Homework 4

Mark Archual | MTH 515

Dr. Scott | Real Analysis

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Problem 2.2.2

Original Solution

Part A

$\lim \frac{2n+1}{5n+4} = \frac{2}{5}$

> Rough Work

$$|\frac{2n+1}{5n+4} - \frac{2}{5}| < \epsilon$$

$$|\frac{2n+1-2(n+4)}{5n+4}| < \epsilon$$

$$|\frac{2n+1-2n-8}{5n+4}| < \epsilon$$

$$|\frac{-7}{5n+4}| < \epsilon$$

Notice then that

$$\frac{1}{5n+4} < \frac{1}{5n}$$

Solving for epsilon

$$\frac{7}{5n} < \epsilon$$

$$\frac{7}{5\epsilon} < n$$

> Formal Proof

Let $\epsilon > 0$ be arbitrary
Let $N \in \mathbb{N}$ be greater than $\frac{7}{5\epsilon}$
Suppose $n \geq N$, then

$$|\frac{2n+1}{5n+4} - \frac{2}{5}| = \frac{7}{5n} \leq \frac{7}{5N} < \frac{7}{5\frac{7}{5\epsilon}} = \epsilon$$

Part B

$\lim \frac{2n^2}{n^3+3} = 0$

> Rough Work

Denominator

$$n^3 + 3 > n^3$$

so

$$\frac{1}{n^3+3} < \frac{1}{n^3}$$

Numerator

$$2n^2 > n^2$$

so

$$|\frac{2n^2}{n^3+3}| < \frac{2n^2}{n^3}$$

$$\frac{2}{n} < \epsilon$$

$$n > \frac{2}{\epsilon}$$

> Formal Proof

Let $\epsilon > 0$ be arbitrary
Let $N \in \mathbb{N}$ be greater than $\frac{2}{\epsilon}$

Suppose $n \geq N$, then

$$|\frac{2n^2}{n^3+3}| < \frac{2n^2}{n^3} < \frac{2}{n} \leq \frac{2}{N} < 2 * \frac{\epsilon}{2} = \epsilon$$

Part C

$lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$

> Rough Work

Numerator: Notice that $\sin(n^2)$ is bounded by 1. So $\sin(n^2) < 1$

This simplifies the problem to

$$\frac{1}{\sqrt[3]{n}} < \epsilon$$

$$\frac{1}{n} < \epsilon^3$$

$$\frac{1}{\epsilon^3} < n$$

> Formal Proof

Let $\epsilon > 0$ be arbitrary
Let $N \in \mathbb{N}$ be greater than $\frac{1}{\epsilon^3}$

$$|\frac{\sin(n^2)}{\sqrt[3]{n}}| < \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} \frac{1}{\sqrt[3]{\frac{1}{\epsilon^3}}} = \epsilon$$

Self-Evaluation

Problem 2.2.6

Original Solution

If $(a_n) - > a$, then, by the definition of convergence $\exists N_1 \in \mathbb{N}$ for $n \geq N_1$

$$|a_n - a| < \frac{\epsilon}{2}, \forall \epsilon > 0$$

If $(a_n) - > b$, then, by the definition of convergence $\exists N_2 \in \mathbb{N}$ for $n \geq N_2$

$$|a_n - b| < \frac{\epsilon}{2}, \forall \epsilon > 0$$

Using the triangle inequality

$$|a_n - a| + |b - a_n| \geq |a_n - a + b_n - b| \geq |b - a|$$

$$|b - a| < \epsilon \, \forall \epsilon > 0$$

Therefore $b = a$, otherwise if $b \neq a$, then an ϵ may be found such that $\epsilon < b - a < \epsilon$, which is not possible.

Self-Evaluation

Problem 2.3.3

Original Solution

By the Order Limit Theorem if $a_n \leq b_n \forall n$ then $a \leq b$. Applying this idea here, we have

$$x_n \leq y_n$$

so $l \leq y$. Similarly

$$y_n \leq z_n$$

so $y \leq l$. Notice then that

$$l \leq y \leq l$$

Thus $l = y$

Self-Evaluation

Problem 2.3.7

Original Solution

Part A

Let $(x_n) = \sin(x_n)^2$, and $(y_n) = \cos(y_n)^2$.

Both sequences oscillated between 0 and 1, however their sum is 1 for all values of n.

Part B

This is not possible by tenant (ii) of the Algebraic Limit Theorem. If (y_n) diverges, then it is not possible that $(x_n) + (y_n)$ converges, as it would contradict the assumption of the A.L.T. that each sequence must converge.

Part C

Let $(b_n) = \frac{1}{x_n}$ then $\frac{1}{b_n} = x_n$, which diverges.

Part D

This is not possible. If (a_n) is unbounded, then $(a_n - b_n)$ is unbounded as well, as it would be impossible to find a value that satisfies the definition of boundedness for this difference.

Part E

$(a_n) = \frac{1}{\sqrt{x_n}}$ and $(b_n) = \sqrt{x_n}$ then
$(a_n * b_n) = 1$ which converges.

Self-Evaluation

Problem 2.3.9

Original Solution

If the $\lim b_n = 0$, then the expression $\lim(a_n * b_n) = 0$ because if we know that all of the possible values of the sequence (a_n) fall within a definable range it will follow the convergence of (b_n) to 0. If a_n was not bounded, this could not be guaranteed.

We cannot use the Algebraic Limit Theorem because it assumes that the sequence (a_n) converges.

Self-Evaluation

External References