Modeling and Control of a Two DOF Helicopter Using a Robust Control Design Based on DK Iteration

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***Abstract*— An alternative way to control a nonlinear system, such as a two Degree Of Freedom helicopter, is the linearization of the model around an equilibrium point, in this case, an equilibrium point around the Pitch angle. Subsequently, the design a Robust Controller for this system is done, using a the DK iteration method.**

However, when a variation of an angle such as its uncertainty is modeled, the performance of the controller is affected and it might turn itself into an unstable system even when minimal perturbations are presented.

In this paper, a nonlinear mathematical model of a two Degree of Freedom helicopter is used. This model is linearized around one specific equilibrium point. Consequently, the uncertainties of the systems are modeled, and finally, the Robust controller by the DK iteration is designed and simulated.

***Keywords*— Robust controller, two Degrees of Freedom Helicopter, Pitch angle, linearization, DK iteration.**

I. Introduction

A Helicopter of DOF (two Degree Of Freedom) is an unstable nonlinear system MIMO (Multiple-Input Multiple-Output), which is prone to multiple uncertainties, and is an attractive candidate for the design of linear robust controllers. In previous works a helicopter of three DOF is linearized around an equilibrium point for the design of H∞ controller, this controller is compared with an LQR (Linear Quadratic Regulator) controller. The experimental results indicate that comparing with the second controller, the H∞ controller is not only more robust stable and anti-disturbance, but also provides asymptotic tracking for given reference input [1]. In [2] the design of a control for a two DOF helicopter is shown, using optimal linear quadratic regulator for the control of Pitch and Yaw angles, and sliding model to guarantee robustness. In [3] a robust controller for the two DOF helicopter is proposed. It provides a semi-global stabilization of uniform ultimate boundedness in order to achieve the desired altitude. In [4] a two DOF helicopter is modeled, a linear time-varying representation is obtained, and a predictive control is designed, which is capable of stabilizing the system for maneuvers for which its linear counterpart fails. In [5] is presented the design and experimental validation of a nonlinear multivariable predictive controller for a three DOF controller. An APC (approximate Predictive Control) extended to MIMO system, is based on a Neural Network model of the nonlinear plant and its linearization in each sample instant. Experimental results demonstrate good tracking and disturbance rejection performance.

In the following paper a design of robust controller using DK iteration is presented, modeling parametric and structured uncertainty. The original model of the system is a nonlinear one, and then it is linearized around an equilibrium point.

In II the mathematical model of the system is presented, In III the controller by DK iteration is commented, in IV the uncertainties of the system are shown, in V the simulations of the system with the controller are presented and finally, the conclusions are shown in VI.

II. Mathematical model for Two DOF helicopter

In Figure 1 the Quanser helicopter of two DOF is shown. It is mounted on a fixed base with two propelled that are driven by DC motors. The front propeller controls the elevation of the nose over the pitch axis. The tail propeller guides the rotation motion around the yaw axis. The motors correspond to each one of the actuators of the propellers. The pitch and yaw motors voltage are ±24V and ±15V respectively. Some others specifications like masses, torques, inertias, and others, are presented in [6].

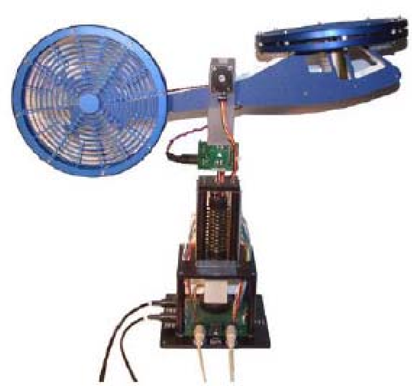

Fig. 1. Quanser Helicopter of two DOF

In order to obtain the mathematical model of the system,

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the angle θ in the pitch axis and the angle ψ in the yaw axis represent two DOF. The pitch axis is positive when the nose of the helicopter goes up, the yaw axis is positive for a clockwise rotation. Also in the Figure 2, there are thrust forces F_p and F_y for everyone for their respective axis, the torque of pitch is being applied at a distance r_p from the pitch axis and a yaw torque is being applied at a distance r_y from the yaw axis. The gravitational force F_g pulls down on the helicopter nose. The center of mass is at a distance of l_{cm} from the pitch axis along the helicopter body length.

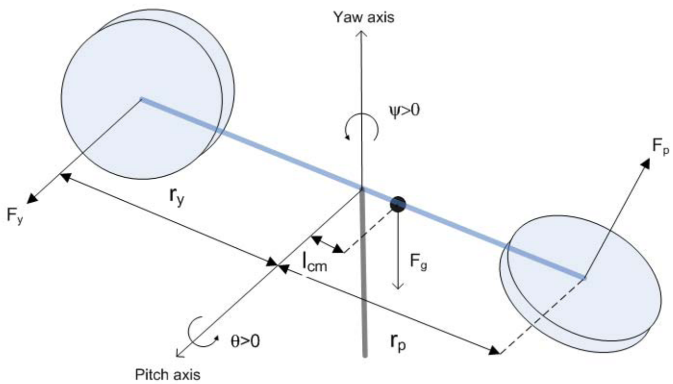

Fig. 2. Dynamics of DOF Helicopter

The center of mass of the aircraft, after the transformation of the coordinates, using the pitch and yaw rotation matrices, is given by:

$$\begin{aligned} x_{cm} &= l_{cm}\cos\psi\cos\theta \\ y_{cm} &= -l_{cm}\sin\psi\cos\theta \\ z_{cm} &= l_{cm}\sin\theta \end{aligned} \tag{1}$$

Where l_{cm} is the distance between the center of mass and the intersection of the pitch and yaw axes. The center of mass is represented in Cartesian coordinates with respect to angles θ and ψ [6]. The direct kinematic is represented by:

$$l_{cm} = \frac{(m_{m,p} + m_{shield})r_p + (m_{m,y} + m_{shield})r_y}{m_{m,p} + m_{m,y} + 2m_{shield}} \tag{2}$$

Where the Pitch motor mass is $m_{m,p}$, the Yaw motor mass is $m_{m,y}$, and m_{shield} is the propeller and shield mass.

The potential energy, due to gravity, is:

$$V = m_{heli} g l_{cm} \sin\theta \tag{3}$$

The total kinetic energy is:

$$T = \frac{1}{2}J_{eq,p}\dot{\theta}^2 + \frac{1}{2}J_{eq,y}\dot{\psi}^2 + \frac{1}{2}m_{heli}\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2\right) \tag{4}$$

It is the sum of the rotational kinetic energies acting from the Pitch $T_{r,p} = \frac{1}{2}J_{eq,p}\dot{\theta}^2$, and Yaw $T_{r,y} = \frac{1}{2}J_{eq,y}\dot{\psi}^2$, along the translational kinetic energy produced by the center of mass $\frac{1}{2}m_{heli}\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2\right)$, this one is obtained used the derived of (1), $J_{eq,p}$ and $J_{eq,y}$ are the moments of inertia of the respective motor. For this case, the Euler Lagrange equations are used, they are given by:

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial}{\partial q_1}L = Q_1 \\ \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial}{\partial q_2}L = Q_2 \end{aligned} \tag{5}$$

Where L is the Lagrange variable, which corresponds to the difference between the kinetic and potential energy of the system L=T-V. The generalized coordinates are:

$$q = [q_1 \quad q_2 \quad q_3 \quad q_4]^T = [\theta \quad \psi \quad \dot{\theta} \quad \dot{\psi}]^T \tag{6}$$

The generalized forces are given by:

$$\begin{aligned} Q_1 &= K_{pp}V_{m,p} + K_{py}V_{m,y} + B_p\dot{\theta} \\ Q_2 &= K_{yp}V_{m,p} + K_{yy}V_{m,y} + B_y\dot{\psi} \end{aligned} \tag{7}$$

Where K_{pp} is the torque constant of the pitch motor propeller actuators above pitch axis, K_{py} is the torque constant of the yaw motor propeller actuators above pitch axis, K_{yp} is the torque constant of the pitch motor propeller actuators above yaw axis, K_{yy} is the torque constant of the yaw motor propeller actuators above yaw axis, $V_{m,p}$ is the voltage of the pitch motor, $V_{m,y}$ is the voltage of the yaw motor, B_p is the viscous rotary friction acting about the pitch axis, and B_y is the viscous rotary friction acting about the yaw axis [6].

To evaluate the Euler Lagrange expressions of (5), using the coordinates defined in (6), and the forces defined in (7), the results obtained of the equations of motion are:

$$\begin{aligned} &\left(J_{eq,p} + m_{heli}l_{cm}^2\right)\ddot{\theta} = K_{pp}V_{m,p} + K_{py}V_{m,y} \\ &-m_{heli}gl_{cm}\cos\theta - B_p\dot{\theta} - m_{heli}l_{cm}^2\sin\theta\cos\theta\dot{\psi}^2 \\ &\left(J_{eq,y} + m_{heli}l_{cm}^2\cos^2\theta\right)\ddot{\psi} = K_{yy}V_{m,y} \\ &+K_{yp}V_{m,p} - B_y\dot{\psi} + 2m_{heli}l_{cm}^2\sin\theta\cos\theta\dot{\psi}\dot{\theta} \end{aligned} \tag{8}$$

Where m_{heli} is the total mass of the helicopter.

The State Space model obtained is:

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= (K_{pp}V_{m,p} + K_{py}V_{m,y} - m_{heli}gl_{CM}\cos x_1 - B_p x_3 \\ &\quad - m_{heli}l_{CM}^2\sin x_1\cos x_1 x_4^2)/(J_{eq,p} + m_{heli}l_{CM}^2) \\ \dot{x}_4 &= (K_{yy}V_{m,y} + K_{yp}V_{m,p} - B_y x_4 \\ &\quad + 2m_{heli}l_{CM}^2\sin x_1\cos x_1 x_3 x_4)/(J_{eq,y} + m_{heli}l_{CM}^2\cos^2 x_1) \end{aligned} \tag{9}$$

A linearized model of the non lineal state space model is required obtain about the equilibrium point [7]. The equilibrium point for this system is:

$$
\begin{aligned}
&K_{pp}V_{m,p} + K_{py}V_{m,y} = m_{heli}gl_{CM}\cos x_1 \\
&K_{yy}V_{m,y} + K_{yp}V_{m,p} = 0 \\
&x_3 = 0 \\
&x_4 = 0
\end{aligned} \tag{10}
$$

III. µ SYNTHESIS AND DK ITERATION

Given a loop shown in the Figure 3, for $M \in C^{n\times n}$, the structured singular value µ is defined by:

$$\mu_\Delta(M) = \frac{1}{\min\{\sigma(\Delta)|\Delta \in \Lambda, \det(I - M\Delta) = 0\}} \tag{11}$$

Unless no one $\Delta \in \Lambda$ makes $det(I - M\Delta) = 0$, it said that $\mu_\Lambda := 0$. Where

$$\Lambda = \{diag(\delta_1 I_{r,1},\ldots,\delta_s I_{r,s},\Delta_1,\ldots,\Delta_f | \delta_i \in C, \Delta_j \in C^{n\times n})\} \tag{12}$$

$$\sum_{i=1}^{s} r_i + \sum_{j=1}^{F} m_j = n \tag{13}$$

Conceptually, the structured singular value is the stability margin M, where the reciprocal of the largest singular value of M is a measure of the smallest structured of Δ, which causes instability of feedback system [8].

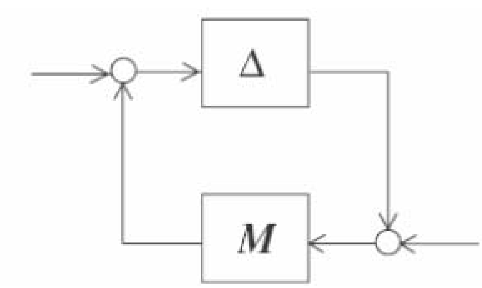

Fig. 3. Closed-loop system with structured uncertainty.

M corresponds to transfer matrix given as:

$$M = G_{1,1} + G_{1,2}K(I + G_{2,2})^{-1}G_{2,1} \tag{14}$$

Where $G = \begin{bmatrix} G_{1,1} & G_{1,2} \\ G_{2,1} & G_{2,2} \end{bmatrix}$ is chosen as:

Nominal performance only (Δ=0):

$$G = \begin{bmatrix} P_{2,2} & P_{2,3} \\ P_{3,2} & P_{3,3} \end{bmatrix} \tag{15}$$

Robust stability only:

$$G = \begin{bmatrix} P_{1,1} & P_{1,3} \\ P_{3,1} & P_{3,3} \end{bmatrix} \tag{16}$$

Robust performance:

$$G = P = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} \tag{17}$$

Where *P* is the open-loop nominal function, shown in the Figure 4.

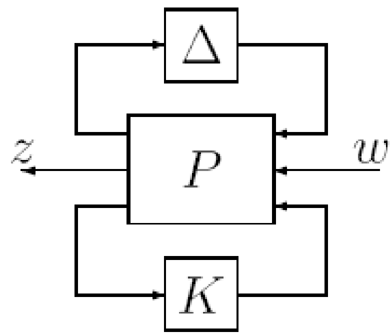

Fig.4. General System

Each case then leads to the system problem

$$\min_K \| F_l(G,K) \| \tag{18}$$

µ does not correspond to norm, because it does not satisfy the triangle inequality [9], a good approximation consists to

$$\min_K \inf_{D, D^{-1} \in H_\infty} \| DF_l(G,K)D^{-1} \|_\infty \tag{19}$$

Resolving D and K by iterating [9].

For a stabilizing K controller, a chosen D(s) matrix that satisfies the condition D(s)Δ(s)= Δ(s) D(s) is searched. The algorithm to develop this method seeks to minimize sequentially two parameters: first the minimization of K over a fixed D, then the minimization of D over a given K, successively, to convert it in an optimization problem over a H∞ space. The details of this process are presented as follow [10]:

1. Fix and starting value of the scaling matrix D pointwise across frequency.
2. Find a scalar transfer function $d_i(s)$, $d_i^{-1}(s) \in RH_\infty$, for i = 1,2,…,(F-1) such that $|d_i(j\omega)| \approx d_i$.
3. Let $D(s) = diag(d_1(s)I,\ldots,d_{F-1}(s)I, I)$. Construct a state space model for system

$$\hat{G}(s) = \begin{bmatrix} D(s) & \\ & I \end{bmatrix} G(s) \begin{bmatrix} D^{-1}(s) & \\ & I \end{bmatrix} \tag{20}$$

As it is presented in the Figure 5.

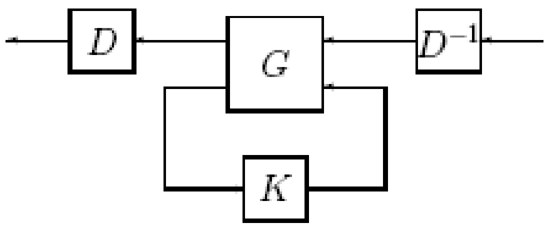

Fig. 5. μ Synthesis via scaling.

4. Solve an H_∞ optimization problem to minimize $\left\| F_l\left(\hat{G}, K\right) \right\|_\infty$ over all stabilizing K's.
5. Minimize $\bar{\sigma}\left[DF_l\left(G, \hat{K}\right)D^{-1}\right]$ over D, pointwise across frequency.
6. Compare D with the previous estimate D. Stop if they are close, but other otherwise replace D with the previous estimate D, and return to step 2 [10].

IV. Model Uncertainty

A. First model.

Based on the equation (10), a linearized model around 30° or pi/6 radians is obtained, the parameters of the system were taken from [6]. The modeled of parametric uncertainties were the moment of inertia around yaw axis and the pitch and yaw viscous friction, which have 5%, 10% and 10% of absolute error, respectively.

For modeling the weights, the structure shown in the Figure 6 was chosen. In this model, weights were introduced in the reference signal and in the output of the actuator signal, thus modeling the disturbances in the input of the actuator, and performance weight.

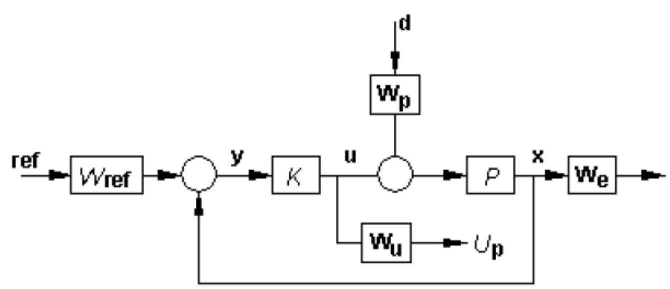

Fig. 6. Close loop system.

The system has two reference signals, the first one for θ and the other one for Ψ. θ is the pitch angle, consequently this angle must move a little off its linear model. Ψ is the yaw angle and this one can move around 2π radians, remember than this angle can not affect the linearization of the equation (10).

Wref corresponds to normalization of the pitch angle for a step input, with a constant value of π/8.

Wu is a filter adapted to limit the actuator's signal for a more adequate response from the motors, a first order filter is used. The Figure 7 shows the frequency response of a first order filter for modeling this weight, which corresponds to the inverse of the control signal. For the actuators, *ωc* was modeled like the cutoff frequency of both motors, the frequency was located in 55rad/seg approximately, and the Mu gain of 24 volts for pitch motor, and 15 volts for yaw motor. The filter is modeled by the following equation:

$$W = \frac{S + \omega_C}{\varepsilon S + M_U \omega_C} \tag{21}$$

Where *ε* is the high frequency gain of the motor, which should be the minimal possible, in this case, the frequency is equal to 1 rad. The equations of the filters for the motors pitch and yaw are, respectively:

$$W_{u1} = \frac{S + 50}{0.01S + 1200}$$
$$W_{u2} = \frac{S + 50}{0.01S + 750} \tag{22}$$

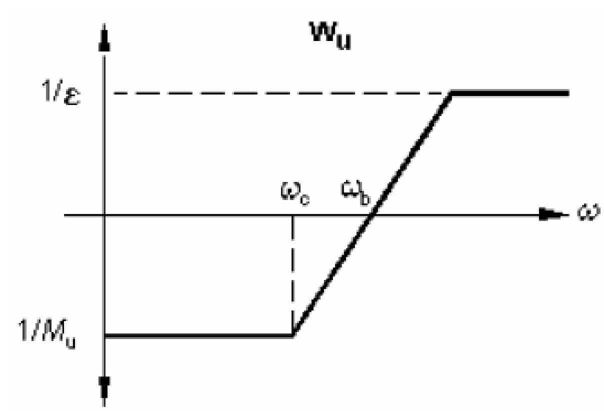

Fig. 7. Frequency response of the weight for the control signal.

The disturbance in the input of the system is considered from the low frequency, and it corresponds to the torques produced by the wind, for this the approximation of 10% of maximal value of motor torque is taken, this one is assumed by 40mNm, and this is converted to voltage value, like it is shown in the Figure 8.

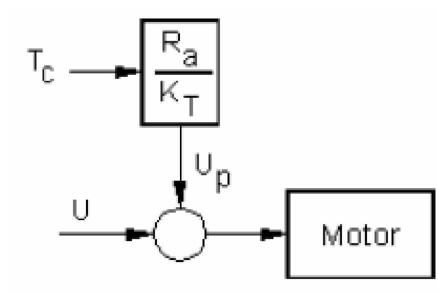

Fig. 8. Disturbances in the output of the system.

R_a is the electrical resistance of the motor, T_c is the torque of the disturbance and K_T is the constant of the torque of the motor, the final value obtained for each motor is:

$$U_{P1} = \frac{R_A T_C}{K_T} = \frac{1.6*0.004}{0.0182} = 0.1824$$
$$U_{P2} = \frac{R_A T_C}{K_T} = \frac{1.6*0.004}{0.0109} = 0.5871 \tag{23}$$

For both cases ω_c =0.05 rad/seg, the value presented in the equation (23) is the gain in low frequency. Finally, a first order filter was modeled, being for θ and Ψ respectively:

$$W_{P1} = \frac{\varepsilon S + U_{P1}\omega_C}{S+\omega_C} = \frac{0.01S+0.01}{S+0.05}$$
$$W_{P2} = \frac{\varepsilon S + U_{P2}\omega_C}{S+\omega_C} = \frac{0.01S+0.03}{S+0.05} \tag{24}$$

For the performance weight a first order filter was modeled, this is shown in the Figure 9, using the following equation:

$$W_{e1,2} = \frac{\frac{1}{M_U}S + \frac{1}{\varepsilon}\omega_C}{S+\omega_C} = \frac{0.95S+50}{S+0.5} \tag{25}$$

The filter is looking for a minimal gain in high frequency, unit gain in low frequency, and a cutoff frequency of 0.5 rad/seg, this weight was used for both outputs of angle.

B. Second model.

A second model is presented with exactly all the last specifications, and add a multiplicative weight in the output, it represents the variation of the output angles above the linearized model, the structure is presented in the Figure 10, where the output uncertainty is represented by:

$$P_o = (I + W\Delta)P \tag{26}$$

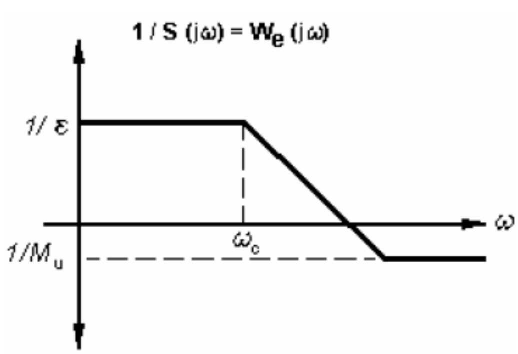

Fig. 9. Frequency response of the weight for the output.

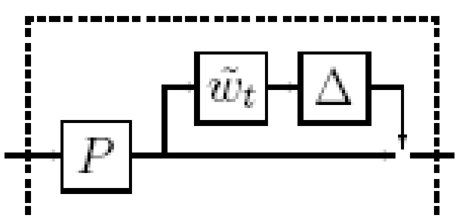

Fig. 10. Multiplicative uncertainty of the output.

P corresponds to the nominal system, in the angle Δ normalized to 1, W is the weight to model. W is obtained by:

$$W = \frac{P_o - P}{P} \tag{27}$$

The nominal value of the system is around 30° and the least cases are taken in ±10° of the point of linearization of θ, the bode plot for each angle θ and Ψ was obtained and points on the worst case were taken for each output, finally a second order filter is designed for these representations, obtaining respectively for θ and Ψ:

$$W_{t\theta} = \frac{89.8863(S+80.54)(S+1.328)}{(S+159.7)(S+16.96)}$$
$$W_{t\psi} = \frac{97.8673(S+115.2)(S+0.4867)}{(S+253.7)(S+16.96)} \tag{28}$$

V. DK Iteration Controller

Given a generalized system of the equations (13-17) as shown in the Figure 11, it is necessary to make a model as the present in the Figure 12. The inputs of this model are the references, uncertainties of the inputs of the helicopter, and control signals. The outputs of this model are the weights of the actuators, and the error signals between the references and the output f the angles. The outputs go either to inputs of the controller.

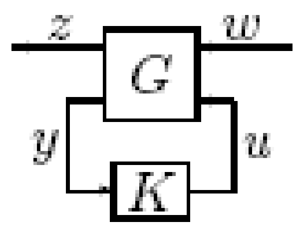

Fig. 11. Synthesis of the system.

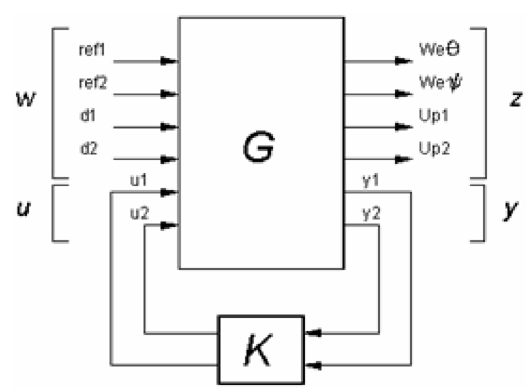

Fig. 12. System generalized.

This process is researched for the first and the second model. Subsequently both controllers were synthesized by DK iteration, through the command dksyn belongs of the *Robust Control toolbox* of *Matlab* 7.3 [9].

In the first model, the robust performance margin obtained was 0.01906, the variation of the 25% in the parametric uncertainties does not modify the margin, this score lets conclude the controlled system holds robust performance. To evaluate the robust stability, the results showed that for this system the uncertainties are possible to change to 300%, furthermore the system is robust stable. One simulation with disturbances in the input is shown in the Figure 13, where the superior image corresponds to angle θ and the inferior image corresponds to angle Ψ, both in radians.

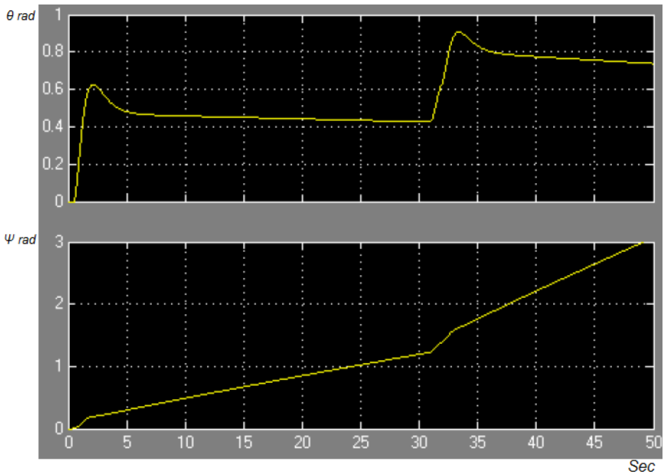

Fig. 13. Step response to uncertainties of first model.

The responses show a high overshoot, but the response to input perturbations is adequate.

The controller for the second model is synthesized, the performance margin obtained for the second case was 0.003, the margin holds with variations of 25% in the structured uncertainties, the margin for the robust stability was 0.0195, but the uncertainties are changed slightly, the system is not robust stable, because few variations change the margin.

The last case is shown in the Figure 14, where the superior image corresponds to angle θ and the inferior image corresponds to angle Ψ, both in radians, the overshoot is less than the first case, but the controller in the case of the angle Ψ does not present reject of the uncertainties, the axis Ψ becomes unstable.

VI. Conclusion

The linearization of the two DOF helicopter depends on the pitch axis, being a decoupled of the yaw axis. It allows a linearization above the pitch axis, with completely freedom in the control about yaw axis.

The design of a robust controller by DK iteration allows to obtain, in the case of two DOF helicopter, a stable robust controller for a linearization model above a region of pich axis, which can control structured uncertainties of 300%. The addition of weight that model multiplicative uncertainties above the linearization axis restricts the robust stability, and the system becomes unstable with few variations of the parameters. It eliminates the reject of uncertainties.

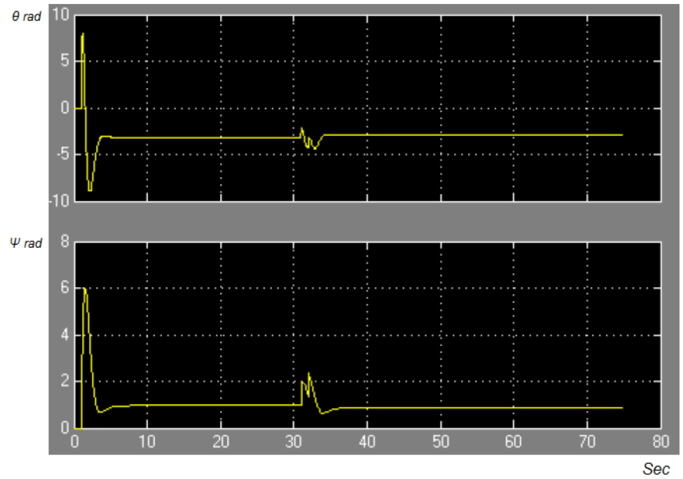

Fig. 14. Step response to uncertainties of second model.

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