Solutions: Homework 1

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January 23, 2019

1. (1.2.1)

   (a) We prove that $\sqrt{3}$ is irrational by contradiction. Suppose that there exist relatively prime integers a and b such that $a/b = \sqrt{3}$. Then $a^2 = 3b^2$, so 3 divides a^2. Since 3 is prime, 3 divides a, so we can write $a = 3c$ for some integer c. Therefore $(3c)^2 = 3b^2$. After some cancellation, we find that $b^2 = 3c^2$. By the same argument as above, 3 divides b. This contradicts our assumption that a and b have no common divisors other than 1. Therefore, no such integers a and b exist.

   The same argument works for $\sqrt{6}$.

   (b) Replacing 3 by 4, we get $a^2 = 4b^2$, so 4 divides a^2. Since 4 is not prime, we cannot conclude that 4 divides a; for example, we could have $a = 2$.

2. (1.2.8)

   (a) Say, $f(n) = 2n$. If $f(n_1) = f(n_2)$, then $2n_1 = 2n_2$, so $n_1 = n_2$, and f is 1-1. Since, for example, 1 is not divisible by 2, f is not onto

   (b) For example, set $f(n) = n/2$ if n is even, and $f(n) = f(n-1)$ if n is odd. Then by construction, f is not 1-1, but for all $n \in \mathbf{N}$, $n = f(2n)$, so f is onto.

   (c) We can enumerate the integers as follows: $0, 1, -1, 2, -2, 3, -3, \ldots$. Define $f(n)$ to be the nth term in this sequence. A fancier way to define the same function f is as follows. For every $n \in \mathbf{N}$, we can write n uniquely as $n = 2m + k$ where $k = 0$ or 1 and $m \in \mathbf{N}$. Set $f(n) = (-1)^k m$.

3. (1.2.11)

   (a) Let $S \subset \mathbf{R}^2$ be the set, $S = \{(a, b) \in \mathbf{R}^2 : a < b\}$. Then the statement becomes: if $(a, b) \in S$, then there exists $n \in \mathbf{N}$ such that $a + \frac{1}{n} < b$. The negation then is: there exists $(a, b) \in S$ and for all $n \in \mathbf{N}$, $a + \frac{1}{n} \geq b$. Better: there exist real numbers with $a < b$ such that for all $n \in \mathbf{N}$, $a + \frac{1}{n} \geq b$. The original statement is true.

   (b) For all real numbers $x > 0$, there exists $n \in \mathbf{N}$ such that $\frac{1}{n} \leq x$. (The negation is true.)

   (c) There exists two distinct real numbers such that every number between them is irrational. (The original is true.)

4. (1.2.12)

   (a) For $n = 1$, $y_1 = 6 > -6$, so the induction is grounded. Suppose now that $y_n > -6$; we will show that $y_{n+1} > -6$. By definition,

   $$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = \frac{-18}{3} = -6.$$

   This completes the inductive step and the proof.

   (b) We must prove: for all $n \in \mathbf{N}$, $y_{n+1} < y_n$. A calculation shows that $y_2 = 2 < 6 = y_1$, so the induction is grounded.

   Suppose now that $y_{n+1} < y_n$. Then multiplying by 2 from both sides of the inequality, subtracting 6, and then dividing by 3, we find that $y_{n+2} < y_{n+1}$, completing the inductive step and the proof.

5. (1.2.7)

   (a) Since $f(x) = x^2$ is an increasing function for $x \geq 0$, if $0 \leq x \leq 2$, then $0 \leq x^2 \leq 4$, so $f(A) \subseteq [0, 4]$. Since the square-root function is also increasing, if $0 \leq y \leq 4$, then $y = f(\sqrt{y})$ with $0 \leq \sqrt{y} \leq 2$, so $y \in f(A)$. Therefore $f(A) = [0, 4]$.

   Using the same argument (since 1 and 4 are positive), $f(B) = [1, 16]$, and $f(A \cap B) = f([1, 2]) = [1, 4]$. Since $[0, 4] \cap [1, 16] = [1, 4]$, $f(A \cap B) = f(A) \cap f(B)$.

   Similarly, $A \cup B = [0, 4]$, so $f(A \cup B) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B)$.

   (b) If $A = [-2, -1]$ and $B = [1, 2]$, then $f(A \cap B) = f(\varnothing) = \varnothing$, while $f(A) \cap f(B) = [1, 4] \cap [1, 4] = [1, 4] \neq \varnothing$.

   (c) Suppose $y \in g(A \cap B)$. Then $y = g(x)$ for some $x \in A \cap B$. Since $x \in A$, $y \in g(A)$. Since $x \in B$, $y \in g(B)$. Therefore $y \in g(A) \cap g(B)$. It follows that $g(A \cap B) \subseteq g(A) \cap g(B)$.

   (d) (It is okay to suppose that $g : \mathbf{R} \to \mathbf{R}$.) Suppose that X and Y are sets. Let $g : X \to Y$. If $A, B \subseteq X$, then we claim that

   $$g(A \cup B) = g(A) \cup g(B).$$

   Indeed, suppose that $y \in g(A \cup B)$. Then for some $x \in A \cup B$, $y = g(x)$. If $x \in A$, then $y \in g(A)$; if $x \in B$, then $y \in g(B)$. Since $x \in A$ or $x \in B$, we have that $y \in g(A) \cup g(B)$. Therefore $g(A \cup B) \subseteq g(A) \cup g(B)$.

   Now suppose that $y \in g(A) \cup g(B)$. If $y \in g(A)$, then $y = g(x)$ for some $x \in A$. Since $A \subseteq A \cup B$, $x \in A \cup B$, so $y \in g(A \cup B)$. The same argument shows that if $y \in g(B)$, then $y \in g(A \cup B)$. So in either case, $y \in g(A \cup B)$, so $g(A) \cup g(B) \subseteq g(A \cup B)$. Therefore the two sets are equal, completing the proof of the claim.