis by the angle $\boldsymbol{\theta}$



$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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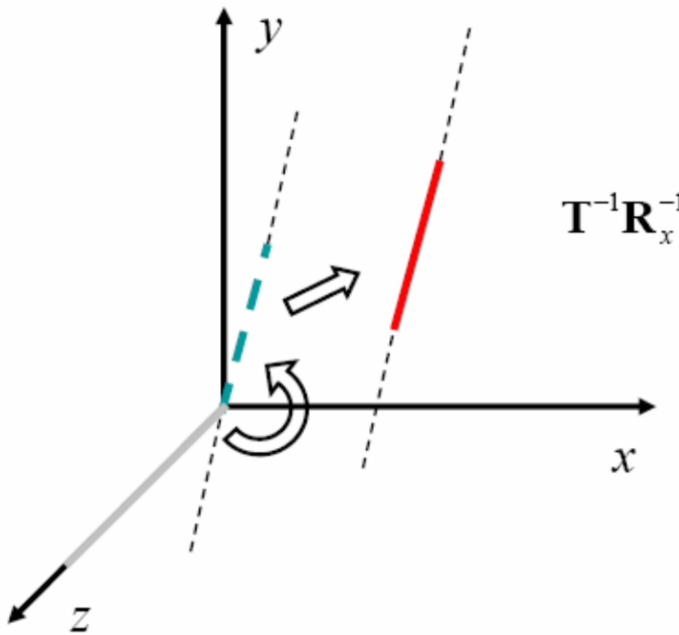
(b) Rotation about an axis not parallel to any of the co-axis:

- Step 5. Reverse transformation

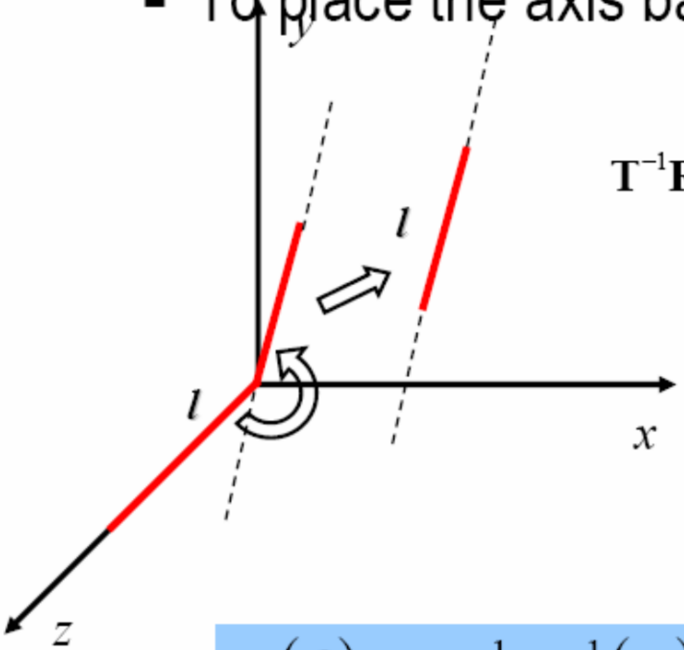


$$\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)=\begin{bmatrix}1&0&0&x_1\\0&1&0&y_1\\0&0&1&z_1\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\cos\alpha&\sin\alpha&0\\0&-\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\cos\beta&0&\sin\beta&0\\0&1&0&0\\-\sin\beta&0&\cos\beta&0\\0&0&0&1\end{bmatrix}$$

$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

(b) Rotation about an axis not parallel to any of the co-axis:

- Step 5. Reverse transformation
  - To place the axis back in its initial position



$$\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)=\begin{bmatrix}1&0&0&x_1\\0&1&0&y_1\\0&0&1&z_1\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\cos\alpha&\sin\alpha&0\\0&-\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}\cos\beta&0&\sin\beta&0\\0&1&0&0\\-\sin\beta&0&\cos\beta&0\\0&0&0&1\end{bmatrix}$$

$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

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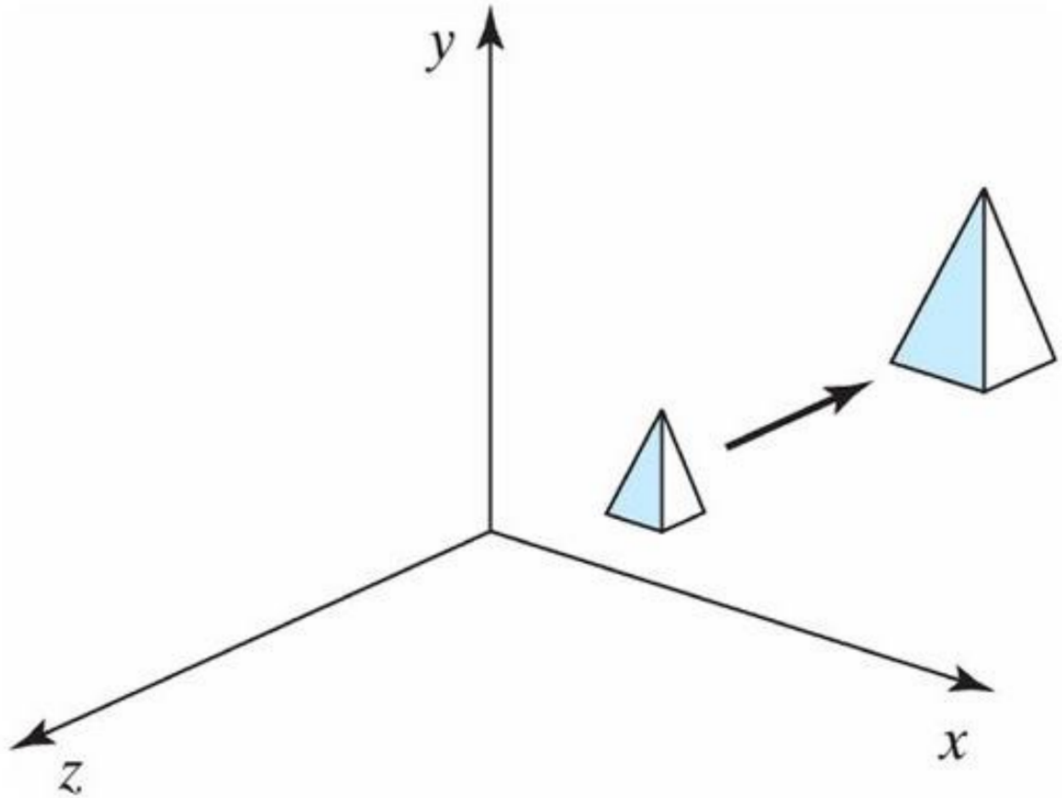
3D Scaling

Scaling about origin

X' = X . Sx

Y' = Y . Sy

Z' = Z . Sz

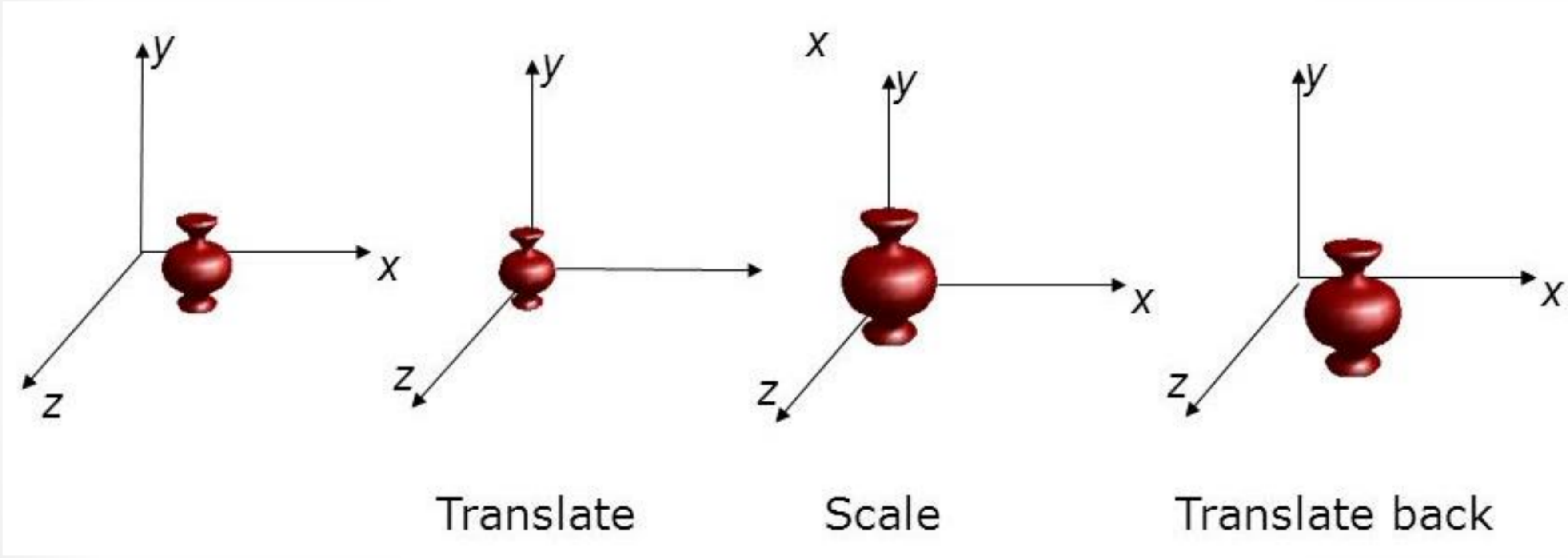


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

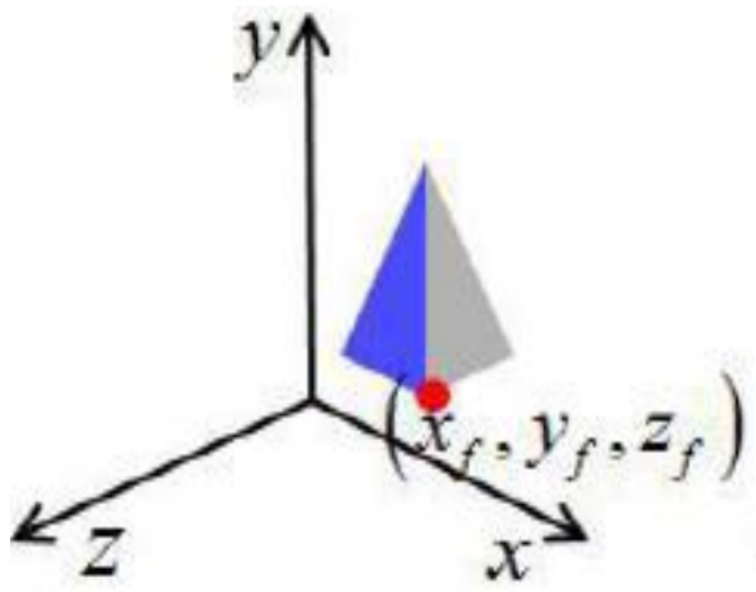
3D Scaling

- *Scaling about an arbitrary point or Fixed point (xf, yf, zf)*



3D Scaling

- *Scaling about an arbitrary point or Fixed point (xf, yf, zf)*



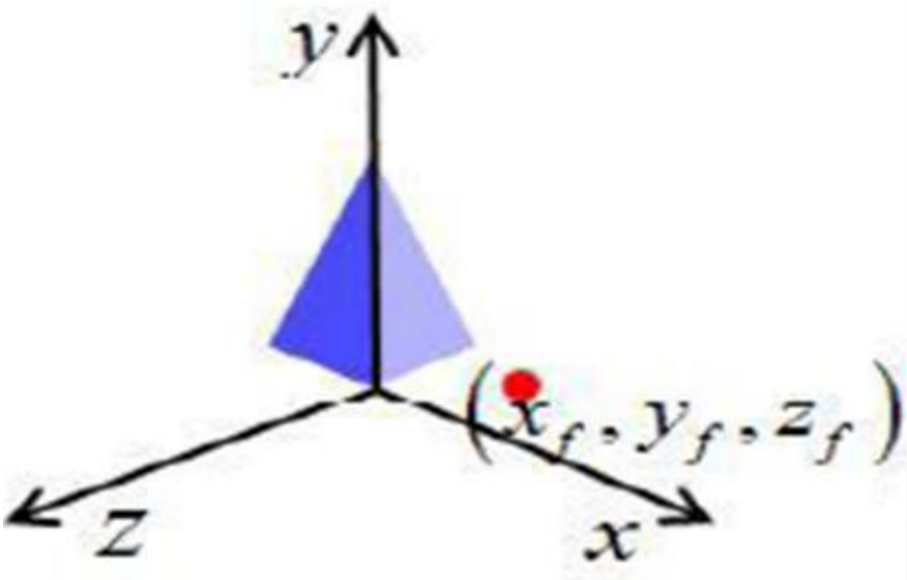
3D Scaling

- ***Scaling about an arbitrary point or Fixed point (xf, yf, zf)***

Step 1

- Translate the fixed point to the origin

$$= \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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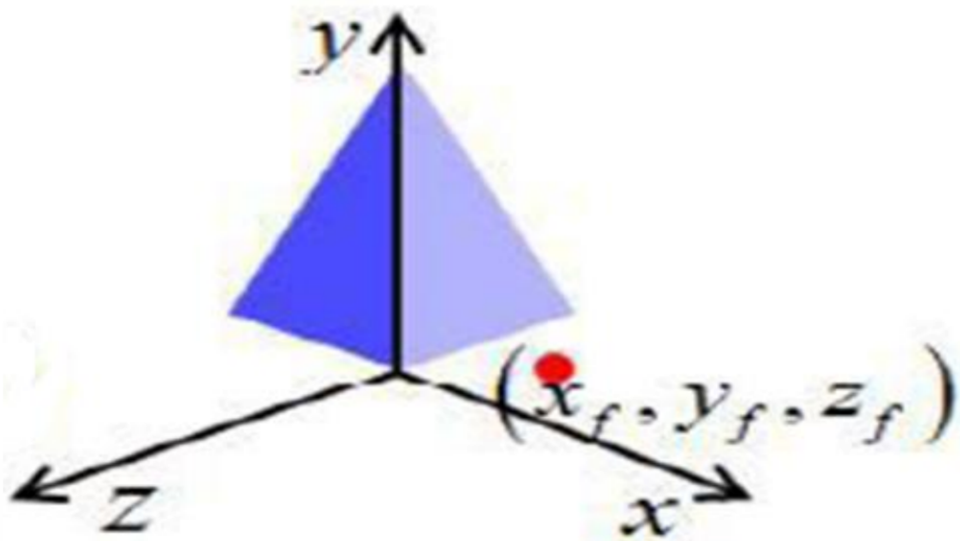
3D Scaling

- *Scaling about an arbitrary point or Fixed point (xf, yf, zf)*

Step 2

- Scale the object relative to the coordinate origin.

$$= \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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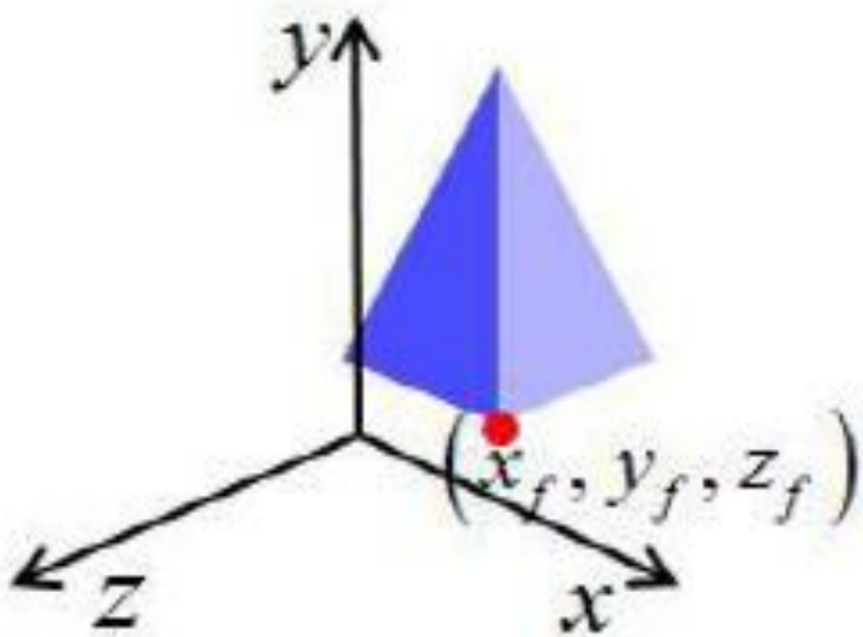
3D Scaling

- ***Scaling about an arbitrary point or Fixed point (xf, yf, zf)***

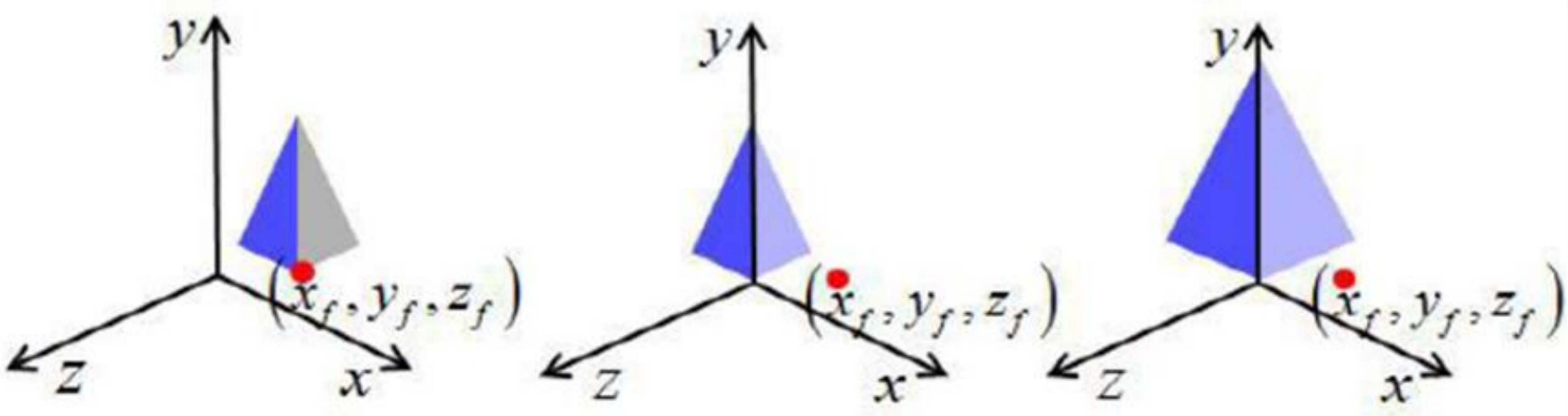
Step 3

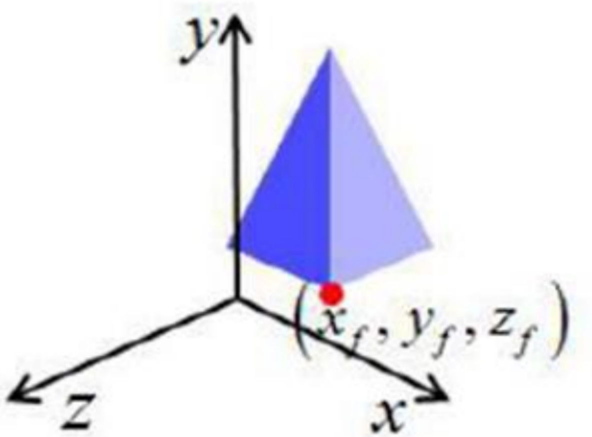
- Translate the fixed point back to its original position.

$$= \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Scaling





$$\mathbf{T}\cdot\mathbf{S}\cdot\mathbf{T}^{-1} = \begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f \\ 0 & S_y & 0 & (1-S_y)y_f \\ 0 & 0 & S_z & (1-S_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

C.M. $= T(x_f, y_f, z_f).S(s_x, s_y, s_z).T(-x_f, -y_f, -z_f)$

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3D Reflection

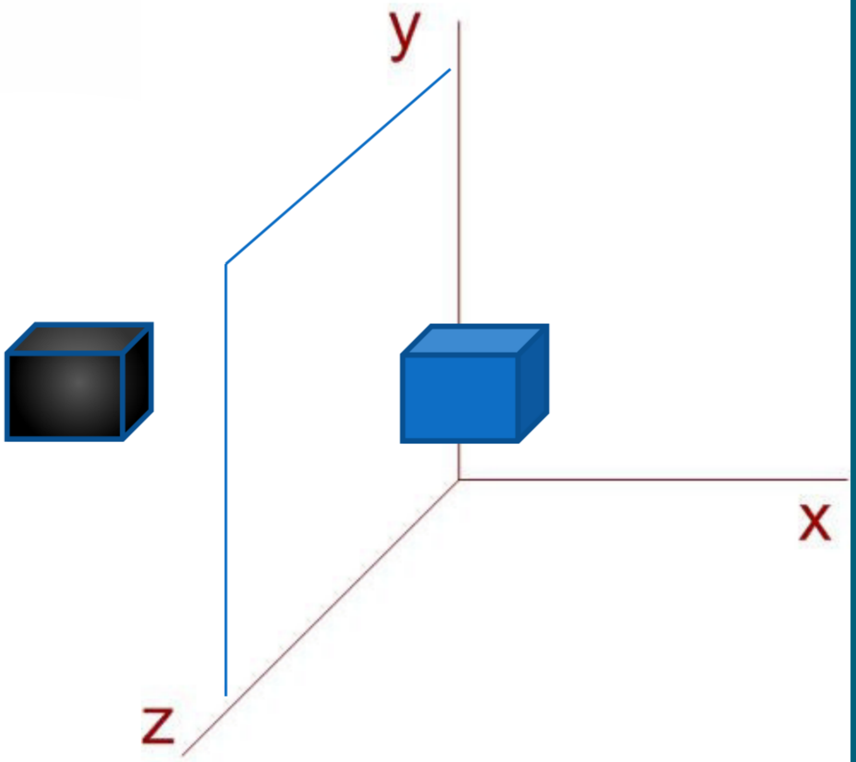
(i) Reflection about yz plane

X’ = -X

Y’ = Y

Z’ = Z

$$T_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



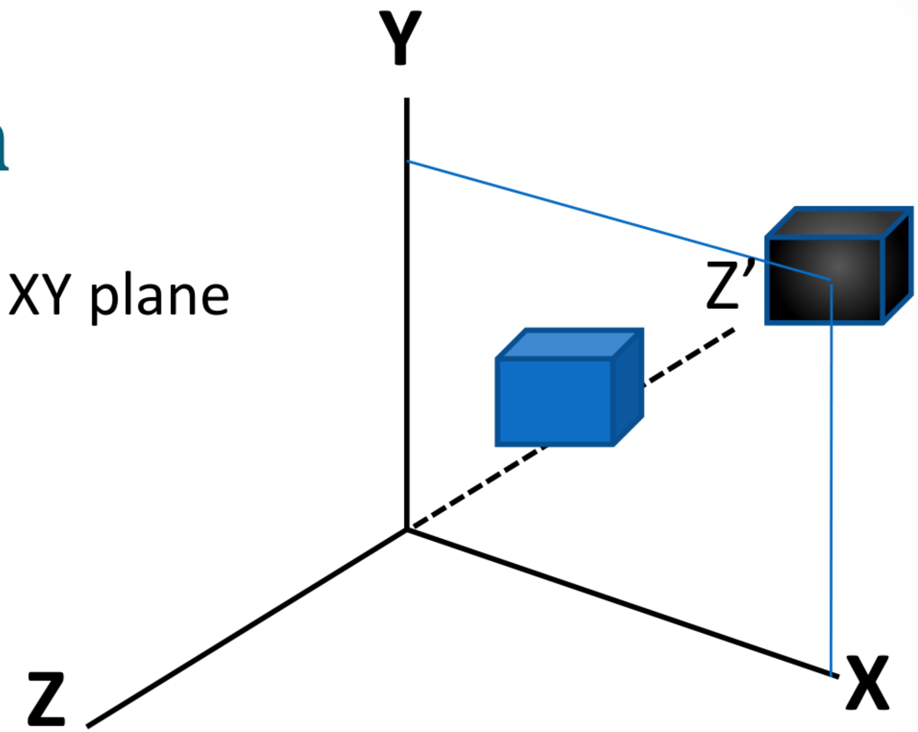
3D Reflection

(ii) Reflection about XY plane

X’ = X

Y’ = Y

Z’ = -Z



$$T_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

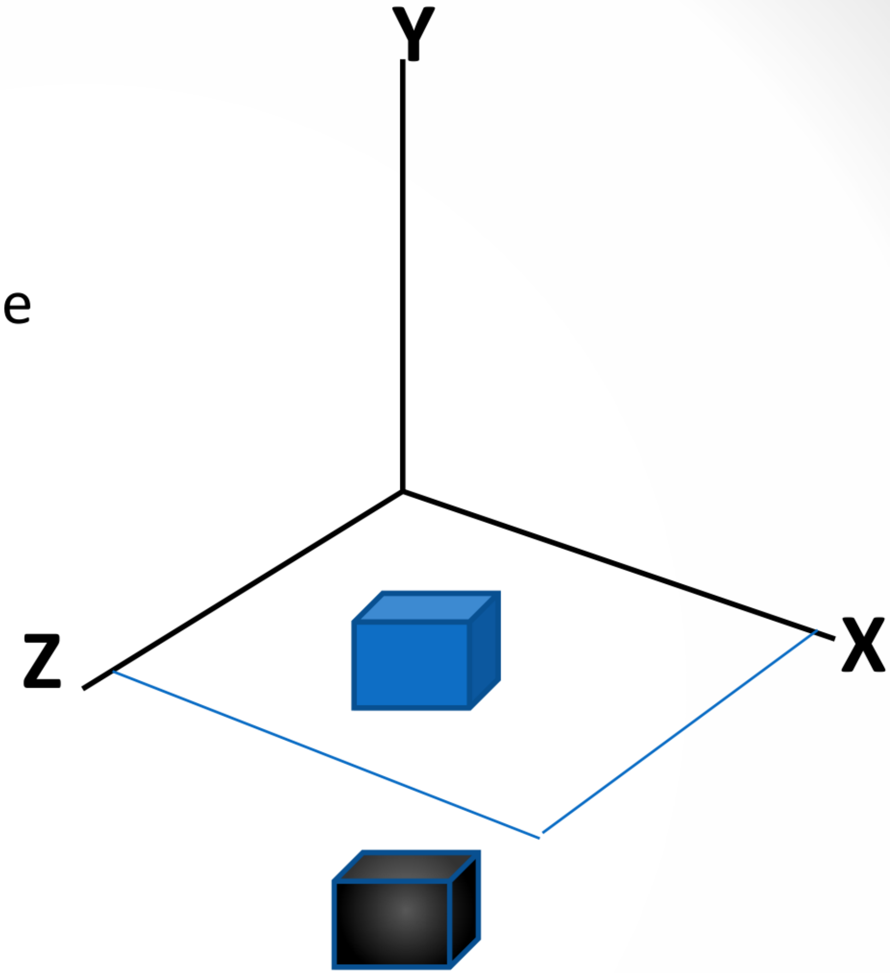
3D Reflection

(iii) Reflection about XZ plane

X’ = X

Y’ = -Y

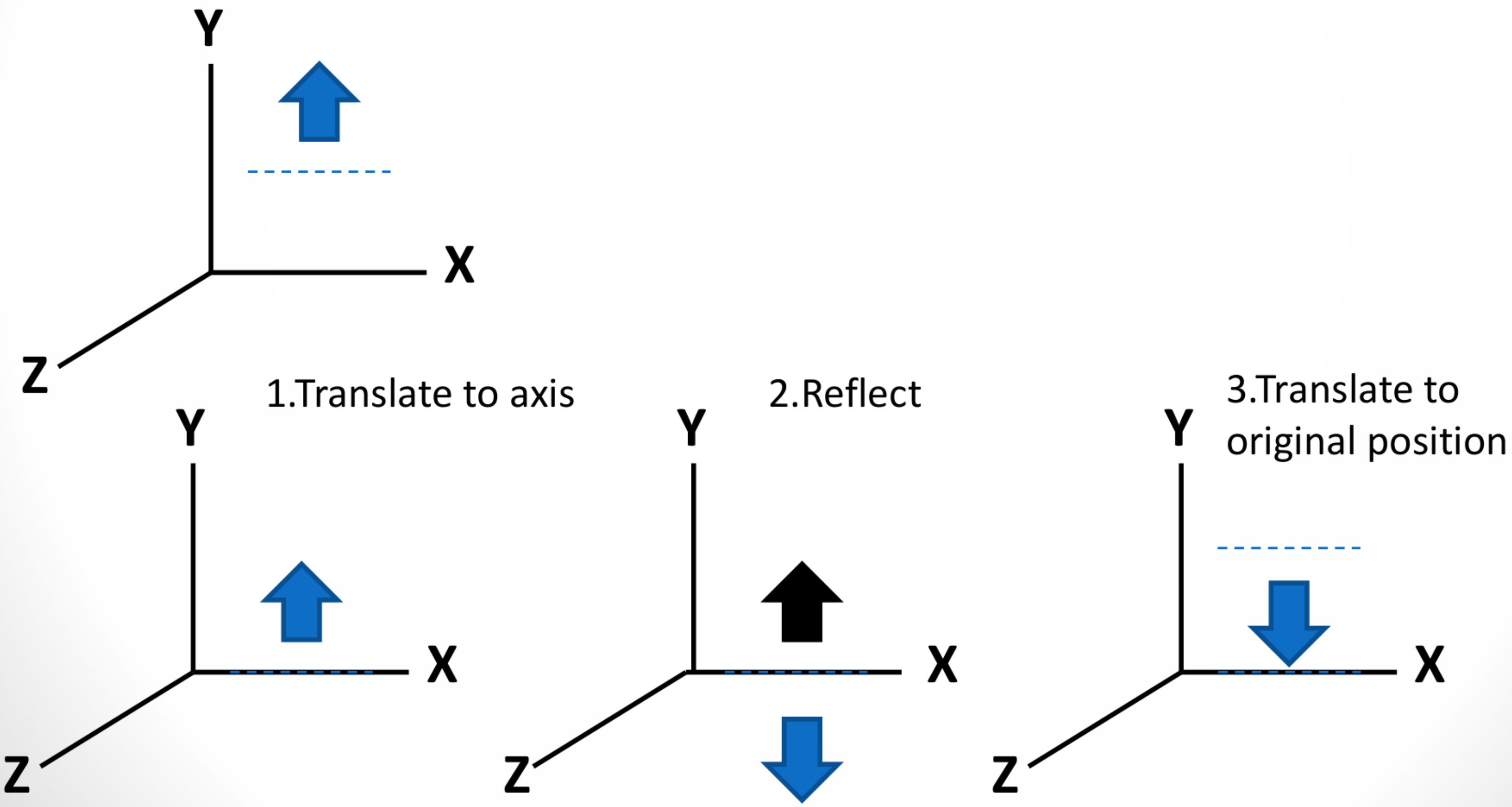
Z’ = Z



$$T_{xz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Reflection

- Reflection of an object about a line that is parallel to one of the major coordinate axes



3D Shear

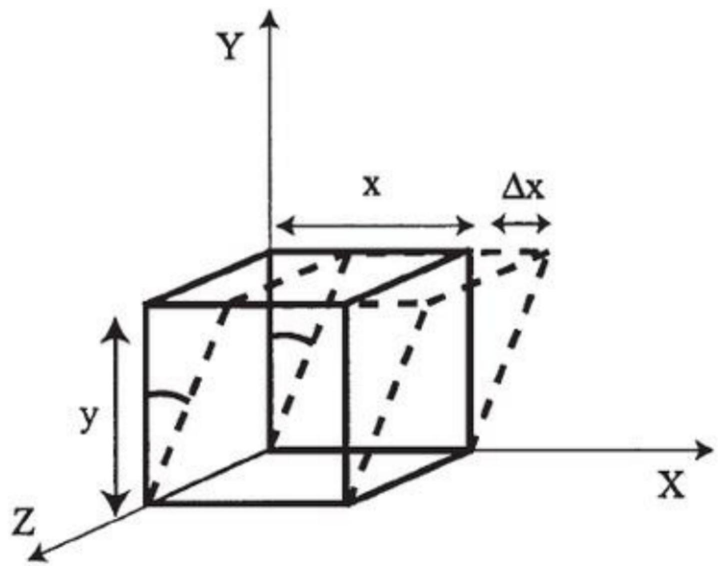
- Shearing transformations can be used to modify object shapes.

Z-axis Shear

- This transformation alters x- and y-coordinate values by an amount that is proportional to the z value, while leaving the z coordinate unchanged, i.e.

  $x' = x + S_{hx} . z$
  $y' = y + S_{hy} . z$
  $z' = z$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & S_{hx} & 0 \\ 0 & 1 & S_{hy} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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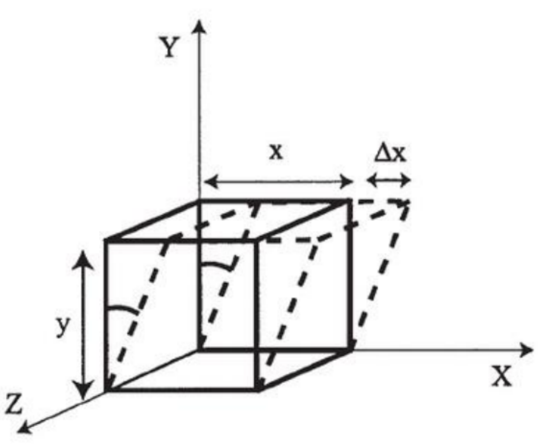
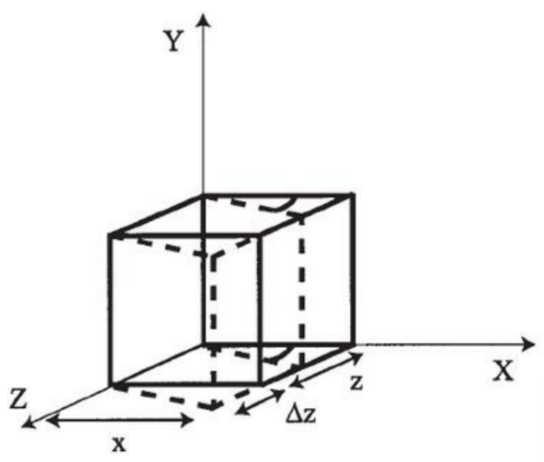
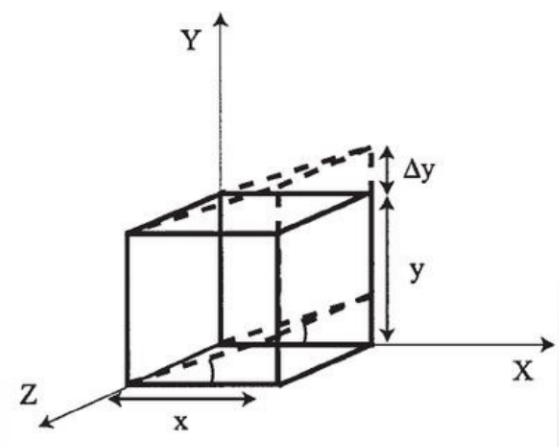
3D Shear

- Similarly, we can find X-axis shear, Y-axis shear and Z-axis shear.

$$SH_z = \begin{bmatrix} 1 & 0 & S_{hx} & 0 \\ 0 & 1 & S_{hy} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad SH_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_{hx} & 1 & 0 & 0 \\ S_{hx} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad SH_y = \begin{bmatrix} 1 & S_{hx} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & S_{hx} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$







Board Exam Questions

1. Define reflection transformation and derive the 2D reflection matrix along x-axis and y-axis in homogeneous coordinate. (2076 Batch)
2. What is the need of homogeneous coordinate system in geometric transformation system? Find the new coordinate of rectangle ABCD whose center is at (4,2) is reduced to half of its size and center will remain same. The coordinate of ABCD are A(0,0), B(0,4), C(8,4), and D(8,0). (2076 Batch)
3. Define scaling transformation? Prove that two successive scaling are multiplicative. (2075 Batch)

Board Exam Questions

4. Reflect a prism A(0,0,0), B(1,1,0), C(1,2,2) and (0,2,0) about YZ-plane which has been rotated previously with +90 degree about y-axis. (2075 Batch)
5. Explain 3D basic geometric transformation with an example. (2074 Batch)
6. Given a triangle with vertices A(2,3), B(5,5), C(4,3) by rotating 90 degree about the origin and then translating two unit in each direction. Use homogenous transformation matrix to find the new vertices of the triangle. (2074 Batch)