Solutions: Homework 4

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1. (2.2.2) We will only show the "polished versions" below.

   (a) Let $\epsilon > 0$ be arbitrary, and set N be a natural number greater than $3/25\epsilon$. Suppose $n \geq N$. Then

   $$\begin{aligned} \left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| &= \frac{3}{25n+20} \\ &< \frac{3}{25n} \\ &< \frac{3}{25\left(\frac{3}{25\epsilon}\right)} \\ &= \epsilon. \end{aligned}$$

   Therefore, $\left(\frac{2n+1}{5n+4}\right) \to \frac{2}{5}$.

   (b) Let $\epsilon > 0$ be arbitrarry, and set N be an integer greater than $2/\epsilon$. Suppose $n \geq N$. Then

   $$\left| \frac{2n^2}{n^3+3} \right| < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon.$$

   Therefore, $\left(\frac{2n^2}{n^3+3}\right) \to 0$.

   (c) Let $\epsilon > 0$ be arbitrarry, and set N be an integer greater than $1/\epsilon^3$. Suppose $n \geq N$. Then

   $$\left| \frac{\sin(n^2)}{\sqrt[3]{n}} \right| \leq \frac{1}{\sqrt[3]{n}} < \epsilon.$$

   Therefore, $\left(\frac{\sin(n^2)}{\sqrt[3]{n}}\right) \to 0$.

2. (2.2.6) Suppose $(a_n) \to a$ and $(a_n) \to b$. Let $\epsilon > 0$ be arbitrary. Then there exist $N, M \in \mathbf{N}$ such that whenever $n \geq N$, we have $|a_n - a| < \epsilon/2$, and whenever $n \geq M$, we have $|a_n - b| < \epsilon/2$. Then if $n \geq \max\{M, N\}$, we have

   $$\epsilon > |a - a_n| + |a_n - b| \geq |a - b|$$

   by the Triangle Inequality.

   Since $|a - b| < \epsilon$ for all $\epsilon > 0$, we must have that $|a - b| = 0$. Therefore $a = b$.

3. (2.3.3) By setting $a_n = y_n - x_n$ and $b_n = z_n - x_n$, we see that if suffices to prove that if $0 \leq a_n \leq b_n$ for all n, and $\lim b_n = 0$, then $\lim a_n = 0$. Let $\epsilon > 0$ be arbitrary. Since $\lim b_n = 0$, there exists some $N \in \mathbf{N}$ such that whenever $n \geq N$, we have $b_n < \epsilon$. (We do not worry about absolute-value bars because $b_n \geq 0$ by hypothesis.) For the same N, whenever $n \geq N$, we then have $0 \leq a_n \leq b_n < \epsilon$. Therefore $\lim a_n = 0$ as desired.

   Note: we cannot assume that (y_n) converges! If we could, then the result would be a straightforward application of the Order Limit Theorem:

$$\lim x_n \leq \lim y_n \leq \lim z_n$$

   and since the outside two quantities are equal, the inequalities must be equalities.

4. (2.3.7)

   (a) Let $x_n = (-1)^n$ and $y_n = -x_n$. Then $x_n + y_n = 0$ for all n, and so converges.

   (b) Impossible by the Algebraic Limit Theorem: If $(x_n + y_n)$ and (x_n) converge, then so must (y_n), because $y_n = (x_n + y_n) - x_n$.

   (c) Let $b_n = 1/n$.

   (d) Impossible, by Theorem 2.3.2 (every convergent sequence is bounded). Since (b_n) is convergent, it is bounded, and so there is some $B \in \mathbf{R}$ such that $|b_n| \leq B$ for all n. Since $(a_n - b_n)$ is bounded by hypothesis, there is some $C \in \mathbf{R}$ such that $|a_n - b_n| \leq C$ for all n. By the Triangle Inequality, $|a_n| \leq |b_n| + |a_n - b_n| \leq B + C$, and so (a_n) must be bounded.

   (e) Let $a_n = 0$ and $b_n = n$ for all n. Then (b_n) diverges, but $(a_n b_n) = (a_n)$ converges.

5. (2.3.9(a)) We may not use the Algebraic Limit Theorem because (a_n) does not necessarily converge. Suppose $|a_n| \leq B$ for some $B \in \mathbf{R}$. Let $\epsilon > 0$ be arbitrary. Since $\lim b_n = 0$, there exists some $N \in \mathbf{N}$ such that $|b_n| < \epsilon/B$ whenever $n \geq N$. Suppose now that $n \geq N$; then $|a_n b_n| = |a_n||b_n| \leq B|b_n| < B(\epsilon/B) = \epsilon$. Therefore, $\lim a_n b_n = 0$.