Homework 1

Mark Archual | MTH 515

Dr. Scott | Real Analysis

1/24/2019

Problem 1.2.1

Original Solution

Part A

Prove that $\sqrt{3}$ is irrational

Assume $\sqrt{3}$ is rational. If $\sqrt{3}$ is rational, then it can be written as a ratio of two relatively prime natural numbers, a and b, such that:

$$\frac{a}{b} = \sqrt{3}$$

Proof:

$$\frac{a}{b} = \sqrt{3}$$

$$\frac{a^2}{b^2} = 3$$

$$a^2 = 3b^2$$

This implies that a^2 and thus a, are divisible by 3. We can rewrite a as a multiple of 3:

$$a = 3 * c$$

where c is some natural number. Subtituting in for a, we see:

$$(3 * c)^2 = 3b^2$$

$$9c^2 = 3b^2$$

$$3c^2 = b^2$$

This is a contradiction as both a and b were assumed to be relatively prime. Thus $\sqrt{3}$ is irrational.

Does a similar argument work to show $\sqrt{6}$ is irrational?

Assume $\sqrt{6}$ is rational. If $\sqrt{6}$ is rational, then it can be written as a ratio of two relatively prime natural numbers, a and b, such that:

$$\frac{a}{b} = \sqrt{6}$$

Proof:

$$\frac{a}{b} = \sqrt{6}$$

$$\frac{a^2}{b^2} = 6$$

$$a^2 = 6b^2$$

This implies that a^2 and thus a, are divisible by 6, as well as its factors 2 and 3. We can rewrite a as a multiple of 2:

$$a = 2 * c$$

where c is some natural number. Subtituting in for a, we see:

$$(2 * c)^2 = 6b^2$$

$$4c^2 = 6b^2$$

$$2c^2 = 3^2$$

This is a contradiction as both a and b were assumed to be relatively prime. Thus $\sqrt{6}$ is irrational.

Part B

Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Assume $\sqrt{4}$ is rational. If $\sqrt{4}$ is rational, then it can be written as a ratio of two relatively prime natural numbers, a and b, such that:

$$\frac{a}{b} = \sqrt{4}$$

Proof:

$$\frac{a}{b} = \sqrt{4}$$

$$\frac{a^2}{b^2} = 4$$

$$a^2 = 4b^2$$

This implies that a^2 and thus a, are divisible by 2. We can rewrite a as a multiple of 2:

$$a = 2 * c$$

where c is some natural number. Subtituting in for a, we see:

$$(2 * c)^2 = 4b^2$$

$$4c^2 = 4b^2$$

$$c^2 = b^2$$

From theorem 1.1.1 we should able to rewrite b^2 at this point in a way that breaks the assumption that b is relatively prime. However, this is not the case and thus we can assume that $\sqrt{4}$ is rational.

Self-Evaluation

Part b

My proof worked to show that both numbers were not relatively prime. Thus I replaced a = 2 in my answer to show that b^2 is in fact relatively prime. I could have stopped earlier in my answer to explain the shortcomings of the proof, but I decided to see the substitution through to prove the point.

Problem 1.2.7

Original Solution

Let $f(x) = x^2, A = [0,2], B = [1,4]$

Part A

Find $f(A)$ ***and*** $f(B)$***. Does*** $f(A \bigcap B) = f(A) \bigcap f(B)$? ***Does*** $f(A \bigcup B) = f(A) \bigcup f(B)$?

$$f(A) = f([0,2])$$

$$f(A) = [0,4]$$

—

$$f(B) = f([1,4])$$

$$f(B) = [1,16]$$

—

$$f(A \bigcap B) = f([0,2] \bigcap [1,4])$$

$$f(A \bigcap B) = f([1,2])$$

$$f(A \bigcap B) = [1,4]$$

—

$$f(A) \bigcap f(B) = f([0,2]) \bigcap f([1,4])$$

$$f(A) \bigcap f(B) = [0,4] \bigcap [1,16]$$

$$f(A)\bigcap f(B) = [1,4] = f(A\bigcap B)$$

—

$$f(A)\bigcup f(B) = f([0,2])\bigcup f([1,4])$$

$$f(A\bigcup B) = f([0,4])$$

$$f(A\bigcup B) = [0,16]$$

—

$$f(A)\bigcup f(B) = f([0,2])\bigcup f([1,4])$$

$$f(A)\bigcup f(B) = [0,4]\bigcup [1,16]$$

$$f(A)\bigcup f(B) = [0,16] = f(A\bigcup B)$$

Part B

Find two sets, A ***and*** B ***for which*** $f(A\bigcap B) \neq f(A)\bigcap f(B)$

Let $A = [-1,0], B = [0,1]$

$$f(A\bigcap B) = f([-1,0]\bigcap [0,1])$$

$$f(A\bigcap B) = f([0])$$

$$f(A\bigcap B) = 0$$

—

$$f(A)\bigcap f(B) = f([-1,0])\bigcap f([0,1])$$

$$f(A)\bigcap f(B) = [0,1]\bigcap [0,1]$$

$$f(A)\bigcap f(B) = [0,1] \neq f(A\bigcap B)$$

Part C

Show that, for an arbitrary function $g : R- > R$ ***it is always true that*** $g(A\bigcap B) \subseteq g(A)\bigcap g(B)$ ***for all sets*** $A, B \subseteq R$

Let $x \in A\bigcap B$ this then implies

$$g(x) \in g(A)\bigcap g(B)$$

The next step in the justification process I struggled with some, as I could not find a good way to justify why $g(A \bigcap B) \subseteq g(A) \bigcap g(B)$ other than the fact it is obvious given the above relationship.

Part D

Form and prove a conjecture about the relationship between $g(A \bigcup B)$ and $g(A) \bigcup g(B)$ ***for an arbitrary function*** g

My conjecture is that $g(A \bigcup B) = g(A) \bigcup g(B)$

My intuition about this comes from a background in boolean logic. My first inclination was to try and prove this with a truth table, but I could not make it generic enough for this process.

Let $x \in A \bigcup B$ this then implies

$$g(x) \in g(A \bigcup B) = g(A) \bigcup g(B)$$

Again I somewhat struggled with the last step of this process as I could not find a clear way to communicate the equality of this relationship other than the fact that I feel it is intuitively obvious.

Self-Evaluation

Part A

My answers for finding $f(A)$ and $f(B)$ lack verification that the solution range maps back to the original domain. I did not think this was needed, due to the nature of f given in the problem.

Part B

My answer differs, but is logically consistent with the solution presented.

Part C

Reviewing the solution, I see that the component I was missing in my answer was working backwards from the assumption that $y \in g(A \bigcap B)$ as the start of my proof. I hit a wall when trying to connect the relationship between $g(A \bigcap B)$ and $g(A) \bigcap (B)$ but I now see how that is accomplished.

Part D

The beginning part of my answer lacks some of the formality shown in setting up the conjecture. I assumed that the nature of the sets, A and B, were obvious given the context of the problem.

As for the proof itself, I made a similar mistake as I did in part C. Had I started my proof with the assumption shown, I feel I would have been able to work through the proof more effectively. My answer also originally stated $g(x) \in g(A \bigcup B) = g(A \bigcup B)$ but this has been updated in this submssion to be:
$g(x) \in g(A \bigcup B) = g(A) \bigcup g(B)$.

Problem 1.2.8

Original Solution

Part A

$f: N -> N$ that is 1-1 but not onto

Let $f(x) = 2(x) + 1$

then

$$f(1) = 3$$

$$f(2) = 5$$

$$\cdots$$

so the mapping is 1-1, in that it maps every element uniquely, but not all of B is represented, i.e. there are no even numbers in the mapping.

Part B

$f: N -> N$ that is onto but not 1-1

Let

$$f(x) = \begin{cases} x & x < 5 \\ x - 5 & x \geq 5 \end{cases}$$

The mapping is onto, in that all of B is represented. However it is not 1-1. For example $f(6) = f(1)$.

Part C

$f: N -> Z$ that is 1-1 and onto

This is not possible. Because $N \subseteq Z$, a function that is both 1-1 and onto cannot be achieved.

Self-Evaluation

Part A

My answer differs, but I believe it is still correct for the same reasoning presented in the solutions.

Part B

The same as above, my answers differs but I believe it is still a valid solution.

Part C

My logic here was flawed about the relationship between N and Z. I was struggling to come up with a well behaved function that met the requirements of the problem statement, so I took a guess that it was not possible.

Problem 1.2.11

Form the logical negation of each claim. In each case, make an intuitive guess as to wether the claim or its negation is the true statement.

Original Solution

Part A

***For all real numbers satisfying* $a < b$*, there exists an* $n \in N$ *such that* $a + \frac{1}{n} < b$**

There exists real numbers $a < b$ such that for all $n \in N a + \frac{1}{n} \geq b$

The negated statement falls apart when trying to prove the statement holds *for all* n. For example if we let $a = 1, b = 2,$ and $n = 10$, the negated statement is false (for all $n > 1$). Conversely, n can always be chosen in such a way that the original statement is true, regardless of the values for a and b.

Part B

***There exists a real number* $x > 0$ *such that* $x < 1/n$ *for all* $n \in N$**

For all real numbers, $x > 0$, there exists an n ($n \in N$) such that $x \geq \frac{1}{n}$

I would hypothesize that the orignal statement is correct. The negated statement does not hold across R for all possible x. For example if $x = 0.1,$ and $n = 1$ the orignal statement would hold, for all n, while the negated statement does not.

Part C

Between every two distinct real numbers there is a rational number

Between every two distinct real numbers, there is not a rational number.

The original statement is correct. R exists as an extension of Q. As the book describes it, R “fills in the gaps” left behind by Q. While these “gaps” may not be as literal as I am assuming, the expression helps visualize how the two sets coexist, which leads me to my conclusion.

Self-Evaluation

Part B

My answer differed in that I switched the direction of the second inequality from $x < 1/n$ to $x \geq \frac{1}{n}$. I believe this is correct and that the answer given in the solutions is incorrect. That being said, my answer about which statement is correct (the original or negated statement) would change depending on the direction of this inequality. I stand by my original answer.

Part C

My solution is worded differently but is logically equivalent.

Problem 1.2.12

Let $y_1 = 6$ and for each $n \in N$ define $Y_{n+1} = \frac{(2y_n - 6)}{3}$

Original Solution

Part A

Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in N$

For $n = 1$

$$y_1 = 6 > -6$$

The induction is grounded

Suppose $y_n > -6$

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = -6$$

Part B

Use another induction argument to show the sequence is decreasing

For $n = 1$

$$y_2 = \frac{2(y_1) - 6}{3} = \frac{2(6) - 6}{3} = 2 < 6$$

The induction is grounded

For $y_{n+1} < y_n$

$$2y_{n+1} < 2y_n$$

$$2y_{n+1} - 6 < 2y_n - 6$$

$$\frac{2y_{n+1} - 6}{3} < \frac{2y_n - 6}{3}$$

It then follows that $y_{n+2} < y_{n+1}$, etc.

Self-Evaluation

No comments.

External References