UC San Diego
Halıcıoğlu Data Science Institute
DSC140A Probabilistic Modeling and Machine Learning

Problem Set 1
Due: 11:59 PM, 04/11/2025

Additional References

- ⋆ Lecture 2 & Discussion 1
- ⋆ Estimating Probabilities: MLE & MAP. Tom Mitchell, 2016.
- ⋆ PML Ch 4.2
- ⋆ BRML Ch 8.1, 8.3, 8.6
- ⋆ MIT OCW Reading 10b and Reading 24

1. MLE Basics

Suppose that we are given n data points drawn independently from a Normal distribution:

$$x_1, x_2, \ldots, x_n \overset{\text{iid}}{\sim} \mathsf{Normal}(\mu, \sigma).$$

(a) What is the likelihood function $L(\mu, \sigma \mid x_1, \ldots, x_n)$?

(b) What is the maximum likelihood estimator $\hat{\mu}_{\text{MLE}}$?

(c) What is the expected value of the estimator $\mathrm{E}(\hat{\mu}_{\text{MLE}})$? Is it biased?

(d) What is the variance of estimator $\mathrm{Var}(\hat{\mu}_{\text{MLE}})$? Is it consistent?

(e) What is the maximum likelihood estimator $\hat{\sigma}_{\text{MLE}}$?

2. MLE Basics (Matrix version)

Suppose that we are given n data points drawn independently from a d-dimensional normal distribution with known covariance matrix Σ:

$$x_1, x_2, \ldots, x_n \overset{\text{iid}}{\sim} \mathsf{Normal}(\theta, \Sigma).$$

(a) What is the likelihood function $L(\theta, \Sigma \mid x_1, \ldots, x_n)$?

(b) What is the maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$?

(c) What is the expected value of the estimator $\mathrm{E}(\hat{\theta}_{\text{MLE}})$? Is it biased?

(d) What is the variance of estimator $\mathrm{Var}(\hat{\theta}_{\text{MLE}})$? Is it consistent?

(e) What is the maximum likelihood estimator $\hat{\Sigma}_{\text{MLE}}$?

3. MLE for the Exponential Distribution

Suppose we are given n data points drawn independently from an Exponential distribution

$$x_1, x_2, \ldots, x_n \overset{\text{iid}}{\sim} \mathsf{Exponential}(\theta).$$

(a) Write the probability density function for a single point x_i.

(b) Describe a real-world outcome that we could model using this distribution.

(c) Derive an expression for $\hat{\theta}_{\text{MLE}}$, the maximum likelihood estimator of θ?

(d) Describe how you could compute $\hat{\theta}_{\text{MLE}}$ using only a summary statistic of the data?

(e) Say you compute the MLE estimator using n data points, and then delete the data. Describe how you can update the MLE estimator with another point x_{n+1}?

4. MLE without First-Derivatives

(a) Suppose we have 5 data points drawn independently from a $\mathsf{Uniform}(a, b)$ distribution: 1.2, 2.1, 1.3, 10.5, 5. What are the maximum likelihood estimates for the parameters a and b?

(b) Suppose that we have n data points $x_1, x_2, \ldots, x_n$ drawn independently from a $\mathsf{Uniform}(a, b)$ distribution. What are the maximum likelihood estimators for a and b?

5. Simple Linear Regression via MLE

In this problem, we will use maximum likelihood estimation to fit the "most likely" line between x and y. We assume that x and y are related to one another through a model of $y = ax + b$ for some constants a and b. Our goal is to estimate the values of a and b from n data points $(x_1, y_1), \ldots, (x_1, y_n)$.

We can estimate these values using MLE if we assume that each y_i is observed with "measurement error" $\epsilon_i \overset{\text{iid}}{\sim} \mathsf{Normal}(0, \sigma^2)$. Each ϵ_i is sampled from a Normal distribution with mean 0 and variance σ^2, where σ is a positive value like $\sigma = 1$.

Putting everything together, this means that each y_i is an independent draw from a random variable

$$Y_i \sim ax_i + b + \epsilon_i,$$

where $\epsilon_i \sim \mathsf{Normal}\left(0, \sigma^2\right)$.

(a) Write an expression for the likelihood function $L(y_i \mid a, b, x_i, \sigma)$ for one observation of y_i.

(b) Suppose that we are given 3 data points $(1, 8), (3, 2), (5, 1)$. Write down the expressions for the likelihood function and the log-likelihood function as functions of a, b, and σ.

(c) Assume that σ is a constant, known value. For the data in part (b), find the maximum likelihood estimates for a and b.

(d) For general data $(x_1, y_1) \ldots (x_n, y_n)$, write down the expression for the likelihood and the log-likelihood functions as functions of a, b, σ.

(e) Use Python to plot the data and the line from part (b). Print the plot and turn it in.

6. Complete the Google form at the following link:

https://forms.gle/XLctPCtFurMynQS37