Homework 2

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Problem 1.3.3

Original Solution

Part A Show that $\sup B = \inf A$

Because A is nonempty and bounded below, then we know an infimum exists. Let $x = \inf(A)$. By the definition of $\inf(A)$, then we know $x \in B$. Thus B is nonempty and due to the relationship defined in the problem statement, we know it has an upper bound. Let $y = \sup(B)$.

Assume $y \quad x$

We know $x \in B$ and x must be an upper bound for B. If x is an upper bound for B, then $x \geq y$. However, we know x is not greater than y, otherwise it would not be the $\inf(A)$ because it would no longer be a lower bound. Thus

$$y = x$$

$$\sup B = \inf A$$

Part B Use Part A to explain why there is no need to assert that greatest lower bounds exist as part of the AOC

As shown in part A, the greatest lower bound of one set, can be thought of as the least upper bound of another. This idea is further illustrated by the Nested Interval Property.

Self-Evaluation

Part A

My answer falls apart at the following statement "However, we know x is not greater than y, otherwise it would not be the $\inf(A)$ because it would no longer be a lower bound". This is not true. It would simply mean that x is no longer the infimum. I should have taken the approach shown in the solution and shown $y \geq x$.

Problem 1.3.4

Original Solution

Part A Find a formula for $\sup\left(A_1 \cup A_2\right)$***. Extend this to*** $\sup\left(\cup_{k=1}^{n} A_k\right)$

Define $max(A)$ to be the maximum value of a set. Then

$$\sup(A_1 \cup A_2) = \max(A_1 \cup A_2)$$

Because the set $\cup_{k=1}^{n} A_k$ is still bounded above, we know that whatever value is the maximum of the set, is also the supremum.

$$\sup\left(\cup_{k=1}^{n} A_k \right) = \max(\cup_{k=1}^{n} A_k)$$

Part B Consider $\sup\left(\cup_{k=1}^{\infty} A_k \right)$***. Does the formula in Part A extend to the infinite case?***

The formula does not extend to the infinite case. This case is not bounded. Therefore one cannot assume that the supremum and the maximum value are contained in the given intersection, as it is possible that supremum exists outside the bounds of the interval of the intersection.

Self-Evaluation

I did not take the time explain why the maximum of a finite set is equal to its least upper bound. I assumed we could use this fact as it is stated in the book. Also my answer is $\max(A_1 \cup A_2)$, which I believe is equivalent to $\max(\sup(A_1) \cup \sup(A_2))$ becuase, as stated in the book for bounded sets, “when the maximum exists, then it is also the supremum”.

Problem 1.3.5

Original Solution

Part A If $c \geq 0$***, show that*** $\sup\,(cA) = c * \sup(A)$

Let $s = \sup\,(A)$ then

$$\forall a \in A, a \leq s$$

so

$$c * a \leq c * s$$

thus, s in an upper bound.

Let b be any upper bound for $c * A$

Let $a \in A$ so $c * a \in c * A$. Then

$$c * a \leq b$$

$$a \leq b/c$$

If b/c is an upper bound for a, then

$$s \leq b/c$$

$$c * s \leq b$$

$$c * s = \sup(cA)$$

Part B Postulate a similar type of statement for $\sup(cA)$ ***for the case*** $c < 0$

$\sup(cA) = c * \inf(A)$

Self-Evaluation

For the first part of the proof, I neglected to state the case for $c = 0$. Similarly, I should have called out the fact that $c * a \in A$. Otherwise, the comparison $c * a \leq c * s$ is not necessarily contained in A.

Problem 1.3.6

Original Solution

Part A Let $s = \sup A$ ***and*** $t = \sup B$***. Show*** $s + t$ ***is an upper bound for*** $A + B$***.***

By the definition of supremum…

$$s \geq a, \forall a \in A$$

$$t \geq b, \forall b \in B$$

$$s + t \geq A + B$$

Part B Now let u ***be an arbitrary upper bound for*** $A + B$***, and temporarily fix*** $a \in A$***. Show*** $t \leq u - a$***.***

$$u \geq a + b$$

$$u - a \geq b, \forall b \in B$$

$$u - a \geq t$$

Part C Finally, show $\sup\,(A + B) = s + t$

From *Part B* we know that u is u.b for $A + B$. We also know

$$u - a \geq t$$

Similarly, it follows that

$$u - b \geq s$$

Combining these two expressions, we get

$$s + t \leq 2 * u - a - b$$

$$s + t + a + b \leq 2 * u$$

Because u is an upper bound for $A + B$,

$$u \geq a + b$$

which helps resolve the previous expression to

$$s + t \leq u$$

which implies that $s + t = \sup\ (A + B)$.

Part D Construct another proof of this same fact using Lemma 1.3.8

Assume $s + t$ is an upper bound for $A + B$. Then, by Lemma 1.3.8

$$s + t < a + b - \epsilon, \forall \epsilon > 0$$

Therefore $s + t = \sup(A + B)$

Self-Evaluation

Part C

My answer differs, but I believe is logically consistent and also proves the stated question. I could see there being an issue with just assuming $u - b \geq s$ based on the fact $u - a \geq t$, without proving it. In which case, my answer would not stand.

Part D

My answer was a little quick to assume that the facts surrounding $\sup(A)$ and $\sup(B)$ individually being a least upper bound for their respective sets. I should have included the statements $s - \frac{\epsilon}{2} < a$ and $t - \frac{\epsilon}{2} < b$.

Problem 1.3.8

Original Solution

Part A $\frac{m}{n} : m, n \in \boldsymbol{N} \text{ with } m < n$

Sup = 1

Inf = 0

Part B $\frac{-1^m}{n} : m, n \in \boldsymbol{N}$

Sup = 1

Inf = -1

Part C $\frac{n}{3n+1} : n \in \boldsymbol{N}$

Sup = 1/3

Inf = 1/4

Part D $\frac{m}{m+n} : m, n \in \boldsymbol{N}$

Sup = 1

Inf = 0

Self-Evaluation

No comments.

External References