Homework 9

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Problem 7.2.4

Original Solution

The upper sum of g is equal to the lower sum so $\inf(g, P) = \sup(g, P)$. If the infimum of a function is equal to the supremum, the the function's value is constant across its domain.

g is integrable. The criterion for Riemann Integrability requires that $U(g, P) - L(g, P) < \epsilon$ In this case, $U(g, P) - L(g, P) = 0$ so g is Riemann Integrable.

The value of $\int_a^b g(x) = \gamma * (b - a)$ where γ is the constant value of the function, g.

Self-Evaluation

My solution agrees with the one presented.

Problem 7.2.6

Original Solution

If f satisfies the given definition, then

$$L(f, P_o) \leq R(f, P_o) \leq U(f, P_o)$$

because $m_k \leq c_k \leq M_k$. This means that P_o is the result of a common refinement, so ∃ partitions, P_1, P_2 s.t. $P_o = P_1 \cup P_2$. It then follows that for these partitions,

$$U(f, P_1) \leq U(f) + (\frac{\epsilon}{2} - A)$$

$$L(f, P_2) \geq L(f) + (\frac{\epsilon}{2} - A)$$

Then ...

$$U(f, P_o) - L(f, P_o) \leq U(f, P_1) - L(f, P_2) < U(f) + (\frac{\epsilon}{2} - A) - (L(f) + (\frac{\epsilon}{2} - A)) < U(f) - L(f) + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

By definition 7.2.7 we know $U(f) - L(f) = 0$ so the above expression reduces to

$$U(f, P_o) - L(f, P_o) < \epsilon$$

$\therefore f$ is integrable.

Self-Evaluation

I had the right idea here, but I think using the idea of a refinement was a little misguided. I was trying to find a way to bound $U(f, P) - L(f, P)$ using $R(f, P)$ but I think that the additional partitions don't quite do that because $R(f, P)$ could simply be shifted based on the partition selected and then the relationships between the upper and lower sums wouldn't quite hold.

Problem 7.3.2

Original Solution

Part A

By the Axiom of Completeness we know that every rational number is "surrounded" by two irrational numbers. Thus, for any partition, P, the $\inf(f(x)), x \in P = 0$

Part B

Part B I struggled with because I wasn't sure how to go about finding the size of the set. I understood what values would be a part of the set, but I could not find a way to succinctly describe its size. I felt like there was maybe some implied information about the set that could have helped me get there, but I had a hard time seeing it.

Part C

This part depends on the question before it, so I am also not able to provide a complete answer. I can tell that the partition, P_ϵ depends on the insights gained into the set from part B as knowing when $t(x)$ exceeds $\frac{\epsilon}{2}$ would be key for understanding how to construct P_ϵ.

Self-Evaluation

Part B

I knew how to find what values of x would be in the given set, but I mainly struggled with quantifying it. I see how to do that now.

Part C

The solution presented makes sense. I see that the key is really the selection of δ, which would have been difficult to see without an insight into the answer from *Part B*.

Problem 7.4.5

Original Solution

Part A

$U(f+g,P) = \sum(M_k)\Delta x_k$, where $M_k = \sup(f(x)+g(x)))$

Then by the triangle inequality,

$$\sum \sup(f(x)+g(x))\Delta x_k \leq \sum \sup(f(x))\Delta x_k + \sum \sup(g(x))\Delta x_k = U(f,P) + U(g,P)$$

for some partition, P.

For lower sums,

$$L(f+g,P) \geq L(f,P) + L(g,P)$$

The inequality is strict if f and g have counteracting behaviors over P. For example, if f is monotonically decreasing while g is monotonically increasing.

Part B

$\int_a^b (f+g) = \int_a^b f + \int_a^b g$

$\int_a^b (f+g)$ implies that $U(f+g) = L(f+g)$ for some partition, P.

Then, using the relations found in *Part A* $\ldots$

$$U(f,P) + U(g,P) \geq U(f+g,P) \geq L(f+g,P) \geq L(f,P) + L(g,P)$$

Because f and g are known to be integrable,

$$U(f,P) = L(f,P)$$

and

$$U(g, P) = L(g, P)$$

which implies that

$$\int_a^b (f+g) = U(f+g, P) = L(f+g, P) = \int_a^b f + \int_a^b g$$

Self-Evaluation

My solution agrees with the one presented.

Problem 7.4.8

Original Solution

Using the Weierstrauss M-test, we see that

$$|h_n(x)| \leq \frac{1}{2^n} \, \forall x \in A$$

.

The geometric sum $\sum \frac{1}{2^n}$ converges to 1, so $\sum h_n(x)$ converges uniformly to H and is integrable.

$$\int_0^1 H = \sum \int_0^1 h_n(x) = \sum \frac{1}{2^n} * \frac{1}{2^n} = \sum \frac{1}{4^n} = \frac{1}{3}$$

Self-Evaluation

My approach agrees with the solution, however we arrived at different answers for the integral. I am pretty confident that my solution is correct and that the $(1 - \frac{1}{2^n})$ term is incorrect. Integrating h_n given $(\frac{1}{2^n} * x)$. Inserting the terms of integration, when x = 0, the whole term is zero, and when x = 1, an additional $\frac{1}{2^n}$ term is earned, as shown in my solution.

External References