Homework 7

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Problem 3.3.2

Original Solution

Part A

N is not compact as it does not contain any convergent subsequences.

Part B

$\mathrm{Q} \cap [0, 1]$ is not compact. For $0 < q_n < 1$, a subsequence of qn can be found such that $q_{n_k} - > \frac{1}{\pi}$ which $\notin \mathrm{Q} \cap [0, 1]$.

Part C

The Cantor set is compact as it is closed and bounded (Heine-Borel Theorem).

Part D

This set is not compact as it does not contain all of its limit points, i.e. $\frac{\pi^2}{6}$

Part E

This set is compact as it is bounded by $[\frac{1}{2}, 1]$ and it contains both of its limit points, $\frac{1}{2}, 1$.

Self-Evaluation

Problem 3.3.5

Original Solution

Part A

True. We know that the arbitrary intersection of closed sets remains closed (Thm 3.2.14) so the arbitrary intersection of compact sets retain all of their limit points. Compact sets must also be bounded, and the intersection of compact sets would remain bounded.

Part B

This is false as it is essentially the negation of the statement of Part A which we know to be true.

Part C

False, for example consider the $(0, 1) \cap [0, 1]$. The result is not compact, $(0, 1)$ as it is not closed.

Part D

This is false. An example is given in the proof of theorem 3.4.3. In general, the arbitrary intersection of closed subsections of $\mathbb{N}$ can only be $\varnothing$.

Self-Evaluation

Problem 4.2.5

Original Solution

Part A

> Rough Work

Solve $|(3x + 4) - 10|$ in terms of $|x - 2|$.

$$|(3x - 6)| < \epsilon$$

$$3|(x - 2)| < \epsilon$$

$$|(x-2)| < \frac{\epsilon}{3}$$

Thus $\delta = \frac{\epsilon}{3}$.

> Proof

Let $\epsilon > 0$ be arbitrary, and $\delta = \frac{\epsilon}{3}$ If 0 < $|x-2| < \delta$ Then …

$$\therefore |(3x+4)-10| = 3|(x-2)| < 3\delta = 3(\frac{\epsilon}{3}) = \epsilon$$

Part B

> Rough Work

Solve $|x^3|$ in terms of x.

$$x^3 < \epsilon$$

$$x < \sqrt[3]{\epsilon}$$

So $\delta = \sqrt[3]{\epsilon}$.

> Proof

Let $\epsilon > 0 bearbitrary$ and set $\delta = \sqrt[3]{\epsilon}$. If $0 < |x| < \delta$ then …

$$\therefore |x^3| = x < \delta^3 = (\sqrt[3]{\epsilon})^3 = \epsilon$$

_Part C

> Rough Work

Solve $|(x^2+x-1)-5$ in terms of $(x-2)$.

$$|x^2+x-6| < \epsilon$$

$$|x+3||x-2| < \epsilon$$

$$(x-2) < \frac{\epsilon}{x+3}$$

$x+3 needs a lower bound.

Suppose $\delta \leq 1$. So $|x-2| < 1 \ldots$

$$-1 < |x-2| < 1$$

$$5 < |x+3| < 7$$

Taking $|x+3| < 7$ and resolving for $\epsilon \ldots$

$$|x+3||x-2| < 6 * |x-2| < \epsilon$$

$$x-2 < f\frac{\epsilon}{6}$$

So $\delta = \min(1, \frac{\epsilon}{6})$

> Proof

Let $\epsilon > 0$ be arbitrary and $\delta = \min(1, \frac{\epsilon}{6})$. If $0 < x-2 < \delta$ then $\ldots$

$$\therefore\ |x^2+x-6| = |x+3||x-2| < 6|x-2| \leq 6(\frac{\epsilon}{6}) = \epsilon$$

Part D

This one I had some trouble with the algebra and trying to solve for epsilon in terms of the limit. I could not make any significant progress to generate a proof.

Self-Evaluation

Problem 4.2.8

Original Solution

Part A

The limit does not exist as the denominator is undefined at the limit point, so $f(x) - 2 > \epsilon$ as $x -> 2$.

Part B

$$\lim_{x \to \frac{7}{4}} \frac{|x-2|}{x-2} = -1$$

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Set $\delta = \frac{7}{4} - 2 = \frac{1}{4}$ then for $0 < |x - \frac{7}{4}| < \delta \ldots$

$\therefore |\frac{|x-2|}{x-2} + 1| = 0 < \epsilon$

Part C

This limit does not exists. Consider separate subsequences x_{n_e} for the even valued terms and x_{n_o} for the odd valued terms. The even valued terms would converge to 1 whereas the odd valued terms converge to -1, thus by corollary 4.2.5 the limit d.n.e.

Part D

> Rough Work

Solve $\sqrt[3]{x}(-1)^{\frac{1}{x}}$ in terms of x.

$$|\sqrt[3]{x}(-1)^{\frac{1}{x}}| < \epsilon$$

We can simplify this expression by recongnizing that $(-1)^{\frac{1}{x}}$ reduces to 1 for our purposes. So $\ldots$

$$|\sqrt[3]{x}| < \epsilon$$

Thus $\delta = \epsilon^3$.

> Proof

Let $\epsilon > 0$ be arbitrary and set $\delta = \epsilon^3$. If $0 < x < \delta$ then $\ldots$

$$|\sqrt[3]{x}(-1)^{\frac{1}{x}}| < |\sqrt[3]{x}| < \sqrt[3]{\delta} = \sqrt[3]{\epsilon^3} = \epsilon$$

Self-Evaluation

Problem 4.3.1

Original Solution

Part A

Prove g is continuous at $c = 0$.

> Rough Work

We want $|\sqrt[3]{x} - 0| < \epsilon$

$$|\sqrt[3]{x} < \epsilon|$$

$$x < \epsilon^3 = \delta$$

> Clean

Let $\epsilon > 0, \delta = \epsilon^3$ whenever $x \geq 0$ and $|x - 0| < \delta, x < \epsilon^3$ and $\sqrt[3]{x} < \epsilon$, $\sqrt[3]{x}$ is continuous at $x = 0$.

Part B

> Rough Work

$|\sqrt[3]{x} - \sqrt[3]{c}| < \epsilon$ when $|x - c| < \delta$

Let $a = \sqrt[3]{x}$ and $b = \sqrt[3]{c}$. Then …

$$|\sqrt[3]{x} - \sqrt[3]{c}| = |\frac{(a-b)(a^2 + ab + b^2)}{a^2 + ab + b^2}$$

$$= \frac{a^3 - b^3}{a^2 + ab + b^2}$$

$$= \frac{x - c}{\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2} < \frac{x - c}{-\sqrt[3]{c}} < \epsilon$$

So $x - c < \epsilon * \sqrt[3]{c}^2 = \delta$

> Clean

Let $\epsilon > 0$ be arbitrary, $\delta = \epsilon^3$ whenever $x \geq 0$ and $|x - c| < \delta$ then …

$|\sqrt[3]{x} - \sqrt[3]{c}| = \frac{x-c}{\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{c} + \sqrt[3]{c}^2} \leq \frac{x-c}{-\sqrt[3]{c}} < \epsilon * \sqrt[3]{c}^2 = \epsilon$

Self-Evaluation

Problem 4.3.4

Original Solution

Part A

Let $f(x) = \frac{1}{x}$ and $g(x) = \cos(x)$.

Then …

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to 0} \cos(x) = 1$$

But …

$$\lim_{x \to \infty} \frac{1}{\cos(x)} = \text{d.n.e.}$$

Part B

If f and g are continuous then they must be continuous on all of $\mathbb{R}$ per the definition of the problem. Following the proof for composite functions shows that $g(f(x_n))$ has limit $g(f(c))$ if (x_n) is a sequence in $\mathbb{R}$ with limit c.

Part C

If f is continuous and not g then it does not hold because g must be continuous at $f(c)$ per *Part B*.

If g is continuous and not f then the validity depends on the nature of gof.

Self-Evaluation

External References