Solutions: Homework 6

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1. (2.7.1 (a)) We will start with a few observations that should be familiar from Calc II.

   Observation 1: (s_{2n+1}) forms a decreasing sequence, since $s_{2n+3} = s_{2n+1} - (a_{2n+2} - a_{2n+3})$ and $a_{2n+3} \leq a_{2n+2}$.

   Observation 2: (s_{2n}) forms an increasing sequence, since $s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2})$ and $a_{2n+2} \leq a_{2n+1}$.

   Observation 3: For all m and n, $s_{2m} \leq s_{2n+1}$. (You may have seen this as $s_{\text{even}} \leq s_{\text{odd}}$.) Indeed, if $m \leq n$, then

   $$s_{2m} \leq s_{2n} \leq s_{2n} + a_{2n+1} = s_{2n+1},$$

   while if $m \geq n$, then

   $$s_{2m} \leq s_{2m} + a_{2m+1} = s_{2m+1} \leq s_{2n+1}.$$

   The above uses Observations 1 and 2.

   Observation 4: Putting the three previous Observations together, we see that if m is even, then $s_m \leq s_n$ for all $n \geq m$. If m is odd, then $s_m \geq s_n$ for all $n \geq m$.

   Now we are ready to prove the Hint from class, namely, that if $n > m$, then $|s_n - s_m| \leq a_{m+1}$. If m is even, then $s_n - s_m \geq 0$ (Obs. 4), and

   $$s_n - s_m = (s_n - s_{m+1}) + (s_{m+1} - s_m) \leq a_{m+1}$$

   since $s_n - s_{m+1} \leq 0$ by Obs. 4 and $s_{m+1} - s_m = a_{m+1}$.

   If m is odd, then $s_m - s_n \geq 0$ (Obs. 4), and

   $$s_m - s_n = (s_m - s_{m+1}) + (s_{m+1} - s_n) \leq a_{m+1}$$

   since $s_{m+1} - s_n \leq 0$ by Obs. 4 and $s_m - s_{m+1} = a_{m+1}$.

   Okay, we can finally show that (s_n) is a Cauchy sequence. Let $\varepsilon > 0$. Since $\lim a_n = 0$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies that $a_n < \varepsilon$. Therefore if $n > m \geq N$, we have

   $$|s_n - s_m| \leq a_{m+1} < \varepsilon.$$

2. (2.7.2)

(a) Since $2^n + n > 2^n$ and $\sum_{n=1}^{\infty}(1/2)^n$ is a convergent geometric series, the given series converges by the Comparison Test.

(b) Since $|\sin(n)| \leq 1$ for all n, the series converges absolutely by Comparison with the convergent p-series $\sum_{n=1}^{\infty} 1/n^2$. Since the series converges absolutely, it converges.

(c) The given series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{2n}.$$

The sequence $((-1)^{n+1}(n+1)/2n)$ diverges since the even terms converge to $-1/2$ while the odd terms converge to $1/2$. By Theorem 2.7.3, the series diverges.

(d) Write the series as $\sum_{n=1}^{\infty} a_n$. Notice that the negative terms "add up" to $-\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}$. That is, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n} - \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}$, and so diverges.

(e) Suppose the series $\sum_{n=1}^{\infty} a_n$ converges. Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{2n-1} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2}.$$

The second series converges since it equals $\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$. The first series, however, diverges by comparison to $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$. Since the sum of a convergent and a divergent series is divergent, the original series could not have been convergent to begin with.

3. (2.7.9)

(a) Let $\varepsilon = r' - r$. There exists $N \in \mathbf{N}$ such that $n \geq N$ implies that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| < \varepsilon$$

or equivalently,

$$-\varepsilon < \left|\frac{a_{n+1}}{a_n}\right| - r < \varepsilon.$$

Adding r and then multiplying by $|a_n|$ throughout, we obtain

$$|a_n|(r-\varepsilon) < |a_{n+1}| < |a_n|(r+\varepsilon)$$

Since $r + \varepsilon = r'$, we are done.

(b) Since $0 \leq r < r' < 1$, the series shown is a convergent geometric series.

(c) We will prove by induction that $|a_{N+k}| < |a_N|(r')^k$ for $k \geq 1$. The case $k = 1$ was part (a) above. Suppose true for k. Then by (a) and the inductive hypothesis,

$$|a_{N+k+1}| < |a_{N+k}|r' < (|a_N|(r')^k)r' = |a_N|(r')^{k+1}.$$

This completes the inductive step and establishes our claim.

By the Comparison Test and part (b), we now have that $\sum_{n=N}^{\infty}|a_n|$ converges. Since $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N-1}|a_n| + \sum_{n=N}^{\infty}|a_n|$, the series $\sum_{n=1}^{\infty}|a_n|$ converges. By the Absolute Convergence Test, $\sum_{n=1}^{\infty} a_n$ converges.

4. (3.2.2)

   (a) For A: The set A is the disjoint union of $A' = \{1 + \frac{2}{2n} : n = 1, 2, 3, \ldots\}$ and $A'' = \{-1 + \frac{2}{2n-1} : n = 1, 2, 3, \ldots\}$. For A', notice that

   $$1 + \frac{1}{n} = \frac{n+1}{n} \to 1$$

   so 1 is the only limit point of A'.

   For A'', we have

   $$-1 + \frac{2}{2n-1} = \frac{3-2n}{2n-1} \to -1$$

   so -1 is the only limit point of A''. In conclusion, the limit points for A are 1 and -1.

   For B, as discussed in class, density of $\mathbf{Q}$ implies that the limit points for B are all of the real numbers in $[0, 1]$.

   (b) It is worth writing out the first few terms of A:

   $$1, 2, -\frac{1}{3}, \frac{3}{2}, -\frac{3}{5}, \frac{4}{3}, -\frac{5}{7}, \ldots$$

   Let us pick $2 \in A$. Then A consists of one sequence $(2, 3/2, 4/3, 5/4, \ldots)$ decreasing to 1 and another, $(1, -1/3, -3/5, -5/7, \ldots)$ decreasing to -1. Thus the nearest point in A to 2, other than 2 itself, is $3/2$. So $V_{1/2}(2)$ is not contained in A, so A is not open. While A contains the limit point 1, it does not contain -1, so it is not closed.

   B is not open because of density. Since it does not contain all of its limit points, it is not closed.

   (d) From the above, $\overline{A} = A \cup \{-1\}$ and $\overline{B} = [0, 1]$.

5. (3.2.3)

   (a) $\mathbf{Q}$ is neither open nor closed. For example, no neighborhood of 0 is contained in $\mathbf{Q}$, by density. And $\sqrt{2} \notin \mathbf{Q}$ but $\sqrt{2}$ is a limit point.

   (b) $\mathbf{N}$ is closed since its complement is the union of open intervals; it is not open since no $V_\varepsilon(1) \subseteq \mathbf{N}$.

   (c) The given set is open since its complement consists of a single point, which is closed. It is not closed because $(1/n)$ is a sequence in the set that converges to 0.

   (d) Not open, because all elements other than $n = 1$ are $\geq 1 + \frac{1}{4}$, so $V_{1/4}(1)$ is not contained in the set. Not closed, because we know the elements can be ordered into a sequence of partial sums for the convergent series $\sum(1/n^2)$, and so that limit point $(\pi^2/6)$ is not in the set (it is strictly greater than every element of the set).

   (e) Not open since $V_{1/2}(1)$ is not contained. Closed because there are no limit points!