Solutions: Homework 9

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(7.2.4) With notation from class: we always have $M_k \geq m_k$, so if $M_k > m_k$ on the kth interval of P, then $U(g, P) > L(g, P)$. This means that if $U(g, P) = L(g, P)$, then g is constant on each of the subintervals of P. (That is, it looks like a histogram.) For all $\varepsilon > 0$, $U(g, P) - L(g, P) < \varepsilon$, so g is integrable, and $\int_a^b g = U(g, P) = L(g, P)$.

(7.2.6) Suppose f is Riemann-Original-integrable on $[a, b]$. Let $\varepsilon > 0$ be arbitrary, and let $\delta > 0$ be as given in the R.O. definition corresponding to $\varepsilon/3$. Let P be any partition with $\Delta x_k < \delta$ for all k. We choose two taggings on P: for each k, there exists $c_k \in I_k$ that satisfies $f(c_k) > M_k - \frac{\varepsilon}{6(b-a)}$. Then

$$M_k \Delta x_k < f(c_k)\Delta x_k + \frac{\varepsilon \Delta x_k}{6(b-a)}.$$

Therefore

$$U(f, P) = \sum_k M_k \Delta x_k < R(f, P, \{c_k\}) + \frac{\varepsilon}{6}.$$

Similarly, there exists a tagging $\{b_k\}$ of P such that

$$L(f, P) > R(f, P, \{b_k\}) - \frac{varepsilon}{6}.$$

Using the triangle inequality and the R.O. integrability of f on $[a, b]$, we get

$$\begin{aligned} U(f, P) - L(f, P) &< |R(f, P, \{c_k\}) - A| + |A - R(f, P, \{b_k\})| + \frac{2\varepsilon}{6} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

(7.3.2) (a) Since every subinterval I_k of P will contain irrational numbers, $m_k = 0$, so $L(t, P) = 0$.

(b) $x \in D_{\varepsilon/2}$ if and only if $x = 0$ or $x = m/n$ in lowest terms with $\frac{1}{n} \geq \frac{\varepsilon}{2}$, that is, if $n \leq \frac{2}{\varepsilon}$. Let N be the greatest integer less than $2/\varepsilon$. For each $n \leq N$, there are n such x's $(1/n, 2/n, \ldots, (n-1)/n)$ distinct from 1. This means that $D_{\varepsilon/2}$ has finitely many elements.

(c) Suppose $D_{\varepsilon/2} = \{0 = c_0, \ldots, c_{n-1}, c_n = 1\}$. Let

$$\delta_k = \min\left\{\frac{\varepsilon}{4nt(c_k)}, \frac{c_k - c_{k-1}}{4}, \frac{c_{k+1} - c_{k-1}}{4}\right\}$$

(with obvious modifications for $k = 0$ and n). Let

$$P = \{0, \delta_0, c_1 - \delta_1, c_1 + \delta_1, c_{n-1} - \delta_{n-1}, c_{n-1} + \delta_{n+1}, \ldots, 1 - \delta_n, 1\}.$$

Thanks to our choice of δ_k, there is no repetition in the above list, and the numbers are listed in order. Now, there is precisely one element of $D_{\varepsilon/2}$ on each subinterval of the form $[0, \delta_0]$ or $[c_k - \delta_k, c_k + \delta_k]$ or $[1 - \delta - n, 1]$, namely c_k, so for $k \neq 0, n$,

$$M_k = t(c_k)\Delta x_k \leq \frac{t(c_k)\varepsilon}{2nt(c_k)} = \frac{\varepsilon}{2n}.$$

while

$$M_0, M_1 \leq \frac{\varepsilon}{4n}.$$

Adding, we get a contribution to $U(t, P)$ of at most $\varepsilon/2$.

The other subintervals contain no elements of $D_{\varepsilon/2}$, and so the supremum of t is $\leq \varepsilon/2$. Since these intervals have total length less than one, the contribution to $U(t, P)$ is again at most $\varepsilon/2$. Thus $U(t, P) \leq \varepsilon$.

(7.4.5) (a) It suffices to show that, for an interval I,

$$\sup_{x\in I}(f(x) + g(x)) \leq \sup_{x\in I} f(x) + \sup_{x\in I} g(x).$$

We do this by showing that the right-hand side is an upper bound for $\{f(x) + g(x) : x \in I\}$. This is straightforward: for all $x \in I$, $f(x) \leq \sup_{x\in I} f(x)$ and $g(x) \leq \sup_{x\in I} g(x)$, and now we just add.

To see that the inequality may be strict, let $f(x) = x$ and $g(x) = -x$ on $[0, 1]$. Then $\sup(f(x) + g(x)) = 0$ while $\sup f(x) + \sup g(x) = 1 + 0 = 1$.

For lower sums, we get

$$L(f + g, P) \geq L(f, P) + L(g, P).$$

(b) Since f and g are integrable on $[a, b]$, there exists a sequence of partitions P_n (we need to take a common refinement to get both of the following) such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$$

and

$$\lim_{n\to\infty} [U(g, P_n - L(g, P_n)] = 0.$$

From the result of the previous section,

$$U(f + g, P_n) - L(f + g, P_n) \leq [U(f, P_n) - L(f, P_n)] + [U(g, P_n) - L(g, P_n)]$$

The result now follows by taking limits and using the Order Limit Theorem.

(7.4.8) Let $H_n(x) = \sum_{k=1}^{n} h_k(x)$. Since H_n has finitely many step discontinuities, it is integrable. Since $|h_n(x)| \leq 1/2^n$ and $\sum_{n=0}^{\infty}(1/2^n)$ converges, H converges uniformly on $[0,1]$ by the Weierstrass M-Test. By the Integrable Limit Theorem, H is integrable on $[0,1]$. It is clear that

$$\int_0^1 h_n = \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right) = \frac{1}{2^n} - \frac{1}{4^n}.$$

Summing, we get

$$\int_0^1 H = \frac{1/2}{1-\frac{1}{2}} + \frac{1/4}{1-\frac{1}{4}} = \frac{2}{3}$$