Homework 1

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Problem 1.2.1

Original Solution

Part A

Prove that $\sqrt{3}$ is irrational

Assume $\sqrt{3}$ is rational. If $\sqrt{3}$ is rational, then it can be written as a ratio of two relatively prime natural numbers, a and b, such that:

$$\frac{a}{b} = \sqrt{3}$$

Proof:

$$\frac{a}{b} = \sqrt{3}$$

$$\frac{a^2}{b^2} = 3$$

$$a^2 = 3b^2$$

This implies that a^2 and thus a, are divisible by 3. We can rewrite a as a multiple of 3:

$$a = 3 * c$$

where c is some natural number. Subtituting in for a, we see:

$$(3 * c)^2 = 3b^2$$

$$9c^2 = 3b^2$$

$$3c^2 = b^2$$

This is a contradiction as both a and b were assumed to be relatively prime. Thus $\sqrt{3}$ is irrational.

Does a similar argument work to show $\sqrt{6}$ is irrational?

Assume $\sqrt{6}$ is rational. If $\sqrt{6}$ is rational, then it can be written as a ratio of two relatively prime natural numbers, a and b, such that:

$$\frac{a}{b} = \sqrt{6}$$

Proof:

$$\frac{a}{b} = \sqrt{6}$$

$$\frac{a^2}{b^2} = 6$$

$$a^2 = 6b^2$$

This implies that a^2 and thus a, are divisible by 6, as well as its factors 2 and 3. We can rewrite a as a multiple of 2:

$$a = 2 * c$$

where c is some natural number. Subtituting in for a, we see:

$$(2 * c)^2 = 6b^2$$

$$4c^2 = 6b^2$$

$$2c^2 = 3^2$$

This is a contradiction as both a and b were assumed to be relatively prime. Thus $\sqrt{6}$ is irrational.

Part B

Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Assume $\sqrt{4}$ is rational. If $\sqrt{4}$ is rational, then it can be written as a ratio of two relatively prime natural numbers, a and b, such that:

$$\frac{a}{b} = \sqrt{4}$$

Proof:

$$\frac{a}{b} = \sqrt{4}$$

$$\frac{a^2}{b^2} = 4$$

$$a^2 = 4b^2$$

This implies that a^2 and thus a, are divisible by 2. We can rewrite a as a multiple of 2:

$$a = 2 * c$$

where c is some natural number. Subtituting in for a, we see:

$$(2 * c)^2 = 4b^2$$

$$4c^2 = 4b^2$$

$$c^2 = b^2$$

From theorem 1.1.1 we should able to rewrite b^2 at this point in a way that breaks the assumption that b is relatively prime. However, this is not the case and thus we can assume that $\sqrt{4}$ is rational.

Self-Evaluation

Problem 1.2.7

Original Solution

***Let** $f(x) = x^2, A = [0, 2], B = [1, 4]$*

Part A

Find** $f(A)$ **and** $f(B)$**. Does** $f(A \bigcap B) = f(A) \bigcap f(B)$**? Does** $f(A \bigcup B) = f(A) \bigcup f(B)$**?

$$f(A) = f([0, 2])$$

$$f(A) = [0, 4]$$

—

$$f(B) = f([1, 4])$$

$$f(B) = [1, 16]$$

—

$$f(A \bigcap B) = f([0, 2] \bigcap [1, 4])$$

$$f(A \bigcap B) = f([1, 2])$$

$$f(A \bigcap B) = [1, 4]$$

—

$$f(A) \bigcap f(B) = f([0, 2]) \bigcap f([1, 4])$$

$$f(A) \bigcap f(B) = [0, 4] \bigcap [1, 16]$$

$$f(A) \bigcap f(B) = [1,4] = f(A \bigcap B)$$

—

$$f(A) \bigcup f(B) = f([0,2]) \bigcup f([1,4])$$

$$f(A \bigcup B) = f([0,4])$$

$$f(A \bigcup B) = [0,16]$$

—

$$f(A) \bigcup f(B) = f([0,2]) \bigcup f([1,4])$$

$$f(A) \bigcup f(B) = [0,4] \bigcup [1,16]$$

$$f(A) \bigcup f(B) = [0,16] = f(A \bigcup B)$$

Part B

Find two sets, A and B for which $f(A \bigcap B) \neq f(A) \bigcap f(B)$

Let $A = [-1,0], B = [0,1]$

$$f(A \bigcap B) = f([-1,0] \bigcap [0,1])$$

$$f(A \bigcap B) = f([0])$$

$$f(A \bigcap B) = 0$$

—

$$f(A) \bigcap f(B) = f([-1,0]) \bigcap f([0,1])$$

$$f(A) \bigcap f(B) = [0,1] \bigcap [0,1]$$

$$f(A) \bigcap f(B) = [0,1] \neq f(A \bigcap B)$$

Part C

Show that, for an arbitrary function $g : R- > R$ it is always true that $g(A \bigcap B) \subseteq g(A) \bigcap g(B)$ for all sets $A, B \subseteq R$

Let $x \in A \bigcap B$ this then implies

$$g(x) \in g(A) \bigcap g(B)$$

The next step in the justification process I struggled with some, as I could not find a good way to justify why $g(A \bigcap B) \subseteq g(A) \bigcap g(B)$ other than the fact it is obvious given the above relationship.

Part D

Form and prove a conjecture about the relationship between $g(A \bigcup B)$ and $g(A) \bigcup g(B)$ ***for an arbitrary function*** g

My conjecture is that $g(A \bigcup B) = g(A) \bigcup g(B)$

My intuition about this comes from a background in boolean logic. My first inclination was to try and prove this with a truth table, but I could not make it generic enough for this process.

Let $x \in A \bigcup B$ this then implies

$$g(x) \in g(A \bigcup B) = g(A \bigcup B)$$

Again I somewhat struggled with the last step of this process as I could not find a clear way to communicate the equality of this relationship other than the fact that I feel it is intuitively obvious.

Self-Evaluation

Problem 1.2.8

Original Solution

Part A

$f : N- > N$ that is 1-1 but not onto

Let $f(x) = 2(x) + 1$

then

$$f(1) = 3$$

$$f(2) = 5$$

$$\ldots$$

so the mapping is 1-1, in that it maps every element uniquely, but not all of B is represented, i.e. there are no even numbers in the mapping.

Part B

$f : N- > N$ that is onto but not 1-1

Let

$$f(x) = \begin{cases} x & x < 5 \\ x - 5 & x \geq 5 \end{cases}$$

The mapping is onto, in that all of B is represented. However it is not 1-1. For example $f(6) = f(1)$.

Part C

$f : N- > Z$ that is 1-1 and onto

This is not possible. Because $N \subseteq Z$, a function that is both 1-1 and onto cannot be achieved.

Self-Evaluation

Problem 1.2.11

Form the logical negation of each claim. In each case, make an intuitive guess as to wether the claim or its negation is the true statement.

Original Solution

Part A

For all real numbers satisfying $a < b$***, there exists an*** $n \in N$ ***such that*** $a + \frac{1}{n} < b$

There exists real numbers $a < b$ such that for all $n \in N a + \frac{1}{n} \geq b$

The negated statement falls apart when trying to prove the statement holds *for all* n. For example if we let $a = 1, b = 2,$ and $n = 10$, the negated statement is false (for all $n > 1$). Conversely, n can always be chosen in such a way that the original statement is true, regardless of the values for a and b.

Part B

There exists a real number $x > 0$ ***such that*** $x < 1/n$ ***for all*** $n \in N$

For all real numbers, $x > 0$, there exists an n ($n \in N$) such that $x \geq \frac{1}{n}$

I would hypothesize that the orignal statement is correct. The negated statement does not hold across R for all possible x. For example if $x = 0.1,$ and $n = 1$ the orignal statement would hold, for all n, while the negated statement does not.

Part C

Between every two distinct real numbers there is a rational number

Between every two distinct real numbers, there is not a rational number.

The original statement is correct. R exists as an extension of Q. As the book describes it, R “fills in the gaps” left behind by Q. While these “gaps” may not be as literal as I am assuming, the expression helps visualize how the two sets coexist, which leads me to my conclusion.

Self-Evaluation

Problem 1.2.12

Let $y_1 = 6$ and for each $n \in N$ define $Y_{n+1} = \frac{(2y_n - 6)}{3}$

Original Solution

Part A

Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in N$

For $n = 1$

$$y_1 = 6 > -6$$

The induction is grounded

Suppose $y_n > -6$

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = -6$$

Part B

Use another induction argument to show the sequence is decreasing

For $n = 1$

$$y_2 = \frac{2(y_1) - 6}{3} = \frac{2(6) - 6}{3} = 2 < 6$$

The induction is grounded

For $y_{n+1} < y_n$

$$2y_{n+1} < 2y_n$$

$$2y_{n+1} - 6 < 2y_n - 6$$

$$\frac{2y_{n+1} - 6}{3} < \frac{2y_n - 6}{3}$$

It then follows that $y_{n+2} < y_{n+1}$, etc.

Self-Evaluation

External References