Solutions: Homework 10

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(7.5.4) Let $F(x) = \int_a^x f(t)dt$. By the secondpart of the Fundamental Theorem of Calculus, $F'(x) = f(x)$ for all $x \in [a, b]$. Since $F(x) = 0$ for all $x \in [a, b]$, $f(x) = 0$ for all $x \in [a, b]$.

If f is not continuous, we could have something like $f(x) = 0$ everywhere on $[0, 2]$, except at $x = 1$ where $f(1) = 1$.

(7.5.8) (a) $L(1) = \int_1^1 \frac{1}{t} = 0$. Since $\frac{1}{t}$ is continuous for $t > 0$, the FTC tells us that $L(x)$ is differentiable and $L'(x) = \frac{1}{x}$.

(b) Consider y to be constant and set $g(x) = L(xy)$. By the Chain Rule, $g'(x) = (1/xy)(y) = 1/x = L'(x)$. Thus, $L(xy) - L(x) = C$ for some constant (with respect to x), C. To find C we plug in $x = 1$ to get $L(y) - L(1) = C$. Since $L(1) = 0$, $C = L(y)$. Thus, $L(xy) - L(x) = L(y)$, or $L(xy) = L(x) + L(y)$.

(c) Using the previous part, $L(x) = L((x/y)y) = L(x/y) + L(y)$, so $L(x/y) = L(x) - L(y)$.

(d) For this question, it helps to draw a picture. Interpret the sum $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ as the total area of n rectangles, drawn over the intervals $[1, 2]$, $[2, 3]$, up to $[n, n+1]$, of height 1, 1/2, and so on up to height $1/n$. Meanwhile, $L(n)$ is the area under the curve $y = 1/x$ on $[1, n]$. So, γ_n is the area of the rectangle $[n, n+1] \times [0, 1/n]$, plus the bits of the other rectangles sticking above the curve $y = 1/x$ on $[1, n]$. It follows that each γ_n, being an area, is ≥ 0.

Furthermore, $\gamma_{n+1} - \gamma_n$ is the area of $[n+1, n+2] \times [0, \frac{1}{n+1}]$ less the area under the the curve $y = 1/x$ over $[n, n+1]$. Since $\frac{1}{x}$ is decreasing, $\frac{1}{x} \geq \frac{1}{n+1}$ on this interval, so this second area is $> \frac{1}{n+1}$. Thus, $\gamma_{n+1} - \gamma_n < 0$, so (γ_n) is decreasing. By Monotone Convergence, the sequence converges.

(e) This one is a little tricky. Since (γ_n) converges, the subsequence (γ_{2n}) converges to the same limit, so $\lim(\gamma_{2n} - \gamma_n) = 0$. Write

$$\gamma_n = 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) - L(n).$$

Then, using part (c),

$$\begin{aligned}
\gamma_{2n} - \gamma_n &= \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n}\right) - L(2n) + L(n) \\
&= \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n}\right) - L(2).
\end{aligned}$$

Now take the limit as $n \to \infty$.