A Mechanism-design Approach to Property Rights

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This slide deck:

Motivation

- an agent can make an investment decision, affecting her (**private**) valuation for the resource,
- and then participates in a trading mechanism chosen by a principal in a **sequentially rational** fashion,
- a designer can endow the agent with a menu of **property rights**, to incentivize efficient investment
- the agent can choose one property right of the right menu $\{(x_i, t_i)\}_{i \in I}$ as her outside option, where x_i denotes the allocation probability/fraction and t_i denotes the transfer (standard in screening model)
- two economic frictions: private information and the hold-up problem

What is the optimal design of property rights?

Contribution of This Paper

- motivation perspective: design of property rights (economic significance)
- technical perspective:
  1. a slight extension of ironing inspired by Toikka (2011) (ironing using concavification) and KMS (2021) (optimization under SOSD)
  2. application of infinite extension of Carathéodory Theorem found in Kang (2023)
- methodological perspective: design of type-dependent outside-option in mechanism design
- others: a user's guide of optimization techniques in mechanism design

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1. At time $t = 0$, the designer chooses a contract that determines the agent's rights
2. At time $t = 1$, the agent decides whether to undertake a costly investment
3. At time $t = 2$, the agent's private type and the state are realized, and the principal designs a trading mechanismin a sequentially rational manner, respecting the rights that the designer endowed the agent with at time $t = 0$ (agent's outside options).

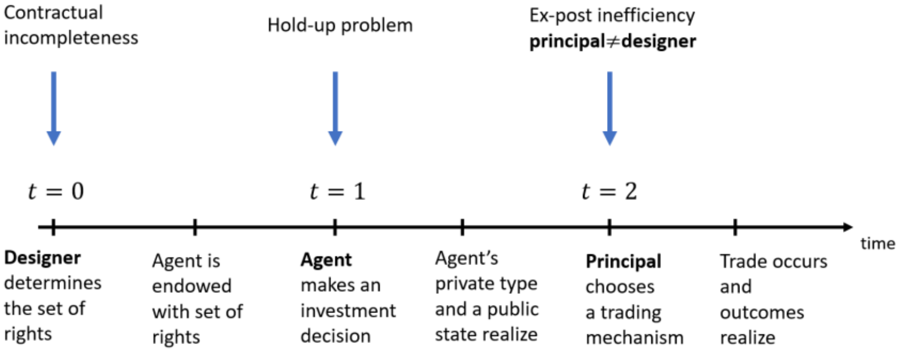

Figure 1: Model overview and timeline.

Contracts of Rights

1. At time t $= 0$, the designer chooses a contract M that is a menu of rights held by the agent in subsequent periods.
2. Specifically, we allow for any contract of the form $M = \{(x_i, t_i)\}_{i \in I}$
   1. $x_i \in [0, 1]$ denotes an allocation
   2. $t_i \in \mathbb{R}$ denotes a payment made by the agent to the principal in period t=2
   3. the set I is arbitrary
3. assume that M is a compact subset of $[0, 1] \times \mathbb{R}$
4. Any right in the menu M can be executed by the agent at t=2 , in the sense that any $(x_i, t_i) \in M$ constitutes an outside option available to the agent when contracting with the principal.

Costly Investment

1. At time t=1 , the agent takes a binary investment decision.
2. Investing is associated with a (sunk) cost $c > 0$.
3. The investment decision determines the joint distribution of the agent's type $\theta \in \Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$ and the public state $\omega \in \Omega \subset \mathbb{R}$
   1. If the agent invests, the public state is drawn from a distribution G, and the agent's type is drawn from a conditional distribution F_ω .
   2. If the agent does not invest, the respective distributions are denoted $\underline{G}$ and $\underline{F}_\omega$.
4. We assume that, for every ω, F_ω and $\underline{F}_\omega$ admit absolutely continuous densities on Θ (denoted f_ω and $\underline{f}_\omega$, respectively).
5. For every ω, F_ω first-order stochastically dominates $\underline{F}_\omega$, so that the primary role of investment is that it increases the agent's type.

Trading Mechanism

- At time t = 2, the agent's private type θ and the public observable state ω are realized.
- The principal then chooses a (direct) trading mechanism $\langle x_\omega, t_\omega \rangle$
  1. $x_\omega \colon \Theta \to [0,1]$ denotes the allocation rule
  2. $t_\omega \colon \Theta \to \mathbb{R}$ denotes the transfer rule

  , satisfying appropriate incentive-compatibility and individual-rationality constraints for for every $\theta \in \Theta$ and $\omega \in \Omega$
  1. $U_\omega(\theta) \equiv \theta x_\omega(\theta) - t_\omega(\theta) \geq \theta x_\omega(\theta') - t_\omega(\theta')$
  2. $U_\omega(\theta) \geq \max\{0, \max_{i \in I}\{\theta x_i - t_i\}\}$
- notice that the contract M chosen by the designer gives rise to an endogenous type-dependent outside option determined by the agent's optimal choice of a right from the menu M at time t=2

Principal's Problem: Trading Mechanism Design

- Principal's problem: Given a realized state ω , the principal solves the problem

$$\begin{aligned} \max_{\langle x_\omega(\cdot), t_\omega(\cdot)\rangle} \int_\Theta \left[V_\omega(\theta)x_\omega(\theta) + \alpha t_\omega(\theta)\right] dF_\omega(\theta) \\ \text{s.t.} \quad (\mathrm{IC}), \quad (\mathrm{IR}) \end{aligned}$$

  where $V_\omega : \Theta \to \mathbb{R}$ is upper semi-continuous in θ , and $\alpha > 0$ is the weight that the principal places on revenue.

- We denote by $\langle x^\star_\omega(\cdot; M), t^\star_\omega(\cdot; M)\rangle$ the optimal mechanism for the principal when the participation constraint is induced by contract M .

- The optimal mechanism is generically unique; in case of indifference by the principal our proofs utilize a particular tie-breaking rule that simplifies exposition

- The results continue to hold under a large class of tie-breaking rules, including designer-preferred selection

Agent's Problem: Investment Decision

- Agent's problem: the agent will invest if and only if

$$\int_\Omega \int_\Theta \left(\theta x^\star_\omega(\theta; M) - t^\star_\omega(\theta; M)\right) dF_\omega(\theta) dG(\omega) - c \geq \underline{U}$$

- Contractibility of the investment matters

  1. In the non-contractible case, $\underline{U} = \int_\Omega \int_\Theta \left(\theta x^\star_\omega(\theta; M) - t^\star_\omega(\theta; M)\right) d\underline{F}_\omega(\theta; M) d\underline{G}(\omega)$
  2. In the contractible case, $\underline{U} = \int_\Omega \int_\Theta \left(\theta \underline{x}^\star_\omega(\theta; \emptyset) - \underline{t}^\star_\omega(\theta; \emptyset)\right) d\underline{F_\omega}(\theta) d\underline{G}(\omega)$

where $\langle \underline{x}^\star_\omega(\cdot; \emptyset), \underline{t}^\star_\omega(\cdot; \emptyset)\rangle$ is the principal's optimal mechanism assuming that $M = \emptyset$ and the agent's type θ is drawn from $\underline{F}_\omega$ given the realized ω

Designer's Problem: Menu of Rights Design

- Designer's problem:

$$\max_M \int_\Omega \int_\Theta \left[V^\star_\omega(\theta) x^\star_\omega(\theta; M) + \alpha^\star t^\star_\omega(\theta; M)\right] dF_\omega(\theta) dG(\omega)$$
$$\text{s.t.} \quad (\mathrm{I-OB}),$$

where $V^\star_\omega : \Theta \to \mathbb{R}$ is continuous in θ , and $\alpha^\star \geq 0$ is the weight that the principal places on transferring a unit of money from the agent to the principal.

- We assume that
  1. the designer prefers to induce investment (which is why we included the investment-obedience constraint in the designer's problem)
  2. there exists some contract M that satisfies (I-OB)
  3. the agent does not invest if $M = \emptyset$

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The Structure of Optimal Menu

Theorem (The Structure of Optimal Menu)

There exists an optimal contract that takes the form $M^\star = \{(1, p), (y, p')\}$ *for some* $p, p' \in \mathbb{R}$ *and* $y \in [0, 1)$.

- the optimal contract M consists of at most two types of rights and takes a simple and economically interpretable form:
  1. an option-to-own, $(1, p)$, gives the agent the right to control the resource by paying a pre-specified price p
  2. a partial right, (y, p'), giving the agent partial control over the resource at a lower price (possibly for free)
  3. however, it can also take the form of a cash payment to the agent (when $y = 0$ and $p' < 0$)

The Structure of Optimal Menu

Theorem (The Structure of Optimal Menu)

There exists an optimal contract that takes the form $M^\star = \{(1, p), (y, p')\}$ *for some* $p, p' \in \mathbb{R}$ *and* $y \in [0, 1)$.

- We can derive a tighter prediction by imposing additional regularity conditions, and considering separately the cases of contractible and non-contractible investment
- A number of configurations can emerge as optimal in applications—the optimal menu could be a singleton containing an option-to-own $(1, p)$, a cash transfer $(0, p)$, or a partial right allocated for free $(y, 0)$

Proof Roadmap

- We proceed backwards, by first solving the principal's problem in period $t = 2$, then considering the agent's investment problem in period $t = 1$, and finally solving the designer's problem in period $t = 0$.
- Then we want to transform the problem to an explicit form of optimization problem

1. transform the outside option as a convex, type-dependent functions $R(\theta) \equiv u_0 + \int_{\underline{\theta}}^{\theta} x_0(\tau)d\tau$ for some $u_0 \geq 0$ and non-decreasing allocation rule $x_0 : \Theta \to [0, 1]$
2. thus transform the problem into a general theory of optimization subject to second-order stochastic dominance constraints with slight modifications
3. given the optimal trading mechanism, we can calculate the agent's utility under a given property right menu
4. with these, the designer's problem can be rewritten as maximization of a linear functional under two constraints of the allocation function: a linear functional constraint and standard non-decreasing monotonicity constraint

Step 1: Formulating the principal's problem

- We first focus on solving the principal's problem, given an arbitrary menu of rights M and a realization $\omega \in \Omega$ of the public state

Lemma

A choice of contract M by the designer is equivalent to choosing an outside option function $R : \Theta \to \mathbb{R}$ for the agent in the second-period mechanism, where R is non-negative, non-decreasing, convext with a right derivative that is bounded above by 1.

- remember that $R(\theta) = \max\left\{0, \max_{i \in I}\left\{x_i\theta - t_i\right\}\right\}$, given a menu $M = \left\{(x_i, t_i)\right\}_{i \in I}$
- the principal's problem therefore reduces to maximizing over the set of type-dependent outside option functions R
- we can decompose the R into $R(\theta) \equiv u_0 + \int_{\underline{\theta}}^{\theta} x_0(\tau) d\tau$

Step 1: Formulating the principal's problem

- a direct mechanism $\langle x, t \rangle$ chosen by the principal is incentive-compatible if and only if x is a non-decreasing function and, for any $\theta \in \Theta$, the agent's utility under truthful reporting is given by $U(\theta) = \underline{u} + \int_{\underline{\theta}}^{\theta} x(\tau) d\tau$ (the envelope theorem)

  where $\underline{u} \in \mathbb{R}$ denotes the utility of the lowest type $\underline{\theta}$
- This implies that U is a convex function with $U'(\theta) = x(\theta)$ almost everywhere.
- for all $\theta \in \Theta$, we have $t(\theta) = \theta x(\theta) - \int_{\underline{\theta}}^{\theta} x(\tau) d\tau - \underline{u}$

Step 1: Formulating the principal's problem

- After standard transformations, this yields

$$\int_\Theta [V(\theta)x(\theta) + \alpha t(\theta)]dF(\theta) = \int_\Theta [V(\theta) + \alpha B(\theta)]x(\theta)dF(\theta) - \alpha \underline{u}$$

where $B(\theta) \equiv \theta - \frac{1-F(\theta)}{f(\theta)}$ is the usual (buyer's) virtual value function.

- Combining this with Lemma 1, the principal's problem (P) can be rewritten as

Principle's Problem

$$\max_{x:\Theta\to[0,1],\underline{u}\geq 0} \int_{\underline{\theta}}^{\bar{\theta}} W(\theta)x(\theta)d\theta - \alpha\underline{u}$$

$$\text{s.t.} \quad x \text{ is non-decreasing, and } U(\theta) = \underline{u} + \int_{\underline{\theta}}^{\theta} x(\tau)d\tau \geq R(\theta), \quad \forall\theta \in \Theta$$

where $W(\theta) \equiv (V(\theta) + \alpha B(\theta))f(\theta)$

- We will refer to the constraint $U(\theta) \geq R(\theta)$ as the outside option constraint.

Step 2: Solving the principal's problem

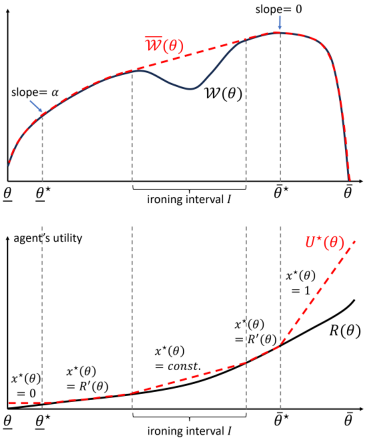


Figure 2: An illustration of the ironing procedure (top panel) and the mapping from the ironing procedure to the optimal indirect utility function U^* (bottom panel).

- ironing procedures

1. concavification of the objective function

$$\mathcal{W}(\theta) = \int_{\theta}^{\bar{\theta}} W(\tau)d\tau$$

$$\overline{\mathcal{W}} = \text{co}(\mathcal{W})$$

2. figure out ironing interval

$$\underline{\theta}^\star = \sup\left\{\left\{\theta \in \Theta : \overline{\mathcal{W}}'(\theta) \geq \alpha\right\} \cup \{\underline{\theta}\}\right\},$$

$$\bar{\theta}^\star = \inf\left\{\left\{\theta \in \Theta : \overline{\mathcal{W}}'(\theta) \leq 0\right\} \cup \{\bar{\theta}\}\right\}$$

3. (heuristically) construct the optimal allocation policy

Step 2: Solving the principal's problem

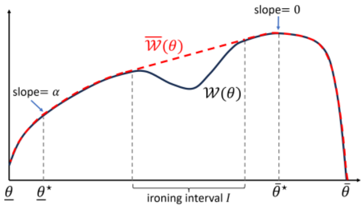
- ironing procedure

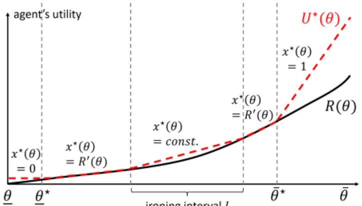
1. concavification of the objective function

2. figure out ironing interval
$$\underline{\theta}^\star = \sup\left\{\left\{\theta \in \Theta : \overline{\mathcal{W}}'(\theta) \geq \alpha\right\} \cup \{\underline{\theta}\}\right\},$$
$$\bar{\theta}^\star = \inf\left\{\left\{\theta \in \Theta : \overline{\mathcal{W}}'(\theta) \leq 0\right\} \cup \{\bar{\theta}\}\right\}$$

3. (heuristically) construct the optimal allocation policy

$$\underline{u}^\star = R(\underline{\theta}^\star) \text{ and } x^\star(\theta) = \begin{cases} 0 & \text{if } \theta \leq \underline{\theta}^\star, \\ \frac{\int_a^b R'(\tau)d\tau}{b-a} & \text{if } \theta \in (a,b) \text{ for some } (a,b) \in \mathcal{I}, \\ R'(\theta) & \text{if } \theta \in [a,b] \text{ for some } [a,b] \in \mathcal{I}^c, \\ 1 & \text{if } \theta \geq \bar{\theta}^\star. \end{cases}$$





Step 2: Solving the principal's problem

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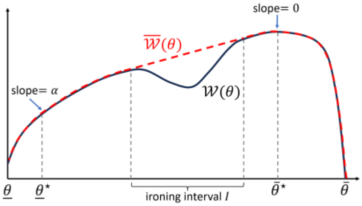
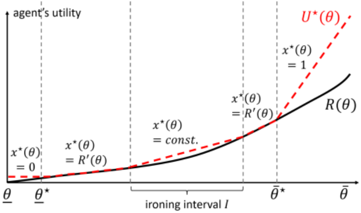
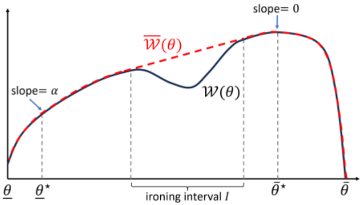
Lemma

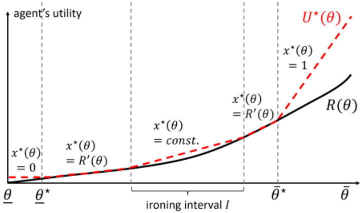
The pair $(x^*, \underline{u}^*)$ as defined in solves problem





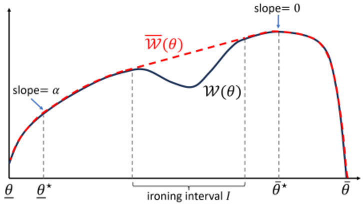
The Intuitions behind the Optimality

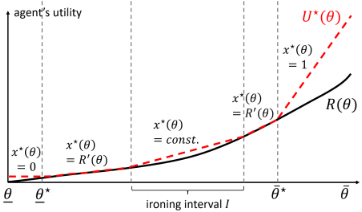




- non-ironing intervals: basic linear programming problem $\rightarrow$ 0 or 1 (dependent on the coefficients of the Cons and the OBJ)
- $\underline{u}^\star = R(\underline{\theta}^\star)$ and $x^\star(\theta) = \begin{cases} 0 & \text{if } \theta \leq \underline{\theta}^\star, \\ 1 & \text{if } \theta \geq \bar{\theta}^\star. \end{cases}$
- the key challenge: what is the optimal policy in the ironing interval

The Intuitions behind the Optimality

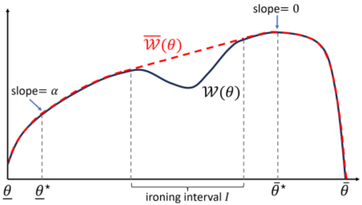


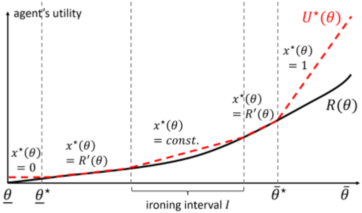


- the optimal solution in this region is thus the "cheapest" way for the principal to satisfy the constraint
- notice that we have two incentizing policies: allocation versus monetary transfer
- $\underline{u}^\star = R(\underline{\theta}^\star)$ and $x^\star(\theta) = \begin{cases} \frac{\int_a^b R'(\tau)d\tau}{b-a} & \text{if } \theta \in (a,b) \text{ for some } (a,b) \in \mathcal{I}, \\ R'(\theta) & \text{if } \theta \in [a,b] \text{ for some } [a,b] \in \mathcal{I}^c. \end{cases}$
- we use them interchangebly in this region!

The Intuitions behind the Optimality

- recall the definition of the boundary point
$$\underline{\theta}^\star = \sup\left\{\left\{\theta \in \Theta : \overline{\mathcal{W}}'(\theta) \geq \alpha\right\} \cup \{\underline{\theta}\}\right\},$$
$$\bar{\theta}^\star = \inf\left\{\left\{\theta \in \Theta : \overline{\mathcal{W}}'(\theta) \leq 0\right\} \cup \{\bar{\theta}\}\right\}$$

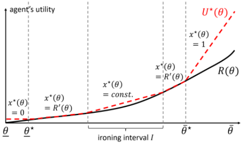




- $\underline{u}^\star = R(\underline{\theta}^\star)$ and $x^\star(\theta) = \begin{cases} \frac{\int_a^b R'(\tau)d\tau}{b-a} & \text{if } \theta \in (a,b) \text{ for some } (a,b) \in \mathcal{I}, \\ R'(\theta) & \text{if } \theta \in [a,b] \text{ for some } [a,b] \in \mathcal{I}^c. \end{cases}$
- affine part: monetary transfer, (strictly) concave part: allocation

Proof Sketch and Some Observations

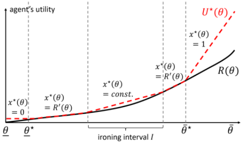
$$x^\star(\theta) = \begin{cases} 0 & \text{if } \theta \leq \underline{\theta}^\star, \\ \frac{\int_a^b R'(\tau)d\tau}{b-a} & \text{if } \theta \in (a,b), (a,b) \in \mathcal{I}, \\ R'(\theta) & \text{if } \theta \in [a,b], [a,b] \in \mathcal{I}^c, \\ 1 & \text{if } \theta \geq \bar{\theta}^\star. \end{cases}$$



- Proof Sketch: interprate the problem into a general theory of optimization subject to second-order stochastic dominance constraints with slight modifications
- allocation rules - CDFs (by proper extension) and outside option function $R(\theta) \equiv u_0 + \int_{\underline{\theta}}^{\theta} x_0(\tau)d\tau$
- the outside option constraint takes a form similar to SOSD of the candidate distribution x by the fixed distribution x_0 defining the outside option

Proof Sketch and Some Observations

$$x^\star(\theta) = \begin{cases} 0 & \text{if } \theta \leq \underline{\theta}^\star, \\ \frac{\int_a^b R'(\tau) d\tau}{b-a} & \text{if } \theta \in (a,b), (a,b) \in \mathcal{I}, \\ R'(\theta) & \text{if } \theta \in [a,b], [a,b] \in \mathcal{I}^c, \\ 1 & \text{if } \theta \geq \bar{\theta}^\star. \end{cases}$$



- ironing - to taking a MPS of the distribution x_0, and thus preserves the constraint that x is second-order stochastically dominated by x_0.
- Obs: The optimal solution $(x^*, \underline{u}^*)$ depends linearly on the outside option R

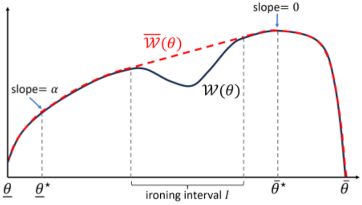
Proof Sketch

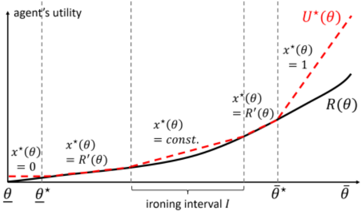
- consider first an auxiliary problem in which we fix u at some level weakly above u_0
- the problem is now to choose a CDF x to maximize

$$\int_{\underline{\theta}}^{\bar{\theta}} \mathcal{W}(\theta) dx(\theta) \text{ subject to } \int_{\underline{\theta}}^{\theta} x(\tau) d\tau \geq (u_0 - \underline{u}) + \int_{\underline{\theta}}^{\theta} x_0(\tau) d\tau, \forall \theta$$

- in particular, if $\mathcal{W}$ is non-decreasing and concave, then the optimal x must satisfy the inequality as an equality (whenever this is feasible)

$$\bar{x}(\theta) \equiv x_0(\theta)\mathbf{1}_{\theta \geq \theta_0}, \text{ where } u_0 - \underline{u} + \int_{\underline{\theta}}^{\theta_0} x_0(\tau) d\tau = 0 \, (\text{and } \theta_0 = \underline{\theta} \text{ if there is no solution })$$





Proof Sketch

- the key idea of the proof (mimicking the logic behind classical “ironing”) is to define a relaxed problem in which the objective is concave non-decreasing
- let $\overline{\mathcal{W}}$ be the concave closure of $\mathcal{W}$, and let $\overline{\mathcal{W}}_+$ be the non-decreasing concave closure of $\mathcal{W}$
- note that $\overline{\mathcal{W}}_+$ differs from $\overline{\mathcal{W}}$ only in that $\overline{\mathcal{W}}_+(\theta)$ is constant for all $\theta \geq \bar{\theta}^\star$
- by our previous argument, we have obtained an upper bound on the value of the problem equal to

$$\int_{\underline{\theta}}^{\bar{\theta}} \overline{\mathcal{W}}_+(\theta) d\bar{x}(\theta)$$

- then construct an allocation rule $x^\star$ that is feasible in the original problem and achieves this upper bound
- finally, optimize the $\underline{u}$

Step 3: Solving the designer's problem

- We can reformulate the designer's problem as

Designer's Problem

$$\max_{u\geq 0, x(\cdot)\text{ non-decreasing}} \int_{\underline{\theta}}^{\bar{\theta}} \Phi(\theta)dx(\theta) - \alpha^\star u \quad \text{subject to} \quad \int_{\underline{\theta}}^{\bar{\theta}} \Psi(\theta)dx(\theta) + \mathbf{1}_{\text{cont}} \cdot u \geq \tilde{c}$$

for some constant $\tilde{c} \geq 0$ and functions $\Phi, \Psi : \Theta \to \mathbb{R}$, where $\mathbf{1}_{\text{cont}}$ is an indicator function that is 1 if investment is observable (the contractible case) and 0 otherwise.

Lemma (Optimal Allocation)

The solution $(x^\star, u^\star)$ such that either (i) $u^\star = 0$ and $x^\star$ takes on at most one value other than 0 or 1 , or (ii) $u^\star > 0$ and $x^\star(\theta) \in \{0, 1\}$ for all $\theta \in \Theta$.