Homework Six

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3/6/2019

Problem 2.7.1 (Part A only)

Original Solution

Let S_n be the partial sum of

$$S_n = \sum_{k=1}^{n} a_k$$

which is a monotonically decreasing sequence that converges to 0.

By the cauchy criterion for series, $\exists N \in \mathbb{N}$ s.t. $n > m \geq N$ implies $|S_n - S_m| < \epsilon$ for an arbitrary $\epsilon > 0$.

Notice that

$$|S_n - S_m| = (a_1 - a_2 + a_3 \ldots \pm a_n) + (-a_1 + a_2 - a_3 \ldots \pm a_m)$$

Because the series is decreasing, we know that $S_m \geq S_n$ so

$$(a_m - a_{m+1} + \ldots \pm a_{n-1}) \geq (a_{m+1} - a_{m+2} \ldots \pm a_n)$$

So

$$2|S_n - S_m| = 2 * |(a_{m+1} - a_{m+2} \ldots \pm a_n)|$$

$$\leq (a_m - a_{m+1} + \ldots \pm a_{n-1}) - (a_{m+1} - a_{m+2} \ldots \pm a_n)$$

$$= |a_m \pm a_n|$$

$$\leq |a_m + a_n| < 2 * \epsilon$$

$\therefore S_n$ is cauchy.

Self-Evaluation

I glossed over the behavior of the alternating sign here but I see that I complete solution requires an understanding of how the relationship between the partial sum changes based on the indices. I did try to use the cancelation of terms to my advantage, but this ended up complicating things as I did not motivate a relationship between the partial sums and the individual terms of the sequence. If I would have taken the approach shown in the solution, I would lose the triangle inequality and multiplication factor and the relationship would not be as convoluted.

Problem 2.7.2

Original Solution

Part A

$\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converges because the sequence $\left(\frac{1}{2^n+n}\right)$ converges to 0.

Part B

$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges as

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and the sequence $\left(\frac{1}{n^2}\right)$ converges to 0 so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and then by the comparison test $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

Part C

The $\sum \frac{(-1)^{n+1}(n+1)}{2n}$ does not converge. We know that $\frac{(n+1)}{2n}$ converges to $\frac{1}{2}$, so by the alternating series test, the sum diverges.

Part D

Let $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots = a_k$. And let b_k be infinite the sum of the harmonic series.

Then $0 \leq a_k \leq b_k \forall k \in \mathbb{N}$

We know b_k diverges, so by the comparison test, a_k diverges as well.

Part E

Let $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} \ldots = a_k$ And let b_k be the infinite sum of the harmonic series.

Then $0 \leq a_k \leq b_k \forall k \in \mathbb{N}$

We know b_k diverges, so by the comparison test, a_k diverges as well.

Self-Evaluation

Problem 2.7.9

Part A

My justification here was a little too loose. Just because the sequence of a sum goes to zero does not justify the convergence of the sum itself.

Part C

I used the alternating series test here, but not correctly. An easy route would have been to show that the positive and negative terms of the sequence converge to different values.

Parts D and E

I got these wrong by incorrectly trying to use the comparison test.

Original Solution

Part A

If $r < r' < 1$, then this implies that r' is an valid upper bound on the value of the ratio, and that it exists between r and 1 by the archimedean property. *Let* $\epsilon = r' - r$.

Because the $\lim |\frac{a_{n+1}}{a_n}| = r$, then we know it never passes r and therefore never passes r'. *This allows us to say:*

$$||\frac{a_{n+1}}{a_n}| - r| = \epsilon$$

Or

$$-\epsilon < |\frac{a_{n+1}}{a_n}| - r < \epsilon$$

Adding r to both sides, and multiplying by $|a_n|$ gives

$$|a_n|r - \epsilon < |a_{n+1}| < \epsilon + r|a_n|$$

Because $r' = \epsilon + r$

$$|a_{n+1}| \leq |a_n|r'$$

Part B

$|a_N| \sum (r')^n$ converges because it is a geometric sequence with a ratio less than 1.

Part C

Using induction to prove this, we have already proved the base case in (A).

By the inductive hypothesis

$$|a_{N+k+1}| < |a_{N+k}|r' < (|a_N|(r')^k)r' = |a_N|(r')^{k+1}$$

This completes the induction.

$\sum a_n$ is a combination of two different series,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

The first summation is finite, so it is guaranteed to converge. The second summation we know to converges because $|a_{N+k}| < r^k |a_N| \forall k$ which allows us to utilize the comparison test, to show

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_N + k|$$

.

$\therefore \sum |a_n|$ converges and by the absolute convergence test, $\sum a_n$ converges.

> Note: I did some additional reading on the Ratio test at the link provided in External References

Self-Evaluation

Part A

I think I had the right idea here, I just had some trouble expressing it. Exploiting the use of an ϵ as a quantifiable expression for $r' - r$ as the indexing term grows makes a lot of sense. The epsilon term captures what I was thinking about when discussing the archimedean property. I have corrected my solution above, the additions are in *italics.*

Part C

I got pretty much followed the solution here, but I left out some details that I haved added above in *italics*.

Problem 3.2.2 (Parts A, B and D Only)

Original Solution

Let $A = (-1)^n + \frac{2}{n}$

Limit Points: -1, 1

The set is not open, because there is no viable epsilon neighborhood for 2, and not closed because it does not contain -1.

$A \cup L = A \cup \{-1\} = \bar{A}$

Let $B = x \in \mathbb{Q}, 0 < x < 1$

The limit points are 0 and 1.

The set is open, as there is a valid epsilon neighborhood for all points in the set. The set is also not closed because it does not contain its limit points.

$B \cup L = B \cup \{0, 1\} = \bar{B}$

Self-Evaluation

I incorrectly stated that B is open. The solution's reasoning for why it is not open, that is the density of $\mathbb{Q} \in \mathbb{R}$, is clear to me. I even stated this as part of my reasoning for 3.2.3 part A.

Problem 3.2.3

Original Solution

Part A

$\mathbb{Q}$ is not closed because every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$, so $\mathbb{Q}$ does not contain any of its limit points.

$\mathbb{Q}$ is also not open because by the Archimedean property, each epsilon neighborhood inherently includes irrational numbers between the rationals, i.e. the density of $\mathbb{Q}$ in $\mathbb{R}$

Part B

$\mathbb{N}$ is closed as it has no limit points, and thus, does not contain them. Similar to $\mathbb{Q}$, $\mathbb{N}$ is not open as the epsilon neighborhoods are essentially isolated to that point.

Part C

$x \in \mathbb{R}, x \not= 0$ is open, and closed, for the same reason as $\mathbb{R}$. That is all epsilon neighborhoods are valid, and it does not have any limit points.

Part D

$\sum \frac{1}{n^2}, n \in \mathbb{N}$ is not closed, as the limit, $\frac{\pi^2}{6} \notin \mathbb{N}$. It is also not open, as each term is a finite point, thus there is no valid epsilon neighborhood for each term.

Part E

$\sum \frac{1}{n}, n \in \mathbb{N}$ is closed as it has no limit points, similar to $\mathbb{N}$ itself. It is not open because each term is a finite point, and thus there is no valid epsilon neighborhood for each term.

Self-Evaluation

Part C

I missed that the set is not open here. I was debating about 0 being a limit point of the set when I was deciding on an answer for this one, but I got caught up on the idea of $\mathbb{R}$ going out to infinity, more so than not containing 0.

In general, I could have improved my answers by giving some examples of invalid limit points.

External References

Used on 2.7.9, parts B and C