Homework 2

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Problem 1.3.3

Original Solution

Part A Show that $\sup B = \inf A$

Because A is nonempty and bounded below, then we know an infimum exists. Let $x = \inf(A)$. By the definition of $\inf(A)$, then we know $x \in B$. Thus B is nonempty and due to the relationship defined in the problem statement, we know it has an upper bound. Let $y = \sup(B)$.

Assume $y \neq x$

We know $x \in B$ and x must be an upper bound for B. If x is an upper bound for B, then $x \geq y$. However, we know x is not greater than y, otherwise it would not be the $\inf(A)$ becaues it would no longer be a lower bound. Thus

$$y = x$$

$$\sup B = \inf A$$

Part B Use Part A to explain why there is no need to assert that greatest lower bounds exist as part of the AOC

As shown in part A, the greatest lower bound of one set, can be thought of as the least upper bound of another. This idea is further illustrated by the Nested Interval Property.

Self-Evaluation

Problem 1.3.4

Original Solution

Part A Find a formula for $\sup\ (A_1 \bigcup A_2)$***. Extend this to*** $\sup\ (\bigcup_{k=1}^{n} A_k)$

Define $max(A)$ to be the maximum value of a set. Then

$$\sup(A_1 \bigcup A_2) = \max(A_1 \bigcup A_2)$$

Because the set $\bigcup_{k=1}^{n} A_k$ is still bounded above, we know that whatever value is the maximum of the set, is also the supremum.

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max\left(\bigcup_{k=1}^{n} A_k\right)$$

Part B Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$***. Does the formula in Part A extend to the infinite case?***

The formula does not extend to the infinite case. This case is not bounded. Therefore one cannot assume that the supremum and the maximum value are contained in the given intersection, as it is possible that supremum exists outside the bounds of the interval of the intersection.

Self-Evaluation

Problem 1.3.5

Original Solution

Part A If $c \geq 0$***, show that*** $\sup(cA) = c * \sup(A)$

Let $s = \sup(A)$ then

$$\forall a \in A, a \leq s$$

so

$$c * a \leq c * s$$

thus, s in an upper bound.

Let b be any upper bound for $c * A$

Let $a \in A$ so $c * a \in c * A$. Then

$$c * a \leq b$$

$$a \leq b/c$$

If b/c is an upper bound for a, then

$$s \leq b/c$$

$$c * s \leq b$$

$$c * s = \sup(cA)$$

Part B Postulate a similar type of statement for $\sup(cA)$ ***for the case*** $c < 0$

$\sup(cA) = c * \inf(A)$

Self-Evaluation

Problem 1.3.6

Original Solution

Part A Let $s = \sup A$ ***and*** $t = \sup B$***. Show*** $s + t$ ***is an upper bound for*** $A + B$***.***

By the definition of supremum…

$$s \geq a, \forall a \in A$$

$$t \geq b, \forall b \in B$$

$$s + t \geq A + B$$

Part B Now let u ***be an arbitrary upper bound for*** $A + B$***, and temporarily fix*** $a \in A$***. Show*** $t \leq u - a$***.***

$$u \geq a + b$$

$$u - a \geq b, \forall b \in B$$

$$u - a \geq t$$

Part C Finally, show $\sup\,(A + B) = s + t$

From *Part B* we know that u is u.b for $A + B$. We also know

$$u - a \geq t$$

Similarly, it follows that

$$u - b \geq s$$

Combining these two expressions, we get

$$s + t \leq 2 * u - a - b$$

$$s + t + a + b \leq 2 * u$$

Because u is an upper bound for $A + B$,

$$u \geq a + b$$

which helps resolve the previous expression to

$$s + t \leq u$$

which implies that $s + t = \sup\ (A + B)$.

Part D Construct another proof of this same fact using Lemma 1.3.8

Assume $s + t$ is an upper bound for $A + B$. Then, by Lemma 1.3.8

$$s + t < a + b - \epsilon, \forall \epsilon > 0$$

Therefore $s + t = \sup(A + B)$

Self-Evaluation

Problem 1.3.8

Original Solution

Part A $(\frac{m}{n} : m, n \in N \text{ with } m < n)$

Sup = 1

Inf = 0

Part B $(\frac{-1^m}{n} : m, n \in N)$

Sup = 1

Inf = -1

Part C $(\frac{n}{3n+1} : n \in N)$

Sup = 1/3

Inf = 1/4

Part D $(\frac{m}{m+n} : m, n \in N)$

Sup = 1

Inf = 0

Self-Evaluation

External References