# Solutions: Homework 5

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1. (2.4.1)

   (a) We will show that $(x_n)$ is decreasing and bounded below. We proceed by induction.

   First, let us prove that $(x_n)$ is strictly decreasing, that is, for all $n \in \mathbf{N}$, $x_{n+1} < x_n$. To ground the induction, we remark that $x_2 = 1 < 3 = x_1$. Now suppose that for some $n \geq 1$, we have $x_{n+1} < x_n$. Then

   $$\begin{array}{rcl} 4 - x_{n+1} & \geq & 4 - x_n \\ \dfrac{1}{4 - x_{n+1}} & \leq & \dfrac{1}{4 - x_n} \\ x_{n+2} & \leq x_{n+1} & \end{array}$$

   This completes the inductive proof that $(x_n)$ is decreasing.

   Now we will prove that $x_n > 0$ for all $n \in \mathbf{N}$. The induction is grounded since $x_1 = 3 > 0$. Now suppose that for some $n \geq 1$, we have $x_n > 0$. Then $4 - x_n < 4$; taking the reciprocal, we find that $x_{n+1} > 1/4 > 0$. Therefore, $x_n > 0$ for all $n \in \mathbf{N}$.

   By the Monotone Convergence Theorem (MCT), the sequence converges.

   (b) Suppose $\lim x_n = \ell$. Given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - \ell| < \epsilon$. Therefore $n \geq N - 1$ implies $|x_{n+1} - \ell| < \epsilon$. Thus $\lim x_{n+1} = \ell$.

   (c) Taking limits of both sides, we get

   $$\ell = \frac{1}{4 - \ell}.$$

   Cross-multiplying, we get $4\ell - \ell^2 = 1$, that is, $\ell^2 - 4\ell + 1 = 0$. By the quadratic formula, $\ell = 2 \pm \sqrt{3}$. Since the sequence is decreasing and $x_2 = 1$, we must have $\ell \leq 1$, so we take the smaller root, namely, $\ell = 2 - \sqrt{3}$.

2. (2.4.3(b)) Define the sequence recursively by $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2a_n}$ for $n \geq 1$. We will show that $(a_n)$ converges by showing that it is increasing and bounded above, then appealing to the MCT.

We will prove by induction that $a_{n+1} > a_n$ for all $n \geq 1$. Note that since $f(x) = \sqrt{x}$ is an increasing function, $a < b$ implies that $\sqrt{a} < \sqrt{b}$. If $n = 1$, then $2 > 1$ implies that $\sqrt{2} > 1$. Multiplying by 2 and taking square roots, we get $a_2 > a_1$.

Now suppose that $a_{n+1} > a_n$ for some $n \geq 1$. Multiply by 2 and take square roots to get $a_{n+2} > a_{n+1}$. This completes the inductive step, so $(a_n)$ is indeed increasing.

Now we will prove, again by induction, that $(a_n)$ is bounded above by 2. (Of course, any number $\geq 2$ will also work.) If $n = 1$, then $2 < 4$, so $a_1 = \sqrt{2} < \sqrt{4} = 2$.

Suppose that $a_n < 2$ for some $n \geq 1$. Then $a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2$, completing the inductive step.

Since $(a_n)$ is increasing and bounded above, it converges by the MCT.

To find the limit, we play the same game as before: if $\lim a_n = \ell$, then $\lim a_{n+1} = \ell$. Thus, $\ell = \sqrt{2\ell}$, so $\ell^2 - 2\ell = 0$, so $\ell = 0$ or 2. Since $(a_n)$ is increasing, we must have that $\ell = 2$.

3. (2.4.6) (515 students only)

   (a) Note that
   $$\sqrt{xy} \leq \frac{x+y}{2}$$
   if and only if
   $$4xy \leq (x+y)^2.$$
   But $(x+y)^2 - 4xy = (x-y)^2 \geq 0$.

   (b) Note that by part (a), we have $x_n \leq y_n$ for all $n \in \mathbf{N}$.
   For all $n \in \mathbf{N}$, we then have $x_{n+1} = \sqrt{x_n y_n} \geq \sqrt{x_n \cdot x_n} = x_n$, so $(x_n)$ is increasing.
   For all $n \in \mathbf{N}$, $y_{n+1} = (x_n + y_n)/2 \leq (y_n + y_n)/2 = y_n$, so $(y_n)$ is decreasing.
   Thus, $x_1 \leq x_n \leq y_n \leq y_1$, so $(x_n)$ and $(y_n$ are both bounded. By the MCT, both sequences converge. Let $\ell = \lim x_n$ and $m = \lim y_n$. Since $\lim x_{n+1} = \ell$, we have $\ell = \sqrt{\ell m}$, so $\ell^2 = \ell m$, so $\ell = 0$ or $\ell = m$. Since $(x_n)$ is increasing, $\ell \neq 0$, so $\ell = m$ as desired.

4. (2.5.2)

   (a) True. Let $n_k = k + 1$. Then $(x_{k+1})$ is a proper subsequence of $(x_k)$, and we have seen that $\lim x_{k+1} = \lim x_k$.

   (b) True; this is the contrapositive form of Theorem 2.5.2.

   (c) True. Since $(x_n)$ is bounded, it contains a convergent subsequence by Bolzano-Weierstrass. Suppose that $(x_{n_r}) \to L$. Since $(x_n)$ diverges, there exists an $\epsilon > 0$ such that for all $N \in \mathbf{N}$, there exists an $n \geq N$ that satisfies $|x_n - L| \geq \epsilon$. Thus we can construct a subsequence $(x_{n_k})$ that satisfies $|x_{n_k} - L| \geq \epsilon$ for all $k \in \mathbf{N}$. This subsequence is still bounded, and so contains a convergent subsequence. In fact, we may assume that $x_{n_k}$ itself converges to, say, $M$. There exists an $N \in \mathbf{N}$ such that $k \geq N$ implies that $|x_{n_k} - M| < \epsilon/2$. Therefore
   $$\epsilon \leq |x_{n_k} - L| = |x_{n_k} - M + M - L| \leq |x_{n_k} - M| + |M - L| < \frac{\epsilon}{2} + |M - L|.$$

Therefore $|M - L| \geq \epsilon/2$, so in particular, $M = L$.

(d) False. Consider, for example, the sequence $(x_n)$, where

$$a_n = \begin{cases} n & \text{if } n \text{ odd}, \\ 1 - \frac{1}{n} & \text{if } n \text{ even}. \end{cases}$$

Then $(x_n)$ is monotone increasing; $(x_{2n})$ is a convergent subsequence; but $(x_n)$ is unbounded and so diverges. (One could also interleave two sequences that increase to different limits.)

5. (2.6.4)

(a) True. Since $(a_n)$ and $(b_n)$ are Cauchy, they converge, so $(|a_n - b_n|)$ converges by the Algebraic Limit Theorem (notice that $|x| = \sqrt{x^2}$). By the Cauchy Criterion, $(c_n)$ is Cauchy.

(b) False. Consider, for example, the constant sequence $(a_n) = (1)$. Then $c_n = (-1)^n$, which defines a divergent, hence non-Cauchy, sequence.

(c) False. Consider, for example,

$$a_n = 1 + \frac{(-1)^n}{n}.$$

Then $(a_n) = (0, 3/2, 2/3, 5/4, 4/5, \ldots) \to 1$, so is Cauchy. However, $(c_n) = (0, 1, 0, 1, 0, 1, \ldots)$ diverges since it contains subsequences converging to 0 and 1, and so is not Cauchy.