Homework Six

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Problem 2.7.1 (Part A only)

Original Solution

Let S_n be the partial sum of

$$S_n = \sum_{k=1}^{n} a_k$$

which is a monotonically decreasing sequence that converges to 0.

By the cauchy criterion for series, $\exists N \in \mathbb{N}$ s.t. $n > m \geq N$ implies $|S_n - S_m| < \epsilon$ for an arbitrary $\epsilon > 0$.

Notice that

$$|S_n - S_m| = (a_1 - a_2 + a_3 \ldots \pm a_n) + (-a_1 + a_2 - a_3 \ldots \pm a_m)$$

Because the series is decreasing, we know that $S_m \geq S_n$ so

$$(a_m - a_{m+1} + \ldots \pm a_{n-1}) \geq (a_{m+1} - a_{m+2} \ldots \pm a_n)$$

So

$$2|S_n - S_m| = 2 * |(a_{m+1} - a_{m+2} \ldots \pm a_n)|$$

$$\leq (a_m - a_{m+1} + \ldots \pm a_{n-1}) - (a_{m+1} - a_{m+2} \ldots \pm a_n)$$

$$= |a_m \pm a_n|$$

$$\leq |a_m + a_n| < 2 * \epsilon$$

$\therefore S_n$ is cauchy.

Self-Evaluation

Problem 2.7.2

Original Solution

Part A

$\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converges because the sequence $\left(\frac{1}{2^n+n}\right)$ converges to 0.

Part B

$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges as

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}$$

and the sequence $(\frac{1}{n^2})$ converges to 0 so $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, and then by the comparison test $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ converges.

Part C

The $\sum\frac{(-1)^{n+1}(n+1)}{2n}$ does not converge. We know that $\frac{(n+1)}{2n}$ converges to $\frac{1}{2}$, so by the alternating series test, the sum diverges.

Part D

Let $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\ldots=a_k$. And let b_k be infinite the sum of the harmonic series.

Then $0\leq a_k\leq b_k\forall k\in\mathbb{N}$

We know b_k diverges, so by the comparison test, a_k diverges as well.

Part E

Let $1-\frac{1}{2^2}+\frac{1}{3}-\frac{1}{4^2}+\frac{1}{5}-\frac{1}{6^2}\ldots=a_k$ And let b_k be the infinite sum of the harmonic series.

Then $0\leq a_k\leq b_k\forall k\in\mathbb{N}$

We know b_k diverges, so by the comparison test, a_k diverges as well.

Self-Evaluation

Problem 2.7.9

Original Solution

Part A

If $r<r'<1$, then this implies that r' is an valid upper bound on the value of the ratio, and that it exists between r and 1 by the archimedean property.

Because the $\lim|\frac{a_{n+1}}{a_n}|=r$, then we know it never passes r and therefore never passes r'. Thus we can say, that for $n\geq N$, where N is the minimum value to approach the limit of the sum,

$$|a_{n+1}|\leq|a_n|r'$$

Part B

$|a_N|\sum(r')^n$ converges because it is a geometric sequence with a ratio less than 1.

Part C

$\sum a_n$ is a combination of two different series,

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{N}|a_n|+\sum_{n=N+1}^{\infty}|a_n|$$

The first summation is finite, so it is guaranteed to converge. The second summation we know to converges because $|a_{N+k}|<r^k|a_N|\forall k$ which allows us to utilize the comparison test, to show

$$\sum_{n=N+1}^{\infty}|a_n|=\sum_{k=1}^{\infty}|a_N+k|$$

.

$\therefore \sum |a_n|$ converges and by the absolute convergence test, $\sum a_n$ converges.

> Note: I did some additional reading on the Ratio test at the link provided in External References

Self-Evaluation

Problem 3.2.2 (Parts A, B and D Only)

Original Solution

Let $A = (-1)^n + \frac{2}{n}$

Limit Points: -1, 1

The set is not open, because there is no viable epsilon neighborhood for 2, and not closed because it does not contain -1.

$A \cup L = A \cup \{-1\} = \bar{A}$

Let $B = x \in \mathbb{Q}, 0 < x < 1$

The limit points are 0 and 1.

The set is open, as there is a valid epsilon neighborhood for all points in the set. The set is also not closed because it does not contain its limit points.

$B \cup L = B \cup \{0, 1\} = \bar{B}$

Self-Evaluation

Problem 3.2.3

Original Solution

Part A

$\mathbb{Q}$ is not closed because every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$, so $\mathbb{Q}$ does not contain any of its limit points.

$\mathbb{Q}$ is also not open because by the Archimedean property, each epsilon neighborhood inherently includes irrational numbers between the rationals, i.e. the density of $\mathbb{Q}$ in $\mathbb{R}$

Part B

$\mathbb{N}$ is closed as it has no limit points, and thus, does not contain them. Similar to $\mathbb{Q}$, $\mathbb{N}$ is not open as the epsilon neighborhoods are essentially isolated to that point.

Part C

$x \in \mathbb{R}, x \neq 0$ is open, and closed, for the same reason as $\mathbb{R}$. That is all epsilon neighborhoods are valid, and it does not have any limit points.

Part D

$\sum \frac{1}{n^2}, n \in \mathbb{N}$ is not closed, as the limit, $\frac{\pi^2}{6} \notin \mathbb{N}$. It is also not open, as each term is a finite point, thus there is no valid epsilon neighborhood for each term.

Part E

$\sum \frac{1}{n}, n \in \mathbb{N}$ is closed as it has no limit points, similar to $\mathbb{N}$ itself. It is not open because each term is a finite point, and thus there is no valid epsilon neighborhood for each term.

Self-Evaluation

External References

Used on 2.7.9, parts B and C