Homework 8

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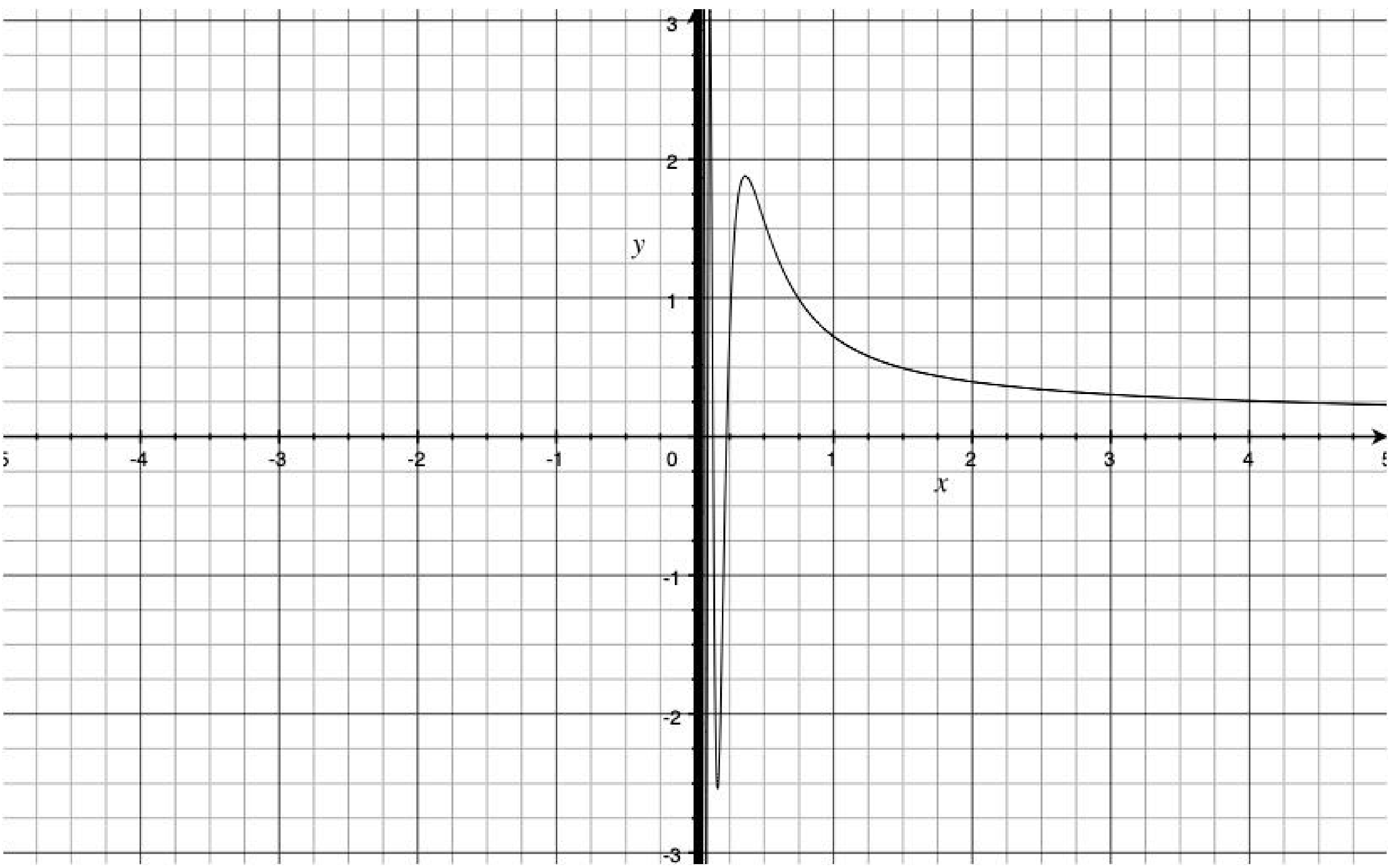
Problem 5.2.7

Original Solution

Part A

Let $a = 3/2$, then $g_a^{'}$ is unbounded on [0,1]. See the plot below.



Part B

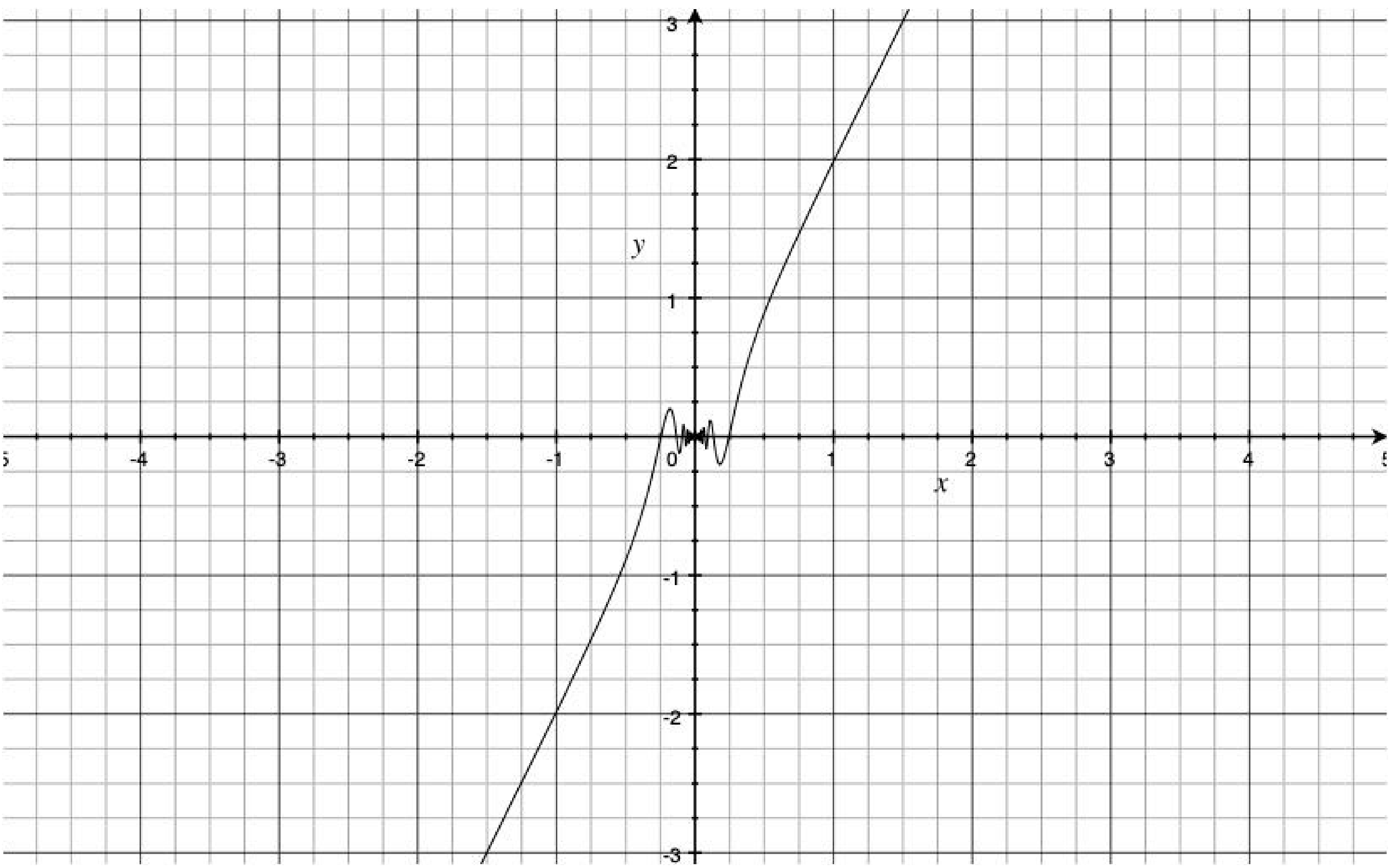
Let $a = 3$, then g_a' is

$$x(3x\sin(1/x) - \cos(1/x))$$

which from the plot we can see is continuous, but the limit as x approaches 0 does not exist due to the dense oscillation seen near the origin. This is confirmed by observing g_a'':

$$\frac{((6x^2 - 1)\sin(1/x))}{x} - 4\cos(1/x)$$

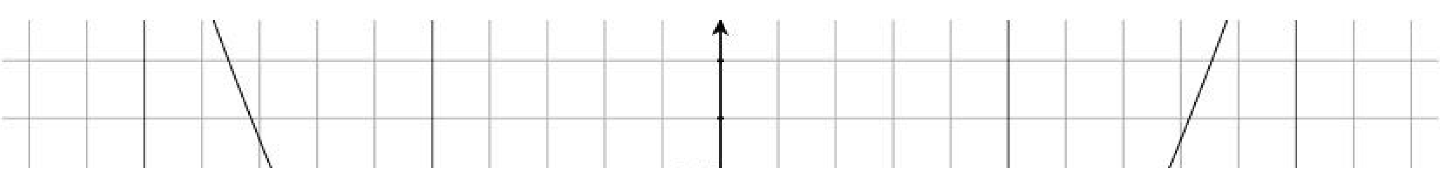
the x term in the denominator renders the whole expression as not defined at 0, thus g_a' is not differentiable at 0.

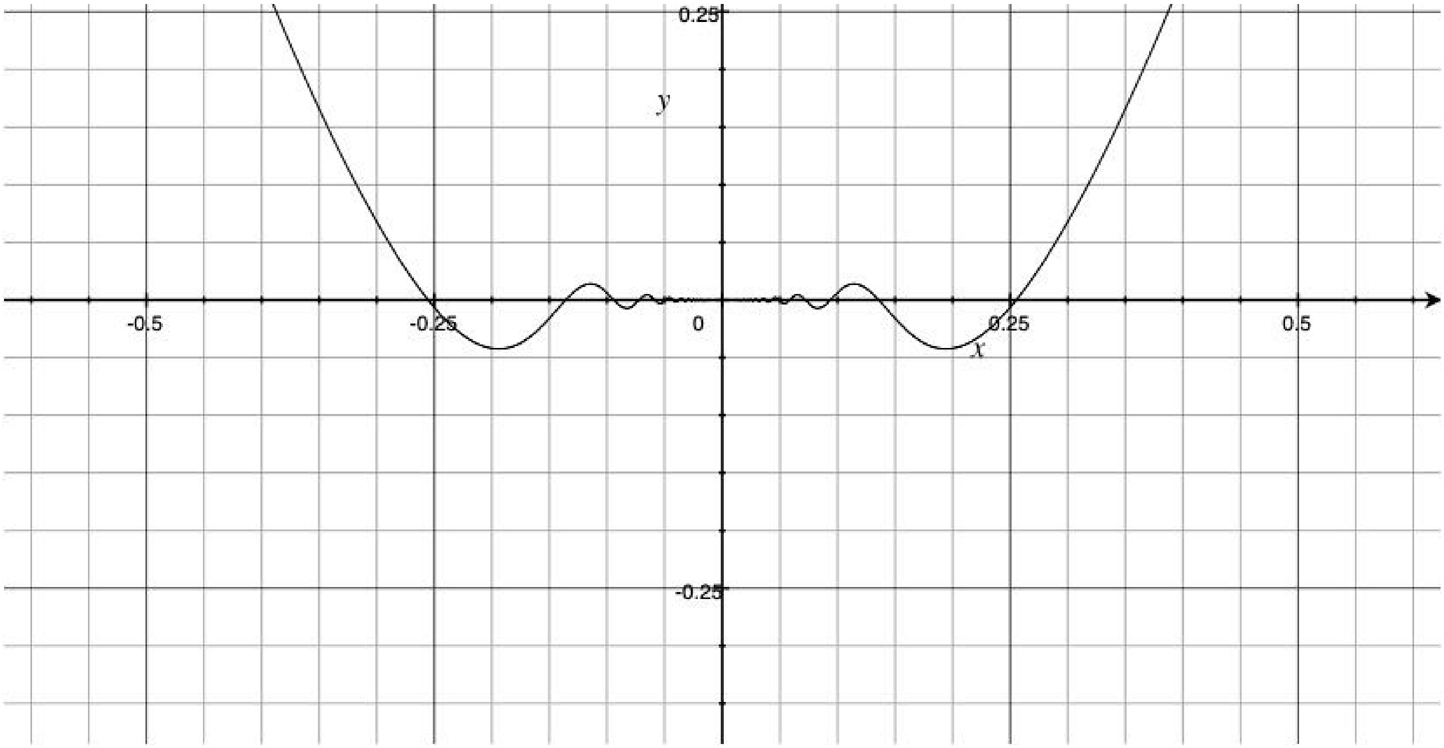


Part C

Let $a = 4$. Following a similar approach shown in *Part B* we see that g_a' appears to be reasonably well behaved on R

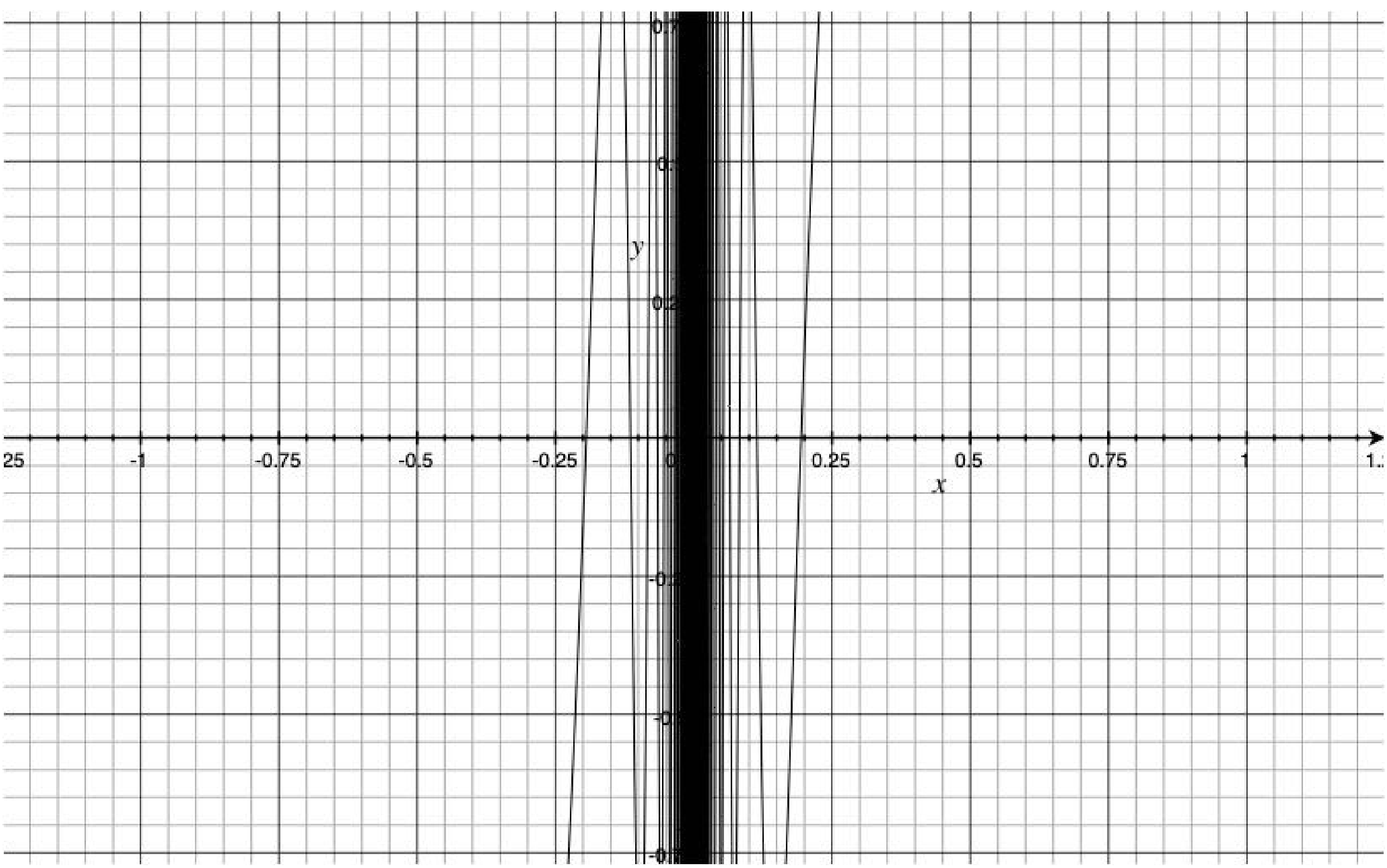
$$g_a' = x^2(4x\sin(1/x) - \cos(1/x))$$





Taking g_a'' yields oscillations near the origin that hint that differentiability may not be well defined on all of R

$$g_a'' = (12x^2 - 1)\sin(1/x) - 6x\cos(1/x)$$



Taking g_a''' we see a value of x^2 in the denominator indicating that the limit d.n.e. at 0.

$$g_a''' = \frac{(6x(4x^2 - 1)sin(1/x) + (1 - 18x^2)cos(1/x))}{x^2}$$

Self-Evaluation

My approach was more graphically driven than the solution, nonetheless the answers agree.

Problem 5.3.2

Original Solution

Let f be differentiable on A. If $f'(x) \neq 0$ on A then f is one-one

Using a corollary of the Mean Value Theorem, because $f'(x) \neq 0$ on A then we know $f(x)$ is not constant. By the Interior Extremum Theorem because $f'(c) \neq 0 \forall x \in A$ then we know that f does not attain a minimum or maximum on A. Therefore f must be a monotonic function on A. Monotonic functions fulfill the requirements of being one-one.

$\therefore f$ is one-one.

As an example of the converse not being true, consider $f = x^3$ we know x^3 is one to one, as it is monotonic, however $f' = x^2 = 0$ for $x = 0$.

Self-Evaluation

My reasoning for why f is one-one takes a different approach but I believe is still valid.

Problem 6.2.3

Original Solution

For $g_n \ldots$

Part A

For a fixed x as $n -> \infty$

$$\lim g_n(x) = \begin{cases} 0 & : x > 1 \\ x & : x < 1 \\ 1/2 & : x = 1 \end{cases}$$

Part B

g_n is not uniformly convergent because for the convergence of the function depends on the value of x chosen.

Part C

$(1, \infty) \forall x \in A, \lim g_n -> 0$ regardless of x.

For $h_n \ldots$

Part A

For a fixed x as $n -> \infty$

$$\lim h_n(x) = \{ \begin{matrix} 1 & : x > 1/n \\ nx & : 0 \le x < 1 \end{matrix}$$

Part B

h_n is not uniformly convergent because for the convergence of the function depends on the value of x chosen.

Part C

$[1, \infty) \forall x \in A, \lim h_n -> 1$ regardless of x.

Self-Evaluation

Looks like I did not have a strong grasp on this topic as my answers for both parts of this question are off. My expressions for the limits of the function are incorrect, which lead to *Part C* of each function not quite lining up with the solution. I reviewed the solution and can now see how to evaluate the limits properly.

Problem 6.3.3

Original Solution

Part A

$f_n(x) = \frac{x}{1+nx^2}$. When n = 1, $f_1(x) = \frac{x}{1+x^2}$, which has roots of -1, 1 and a max/min of -0.5, 0.5 respectively. As $n- > \infty, f_n(x)- > 0$, regardless of x. Therefore we know, we are safe to assume that the behavior of the function as n changes is not dependent on x.

With this knowledge we can then move to find the max and min values for x in terms of n. The roots of the derivative show that x is a max/min when:

$$x = \pm\sqrt{\frac{1}{n}}$$

Therefore we can apply the cauchy criterion to show that

$$|f_m - f_n| = \max(\pm\sqrt{\frac{1}{n}}, \pm\sqrt{\frac{1}{m}}) < \epsilon$$

$$\therefore \lim_{n->\infty} \frac{x}{1+nx^2} = 0$$

Part B

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$$

$f' = \lim f_n'$ when $x \neq 0$

Self-Evaluation

Part A

For this question, I was on the right track. The big difference between my answer and the solution is that I deferred to trying to apply the cauchy criterion, with the incorrect value, when I should have found the expression for the function at the given value for x in terms of n and then applied the criterion.

Problem 6.4.3

Original Solution

Part A

We know that $\cos(2^n * x) \leq 1 \, \forall x, n$. This reduces the summation to

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

$\frac{1}{2^n}$ is a geometric series with a ratio of $\frac{1}{2}$ therefore the infinite sum converges. This allows us to use the weierstrauss M-test to show that

$$|f_n(x)| \leq \sum_{n=0}^{\infty} \frac{1}{2^n}$$

thus f converges uniformly and f is continuous.

Part B

Theorem 6.4.3 states that $\sum g_n^{'}(x)$ must converge uniformly. In this case, $g^{'}(x) = 2^n \sin(2^n x)$, which $\sum_{n=0}^{\infty} \frac{1}{2^n}$ does not converge uniformly, as it is unbounded. Thus g is not differentiable.

Self-Evaluation

No changes here other than removing an extra summation symbol from the solution.

Problem 6.5.4

Original Solution

I struggled with this question. I had a hard time seeing how to get started. I think I would have been able to succeed if the book or our lecture notes provided more examples.

Self-Evaluation

Problem 6.6.2

Original Solution

Part A

$x\cos(x^2) = x^3 - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!} \dots$

Part B

This one I struggled with, mainly for algebraic reasons. I had trouble seeing how to do the substitutions.

Part C

$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} \dots$

Self-Evaluation

Part A

I somehow managed to get the first term of this series wrong, but the rest of the terms are correct.

Part B

I see that the key idea I was missing was executing term by term differentiation.

External References