Solutions: Homework 7

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(3.3.2) (a) Since $\mathbf{N}$ is not bounded, it is not compact.

(b) Let $\alpha \in [0,1]$ be an irrational number (say, $1/\sqrt{2}$). Let (a_n) be the sequence of decimal expansions of α truncated after n digits. Then (a_n) is sequence in $\mathbf{Q} \cap [0,1]$ that converges to α. Since $\alpha \notin \mathbf{Q} \cap [0,1]$, $\mathbf{Q} \cap [0,1]$ is not closed and so not compact.

(c) As discussed in class, the Cantor set is closed and bounded, therefore compact.

(d) This is the set of partial sums of the convergent series $\sum_{n=1}^{\infty}$. If we arrange them in order, they form a sequence that converges to $\pi^2/6$, which is not in the set. Therefore the set is not closed, so not compact.

(e) Any sequence in this set will contain a subsequence of $((n-1)/n)$, which converges to 1, an element of the set. Thus the set is closed. Since the set is also bounded (above by 1, below by 1/2), it is compact.

(3.3.5) (a) True, because the arbitrary intersection of compact sets is closed and, since contained in any one of the compact sets, bounded.

(b) False. Consider the collection $\{[-n,n] : n \in \mathbf{N}\}$. The union is $\mathbf{R}$, which is not bounded and so not compact.

(c) False. For example, $A = (0,1)$, $K = [0,1]$. Then $A \cap K = A$ is not closed and so not compact.

(d) False. For example, $F_n = [n, \infty)$. Then each F_n is closed, but $\cap_{n=1}^{\infty} F_n = \varnothing$.

(4.2.5) (a) Given $\varepsilon > 0$, let $\delta = \varepsilon/3$. Then whenever $0 < |x-2| < \delta$, we have

$$|(3x+4) - 10| = |3x-6| = 3|x-2| < 3\delta = \varepsilon.$$

Therefore $\lim_{x\to 2}(3x+4) = 10$.

(b) Given $\varepsilon > 0$, let $\delta = \varepsilon^{1/3}$. Then whenever $0 < |x| < \delta$, we have

$$|x^3 - 0| = |x|^3 < \delta^3 = \varepsilon.$$

Therefore $\lim_{x\to 0} x^3 = 0$.

(c) Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/6\}$. Then whenever $0 < |x-2| < \delta$, we have $-1 < x - 2 < 1$, so adding 5, we get $4 < x + 3 < 6$, so $|x+3| < 6$. Therefore

$$|(x^2 + x - 1) - 5| = |x+3||x-2| < 6|x-2| < 6\delta \leq \varepsilon.$$

Therefore $\lim_{x\to 2}(x^2 + x - 1) = 5$.

(d) Given $\varepsilon > 0$, let $\delta = \min\{1, 6\varepsilon\}$. Whenever $0 < |x-3| < \delta$, we have $-1 < x-3 < 1$, so $2 < x < 4$. Taking reciprocals, we get $\frac{1}{2} > \frac{1}{x} > \frac{1}{4}$. In particular, $\frac{1}{|x|} < \frac{1}{2}$, and so

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|x-3|}{3|x|} < \frac{|x-3|}{6} < \frac{\delta}{6} \leq \varepsilon.$$

Therefore $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$.

(4.2.8) (a) The limit does not exist. Consider the sequences $a_n = 2 + \frac{1}{n}$ and $b_n = 2 - \frac{1}{n}$. Then $\lim \frac{|a_n - 2|}{a_n - 2} = 1$ but $\lim \frac{|b_n-2|}{b_n-2} = -1$.

(b) Near $x = \frac{7}{4}$, $x - 2 < 0$, so $\lim_{x\to\frac{7}{4}} \frac{|x-2|}{x-2} = -1$.

(c) The limit does not exist. For example, if $a_n = \frac{1}{n}$, then $\lim(-1)^{1/a_n} = \lim(-1)^n$ which we have seen does not exist.

(d) Since $|\sqrt[3]{x}(-1)^{[[1/x]]}| = |\sqrt[3]{x}|$, the limit exists and equals zero.

(4.3.1) (a) Given $\varepsilon > 0$, let $\delta = \varepsilon^3$. Then whenever $|x| < \delta$, we have

$$|g(x) - g(0)| = |x^{\frac{1}{3}}| = |x|^{\frac{1}{3}} < \delta^{\frac{1}{3}} = \varepsilon.$$

(b) We will use Example 4.3.8 as a template. Since g is an odd function, we may suppose that $c > 0$.

$$|x^{\frac{1}{3}} - c^{\frac{1}{3}}| = |x^{\frac{1}{3}} - c^{\frac{1}{3}}| \left(\frac{x^{\frac{2}{3}} + x^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}}}{x^{\frac{2}{3}} + x^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}}} \right) = \frac{|x-c|}{x^{\frac{2}{3}} + x^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}}} \leq \frac{|x-c|}{c^{\frac{2}{3}}}.$$

Let $\varepsilon > 0$ be arbitrary and set $\delta = c^{\frac{2}{3}}\varepsilon$. Then $|x - c| < \delta$ implies that

$$|x^{\frac{1}{3}} - c^{\frac{1}{3}}| \leq \frac{|x-c|}{c^{\frac{2}{3}}} < \frac{\delta}{c^{\frac{2}{3}}} = \varepsilon.$$

(4.3.4) (a) Suppose g is zero everywhere except at 0, where $g(0) = 1$. Let f be the constant function $f(x) = 0$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{y\to 0} g(y) = 0$, but $\lim_{x\to 0} g(f(x)) = 1$.

(b) Done in class.

(c) In our counterexample above, f is continuous, so (a) does not hold if only f is continuous. It does hold if only g is continuous, since if (x_n) is any sequence that converges to p, with $x_n \neq p$ for all n, then by Theorem 4.2.3, $f(x_n) \to q$. Since g is continuous, $g(f(x_n)) \to r$.