Homework 5

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2/20/2019

Problem 2.4.1

Original Solution

Part A

Show the sequence is bounded and monotone.

> Monotone

$$x_2 = \frac{1}{4-3} = 1$$

Thus $x_2 < x_1$ and the induction is grounded.

$$x_{n+1} \le x_n$$

$$4 - x_{n+1} \ge 4 - x_n$$

$$\frac{1}{4 - x_{n+1}} \le \frac{1}{4 - x_n}$$

Thus $x_{n+1} \le x_n$. Inductive step complete and the sequence is decreasing.

> Bounded

Try $a_n > \frac{1}{4}$

$$a_1 = 3 > \frac{1}{4}$$

Suppose $a_n > \frac{1}{4}$ for some $n \geq 1$.

$$\frac{1}{4 - a_n} > \frac{1}{4}$$

Thus the sequence is bounded.

Part B

If x_n exists, then we can use induction to show x_{n+1} exists. Alternatively, x_{n+1} can be thought of as a subsequence of x_n and because x_n converges, x_{n+1} converges as well.

Part C

$$lim x_{n+1} = lim \frac{1}{4 - x_n}$$

Let $lim x_{n+1} = l$.

$$l = \frac{1}{4 - lim x_n}$$

$$l = \frac{1}{4 - l}$$

$$l(4 - l) = 1$$

$$l^2 - 4l + 1 = 0$$

$$(l + 2)^2 - 3 = 0$$

$$l = \sqrt{3} - 2$$

Self-Evaluation

Part A

A small mistake in the conclusion, I should have stated $x_{n+2} \leq x_{n+1}$

Part B

My answer here differs, but I believe the fact about subsequences would hold here and be correct.

Part C

Bad algebra mistake here on finding the square root. I was close!

Problem 2.4.3

Original Solution

Part B

Let $(a_n) = \sqrt{2}, \sqrt{2\sqrt{2}}, etc.$

If (a_n) converges, then it is bounded and monotone.

> (a_n) is monotone

$$a_2 = \sqrt{2\sqrt{2}} > \sqrt{2}$$

The induction is grounded.

Suppose by way of induction that

$$a_{n+1} > a_n$$

for some $n \geq 1$.

$$2 * a_{n+1} > 2 * a_n$$

$$\sqrt{2 * a_{n+1}} > \sqrt{2 * a}$$

$$a_{n+2} > a_{n+1}$$

Inductive step complete.

Find the limit.

$$lim(a_{n+1}) = lim(a_n)$$

Let $lim(a_{n+1}) = l$

$$l = lim(\sqrt{2 * a_n})$$

$$l = \sqrt{2 * lim(a_n)}$$

We know that $lim(a_{n+1}) = lim(a_n) = l$

$$l = \sqrt{2 * l}$$

$$l^2 = 2l$$

$$l^2 - 2l = 0$$

Thus l is either 0 or 2. We know the sequence is increasing so 0 does not make sense.

$$l = 2$$

Self-Evaluation

Somehow I neglected to include the evaluation of the limit for the sequence. I have added my original work above.

Problem 2.4.6

Original Solution

Part A

For any two positive real numbers,

$$(x - y)^2 \geq 0$$

$$(x + y)^2 - 4xy \geq 0$$

$$(x + y)^2 \geq 4xy$$

$$(x + y) \geq 2\sqrt{xy}$$

$$\frac{x+y}{2} \geq \sqrt{xy}$$

Part B

Part B I ran out of time and was a bit confused on. I was not sure if the problem was telling me that the sequence values of the one limit, fed into the recursion of the other, or not.

Self-Evaluation

Part B

I see that I was mistaken and that the terms of the two sequences do not interact with each other. I think the book could have maybe done a better job here with the notation, but regardless I could have done some more critical thinking here. I see that the key was utilizing the known relationships between x_n and y_n and using the MCT to show they converge.

Problem 2.5.2

Original Solution

Part A

This is true and is a result of Theorem 2.5.2.

Part B

This is false. To be able to say that (x_n) diverges, at least two divergent subsequences need to be found.

Part C

This is true and is a result of Theorem 2.5.2.

Part D

This is true and is a result of Theorem 2.5.5.

Self-Evaluation

Parts B and C

I got parts B and C a little confused in my analysis. I remebered us going over Part C in class, and I relied on that example a little too heavily for Part B. In Part B, I was thinking that the subsequence just converged to a different term than the overall sequence, not that it diverged all together. I now see why B is simply the contrapositive of Theorem 2.5.2.

Expanding on Part C, I could have given more background to my answer but I assumed that since it was stated pretty explicitly in the text that we could simply refer back to the claim that was made there.

Part D

Looks like I needed to read the question more carefully and see that the sequence was not bounded. The solution presented makes sense.

Problem 2.6.4

Original Solution

Part A

Let $c_n = |a_n - b_n|$

If a_n is cauchy, then $\forall \frac{\epsilon}{2} > 0, \exists N_1 \in \mathbb{N}$ s.t. for $m, n \geq N_1$

$$|a_n - a_m| < \frac{\epsilon}{2}$$

If b_n is cauchy, then $\forall \frac{\epsilon}{2} > 0, \exists N_2 \in \mathbb{N}$ s.t. for $m, n \geq N_2$

$$|b_n - b_m| < \frac{\epsilon}{2}$$

Let $N = max(N_1, N_2)$, then c_n is cauchy if for $n, m \geq N, \exists \epsilon > 0$ s.t.

$$|c_n - c_m| < \epsilon$$

$$|c_n - c_m| = ||a_n - a_m| + |b_n - b_m||$$

$$|c_n - c_m| \leq |a_n - a_m| + |b_n - b_m|$$

$$|c_n - c_m| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus c_n is cauchy.

Part B

c_n is not cauchy because it does not converge. The odd and even terms will converge to different values. For example if a_n was $\frac{n}{n+1}$ the even terms would converge to 1 and the odd terms to -1.

Part C

c_n is not cauchy. If $a_n = \frac{(-1)^n}{n}$ then the even terms converge to 0 and the odd terms converge to -1.

Self-Evaluation

Part A

My answer here takes a different approach, but I believe still holds.

External References