Solutions: Homework 8

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(5.2.7) (a) Since

$$\frac{g_a(x) - 0}{x - 0} = x^{a-1} \sin\left(\frac{1}{x}\right),$$

g_a is differentiable at 0 (hence everywhere) if and only if $a > 1$. In this case, the derivative is given by $g_a'(0) = 0$ and

$$g_a'(x) = ax^{a-1} \sin\left(\frac{1}{x}\right) - x^{a-2} \cos\left(\frac{1}{x}\right)$$

if $x \neq 0$. The first term is bounded on $[0, 1]$ (by $|a|$), while the second term is unbounded if $a - 2 < 0$. Thus we can take any a that satisfies $1 < a < 2$, for example, $a = \frac{1}{2}$.

(b) With the formula from part (a) for the derivative, we see that if $a > 2$, then g_a' is continuous. Its derivative at 0 would be the limit,

$$\lim_{x \to 0} ax^{a-2} \sin(1/x) - x^{a-3} \cos(1/x)$$

so we require $a > 3$ for differentiability at 0. So we need an a such that $2 < a \leq 3$, say $a = 3$.

(c) From our work in part (b), if $a > 3$ then g_a' is differentiable on $\mathbf{R}$ and $g_a''(0) = 0$. If $x \neq 0$, we have

$$g_a''(x) = x^{a-4}(a(a-1)x^2 - 1) \sin(1/x) - 2(a-1)x^{a-3} \cos(1/x)$$

so if we take $a = 4$, the limit of $g_4''(x)$ as $x \to 0$ does not exist and so g_4'' is not continuous at 0.

(5.3.2) Suppose f is not one-to-one on A, that is, there exists $a, b \in A$ with $a \neq b$ such that $f(a) = f(b)$. By Rolle's Theorem, there is some point c between a and b (hence in A, since A is an interval) such that $f'(c) = 0$.

A counterexample is provided by $f(x) = x^3$ on $[-1, 1]$. Then f is one-to-one on $[-1, 1]$, but $f'(0) = 0$.

(6.2.3) First, g_n.

(a) If $0 \leq x < 1$, then $\lim_{n\to\infty} x^n = 0$, so $\lim g_n(x) = x$. If $x = 1$, then $\lim g_n(1) = \frac{1}{2}$. If $x > 1$, then $x^n \to \infty$, so

$$\frac{x}{1+x^n} = \frac{1}{x^{n-1}}\left(\frac{1}{\frac{1}{x^n}+1}\right) \to 0 \cdot 1 = 0.$$

So the pointwise limit is the function

$$g(x) = \begin{cases} x & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 0 & 1 < x. \end{cases}$$

(b) Since each g_n is continuous and g is not, the convergence cannot be uniform.

(c) We could choose the smaller set $[0, 0.9]$. Suppose $\varepsilon > 0$ is arbitrary. Then

$$\left|\frac{x}{1+x^n} - x\right| = \left|\frac{-x^{n+1}}{1+x^n}\right| < |x|^{n+1} \leq (0.9)^{n+1}.$$

Since $(0.9)^{n+1} \to 0$, there exists $N \in \mathbf{N}$ such that, whenever $n \geq N$, we have $|x|^{n+1} \leq (0.9)^{n+1} < \epsilon$.

Now for h_n.

(a) Let h be the limit function. If $x = 0$, then $h_n(0) = 0$, so $h(0) = 0$. If $x > 0$, then there is some $N \in \mathbf{N}$ such that $\frac{1}{n} < x$ whenever $n \geq N$. Thus $n \geq N$ implies that $h_n(x) = 1$. Therefore

$$h(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0. \end{cases}$$

(b) Since h is not continuous but each h_n is, convergence cannot be uniform.

(c) We will take our smaller interval to be $[\frac{1}{100}, \infty)$. Then $h_n = h$ whenever $n \geq 100$, so uniform convergence is automatic.

(6.3.3)

1. Taking the derivative, we find

$$f_n'(x) = \frac{1 - nx^2}{(1+nx^2)^2}$$

so if there are extrema, then they occur at $x = \pm\frac{1}{\sqrt{n}}$. Since $\lim_{x\to\pm\infty} f_n(x) = 0$, these must indeed be extrema, so we calculate to find $f_n(\pm 1/\sqrt{n}) = \pm 1/2\sqrt{n}$. Therefore, for all $x \in \mathbf{R}$, we have

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}}.$$

Let $\varepsilon > 0$ be arbitrary, and let $N \in \mathbf{N}$ be big enough that $N > \frac{1}{4\varepsilon^2}$. Then whenever $n \geq N$, we have

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}} < \varepsilon$$

so (f_n) converges to the limit function (zero) uniformly.

2. From the previous part, if $x \neq 0$, then

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2} \to 0.$$

However, $f_n'(0) = 1$ for all n, so $f'(0) \neq \lim f_n'(0)$.

(6.4.3) (a) Since

$$\left| \frac{\cos(2^n x)}{2^n} \right| \leq \frac{1}{2^n}$$

and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a convergent geometric series, then g converges uniformly by the Weierstrass M-Test. As the uniform limit of continuous functions, g is continuous.

(b) Since $g_n'(x) = -\sin(2^n x)$, we do not have uniform convergence of $\sum g_n'$.

(6.5.4) (a) Let $x \in (-R, R)$, $x \neq 0$. Since f converges for some u with $|x| < |u| < R$, convergence is absolute at x. By the Cauchy Criterion, given $\varepsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$, we have

$$\sum_{k=m+1}^{n} |a_k x^k| < \varepsilon / |x|.$$

Thus

$$\sum_{k=m+1}^{n} \left| \frac{a_k}{k+1} x^{k+1} \right| = \sum_{k=m+1}^{n} \frac{|a_k x^k|}{k+1} |x| < \epsilon.$$

Therefore $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ converges pointwise at x, therefore absolutely on $(-|x|, |x|)$. It follows that the series converges uniformly on $(-R, R)$. The rest follows from Theorem 6.5.7.

(b) Just add a constant of integration.

(6.6.2) (a) Substitute $t = x^2$ in the Taylor series

$$\cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}$$

and multiply by x to obtain

$$x \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+1}$$

valid for all $x \in \mathbf{R}$.

(b) Substitute $t = -4x^2$ into the geometric series

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

to obtain

$$\frac{1}{1+4x^2} = \sum_{n=0}^{\infty} (-4)^n x^{2n}$$

valid for $4x^2 < 1$, or $|x| < \frac{1}{2}$. Differentiating term-by-term inside the radius of convergence, we get

$$\frac{-8x}{(1+4x^2)^2} = \sum_{n=1}^{\infty} (-4)^n (2n) x^{2n-1}.$$

To finish, we divide by -8.

$$\frac{x}{(1+4x^2)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} 4^{n-1} n x^{2n-1}.$$

Again, valid for $|x| < \frac{1}{2}$.

(c) The geometric series with ratio $-x^2$ gives us

$$\frac{2x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$

valid for $|x| < 1$. Integrating term-by-term and using the fact that $\log(1+0) = 0$ to resolve the constant of integration, we find

$$\log(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2}$$

valid for $|x| < 1$.