Selling Training Data*

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Abstract

In this paper, we develop a framework to analyze the design and price of supplemental training dataset for hypothesis testing. A monopolistic seller versions training datasets and associated tariffs to screen data buyers with different private datasets. Three characteristics are relevant in this set-up, the coexistence of both horizontal and vertical differences, the obedience constraints and the possibilities of double deviation. We show that exclusion of double deviation imposes rigidity of menu structure brought by multi-dimension nature of data allocation, reducing dimension of the design problem and leading to two-tier structure as its extreme point. The seller can exploit the horizontal difference to neutralize the vertical difference, through subtly designing the lower-tiered dataset to nullify the impact of private dataset. Such operation can maintain high price for higher tiered dataset without excluding low-valued buyers. The obedience constraints impose the limit of the exploitation.

Keywords: Selling Training Data, Hypothesis Testing, Multi-dimensional Screening, Information Design

1 Introduction

Information asymmetry, one of main reasons leading to market failure, makes a key component for modern economics. It incubates an important profession, i.e. intermediary, whose job is to provide information to market participants to (partially) restore efficiency of resource allocation deteriorated by information asymmetry. Nowadays, the advancement of technology has elevated the value of data to an unprecedented level. On one hand, digitization, connectivity to cloud-based infrastructures, together with cheaper storage and more effective use of data (i.e., improvements in machine learning and artificial intelligence) improves the efficiency and lowers the costs of data usage. On the other hand, as more and more transactions are conducted online, the elimination of menu costs and

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the possibility of personalized pricing and price discrimination enable decision-making to be more responsive to data. Therefore, data brokers, as a crucial "carrier for data transmission", are playing an increasingly vital role in economic activities. In this paper, we characterize revenue-maximizing policies for data brokers to design and price supplementary datasets to buyers with unknown private/baseline datasets. Buyers may have private datasets either from other external sources or collecting by themselves. Private datasets impact evaluation of supplementary datasets by not only altering buyers' outside option but also affecting the way how supplementary datasets are merged (with private datasets) in statistical decision making. Therefore, the mechanism should be subtly designed to screen buyers with various private datasets.

We formalize this problem in the classic Bayesian decision-theoretic model pioneered by Blackwell (1951) and the framework of mechanism design and information design. A single data buyer trains data to conduct a hypothesis testing and his payoff is the probability of correct identification of true states. A data seller maximizes his profits by designing the optimal menu of training data with prices per buyer type.

The data buyer owns a private training dataset, modelled as a statistical experiment, containing two signals about rejection or acceptance. These signals generate a bundle of statistical error, i.e. Type I error α and Type II error β, which are actually the private multi-dimensional preference of the buyer in the data selling mechanism.[1] To reduce the initial statistical error (α, β), the data buyer purchases supplementary training dataset from the data seller. In the design of data selling mechanism, the revelation principle enables us to represent the supplementary training dataset by the ratio of Type I and Type II error it reduces respectively. Therefore the seller allocates the reduction ratio of statistical error (π_1, π_2) and designs the associated price t to achieve revenue-maximization. The value of supplement training data is measured by the reduction of statistical error in the hypothesis testing, i.e. $V = \alpha + \beta - \alpha\pi_1 - \beta\pi_2$. Therefore, the design of a training data sales mechanism is a multi-dimensional mechanism design problem, encompassing multi-dimensional preferences and allocations.

Compared to selling conventional multi-dimensional goods, selling training data presents distinct challenges. The interdependence of Type I and Type II errors in data goods restricts the screening scope by limiting error allocation and weakening differentiation capabilities. First, the allocation of the error reduction ratio (π_1, π_2) is inherently constrained. If the designer significantly reduces Type I error while only marginally reducing Type II error (allocate low π_1 and high π_2), the data buyer may be inclined to commit Type I errors exclusively. Thus, the allocation of statistical errors is inherently limited.

Moreover, the interdependence of Type I and Type II errors neutralizes the product differentiation. When conducting hypothesis testing with combined training data, the data buyer seeks to minimize their overall statistical error. This suggests that a buyer, when attempting to mimic another type within the screening menu, may strategically only commit either Type I or Type II error to reduce their overall error, contingent on their initial statistical error of the private data.

[1] In this paper, we use "Type" to denote the type of statistical error, and use "type" to denote the private type of buyer.

Such error re-minimization neutralizes the product differentiation and increases their net utility, thus enhancing their incentive to mimic another type. Consequently, this re-minimization effect constrains the ability to differentiate data goods in the screening process.

In the binary situation, we explicitly construct four optimal selling schemes. The basic trade-off for the data seller is balancing the reduction of information rent with the extraction of low type surplus. The data seller aims to distort the data goods for low type to reduce the information rent, yet simultaneously seeks to avoid the distortion of data to extract the surplus for low type. When the two-type statistical errors for the low type are significantly lower than or comparable to those of the high type, the designer implements grand bundling policy.

In scenarios where a certain Type of statistical error-for example Type II error β of the low type-is not significantly lower than that of the high type, the designer would sell a fixed partially informative dataset. This dataset only reduces the Type II β error by a fixed ratio c, without altering the Type I error α. Focusing on reducing the Type II error offers dual economic benefits for the designer. Primarily, it allows for a more efficient extraction of the low type's surplus. Additionally, allocation in this dimension would generate less information rent. The determination of the fixed ratio c hinges on the seller's strategy to exploit the reduction of the high type's Type II error, rendering it independent of the low type.

In scenarios where horizontal differences in error Types exist between the two types-for example, when the low type's Type I error exceeds the high type's-the designer can utilize these differences to differentiate the menu and thereby eliminate information rent. This result aligns with the full extraction in Bergemann et al. (2018), although it is based on different reasons that will be detailed subsequently. Notably, the seller may reduce the low type's Type II error in response to an increase in his Type I error, highlighting the interplay between various types of statistical error.

We then generalize our insights from the binary situation to a continuous type space. Our generalized framework accommodates both horizontal and vertical differences in types, capturing the most interesting aspects of the data selling mechanism. Specifically, Type I and Type II error are perfectly and negatively correlated with constant substitution rate. Even if buyers' preferences are single dimension, the nature of data allocation, equivalent to reducing Type I and Type II error respectively, makes it as a multi-dimension mechanism design problem. However, comparing to standard multi-dimension mechanism design, there are two critical differences regarding data selling. On one hand, the revelation principle, while enabling us to represent the supplementary dataset by the ratio of Type I and Type II error it reduces respectively, imposes an obedience constraints to the available supplementary datasets for each type of buyers. On the other hand, strategic buyers may deviate by not only choosing the supplementary dataset not oriented for him (a regular truth-telling constraint) but also not following the seller's recommendation in error reduction. This type of deviation, named after double deviation, prevails in the literature of information design as screening tools. The literature deals with double deviation by first ignoring the second type deviation and then verifying the solution to the related optimization problem also excludes double deviations. However, this is not the case in our model, since we can show that, in the optimal menu, for each supplementary dataset, there always exists some type of buyers, whose incentive compatibility constraint of double deviation (through choosing this dataset) binds.

Whenever a buyer conducts double deviations, he is indifferent to drop his own private dataset and focuses on reducing one type of error in statistical decision making. To exclude double deviations, the price gap between two supplementary datasets should exactly measure differences in the informativeness or the value of these two supplementary datasets themselves in statistical decision making, which is the differences in reduction ratio of a specific type error. Therefore, the exclusion of two step deviation imposes an endogenous link between the reduction ratio of a specific type error and the price, of the supplementary datasets. Such linkage reduces the dimension of the design problem, in which the two tier structure of the menu is the extreme point.

In a single dimension mechanism design with only vertical differences, the classical no-haggling menu excludes low-valued buyers below a threshold, to maintain a high price for the object through emptying the rent of threshold type buyers. Whenever the horizontal differences involve, the seller can subtly design the lower tiered supplementary dataset by setting the ratio of reducing Type I and Type II error as the inverse of the constant substitution rate of private datasets. Buyers' valuations on this supplementary dataset are completely independent of their own private datasets given following seller's recommendation. Therefore, the seller can charge a price at the common value for this lower-tiered supplementary dataset and fully extract the surplus of all types below the threshold. In this sense, the seller can still maintain a high price of higher-tiered dataset (i.e. full supplementary dataset) while covering all the markets. To maximize profits, the seller can improve the informativeness of the lower-tiered dataset, fixing the ratio of reducing Type I and Type II error. This operation can continue until the threshold buyer is indifferent between following seller's recommendation or not. In other words, the obedience constraint puts a limit to the exploitation of horizontal differences.

1.1 Related Literature

Our paper is primarily closely related to the literature on markets for information (a large literature beginning with Admati and Pfleiderer (1986) and Admati and Pfleiderer (1990), see Bergemann and Bonatti (2019) for a comprehensive review), especially those studying the direct sale of information structures to privately informed decision makers that maximize the monopolist's revenue. In these models, three key challenges complicate the analysis: (i) information structure is high-dimensional and flexible to design, (ii) there exist obedience constraints and double deviations in incentive compatibility due to the inseparability between communication and screening,[2] (iii) the outside option is non-negative and type-dependent due to agent's private information.[3]

These challenges make the problem a joint design of multi-dimensional screening and per-

[2] A direct consequence is that the payoff functions are non-linear, complicating the mutual IC analysis commonly used in the literature.

[3] (i) and (iii) come from the direct sale of information structure. (ii) comes from the private interim belief before contracting. For example, Yang (2022b) discuss the direct sale of information to buyers with common prior and private production costs. In his model, (i) and (iii) exist while (ii) does not exist.

suasion, which is difficult and intractable without further assumptions. Present papers focus on different application scenarios and make it tractable. Bonatti et al. (2024) and Rodríguez Olivera (2024) focus on selling information to dominant-strategy games with binary states. The multiplicatively decomposable utilities between action and state in Bonatti et al. (2024) make that incentive compatibility is equivalent to requiring truthfulness and obedience separately, avoiding the possibilities of double deviation. Segura-Rodriguez (2022) and Bonatti et al. (2023) restrict attention to specific information structure and discusses the optimal sale in linear-quadratic-Gaussian settings.[4]

The closest paper to ours is Bergemann et al. (2018).[5] By assuming that the private type of agents is private signal realization before contracting, which is equivalent to assuming that their private type is either Type I error or Type II error in their framework,[6] Bergemann et al. (2018) actually separate the agents into two classes and focus on reducing the corresponding certain Type error, thus reducing the dimensions for design and simplifying the possibilities of double deviation and obedience constraints.

In our framework, we allow for general information structure, the possibilities of double deviation, the obedience constraints, and the coexistence of both horizontal and vertical differences.[7] We provide an explicit and relevant interpretation of the common challenges in the hypothesis testing setting: (i) the multi-dimensions of data come from the possibilities of different Types of statistical error (ii) the interdependence of Type I error and Type II error constrains the screening (iii) the initial bundles of statistical error determine the outside options. Compared with most papers above, we fully characterize the optimal sale menu in closed-form.

On the application side, these models focus on different data selling strategies. In the framework of decision making, current models discuss the sale of input data (Bergemann et al. (2018)),[8] marketing scores (Segura-Rodriguez (2022)), consumer demands or market segementations (Yang (2022b)). In our model, we focus on selling training data in conducting hypothesis problem.

Our result also delivers new insights into multi-dimensional mechanism design problems (dating back to Adams and Yellen (1976), McAfee et al. (1989)). It is difficult to characterize the optimal menu (Armstrong and Rochet (1999), Daskalakis et al. (2017)). The optimal menu tends to be complex and infinite and simple menu would attain only negligible profits (Manelli and Vincent (2007), Hart and Nisan (2019)). The literature imposes additional assumptions or sufficient conditions for the optimality of some certain bundling strategies or some properties of the optimal

[4]Segura-Rodriguez (2022) restrict information structures to elliptical distributions, a family of distributions generalizing normally distributions while preserving two key properties simplifying the Bayes updating process.

[5]Bergemann et al. (2018) focus on the single buyer, binary state and binary action situation. Their framework is extended to many actions (Bergemann et al. (2022)), a modified decision problem (Liu et al. (2021)), multiple players under a coordination game (Rodríguez Olivera (2024)) and information acquisition (Li (2022)).

[6]In Bergemann et al. (2018), the position and informativeness of information can be reinterpreted in this way: the position is the Type of statistical error, while the informativeness of information is the overall error. Therefore, the design problem in Bergemann et al. (2018) is actually a one-dimensional screening under one-dimensional preference with possible incongruent orders.

[7]In the general case, to make the problem tractable, we assume the perfect substitution between the dimensions, preserving all these characteristics. This assumption simplifies the double deviation analysis.

[8]By the definitions in Gans (2024), input data is within consumer data, while training data is across consumer data.

menu, such as robustness concerns (Carroll (2017), Deb and Roesler (2023)), certain correlations or monotonicity between dimensions or in bundles (Haghpanah and Hartline (2021), Yang (2022a), Yang (2023)), certain class of mechanisms (Hart and Reny (2015)).

In our model, we discuss the optimal bundling of statistical error, which is naturally randomized bundling and subject to additional allocation rigidity including double deviation and obedience constraints.[9] This rigidity complicates the analysis but simplifies the menu through introducing correlations between different dimensions of allocation. We echo the insights in Adams and Yellen (1976) that bundling policy trade-offs the exclusion, inclusion and extraction principles,[10] and show how it shapes the four policies, i.e. inclusive/exclusive grand bundling, partial grand bundling and nested bundling, in the binary situation. In a continuous type space with perfectly substitution between dimensions, we show that a two-tiered partial grand bundling policy exploiting the horizontal difference to neutralize the vertical difference is optimal, which is novel in the literature.

On the technical side, the contribution is a novel approach to screening information structure. The commonly used first-order approach and duality approach in multi-dimensional screening (Rochet and Choné (1998), Manelli and Vincent (2006), Manelli and Vincent (2007), Daskalakis et al. (2017)) do not work because the double deviation makes it impossible to give a characterization of implementable allocation rules in a standard way. Instead, we provide a direct approach by constructing two functions to identify the tightness of incentive compatibility with double deviation and the incentive compatibility only conducting one-set deviation (truth-telling). We turn to analyze the two functions and explore the structure of optimal menu.

Broadly speaking, our paper proposes a new framework in the literature of the joint design of information disclosure and screening mechanism.[11] We investigate the screening and persuasion with agents of a private distribution of beliefs. In this framework, we provide an extension of revelation principle in communication games and screening (Myerson (1986), Kamenica and Gentzkow (2011), Taneva (2019)), and characterize the structural properties of optimal experiments. The straight and direct mechanism in this framework is to provide an action profile recommendation to all beliefs in the support. Also, the designer can restrict attention to information structures directly reducing the statistical error under obedience constraints in this framework.

Outline. The rest of the paper is organized as follows. Section 2 presents the setup of data selling mechanism design under hypothesis testing and further explores the structural property of the optimal menu. Section 3 reports our results in the binary situation. In Section 4, we generalize the binary situation and explore the structure of the optimal mechanism. The proof of lemmas and theorems can be found in the Appendix.

[9] The obedience constraint can be re-interpreted as a production constraint.

[10] Adams and Yellen (1976) illustrate the insights through some examples under no information asymmetry, two goods, deterministic allocation, and few types. Also, in most situations the demands for grand bundling is perfectly elastic.

[11] A large literature includes Lizzeri (1999), Bergemann and Pesendorfer (2007), Eső and Szentes (2007), Krähmer and Strausz (2015), Li and Shi (2017), Guo and Shmaya (2019), Bergemann et al. (2022), Wei and Green (2024).

2 Basic Model

Hypothesis Testing. A single data buyer with private statistical experiment conducts a hypothesis testing and faces a decision problem under uncertainty. The set of state of world $\Omega = \{\omega_1, \omega_2\}$ contains a null hypothesis $H_0 = \{\omega_1\}$ and alternative hypothesis $H_1 = \{\omega_2\}$.[12]

The data buyer is a Bayesian decision maker with a prior distribution $\mu_0 = (\frac{1}{2}, \frac{1}{2})$ over the hypotheses. He wishes to test the null hypothesis and distinguish a null hypothesis H_0 from an alternative hypothesis H_1. He can take one of two actions $A = \{a_1, a_2\}$, each of which is optimal under the respective hypothesis. The ex post utility is denoted by $u(\omega_i, a_j) \triangleq u_{ij}$. To simplify the model, we assume that the data buyer cares about the probability of identification of the true state, i.e. $u_{ij} = \mathbb{I}_{[i=j]}$.

Table 1: Payoff Matrix

u	a_1	a_2
ω_1	1	0
ω_2	0	1

Table 2: Private Experiment

E'	s'_1	s'_2
ω_1	π'_1	$1-\pi'_1$
ω_2	$1-\pi'_2$	π'_2

Table 3: Statistical Error

E'	s'_1	s'_2
H_0	$1-\alpha$	α
H_1	β	$1-\beta$

Experiment and Statistical Error. The private type $\theta \in \Theta$ of data buyer is a statistical experiment $E'_\theta = (S', \pi')$,[13] with signal space $S' = \{s'_1, s'_2\}$, and the type-dependent likelihood functions of signal $\pi'_i \equiv \Pr[s'_i|\omega_i]$. Before contracting, the data buyer does not receive any signal s' but knows the likelihood functions of signal realizations. From the perspective of the data seller, these private experiments are distributed according to a distribution $F(\theta) \in \Delta(\Theta)$, which we take as a primitive of our model. Without loss of generality, we assume that $\pi'_i \geq \frac{1}{2}$, which implies that the data buyer identifies the true state as ω_i and chooses a_i upon receiving s'_i without additional information. We can thereby label s'_1 as acceptance and s'_2 as rejection.

The data generates a bundle of initial statistical error (α, β) when conducting a hypothesis testing, where $\alpha = \Pr(s'_2|\omega_1)\mu_0(\omega_1)$ is the Type I error and $\beta = \Pr(s'_1|\omega_2)\mu_0(\omega_2)$ is the Type II error. Within the framework of hypothesis testing, the private type of data buyer can be represented as a bundle of statistical error, i.e. $\theta = (\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+$, $\alpha + \beta \leq \frac{1}{2}$. The value of his private experiment, or the probability of correct identification is $u_\theta = 1 - \alpha - \beta$, showing a clear and simple representation.

Allocation and Reduction of Statistical Error. The data buyer seeks to reduces his initial statistical error by obtaining additional information from the data seller. The data seller designs a menu $\mathcal{M} = \{\mathcal{E}, t\}$ to maximize his profits, where $\mathcal{E}$ is set of experiments $E = (S, \pi)$ with $S = \{s_1, ..., s_K\}$ and $\pi_{ik} \equiv \Pr[s_k|\omega_i]$, and $t : \mathcal{E} \to \mathbb{R}_+$ is the associated price. We assume throughout that the realization of the buyer's private signal s'_i and that of the signal s_k from any experiment E sold by the seller are independent, conditional on the state ω. Also, the realization of the buyer's

[12] In the main body of our work, we concentrate on the hypothesis testing scenario, a situation that holds significant prominence in statistical decision-making, machine learning tasks, and econometrics. This facilitates a clearer exposition of the key findings without sacrificing the integrity of the economic insights. In the discussion A, we introduce a generalized decision-making problem and provide proofs for the lemmas presented in the section 2.

[13] We replace π'_θ with π' to simplify the notation.

private signal s'_i from different private datasets are also independent, conditional on the state ω. In other words, the buyer with different types and the seller draw their information from independent sources.

To simplify the signal space, we can restrict attention to the straight action-recommendation experiment. A key distinction of this framework from the action-recommendation approach in information design, as articulated in Kamenica and Gentzkow (2011), Bergemann and Morris (2016), and Taneva (2019), is that the buyer's private type is a statistical experiment that generates a distribution of interim beliefs. Consequently, the signals within the information structure designed by the seller are profiles of action recommendations that are contingent upon these potential interim beliefs of the buyer. Precisely speaking, by constructing an onto mapping from the private signal to the recommendation outcome, we can without loss of generality represent the signal as a recommendation profile for all possible interim beliefs, i.e $s_k = (a_{k_1}, a_{k_2})$, $a_{k_j} \in A$ for $j = 1, 2$, where a_{k_1} is recommended to interim belief upon receiving s'_1, while a_{k_2} is recommended to the one upon receiving s'_2.

An experiment E is obedient for type θ if every signal $s_k = (a_{k_1}, a_{k_2})$ is obeyed for θ, i.e.

$$a_{k_j} \in \arg\max_{a_{j'} \in A} \mathrm{E}[u_{ij'} | s_k, s'_j] \text{ for all } s_k \text{ and } j = 1, 2.$$

Finally, we define an outcome of a menu as the joint distribution of states, actions, and monetary transfers resulting from every type's optimal choice of experiment and subsequent choice of action.

Lemma 1. *The outcome of every menu $\mathcal{M} = \{\mathcal{E}, t\}$ can be attained by a direct and straight mechanism $\mathcal{M} = \{\mathcal{E}_\Theta, t\}$, where each type θ chooses an obedient experiment E_θ from $\mathcal{E}_\Theta = \{E_\theta\}$, and pays the price $t : \mathcal{E}_\Theta \to \mathbb{R}_+$.*

Lemma 1 establishes that it is without loss of generality to restrict attention to direct mechanism with straight information design in which every experiment E_θ is obedient. Within the framework of hypothesis testing, the selling training data is essentially a bundle of contingent predictions that complement the original data forecasts. Importantly, obedience is only required for every experiment E_θ and for the corresponding type θ. In other words, we do not require this condition to hold if signals in E_θ are evaluated by a different type $\theta' \neq \theta$.

Table 4: Straight Experiment

E	(a_1, a_1)	(a_1, a_2)	(a_2, a_1)	(a_2, a_2)
ω_1	π_{11}	π_{12}	π_{13}	π_{14}
ω_2	π_{21}	π_{22}	π_{23}	π_{24}

Table 5: Type-wise Reduction Experiment

E	(a_1, a_1)	(a_1, a_2)	(a_2, a_1)	(a_2, a_2)
ω_1	$1 - \pi_1$	π_1	0	0
ω_2	0	π_2	0	$1 - \pi_2$

Here we propose a class of experiments (π_1, π_2) called Type-wise reduction, with specific structure as in table 5. An essential economic interpretation of Type-wise reduction experiment is that the designer allocates the reduction ratio (π_1, π_2) of the initial statistical error (α, β) Type-wisely if obeyed. The new Type I error is $\alpha\pi_1$, while the Type II error is $\beta\pi_1$.

The valuation of (π_1, π_2) for (α, β) is its incremental probability of correct identification, i.e. $V(E,\theta) = (1-\alpha\pi_1-\beta\pi_2)-(1-\alpha-\beta) = \alpha+\beta-\alpha\pi_1-\beta\pi_2$ if obeyed. A remarkable point is that the statistical error ratio π_i is damage goods to the data buyer.

The following lemma 2 establishes that it is without loss of generality to restrict attention to direct mechanism with straight information design in which every experiment E_θ is in this form.

Lemma 2. *The revenues can always be weakly improved by replacing a direct and straight mechanism $\mathcal{M} = \{\mathcal{E}_\Theta, t\}$ with an alternative direct and straight mechanism $\mathcal{M} = \{\mathcal{E}'_\Theta, t'\}$, where $E'_\theta \in \mathcal{E}'_\Theta$ is Type-wise reduction for all θ.*

Remark 1. *In our model, the designer allocates damage goods (π_1, π_2) to buyers. Therefore, $\pi_i \neq 1$, in contrast to $\pi_i \neq 0$ in selling non-damage goods, implies that the component π_i is sold.*

The idea behind the proof is that the signal (a_2, a_1), reversing the initial identification (s'_1, s'_2), is never implementable. Also, it is non-profitable for designer to send signal (a_i, a_i) in state ω_{-i}, which means that he never wants to induce both incorrect identification.

With Type-wise Reduction structure, the obedience constraint shows a simple form. The experiment (π_1, π_2) is obedient for type (α, β) if and only if

$$\alpha\pi_1 + \beta\pi_2 \leq \min\{\tfrac{1}{2}\pi_1, \tfrac{1}{2}\pi_2\}$$

Within the framework of hypothesis testing, the obedience constraint comes from the interdependence between Type I error α and Type II error β. The designer can not arbitrarily allocate the reduction ratio (π_1, π_2) due to buyer's re-minimization of statistical error. For example, if designer reduces the Type I error α sharply while does not change the Type II error β, or in other words, allocates a small π_1 and a big π_2, the data buyer would minimize the overall statistical error by only making Type I error with probability $\frac{1}{2}\pi_2$, instead of making both Types with probability $\alpha\pi_1 + \beta\pi_2$.

Here we state other structural properties of the optimal menu to simplify the problem.

Lemma 3. *In the optimal menu,*

1. *the fully informative experiment $\overline{E}$ with $(\pi_1, \pi_2) = (0, 0)$ always exists.*
2. *with binary type, all E_θ is obedient for any type $\theta' \in \Theta$.*

Timeline. By Lemma 1 and 2, the menu of experiments can be described as $\mathcal{M} = \{E_\theta, t_\theta\}_{\theta\in\Theta}$, where E_θ is a Type-wise reduction experiment obedient for type θ and $t_\theta \in \mathbb{R}_+$ is its associated tariff. Our goal is to characterize the revenue-maximizing menu for the seller. The timing of the game is as follows:

1. the seller posts a mechanism $\mathcal{M}$

2. the type θ buyer chooses an experiment E_θ and pays the corresponding price t_θ
3. the true state ω is realized
4. the buyer receive two signals, one from his private experiment E'_θ, another from the experiment E_θ he bought, and he chooses an action a
5. payoffs are realized

Mechanism. The seller's choice of a revenue-maximizing menu of experiments may involve, in principle, designing one experiment per buyer type, i.e. allocating $(\pi_1(\theta), \pi_2(\theta)) \in [0,1] \times [0,1]$ to $\theta = (\alpha, \beta)$. The seller's problem consists of maximizing the expected transfers

$$\max_{\{E_\theta, t_\theta\}} \int_{\theta \in \Theta} t(\theta) dF(\theta)$$

subject to obedience constraints (Ob),

$$\alpha\pi_1(\theta) + \beta\pi_2(\theta) \leq \min\{\tfrac{1}{2}\pi_1(\theta), \tfrac{1}{2}\pi_2(\theta)\}$$

individual-rationality constraints (IR),

$$\alpha + \beta - \alpha\pi_1(\theta) - \beta\pi_2(\theta) - t_\theta \geq 0, \quad \forall \theta \in \Theta.$$

and incentive-compatibility constraints (IC),

$$\alpha + \beta - \alpha\pi_1(\theta) - \beta\pi_2(\theta) - t_\theta \geq \alpha + \beta - \min\{\alpha\pi_1(\theta') + \beta\pi_2(\theta'), \tfrac{1}{2}\pi_1(\theta'), \tfrac{1}{2}\pi_2(\theta')\} - t_{\theta'}, \theta, \theta' \in \Theta,$$

Compared to traditional multi-dimensional goods, training data, or statistical error are inherent interdependence across dimensions. This interdependence necessitates the re-minimization of error, thus introducing two difficulties into the standard model. First, it introduces the obedience constraint. The allocation of statistical error is constrained and lies in a linear production possibility set. Second, there exists double deviation in incentive compatibility. Agents with different initial statistical error may choose different Types of error when combining their private training data with the purchased data to conduct the hypothesis testing. A remarkable point is that the types, who make double deviation when pretending to be some type θ, commit type-independent error $\frac{1}{2}\pi_i$ and share the same incentives, which plays a key role in the optimal screening.

3 Binary Situation

Consider the situation of binary type (α, β), (α', β') drawing from uniform distribution $\Pr((\alpha, \beta)) = \frac{1}{2}$, $i = 1, 2$. Suppose that (α, β) is the H-igh type and (α', β') is the L-ow type in the sense of their vertical valuation for the fully informative experiment/overall statistical error, i.e. $V(\overline{E}, 1) \geqslant$

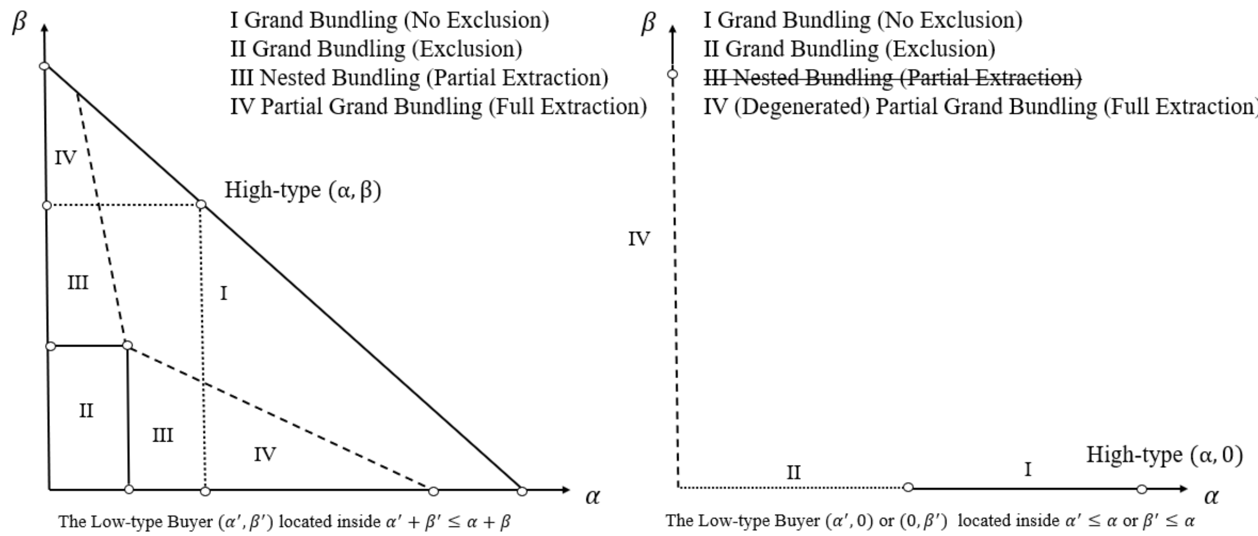

Figure 1: Optimal Selling Schemes. Left: optimal mechanism in selling training data. Right: optimal mechanism in selling information in Bergemann et al. (2018).

$V(\overline{E}, 2)$, or $\alpha + \beta \geqslant \alpha' + \beta'$. By lemma 3 and simple deduction,[14] the designer sells the fully informative experiment to the type-H and only needs to design the experiment for type-L, which is obedient for both types.

Here we state the main result in the binary setting in a qualitative way and then elaborates it in a quantitative way. As the left of figure 1 shows, the optimal selling mechanism features four typically different selling schemes.

Theorem 1. *When low type (α', β') lies*

1. *in region I, the seller implements inclusive grand bundling policy, i.e. selling $\overline{E}$ to both types.*
2. *in region II,[15] the seller implements exclusive grand bundling policy, i.e. only selling $\overline{E}$ to type-H.*
3. *in region III, the seller implements nested bundling policy, i.e. selling $\overline{E}$ to type-H, and fixed $\hat{E}$ to type-L only reducing certain Type of error, and partially reducing the information rent.*
4. *in region IV, the seller implements partial grand bundling policy, i.e. selling $\overline{E}$ to type-H and E^* to type-L reducing both Types of error, and extracting all the surplus.*

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[14]The mechanism where $\overline{E}$ is not allocated to type-H is always not optimal.

[15]The coefficient of the boundary line is determined by the market share between type-H and type-L. Here the same market share induces the $\frac{1}{2}$

3.1 Economic Interpretations of Theorem 1

The designer implements different bundling policies of error reduction ratio (π_1, π_2) to trade-off the economics from exclusion or inclusion of type-L, and extraction of type-L and type-H surplus. The fundamental trade-off in the data selling policy lies in the extraction of information rent versus the extraction of efficient surplus for type-L. On one hand, the data seller distorts the menu for type-L or excludes type-L, to reduce the information rent of type-H. On the other hand, the data seller seeks to avoid such distortion or exclusion to extract the efficient value of type-L. The boundary line, which divides region I from other regions, reflects this trade-off.

In regions I and II, the seller implements inclusive/exclusive grand bundling policy, which is analogous to conventional one-dimensional screening as describe in Riley and Zeckhauser (1983).[16] In region I, type-H itself does not have much information rent because the dimension with large difference in error are relatively close. Therefore, the seller fully discloses the true state and applies no distortion to either type, whereas in other regions, distortion of type-L is employed to reduce the information rent of type-H. In region II, the error of type-L in both Type I and Type II errors is significantly smaller than that of type-H. In this scenario, the seller exclusively sells $\overline{E}$ to type-H to reduce the information rent.

Table 6: Region III (Left)

	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	0	1	0
ω_2	0	π_2^*	$1 - \pi_2^*$

Table 7: Region IV

	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
ω_1	$1 - \pi_1^*$	π_1^*	0
ω_2	0	π_2^*	$1 - \pi_2^*$

In regions III and IV, the multi-dimensional nature of statistical error and its interdependence across dimensions are pivotal in the trade-off. In region III, while both dimensions of type-L are lower than those of type-H, some dimensions come close to the values of type-H. The seller implements a nested bundling policy, i.e. $(\pi_1^*, 1)$ or $(1, \pi_2^*)$, to maximize extraction from type-L by only minimizing the relatively high Type error of type-L.[17] Also, π_1^* or π_2^* are fixed and invariant to (α', β') across the regions,.

For instance, in the left part of region III, β' is not significantly smaller than β, as depicted in left of figure 1. The seller wants to reduce β^* as much as possible to extract more surplus of type-L. Given that $(\alpha, \beta) > (\alpha', \beta')$, the new overall error for type-H, $\alpha\pi_1 + \beta\pi_2$, always exceeds $\alpha'\pi_1 + \beta'\pi_2$, resulting in two consequences. First, it becomes more difficult to allocate a partial reduction to type-H as opposed to type-L. The obedience constraints on type-H are more stringent than those on type-L. Second, the complete elimination of the information rent is unattainable. This implies that (IC-H) must be binding, whereas the (IR-H) must not be, ensuring that only the

[16] Riley and Zeckhauser (1983) argues that "no-haggling" (selling one good with posted price/"take-it-or-leave-it") is optimal in standard one-dimensional screening. Here region I and region II is approximately degenerated to one-dimensional case due to the slight/huge differences between type-L and type-H.

[17] Our model allows probabilistic allocation. Therefore, we use nested bundling to emphasize the strict bundling set inclusion of the two menus, narrower than Bergemann et al. (2021) and Yang (2023). The menu of type-L only contains the reduction of Type i error for some $i = 1, 2$ while the one of type-H contains both.

(IC-H) and (Ob-H) determine the experiment for the low-type buyers.

Consequently, the experiment within this region is determined by type-H, causing it to remain invariant with respect to the initial error of type-L. (Ob-H) establishes the optimal ratio of π_1 to π_2, while (IC-H) is associated with the shadow price that corresponds to this ratio. The seller aims to allocate (π_1, π_2) to minimize the information rent and extract the surplus from type-L. Achieving both objectives is more economical through only reducing the Type II error, as when where β' is slightly smaller than β, the information rent generated from allocation in this dimension is relatively low, yet the potential for extraction is high. Therefore, the designer elects to decrease only the Type II error by a fixed ratio.

In region IV, one type of error for type-L is actually higher than that for type-H. The designer can implement partial grand bundling policy, i.e. (π_1^*, π_2^*), to exploit these horizontal differences to differentiate the products, thereby eliminating all information rent. The most important insight of region IV is that it reflects the interaction of horizontal preference.[18] As shown in Figure 1, even with a fixed α', the Type I error decreases as β' increases. On the one hand, when transitioning from region III to region IV, the optimal menu strategy shifts from reducing one type of error to addressing the reduction of both types. On the other hand, within region IV, the reduction ratio for one dimension increases as the level of predictive power for the other dimension decreases.

The interaction observed in region IV stems from the multi-dimensional nature of statistical error. If the designer in region IV continues to solely reduce a specific type of error for type-L (for example, only reducing β'), it is important to note that with $\beta' > \beta$, the low-type buyer effectively becomes the "type-H" under this one-dimensional scheme. Consequently, the (IR-H) becomes more stringent than the (IC-H), leading to a reduction in the tariffs that can be charged to high-type buyers. By strategically shifting the experiment designed for the lower types along the hyperplane determined by the binding obedience constraint, it is possible to reduce both types of errors and eliminate information rent.

In Bergemann et al. (2018), the private type is the buyer's interim belief, which implies a binary outcome: he commits either a Type I or a Type II error, with vertical preference reflecting the probabilities of such error.[19] Consequently, the types in Bergemann et al. (2018) actually lies in the axes as depicted in the right of figure 1, which is a degenerated situation of our model. Moreover, the two-dimensional allocation is degenerated to separable one-dimensional one, because the designer can only reduce the certain Type of error which the buyer commits. The different positions of the types free the concerns of the interdependence between errors and its consequences, such as double deviation and obedience. In their binary case, the inclusive/exclusive grand bundling policy emerges when both types lie in the same axis,[20] while the degenerated partial grand bundling and full extraction occur when types are distributed across both axes.[21]

[18]Partial grand bundling means that the seller decrease both Types of error of both types

[19]The former corresponds to the position of information while the latter corresponds to the overall informativeness of information as described in Bergemann et al. (2018).

[20]Bergemann et al. (2018) relates it to the no-haggling result in Riley and Zeckhauser (1983), as the allocation is actually one-dimensional in this case.

[21]The degenerated partial grand bundling means that in type-L menu, $\pi_i = 0$ and $\pi_{-i} \in [0, 1)$ for some $i = 1, 2$,

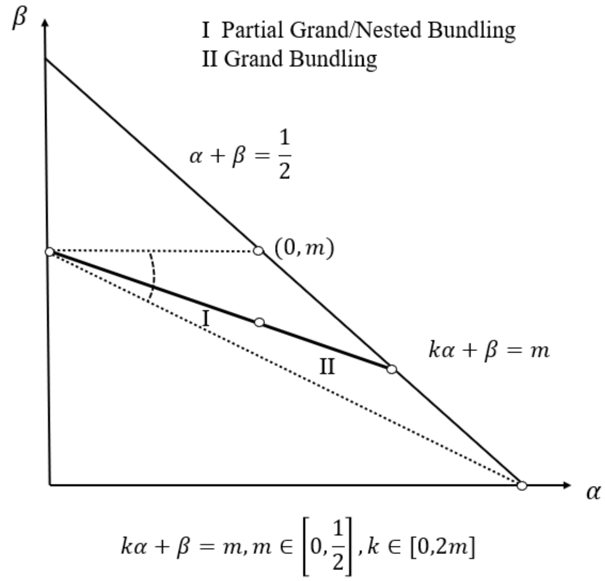

Figure 2: Optimal Selling Schemes

4 General Case

In the general case, we introduce one assumption. This assumption maintains the tractability of the problem while also generalizes insights from the binary situation. Our generalized framework accommodates both horizontal and vertical differences in types, capturing the most interesting aspects of the data selling mechanism.

Assumption 1. *The statistical error of buyer's private data is characterized by a linear relationship: for the private type (α, β), it holds that $k\alpha + \beta = m$, with $m \in [0, \frac{1}{2})$ and $k \in [0, 2m]$.*[22][23]

Under assumption 1, data buyer's type can be represented $(\alpha, m - k\alpha)$, where $\alpha \in \mathcal{A} = [\underline{\alpha}, \overline{\alpha}] = [0, \frac{\frac{1}{2}-m}{1-k}]$ draws from distribution $F(\alpha)$ with a continuous, strictly positive density $f(\alpha)$. Although we assume the one-dimensional preference, the allocation instrument is still two-dimensional. Also, we can discuss the effects of k and m to the optimal menu.

Here we state our main result, as depicted in figure 2.

Theorem 2. *The optimal selling mechanism is two-tiered pricing:*

1. $(E_\alpha, t_\alpha) = (\overline{E}, \overline{t})$ *for* $\alpha \in [\alpha^*, \overline{\alpha}]$
2. $(E_\alpha, t_\alpha) = (E^*, t^*)$ *for* $\alpha \in [\underline{\alpha}, \alpha^*)$, *where* $\pi_1^* = 1 - k(1 - \frac{\alpha^*}{\overline{\alpha}})$, $\pi_2^* = \frac{\alpha^*}{\overline{\alpha}}$

i.e. Type i error is eliminated completely.

[22] We can also solve $\alpha + k\beta = m$, with $m \in [0, \frac{1}{2})$ and $k \in [0, 2m]$. All the proofs and conclusions are symmetric.

[23] The upper bound of k comes from that there exists a buyer with no private data, i.e. the type of him is $(l, \frac{1}{2} - l)$ for some $l \in [0, \frac{1}{2}]$.

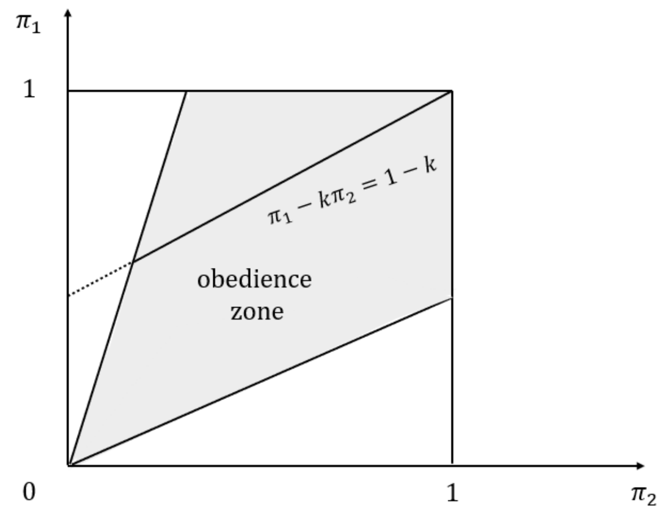

Figure 3: Neutralization Line

3. $\alpha^* \in \arg\max_\alpha \alpha \left((1-k)\overline{\alpha} - \frac{1}{2}F(\alpha)\right)$

The optimal menu coincides with the one of region IV when $k > 0$, and region III when $k = 0$ in the binary situation, reflecting the trade-off of inclusion, exclusion and extraction principles in the general case of data selling.

The two tier structure of the optimal menu echoes with the classical no-haggling menu in single dimension mechanism design. Specifically, fix a threshold α^*, in single dimension with only vertical difference, no-haggling menu sells all the data to buyers with type above the threshold. The shadow cost of truth-telling constraint/screening is the exclusion of low-valued buyers. The main obstacle to potential improvement, i.e. sell some information to these excluded low-valued buyers is that, the IR constraint of $\underline{\alpha}$ leaves strictly positive rent to the type α^* buyers. When both vertical and horizontal differences exist, the seller can exploit the horizontal differences to *neutralize* the vertical difference through subtly designing the dataset E^* to include these low-valued buyers and extract their surplus. By setting E^* along the *neutralization* line

$$\frac{1-\pi_1}{1-\pi_2} = k,$$

buyer's valuations of the dataset E^* is completely indifferent to his type, i.e. his own private dataset. Specifically,

$$V(E^*, \alpha) = \alpha + (m - k\alpha) - \alpha\pi_1^* - (m - k\alpha)\pi_2^* = m(1-\pi_2^*) + \alpha[(1-k) - (\pi_1^* - k\pi_2^*)] = m(1-\pi_2^*)$$

The main intuition is that, given that buyers strictly prefer to utilize their own private dataset, same supplementary dataset reduces strictly more statistical error if merging with baseline/private dataset of lower quality. The supplementary dataset E^* is delicately designed such that, as the quality of baseline dataset is deteriorated, the additional error reduced by the supplementary dataset

E^* exactly compensates the loss in baseline dataset and therefore the difference in their outside options.

Hence, to improve efficiency of the menu, the seller can improve the informativeness of E^* along the neutralization line $\frac{1-\pi_1}{1-\pi_2} = k$ and charge all the additional value it generates, to extract all the rent of buyers below type α^*. Such operation can be continued until it hits the obedient boundary.

Beyond the boundary, type α^* buyer will only minimize one type of error, leaving strictly positive rent to him. The positive rent of type α^* buyer limits the price for $\bar{E}$ to meet the IC constraint for type α^* buyers. Specifically,

$$t(\bar{E}) = V(\bar{E}, \alpha^*) - \underbrace{[V(E^*, \alpha^*) - m(1 - \pi_2^*)]}_{\text{Rent of type } \alpha^* \text{ buyer}}$$

Denote $\hat{\alpha}$ the the threshold type such that the obedience constraint binds. We argue that selling $\bar{E}$ to $[\hat{\alpha}, \alpha^*)$ leads to an improvement (strict if and only if $F(\alpha^*) - F(\hat{\alpha}) > 0$) since the seller can charge same price for $\bar{E}$ while sell $\bar{E}$ to more buyers. Specifically, $t'\left(\bar{E}\right) = V\left(\bar{E}, \hat{\alpha}\right)$ and

$$\begin{aligned} t'\left(\bar{E}\right) - t\left(\bar{E}\right) &= V\left(\bar{E}, \hat{\alpha}\right) - \left[V\left(\bar{E}, \alpha^*\right) - \left(V\left(E^*, \alpha^*\right) - m(1 - \pi_2^*)\right)\right] \\ &= \left[V\left(\bar{E}, \hat{\alpha}\right) - V\left(E^*, \hat{\alpha}\right)\right] - \left[V\left(\bar{E}, \alpha^*\right) - V\left(E^*, \alpha^*\right)\right] = 0 \end{aligned}$$

Hence, the obedience constraint, a relevant constraint in the set-up of data sale, limits the degree of exploitation.

Perhaps, a less obvious property, especially to those not working on (multi-dimension) mechanism design, of the optimal menu is its simplicity. The two-tier structure is a classical result in single dimension mechanism design, but it is always NOT the case when extending to multi-dimension. As we will see in the next subsection, the exclusion of double/two-step deviation imposes an endogenous link between $1 - \pi_2$, i.e. the information of the supplementary dataset on state ω_2, and the price of the supplementary dataset. Such linkage reduces the dimension of the optimization problem, leading to two-tier structure as its extreme point. In other words, the possibility of two-step deviation limits the flexibility of menu structure brought by multi-dimension nature of data allocation.

4.1 Sketch of the Proof

Denote

$$\begin{aligned} V_r(E, \alpha) &= \alpha(1 - \pi_1) + (m - k\alpha)(1 - \pi_2) \\ V_n(E, \alpha) &= \alpha + (m - k\alpha) - \min\{\frac{1}{2}\pi_1, \frac{1}{2}\pi_2\} \\ V(E, \alpha) &= \max\{V_r(E, \alpha), V_n(E, \alpha)\} \end{aligned}$$

We first summarize two properties of the value functions, which will be repeatedly utilized in our proof.

Property 1. $V_n(E', \alpha') - V_n(E, \alpha') = V_n(E', \alpha) - V_n(E, \alpha), \forall E, E', \alpha, \alpha'$.

Note that whenever the buyer does not follow seller's recommendation, he is indifferent to drop his own private data. Therefore, the additional value of improving the supplementary dataset from E to E', merging with any private/baseline datasets buyers have, is exactly the value of the additional supplementary dataset itself.

Property 2. $V_r(E', \alpha') - V_r(E, \alpha') \geq V_r(E', \alpha) - V_r(E, \alpha), \forall \alpha' > \alpha$ if and only if $\pi_1' - k\pi_2' \leq \pi_1 - k\pi_2$, where inequality binds if and only if $\pi_1' - k\pi_2' = \pi_1 - k\pi_2$.

Whenever buyers strictly prefer to utilize their own private dataset when merging with the supplementary dataset in making decisions, improving the supplementary dataset from E to E' generates strictly more value to the buyer with less private dataset, i.e. type α'.

For type α buyer, a supplementary dataset $E = (\pi_1, \pi_2)$ satisfies obedience constraint if $\pi_2/\pi_1 \in [\frac{1/2-\alpha}{m-k\alpha}, \frac{\alpha}{1/2-(m-k\alpha)}]$. If E is obedient to α, then it is also obedient to any $\alpha' < \alpha$. If a supplementary dataset is obedient, then the buyer will utilize his own private dataset in decision making. Based on this, given access to more informative private data, then buyer will also not drop his own private data.

For any supplementary dataset E_α sold to type α buyer, denote $\lambda(\alpha) \geq \alpha$ the type of buyers who are exactly indifferent to following seller's recommendation or not, when merging his own private dataset.[24] Denote IC$[\alpha \to \alpha']$ the incentive compatibility constraint that type α buyers weakly prefer the supplementary dataset E_α over $E_{\alpha'}$.

Lemma 4 (Characterization of Obedience Zone)**.** *Optimal menu (E_α, t_α) satisfies*

1. $\pi_2(\alpha)/\pi_1(\alpha) \leq 1$
2. *There exists a threshold α^* such that*
   (a) *for any $\alpha < \alpha$, $\alpha < \lambda(\alpha)$ and there exists some $\alpha' > \lambda(\alpha)$ such that IC$[\alpha' \to \alpha]$ binds;*
   (b) *$E_\alpha = \bar{E}$ if and only if $\alpha \geq \alpha^*$.*

The first part of the lemma 4 requires that the sold supplementary data should reduce relatively more Type II error. Therefore, whenever buyers choose not to follow seller's recommendation, they are supposed to concentrate on Type II error reduction. The second part implies that, the seller should sell full supplementary dataset only to those buyers with the amount of private dataset below some threshold. Otherwise, the partial supplementary dataset E_α, makes buyers α strictly suboptimal to drop their own relatively richer private dataset. Besides, there always exist buyers α'

[24] Define the function $\lambda(\alpha) : \mathcal{A} \to \mathcal{A}$ as below: (i)$\lambda(\alpha) = \frac{(\frac{1}{2}-m)\pi_2(\alpha)}{\pi_1(\alpha)-k\pi_2(\alpha)}$ if $\pi_1(\alpha) \neq 0$; (ii) $\lambda(\alpha) = \bar{\alpha}$ otherwise. Here we assume the obedience is not binding for any $\alpha \neq \overline{\alpha}$ allocated with $\overline{E}$.

with strictly less private datasets, who are indifferent between choosing their targeted supplementary dataset $E_{\alpha'}$, and choosing the dataset E_α while not following the recommendation. Note that this type of deviation, named as double deviation, prevails in the literature of information design as screening tools. The literature deals with double deviation by first ignoring the second type deviation and then verifying the solution to the related optimization problem also excludes double deviations. However, this approach does NOT work in our model as the incentive compatibility constraints excluding double deviation bind at optimum.

Lemma 4 is derived based on mutual incentive compatibility (IC) constraints, i.e.

$$V(E_\alpha, \alpha) - V(E_\alpha, \alpha') \geq V(E_{\alpha'}, \alpha) - V(E_{\alpha'}, \alpha').$$

There are two classes of mutual IC constraints, one for the exclusion of one-step deviation (i.e. choosing others' supplementary datasets while still following the recommendation), the other for the exclusion of double deviation. To deal with these two classes of mutual IC constraints at the same time, we identify a class of perturbations $\{(-k\Delta\pi, -\Delta\pi : \Delta\pi \geq 0)\}$ on supplementary datasets, which does not change the difference in evaluating the dataset (Property 2) and therefore the mutual IC constraints between two obedient buyers. However, when $\pi_1 < \pi_2$, such perturbation enlarges the difference between obedient and non-obedient buyers and therefore relaxes the mutual IC constraint. The seller should exploit such perturbation of informativeness improvement to the maximal degree, where $\pi_1 \geq \pi_2$ and the IC constraints for some non-obedient buyers bind. To understand the rest of Lemma 4, we illustrate the intuition underlying a weaker argument, i.e. if the obedience constraint binds for type α buyer, then $E_{\alpha'} = \bar{E}$ for any $\alpha' > \alpha$. Note that when both players are indifferent to drop their own private datasets, then the difference in valuing supplementary dataset E' between type α and α' buyers is supposed to coincide with the difference in their outside option, i.e. the deteriorating level of private datasets. Therefore, the difference in valuing supplementary dataset E' must be the same as that in valuing the full dataset $\bar{E}$ (Property 1), accessing which any buyer is indifferent to whether or not follow seller's recommendation. However, such difference is always higher than that in evaluating supplementary dataset E between obedient buyers since better supplementary datasets are more efficient to make up the deterioration of baseline/private datasets (Property 2). Mutual IC between α and α' buyers then implies the argument.

Define $\gamma(\alpha)$ some type who is indifferent between choosing $E_{\gamma(\alpha)}$ and conducting double deviation by choosing E_α.[25] Mathematically,

$$\gamma(\alpha) = \begin{cases} \alpha & \text{if } \alpha = \lambda(\alpha) \\ \tilde{\alpha} \in \{\alpha' > \lambda(\alpha) : \text{IC}[\alpha' \to \alpha] \text{ is binding}\} & \text{if } \alpha < \lambda(\alpha) \end{cases}$$

Lemma 5 (Properties of λ and γ)**.** *In optimal menu,*

1. $\lambda(\alpha) \leq \lambda(\hat{\alpha}) \leq \gamma(\alpha)$ *for* $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$.[26]

[25] By lemma 4,if $\alpha < \lambda(\alpha)$, $\{\alpha' \,|\text{IC}[\alpha' \to \alpha] \text{ is binding}\}$ is a non-empty subset of $[\lambda(\alpha), \overline{\alpha}]$.

[26] A direct corollary is that $\lambda(\alpha) \leq \lambda(\hat{\alpha}) \leq \inf\{\alpha' > \lambda(\alpha) : \text{IC}[\alpha' \to \alpha] \text{ is binding}\}$ for $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$.

2. $\pi(\alpha) := \pi_1(\alpha) - k\pi_2(\alpha)$ *is non-increasing for* $\alpha \in [0, \bar{\alpha}]$*;*

The first argument of Lemma 5 implies that, when merging with supplementary data $E_{\hat{\alpha}}$ sold to any buyer $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$, $\lambda(\alpha)$ buyer weakly prefers to utilize while $\gamma(\alpha)$ buyer weakly prefers to ignore their own private datasets respectively, in statistical decision making. Suppose the amount of private/baseline datasets suffer a loss by one unit, the total error are deteriorated by $1-k$ unit due to imperfect substitution between Type I and Type II error. The term $(1-k) - \pi(\alpha)$ measures how much this deteriorated total error are alleviated by merging supplementary dataset E_α. The second argument requires that, the supplementary dataset sold to buyers with less amount of baseline/private dataset should be more effective in compensating loss in baseline dataset. Note that such monotonicity is derived from mutual IC constraint in standard mechanism design. However, the possibility of double deviation implies this property can only be established within the interval $[\alpha, \lambda(\alpha)]$. We then turn to the first argument to connect separated intervals and extend this monotonicity among the whole range.

To establish the first argument, on one hand, the IC$[\gamma(\alpha) \to \alpha]$ (IC$[\gamma(\hat{\alpha}) \to \hat{\alpha}]$) binding implies that $\gamma(\alpha)$ ($\gamma(\hat{\alpha})$) buyers weakly prefer E_α ($E_{\hat{\alpha}}$) over $E_{\hat{\alpha}}$ (E_α). Therefore, the mutual IC between $\gamma(\alpha)$ and $\gamma(\hat{\alpha})$ buyers implies that

$$\begin{aligned} V(E_{\hat{\alpha}}, \gamma(\alpha)) - V_n(E_{\hat{\alpha}}, \gamma(\hat{\alpha})) \leq V_n(E_\alpha, \gamma(\alpha)) - V_n(E_\alpha, \gamma(\hat{\alpha})) \\ = V_n(E_{\hat{\alpha}}, \gamma(\alpha)) - V_n(E_{\hat{\alpha}}, \gamma(\hat{\alpha})) \end{aligned}$$

where the equality follows from Property 1. The argument that $\gamma(\alpha) \geq \lambda(\hat{\alpha})$ then follows. The intuition of the inequality above is that, when merging with fixed supplementary dataset $E_{\hat{\alpha}}$, if the quality improvement of private datasets from $\gamma(\hat{\alpha})$ to $\gamma(\alpha)$ transforms the buyers to strictly prefer to utilize his private dataset, then the gap between private dataset $\gamma(\hat{\alpha})$ and $\gamma(\alpha)$ are strictly enlarged. *In this sense, the supplementary dataset amplifies the quality gap of baseline/private datasets.* On the other hand, IC$[\hat{\alpha} \to \alpha]$, the tightness of IC$[\gamma(\alpha) \to \alpha]$, and IC$[\gamma(\alpha) \to \hat{\alpha}]$ requires that

$$V_r(E_{\hat{\alpha}}, \hat{\alpha}) - V_r(E_\alpha, \hat{\alpha}) \geq V_n(E_{\hat{\alpha}}, \gamma(\alpha)) - V_n(E_\alpha, \gamma(\alpha)).$$

Property 2 implies that

$$V_r(E_{\hat{\alpha}}, \hat{\alpha}) - V_r(E_\alpha, \hat{\alpha}) \leq V_r(E_{\hat{\alpha}}, \lambda(\hat{\alpha})) - V_r(E_\alpha, \lambda(\hat{\alpha})) = V_n(E_{\hat{\alpha}}, \lambda(\hat{\alpha})) - V_r(E_\alpha, \lambda(\hat{\alpha}))$$

Combining these two inequalities, it is required that

$$V_n(E_{\hat{\alpha}}, \lambda(\hat{\alpha})) - V_n(E_\alpha, \lambda(\hat{\alpha})) \leq V_n(E_{\hat{\alpha}}, \lambda(\hat{\alpha})) - V_r(E_\alpha, \lambda(\hat{\alpha}))$$

Property 1 then implies that $V_n(E_\alpha, \lambda(\hat{\alpha})) \geq V_r(E_\alpha, \lambda(\hat{\alpha}))$, i.e. $\lambda(\hat{\alpha}) \geq \lambda(\alpha)$. The main intuition for the inequality above is that, if the quality deterioration of supplementary dataset from $E_{\hat{\alpha}}$ to E_α transforms the buyer to strictly prefer to utilize his own private dataset, then the gap between the supplementary dataset $E_{\hat{\alpha}}$ and E_α are strictly reduced. *In this sense, the private dataset narrows the quality gap of supplementary datasets.*

Lemma 6 (Equivalent Transformation of Constraints)**.** *In the optimal mechanism, the* IC, IR and Ob conditions are equivalent to

1. $\frac{1}{2}\pi_2(\alpha) + t_\alpha = t^*$ for all $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, where t^* is the associated tariff for $\bar{E}$;
2. $V(E_\alpha, \alpha) = \int_{\underline{\alpha}}^{\alpha}(1 - k - \pi(t))dt + V(E_{\underline{\alpha}}, \underline{\alpha})$
3. $\pi(\alpha) : [\underline{\alpha}, \bar{\alpha}] \to [0, 1 - k]$ is non-increasing;
4. IR$[\hat{\theta}]$ holds for some $\hat{\alpha} = \inf\{\alpha | \pi(\alpha) \leq 1 - k\}$.

The first argument of the lemma 6 implies that the price difference between any pair of supplementary datasets in the menu should exactly equal to their difference in Type II error reduction. It comes directly from the mutual IC between $\gamma(\hat{\alpha})$ and $\gamma(\alpha)$ buyers. The main intuition is that, whenever the buyer commits double deviation, he is indifferent to drop his own private dataset and concentrates on reducing Type II error. Therefore, the predictive power of state ω_2 completely determines the price for supplementary dataset. The endogenous linkage between Type II error reduction and price reduces the multi-dimension nature of data allocation to a single dimension. The second argument is derived from Envelope's Theorem. One caveat is that the term $1-k-\pi(t)$, though being non-decreasing, may be negative when t is close to $\underline{\alpha}$. Therefore, the individual rationality constraint is imposed on $\hat{\alpha}$ buyer instead $\underline{\alpha}$ buyer. However, as we will show in the proof, for any menu π such that $\pi(\alpha) > 1 - k$ for some α, the menu $\hat{\pi} = \min\{\pi, 1 - k\}$ strict benefits the seller.

Finally, by some algebraic operations, the seller's optimization problem can be transformed as

$$\max_{\pi} \int_{\underline{\alpha}}^{\overline{\alpha}} \frac{-1}{1-2m} \left[\int_{\alpha}^{\bar{\alpha}} (1 - F(t) - tf(t))dt + 2m\alpha\right] d\pi(\alpha)$$
$$s.t. \begin{cases} \pi : [\underline{\alpha}, \bar{\alpha}] \to [0, 1-k] \text{ is non-increasing} \\ \pi(\bar{\alpha}) = 0 \end{cases}$$

Theorem 2 then follows from the general extension of Carathéodory's theorem found in Kang (2023).

5 Future Work

In the future work, we will extend our model to situations for $k \in (-\infty, 0)$ and also try to characterize the properties of the optimal menu under general joint distribution of Type I error and Type II error.

Moreover, our paper discusses the optimal sale of training data, complementing the discussion of selling input data in Bergemann et al. (2018). Although our concurrent model can cover their model, we may conduct a more nuanced comparison of these two data sales approaches, and discuss the optimal bundling policy of input data and training data in the future.

A Proof of Generalized Lemmas in Section 2

We prove Lemma 1, Lemma 2 and Lemma 3 in a generalized statistical decision making problem.

A single data buyer with private statistical experiment faces a decision problem under uncertainty. The state of world ω is drawn from a finite set $\Omega = \{\omega_1, \ldots, \omega_i, \ldots, \omega_I\}$ and suppose there is a common prior $\mu \in \Delta(\Omega)$ with full-support. The data buyer chooses an action a from a finite set $A = \{a_1, \ldots, a_j, \ldots, a_J\}$. The ex post utility is denoted by $u(\omega_i, a_j) \triangleq u_{ij} \in \mathbb{R}_+$. Following Bergemann et al. (2018), we assume $I \leq J$ and $u_{ii} > u_{ij}$, $\forall j \neq i$. Denote $\mathcal{U} = \{u_{ij}\}_{I \times J}$ the $I \times J$ payoff matrix in the decision problem. We assume the matching utility function in the following and $u(\omega_i, a_j)$ is then given by $u(\omega_i, a_j) \triangleq \mathbb{I}_{[i=j]} \cdot u_i$, and $u_i \triangleq u_{ii}$.[27] Under this assumption, it is without loss of generality to assume that the sets of actions and states have the same cardinality: $|A| = |\Omega| = I = J$.

The statistical experiment $E'_\theta = (S, \pi'_\theta)$,with signal space $S' = \{s'_1, ..., s'_K\}$ and type-dependent likelihood functions of signal $\pi'_{\theta ik} \equiv Pr[s_k|\omega_i]$, is buyer's private type. Note that $\sum_{k=1}^K \pi'_{ik} = 1$ for all i. The distribution of private type is $F(\theta) \in \Delta(\Theta)$. We can also assume that agent's type is the distribution of posteriors, i.e. $\{\mu_{\theta 1}, ..., \mu_{\theta K}\}$ with $\Pr(\mu_{\theta k}) = \sum_{i=1}^I \mu_i \pi'_{\theta ik}$ for all $k \in 1, ..., K$, where $\mu_{\theta k} \in \Delta(\Omega)$ and $\mu_{\theta ki} \equiv \mu_{\theta k}(\omega_i) = \frac{\mu_i \pi'_{\theta ik}}{\sum_{i'=1}^I \mu_{i'} \pi'_{\theta i' k}}$. Experiment $E = (S, \pi)$ with $S = \{s_1, ..., s_R\}$ and $\pi_{ir} \equiv Pr[s_r|\omega_i]$ are the purchased one.[28]

Type θ data buyer's decision problem is given by conditioning on accepting signal s'_k, choosing an optimal action $a(s'_k \mid E_n)$ that maximizes the expected utility (of states) given the posterior, i.e.,

$$a(s'_k \mid E_\theta) \in \arg\max_{a_j \in A} \{\textstyle\sum_{i=1}^I \mu_{\theta ki} u_{ij}\} \text{ and } u(s'_k \mid E_\theta) \triangleq \max_j \{\textstyle\sum_{i=1}^I \mu_{\theta ki} u_{ij}\}.$$

The expected utility of type θ is therefore given by

$$u_\theta \triangleq \textstyle\sum_{k=1}^K \Pr(\mu_{nk}) u(s'_k \mid E_\theta) = \sum_{k=1}^K \max_j \left\{ \sum_{i=1}^I \mu_i \pi'_{\theta ik} u_{ij} \right\}$$

By the order-invariance of bayesian updating, we can suppose that the buyer receives the signals sequentially after contracting. Consequently, for any signal $s_r \in S$ that occurs with strictly positive probability π_{ir}, an action that maximizes the expected utility of type θ is given by

$$a(s_r, s'_k|E_\theta) \in \arg\max_{a_j \in A} \left\{ \textstyle\sum_{i=1}^I \left(\frac{\mu_{\theta ki} \pi_{ir}}{\sum_{i'=1}^I \mu_{\theta ki'} \pi_{i'r}} \right) u_{ij} \right\}$$

which leads to the following conditional expected utility:

$$u(s_r, s'_k|E_\theta) \triangleq \max_j \left\{ \textstyle\sum_{i=1}^I \left(\frac{\mu_{\theta ki} \pi_{ir}}{\sum_{i'=1}^I \mu_{\theta ki'} \pi_{i'r}} \right) u_{ij} \right\}$$

[27] Our results can be extended to the general payoff matrix. To streamline the presentation of our results and avoid the inclusion of trivial and cumbersome scalars, we have opted for a matching utility form.

[28] The conditional independence assumption is still required as described in the main body.

Integrating over all signal realizations s_k and subtracting the value of prior information, the (net) value of an experiment E for type θ is given by $V(E,\theta)$, where

$$V(\mathrm{E},\theta) \triangleq u(\mathrm{E},\theta) - u_\theta = \sum_{k=1}^{K}(\sum_{r=1}^{R} \max_j \left\{\sum_{i=1}^{I} \mu_i \pi'_{\theta ik}\pi'_{ir}u_{ij}\right\} - \max_j \left\{\sum_{i=1}^{I} \mu_i \pi'_{\theta ik}u_{ij}\right\})$$

E is obedient for θ if every signal $s \in S$ leads type θ to a different optimal action vector. Precisely speaking, we can w.l.o.g represent the signal as a recommendation profile for all possible posteriors, i.e $s_r = (a_{r1}, ..., a_{rK})$, where $a_{rk} \in A$, and the recommendation is obeyed

$$a(s_r, s'_k|E_n) = a_{rk}, \quad \forall s'_k \in S', s_r \in S.$$

A.1 Proof of Lemma 1

Direct Mechanism. By the revelation principle, we can restrict attention to direct mechanisms.

Straight Mechanism. The proof idea comes from Bergemann et al. (2018). Consider any type θ and experiment $E = (S, \pi)$. Without loss of generality, let every posterior of the type choose a single action upon receiving each signal. Let S^a denote the set of signals in experiment E that induces type θ to choose action profile $a \in A^K$ for every posterior. Thus, $\cup_{a\in A^K} S^a = S$. Construct experiment $E' = (S', \pi')$ as a recommendation for type θ based on experiment E, with signal space $S' = A^K$ and $\pi'(a|\omega) = \int_{S^a} \pi(s|\omega)$, for all $\omega \in \Omega$ and $a \in A^K$.

By construction, E' and E induce the same outcome distribution for type θ; hence, $V(E', \theta) = V(E, \theta)$. Moreover, E' is a garbling of E. By Blackwell's theorem, we have $V(E', \theta') = V(E, \theta')$ for all θ'. Therefore, for any incentive-compatible and individually rational direct mechanism E_θ, t_θ, we can construct another direct mechanism E'_θ, t_θ whose experiments lead type θ to take action a after observing recommendation profile s that is also incentive compatible and individually rational, thus yielding weakly larger profits.

A.2 Proof of Lemma 2

We prove Lemma 2 in a generalized case with two states, finite signals and finite actions. We first characterize all implementable signals, thereby excluding some signals like (a_2, a_1) in the main body. We then explore the "non-dispersed" property of the experiments in the optimal menu, like $\Pr((a_i, a_i|\omega_{-i}) = 0$ in the main body.

Notice that in binary state situation, we can reorder the index of posteriors as below:

$$\frac{\mu_{\theta 11}}{\mu_{\theta 12}} \geq \frac{\mu_{\theta 21}}{\mu_{\theta 22}} \geq ... \geq \frac{\mu_{\theta K1}}{\mu_{\theta K2}} \text{ for all } \theta$$

Consider a subset of recommendation profile signal s_r in which $s_{rj} = a_1$ implies that $s_{ri} = a_1$ for all $i \leq j$. Denote the set of these recommendation profiles as $\mathcal{A}^*$. Relabel the recommendation profile such that $a^1 = (a_1, a_1, ..., a_1)$, $a^2 = (a_1, a_1, ..., a_2)$, ..., $a^{K+1} = (a_2, a_2, ..., a_2)$. So $\mathcal{A}^* =$

$\{a^1, ..., a^{K+1}\}$. A direct observation is that the cardinality of recommendation profile reduces to $|\mathcal{A}^*| = K + 1$.

$\mathcal{A}^*$ is actually the implementable signal set. If some signal s with $a_{rj} = a_1$ and $a_{ri} = a_2$ for some pair (i, j) where $i > j$ can be implemented for some type θ, then we have

$$\frac{\mu_{\theta j2}}{\mu_{\theta j1}} \leq \frac{\pi_{1k}}{\pi_{2k}} \leq \frac{\mu_{\theta i2}}{\mu_{\theta i1}},$$

which contradicts that the the order the index of posteriors implies the order of likelihood-ratio between the two state.

For $i = 1, ..., I$ and $j_1, j_2 \in \{1, ..., K+1\}, j_1 \neq j_2$, denote $a^{j_1} \xrightarrow{i} a^{j_2}$ and $E(a^{j_1} \xrightarrow{i} a^{j_2}) = (S, \pi')$ as the adjustment of $E = (S, \pi)$ as followed: $\pi'_{ij_1} = \pi_{ij_1} - \delta$, $\pi'_{ij_2} = \pi_{ij_2} + \delta$ and others unchanged. Considering the adjustment in obedient experiment E for agent n, denote the valuation change under $a^{j_1} \xrightarrow{i} a^{j_2} (i = 1, ..., I)$, $\Delta(a^{j_1} \xrightarrow{i} a^{j_2} \mid E, \theta) \triangleq \frac{V(E(a^{j_1} \xrightarrow{i} a^{j_2}), \theta) - V(E, \theta)}{\delta}$.[29]

Lemma 7 (Properties of Structural Welfare Adjustment)**.** *For any obedient experiment E for type θ, we have the following structural welfare adjustment properties:*

1. $\Delta(a^{j_1} \xrightarrow{i} a^{j_2} \mid E, \theta) = -\Delta(a^{j_2} \xrightarrow{i} a^{j_1} \mid E, \theta)$
2. *the following adjustment always brings constant welfare change*
   (a) $\Delta(a^{K+1} \xrightarrow{1} a^1 \mid E, \theta) = \mu_1$ *and* $\Delta(a^1 \xrightarrow{2} a^{K+1} \mid E, \theta) = \mu_2$
   (b) *there exists* $k^\star$ *such that* $\Delta(a^{K+1} \xrightarrow{1} a^{k^*+1} \mid E, \theta) + \Delta(a^1 \xrightarrow{2} a^{k^*} \mid E, \theta) = u_\theta$

Proof. Statement 1 is obvious because the two adjustments respectively recommend the same subset of posteriors of the type θ to choose converse actions with the same probability in the same state, which induces the converse welfare changes.

The first outcome of statement 2 is to "restore" the valuation with fully informative experiment/without information. Take the first adjustment as example, and the second adjustment is similar.

$$\Delta(a^{K+1} \xrightarrow{1} a^1 \mid E, \theta) \triangleq \frac{V(E(a^{K+1} \xrightarrow{i} a^1), \theta) - V(E, \theta)}{\delta} = \frac{(\sum_{k=1}^K \mu_{\theta k1} \Pr(\mu_{\theta k}) - 0)\delta}{\delta} = \sum_{k=1}^K \mu_{\theta k1} \Pr(\mu_{\theta k}) = \mu_1$$

The last equality comes from the splitting lemma.

We now prove the second outcome of statement 2. For all θ, there exists k^* such that

$$\frac{\mu_{\theta 11}}{\mu_{\theta 12}} \geq \frac{\mu_{\theta 21}}{\mu_{\theta 22}} \geq ... \geq \frac{\mu_{\theta k^* 1}}{\mu_{\theta k^* 2}} \geq 1 \geq \frac{\mu_{\theta k^*+11}}{\mu_{\theta k^*+12}} \geq \frac{\mu_{\theta K1}}{\mu_{\theta K2}}$$

Therefore

[29] In fact, the adjustment is dependent on δ, for simplicity, we omit it because it is useless for our analysis (or assume it is finitely small)

$$
\begin{aligned}
&\Delta(a^{K+1} \xrightarrow{1} a^{k^*+1} \mid E, \theta) + \Delta(a^1 \xrightarrow{2} a^{k^*} \mid E, \theta) \\
=&\frac{V(E(a^{K+1} \xrightarrow{1} a^{k^*+1}), \theta) - V(E, \theta) + V(E(a^1 \xrightarrow{2} a^{k^*}), \theta) - V(E, \theta)}{\delta} \\
=&\sum_{k=1}^{k^*} \mu_{\theta k1} \Pr(\mu_{\theta k}) + \sum_{k=k^*+1}^{K} \mu_{\theta k2} \Pr(\mu_{\theta k}) \\
=&u_\theta
\end{aligned}
$$

So we prove the second outcome. □

An experiment is non-dispersed if and only if $\pi_{1K+1} = 0$ and $\pi_{21} = 0$. Now we prove that the experiment in the optimal menu is always non-dispersed.

sts an experiment E for some type θ in the optimal menu which is not non-dispersed, i.e. $\pi_{1K+1} \neq 0$ or $\pi_{21} \neq 0$. Suppose $\pi_{1K+1} \neq 0$. Now consider the adjustment $\Delta(a^{K+1} \xrightarrow{1} a^1 \mid E, \theta)$ in this experiment. By lemma 7 it brings the net incremental value μ_1 to the type θ while for other types, the net incremental value is weakly less than μ_1 by Blackwell theorem. Meanwhile, the obedience is also more relaxing.

$$
\tfrac{\pi_{11}+\pi_{1K+1}}{\pi_{21}} \geq \tfrac{\pi_{11}}{\pi_{21}} \geq \tfrac{\mu_{\theta K2}}{\mu_{\theta K1}}, \qquad \tfrac{0}{\pi_{2K+1}} \leq \tfrac{\pi_{1K+1}}{\pi_{2K+1}} \leq \tfrac{\mu_{\theta 12}}{\mu_{\theta 11}}
$$

Therefore this adjustment allows the seller charge μ_1 more to the type θ without violating IC, IR and Obedience constraints. Similarily, we can prove that $\pi_{21} = 0$. Therefore the experiment in the optimal menu is always non-dispersed.

A.3 Proof of Lemma 3

Existence of Fully Informative Experiment. If the fully informative experiment $\overline{\mathrm{E}}$ does not lie in the optimal menu, then choose the one charged the highest fee and replace the experiment with $\overline{\mathrm{E}}$, the revenue gets a weakly better improvement.

Obedient for Both Agents. Denote the two type as type 1 and type 2. Suppose the experiment for type 2 is not obedient for type 1. There must exist some recommendation profile signal not obeyed by type 2. By lemma 2, the signal (a_1, a_1) and (a_2, a_2) are obedient for type 1. So the signal (a_1, a_2) is not obedient for type 1.

Suppose (a_1, a_2) actually induces type 1 to choose (a_1, a_1). By adjusting π_2 to $1 - \pi_2$ until the signal (a_1, a_2) is obedient, the designer can charge a strictly higher fee to type 2 without violating other constraints, which contracts the optimality of the menu. The case (a_2, a_2) is similar to (a_1, a_1).

B Proof of Theorem 1

Now the designer's problem is:

$$\max_{E,t_H,t_L} \tfrac{1}{2}\left(t_H + t_L\right)$$

s.t.

$$\begin{array}{ll}
V(\overline{E},1) - t_H \geqslant 0 & \text{(IR-H)} \\
V(E,2) - t_L \geqslant 0 & \text{(IR-L)} \\
V(\overline{E},1) - t_H \geqslant V(E,1) - t_L & \text{(IC-H)} \\
V(E,2) - t_L \geqslant V(\overline{E},2) - t_H & \text{(IC-L)} \\
\max\{\alpha'\pi_1 + \beta'\pi_2, \alpha\pi_1 + \beta\pi_2\} \leq \min\{\frac{1}{2}\pi_1, \frac{1}{2}\pi_2\} & \text{(Ob)} \\
\pi_1, \pi_2 \in [0,1] & \text{(Feasibility)}
\end{array}$$

It is not hard to see that $t_H \geq V(\overline{E},2) \geq t_L$ considering the optimality of the mechanism. Thus, the IC-L is always not binding. Then we can immediately derive that IR-L is binding. Let $T = t_H + V(E,2) - V(\overline{E},1)$, the designer's problem can be reduced as

$$\max_{E,T} \quad T$$

s.t.

$$\begin{array}{ll}
T \leqslant \alpha'\left(1-\pi_1\right) + \beta'\left(1-\pi_2\right) & \text{(IR-H)} \\
T \leqslant \left(2\alpha' - \alpha\right)\left(1-\pi_1\right) + \left(2\beta' - \beta\right)\left(1-\pi_2\right) & \text{(IC-H)} \\
\alpha\pi_1 + \beta\pi_2 \leq \min\{\frac{1}{2}\pi_1, \frac{1}{2}\pi_2\} & \text{(Ob-H)} \\
\alpha'\pi_1 + \beta'\pi_2 \leq \min\{\frac{1}{2}\pi_1, \frac{1}{2}\pi_2\} & \text{(Ob-L)} \\
\pi_1, \pi_2 \in [0,1] & \text{(Feasibility)}
\end{array}$$

Define

$$\begin{aligned}
\Delta &= \text{RHS(IR-H)} - \text{RHS(IC-H)} = (\alpha - \alpha')(1-\pi_1) + (\beta - \beta')(1-\pi_2) \\
\Gamma &= \text{LHS(Ob-H)} - \text{LHS(Ob-L)} = (\alpha - \alpha')\pi_1 + (\beta - \beta')\pi_2
\end{aligned}$$

where Δ measures the information rent for the H-type in this screening problem, Γ measures the difference of statistical error between the H-type and L-type.

Notice that when the two Types of statistical error of L-type are both smaller than the ones of H-type, i.e. $(\alpha,\beta) > (\alpha',\beta')$, the information rent can never be eliminated. Also, the overall statistical error of the H-type always exceeds the L-type. Otherwise when there exists one Type of the low type is higher than type-H, i.e. $\alpha < \alpha'$ or $\beta < \beta'$, the information rent can always be eliminated with proper allocation of (π_1,π_2). In this case, the optimal menu results in both IC-H and IR-H binding.

Case 1: $\alpha' \leq \alpha$ and $\beta' \leq \beta$. In this case, $\Delta, \Gamma \geq 0$. So in the optimal menu, (IC-H) is always binding. Moreover, (Ob-H) is always tighter than (Ob-L). The designer's problem is now

$$\max_{\pi_1,\pi_2} \left(2\alpha' - \alpha\right)\left(1-\pi_1\right) + \left(2\beta' - \beta\right)\left(1-\pi_2\right)$$

s.t.

$$\begin{array}{ll} \alpha\pi_1 + \beta\pi_2 \leq \min\{\frac{1}{2}\pi_1, \frac{1}{2}\pi_2\} & \text{(Ob-H)} \\ \pi_1, \pi_2 \in [0,1] & \text{(Feasibility)} \end{array}$$

Case 1.1: $2\alpha' \geq \alpha$ and $2\beta' \geq \beta$. All coefficients of π_1 and π_2 in the objective function is non-negative. So the optimal policy is $(\pi_1^*, \pi_2^*) = (1,1)$.

Case 1.2: $2\alpha' \leq \alpha$ and $2\beta' \leq \beta$. All coefficients of π_1 and π_2 in the objective function is non-positive. So the optimal policy is $(\pi_1^*, \pi_2^*) = (0,0)$.

Case 1.3: $2\alpha' > \alpha$ and $2\beta' \leq \beta$. The coefficient of π_1 is always non-positive, we can derive that $\alpha\pi_1 + \beta\pi_2 = \frac{1}{2}\pi_1$ in the optimal menu.

Now discuss the choice of optimal π_2

$$\max_{\pi_2} (2\alpha' - \alpha)(1 - \pi_1) + (2\beta' - \beta)(1 - \pi_2) = (2\alpha' - \alpha)(k^* - k_1)\pi_2$$

s.t.

$$0 \leq \pi_2 \leq 1, \quad 0 \leq k_1\pi_2 \leq 1$$

where $k_1 \equiv \frac{\beta}{\frac{1}{2}-\alpha}$ and $k^* \equiv \frac{2\beta'-\beta}{\alpha-2\alpha'}$.

Denote $F(\alpha', \beta') = \beta(2\alpha' - \alpha) - (\beta - 2\beta')(\frac{1}{2} - \alpha)$. $F(\alpha', \beta')$ measures the difference between k_1 and k^*. Notice that $F(\frac{\alpha}{2}, \frac{\beta}{2}) = 0$, $F(\frac{\alpha}{2}, 0) = -\beta(\frac{1}{2} - \alpha) < 0$ and $F(\alpha, \frac{\beta}{2}) = \alpha\beta > 0$ and $F(\alpha, 0) = 2\beta(\beta - \frac{1}{4})$. So in this region, the (α', β') is sold $(\pi_1^*, \pi_2^*) = (0,0)$ when $F(\alpha', \beta') > 0$ while sold $(\pi_1^*, \pi_2^*) = (k_1, 1) = (\frac{\beta}{\frac{1}{2}-\alpha}, 1)$ when $F(\alpha', \beta') \leq 0$, and both situations always exist.

Case 1.4: $2\alpha' < \alpha$ and $2\beta' \geq \beta$. This case is similar to case 1.3 and in this region, the (α', β') is sold either $(\pi_1^*, \pi_2^*) = (0,0)$ or $(\pi_1^*, \pi_2^*) = (1, \frac{\alpha}{\frac{1}{2}-\beta})$, and both situations always exist.[30]

Case 2: $\alpha \leq \alpha'$ and $\beta \geq \beta'$. The coefficients of π_1 in (IC-H) and (IR-H) are both negative. Given fixed level π_2, it is always profitable to reduce π_1 until (Ob-H) or (Ob-L) is binding. It is also easy to verify that $\pi_1 \leq \pi_2$.

We can find that

$$\Gamma = (\alpha - \alpha')\pi_1 + (\beta - \beta')\pi_2 \geq (\alpha - \alpha' + \beta - \beta')\pi_2 \geq 0$$

Therefore the optimal policy is $\alpha\pi_1 + \beta\pi_2 = \frac{1}{2}\pi_1$, or $\pi_1 = \frac{\beta}{\frac{1}{2}-\alpha}\pi_2 = k_1\pi_2$.

$$\max_{\pi_2, T} T$$

[30]Notice that $\alpha + \beta \leq \frac{1}{2}$ so at least one of $(\alpha, 0)$ and $(0, \beta)$ is sold a partially informative experiment $(\pi_1^*, \pi_2^*) = (\frac{\beta}{\frac{1}{2}-\alpha}, 1)$ or $(\pi_1^*, \pi_2^*) = (1, \frac{\alpha}{\frac{1}{2}-\beta})$.

s.t.

$$\begin{aligned}
&T \leqslant (\alpha' + \beta') - (k_1\alpha' + \beta')\,\pi_2 && \text{(IR-H)}\\
&T \leqslant (2\alpha' + 2\beta' - \alpha - \beta) + [(2\beta' - \beta) - k_1\,(2\alpha' - \alpha)]\pi_2 && \text{(IC-H)}\\
&k_1\pi_2, \pi_2 \in [0, 1] && \text{(Feasibility)}
\end{aligned}$$

Recall that $F(\alpha', \beta') = \beta(2\alpha' - \alpha) - (\beta - 2\beta')(\frac{1}{2} - \alpha)$. So in this region, the (α', β') is sold $(\pi_1^*, \pi_2^*) = (0, 0)$ when $F(\alpha', \beta') > 0$. When $F(\alpha', \beta') \leq 0$, notice that $\Delta =$ (IR-H)-(IC-H) $= (\alpha' - \alpha)\pi_1 + (\beta' - \beta)\pi_2 + (\alpha + \beta - \alpha' - \beta') = [(\beta' - \beta) + k_1(\alpha' - \alpha)]\pi_2 + (\alpha + \beta - \alpha' - \beta')$ is positive when $\pi_2 = 0$ while negative when $\pi_2 = 1$, by the continuity and linearity, the optimal π_2^* is the interior point in $[0, 1]$, i.e.

$$[(\beta' - \beta) + k_1(\alpha' - \alpha)]\pi_2 + (\alpha + \beta - \alpha' - \beta') = 0$$
$$\Rightarrow \pi_2^* = \tfrac{\alpha - \alpha' + \beta - \beta'}{\beta - \beta' + k_1(\alpha - \alpha')} \text{ and } \pi_1^* = k_1\pi_2^*$$

So the optimal policy is $(\pi_1^*, \pi_2^*) = (k_1\frac{\alpha - \alpha' + \beta - \beta'}{\beta - \beta' + k_1(\alpha - \alpha')}, \frac{\alpha - \alpha' + \beta - \beta'}{\beta - \beta' + k_1(\alpha - \alpha')})$ when $F(\alpha', \beta') \leq 0$, and $(\pi_1^*, \pi_2^*) = (0, 0)$ otherwise.

Case 3: $\alpha \geq \alpha'$ and $\beta \leq \beta'$. This case is similar to case 1.3 and in this region, the (α', β') is sold either $(\pi_1^*, \pi_2^*) = (0, 0)$ or $(\pi_1^*, \pi_2^*) = (k_2\frac{\alpha - \alpha' + \beta - \beta'}{\beta - \beta' + k_2(\alpha - \alpha')}, \frac{\alpha - \alpha' + \beta - \beta'}{\beta - \beta' + k_2(\alpha - \alpha')})$.

C Proof of Theorem 2 and Lemmas in Section 4

C.1 Proof of Lemma 4

The First Part of Lemma 4

For any buyer α, the obedient zone for the designed $\frac{\pi_1(\alpha)}{\pi_2(\alpha)}$ to be obedient for α is $[\frac{\beta}{\frac{1}{2} - \alpha}, \frac{\frac{1}{2} - \beta}{\alpha}] = [\frac{m - k\alpha}{\frac{1}{2} - \alpha}, \frac{\frac{1}{2} - m + k\alpha}{\alpha}] := [k_1(\alpha), k_2(\alpha)]$. Notice that $k_2(\alpha)$ is decreasing with respect to α since $k_2(\alpha)$ can be expressed as $\frac{\frac{1}{2} - m}{\alpha} + k$. Then, notice that $k_1(\alpha)$ is non-decreasing with respect to α for the derivative $k_1'(\alpha) = \frac{2m - k}{2(\frac{1}{2} - \alpha)^2} \geq 0$. As a result, the obedient zone $[k_1(\alpha), k_2(\alpha)]$ for a given buyer α is strictly shrunk as α increases. Thus, the obedient $E(\alpha)$ designed for α is always obedient for any $\alpha' \geq \alpha$.

Now, we aim to prove that, in the optimal menu, $E(\alpha)$ designed for a given buyer α should satisfied that $\frac{\pi_1(\alpha)}{\pi_2(\alpha)}$ in the optimal obedient zone $[1, k_2(\alpha)]$. To see this, suppose that the designed $E(\alpha)$ satisfied $\frac{\pi_1(\alpha)}{\pi_2(\alpha)} < 1$ in an optimal menu. Then, for any α', if $E(\alpha)$ is obedient for α', the value of $E(\alpha)$ is $V(E(\alpha), \alpha') = (m - k\alpha') + \alpha' - (m - k\alpha')\pi_2 - \alpha'\pi_1$; otherwise, $V(E(\alpha), \alpha') = (m - k\alpha') + \alpha' - \frac{1}{2}\pi_1$.

Consider an adjustment of $E(\alpha)$ to $E'(\alpha)$ where $(\pi_1'(\alpha), \pi_2'(\alpha)) = (\pi_1(\alpha) - k\pi_0, \pi_2(\alpha) - \pi_0)$ and $\pi_0 > 0$ is sufficiently small. Then, for any α', if $E(\alpha)$ is obedient for α', the value of $E(\alpha)$ increase

$\Delta V_r = (m - k\alpha')\pi_0 + \alpha' k\pi_0 = m\pi_0$; otherwise, the value of $E(\alpha)$ increase $\Delta V_n = \frac{1}{2}k\pi_0$. Since $k \leq 2m$, $\Delta V_n \leq \Delta V_r$. Thus, this adjustment can increase the fees charging to α without violating the IC conditions. Finally, we give two subtle illustration about the validity of this adjustment.

First, we check validity of the adjusted $(\pi_1'(\alpha), \pi_2'(\alpha))$. Since we suppose $\frac{\pi_1(\alpha)}{\pi_2(\alpha)} < 1$, $\pi_2(\alpha) > 0$ and $0 < \pi_1(\alpha) < 1$ holds. Thus, it is valid to set $\pi_0 < \min\{\pi_2(\alpha), \frac{\pi_1(\alpha)}{k}\}$.

Second, we check the adjusted $E'(\alpha)$ is still obedient for α when $\pi_0 > 0$ is sufficiently small. A sufficient condition is that $\frac{\pi_1(\alpha) - k\pi_0}{\pi_2(\alpha) - \pi_0} \geq \frac{\pi_1(\alpha)}{\pi_2(\alpha)}$ for π_0 makes $(\pi_1'(\alpha), \pi_2'(\alpha))$ valid. It is equivalent to prove that $\pi_1(\alpha) - k\pi_2(\alpha) \geq 0$. It holds since $\pi_1(\alpha) - k\pi_2(\alpha) = \pi_1(\alpha)[1 - k\frac{\pi_2(\alpha)}{\pi_1(\alpha)}] \geq \pi_1(\alpha)[1 - k\frac{\frac{1}{2} - \alpha}{m - k\alpha}] = \frac{\pi_1(\alpha)}{2(m - k\alpha)}(2m - k) \geq 0$.

The Second Part of Lemma 4

By the first part of 4, $\pi_1(\alpha) - k\pi_2(\alpha) > 0$ when $\pi_1(\alpha) \neq 0$ in the optimal menu. Then, $\lambda(\alpha) \in [\alpha, \bar{\alpha}]$ since $\alpha' \leq \lambda(\alpha)$ is equivalent to $\frac{\pi_1(\alpha)}{\pi_2(\alpha)} \geq k_2(\alpha')$. $\lambda(\alpha)$ is to identify the threshold of the obedience of E_α. The experiment E_α is obedient for $\alpha' \in [0, \lambda(\alpha)]$, and pools the recommendation profile (a_1, a_2) with (a_2, a_2) for $\alpha' \in [\lambda(\alpha), \overline{\alpha}]$. Moreover, the obedience of α is binding if and only if $\lambda(\alpha) = \alpha$.

We first prove that if $E_{\alpha^*} = \overline{E}$ for some $\alpha^* \in \mathcal{A}$, then for all $\alpha > \alpha^*$, $E_\alpha = \overline{E}$. Suppose there exists $\alpha > \alpha^*$, $E_\alpha \neq \overline{E}$. Then considering the IC$[\alpha \to \alpha^*]$ and IC$[\alpha^* \to \alpha]$.

IC$[\alpha \to \alpha^*]$

$$-(m - k\alpha)\pi_2(\alpha) - \alpha\pi_1(\alpha) - t_\alpha \geq -t_{\alpha^*}$$
$$t_{\alpha^*} - t_\alpha \geq m\pi_2(\alpha) + \alpha(\pi_1(\alpha) - k\pi_2(\alpha))$$

IC$[\alpha^* \to \alpha]$

$$-t_{\alpha^*} \geq -(m - k\alpha^*)\pi_2(\alpha) - \alpha^*\pi_1(\alpha) - t_\alpha$$
$$m\pi_2(\alpha) + \alpha^*(\pi_1(\alpha) - k\pi_2(\alpha)) \geq t_{\alpha^*} - t_\alpha$$

Combining the two equations and we get $(\alpha^* - \alpha)(\pi_1(\alpha) - k\pi_2(\alpha)) \geq 0$, which implies that $\pi_1(\alpha) = \pi_2(\alpha) = 0$ since $\alpha > \alpha^*$. Thus, $E_\alpha = \overline{E}$, a contradiction.

By the existence of the fully informative experiment, the set $I_{\overline{E}}$ defined as $\{\alpha | E_\alpha = \overline{E}\}$ is non-empty. Define $\alpha^* = \inf I_{\overline{E}}$. We need then derive the closeness of $I_{\overline{E}}$, which means $\alpha^* = \min I_{\overline{E}}$. We complete the proof of closeness after proving the statement (a).

Then we prove that $\alpha < \lambda(\alpha)$ for all $\alpha < \alpha^*$. Suppose there exist $\alpha < \alpha^*$ whose obedience constraint is binding, i.e $\alpha = \lambda(\alpha) = \frac{(\frac{1}{2} - m)\pi_2(\alpha)}{\pi_1(\alpha) - k\pi_2(\alpha)}$. Therefore, $V(E_\alpha, \alpha) = (m - k\alpha) + \alpha - (m - k\alpha)\pi_2(\alpha) - \alpha\pi_1(\alpha) = (m - k\alpha) + \alpha - \frac{1}{2}\pi_2(\alpha)$ and $V(E_\alpha, \alpha') = (m - k\alpha') + \alpha - \frac{1}{2}\pi_2(\alpha)$ for any $\alpha' \in (\alpha, \alpha^*)$. Consider the IC$[\alpha \to \alpha']$ and IC$[\alpha' \to \alpha]$.

IC$[\alpha \to \alpha']$

$$-\tfrac{1}{2}\pi_2(\alpha) - t_\alpha \geq -(m-k\alpha)\pi_2(\alpha') - \alpha\pi_1(\alpha') - t_{\alpha'}$$
$$t_{\alpha'} - t_\alpha \geq \tfrac{1}{2}\pi_2(\alpha) - (m-k\alpha)\pi_2(\alpha') - \alpha\pi_1(\alpha')$$

IC$[\alpha' \to \alpha]$

$$-(m-k\alpha')\pi_2(\alpha') - \alpha'\pi_1(\alpha') - t_{\alpha'} \geq -\tfrac{1}{2}\pi_2(\alpha) - t_\alpha$$
$$\tfrac{1}{2}\pi_2(\alpha) - (m-k\alpha')\pi_2(\alpha') - \alpha'\pi_1(\alpha') \geq t_{\alpha'} - t_\alpha$$

Combining the two equations and we get $(\alpha-\alpha')(\pi_1(\alpha') - k\pi_2(\alpha')) \geq 0$, which implies that $\pi_1(\alpha) = \pi_2(\alpha) = 0$ since $\alpha < \alpha'$. Thus, $E_{\alpha'} = \overline{E}$, which contradicts that $\alpha' < \alpha^*$.

Motivated by the proof of $\frac{\pi_1(\alpha)}{\pi_2(\alpha)} \geq 1$, we keep the same adjustment to $E(\alpha)$. This adjustment is valid for the same reason in the previous proof. For such a valid adjustment, $\Delta V_r = m\pi_0$ and $\Delta V_r = \frac{1}{2}\pi_0$. Since $m < \frac{1}{2}$, $\Delta V_n \geq \Delta V_r$. Thus, for an optimal menu, there exists $\alpha' > \lambda(\alpha)$, IC $[\alpha' \to \alpha]$ is binding, since otherwise the valid adjustment can be a improvement for the designer, a contradiction for the optimality.

Finally, we can complete the proof of closeness in statement (b), which means $\alpha^* = \min I_{\overline{E}}$. If $\lambda(\alpha^*) > \alpha^*$, then there exist $\alpha' > \alpha^*$, IC$[\alpha' \to \alpha^*]$ is binding, which means that $-\bar{t} = -\frac{1}{2}\pi_2(\alpha^*) - t_{\alpha^*}$, where $\bar{t}$ is the associated tariff of those sold the fully informative one. For some $\alpha^* < \hat{\alpha} < \lambda(\alpha^*)$, IC$[\hat{\alpha} \to \alpha^*]$ implies that $-\bar{t} \geq -\hat{\alpha}\pi_1(\alpha^*) - (m-k\hat{\alpha})\pi_2(\alpha^*) - t_{\alpha^*}$, i.e. $-\frac{1}{2}\pi_2(\alpha^*) \geq -\hat{\alpha}\pi_1(\alpha^*) - (m-k\hat{\alpha})\pi_2(\alpha^*)$. Then we have $\lambda(\alpha^*) \leq \hat{\alpha}$, a contradiction.

If $\lambda(\alpha^*) = \alpha^*$, for $\alpha' > \alpha^*$, IC$[\alpha' \to \alpha^*]$ implies that $-\bar{t} \geq -\frac{1}{2}\pi_2(\alpha^*) - t_{\alpha^*}$, while IC$[\alpha^* \to \alpha']$ implies that $-\frac{1}{2}\pi_2(\alpha^*) - t_{\alpha^*} \geq -\bar{t}$, which means that $t_{\alpha^*} + \frac{1}{2}\pi_2(\alpha^*) = \bar{t}$. Therefore α^* is indifferent between the menu of her own and those of α where $\alpha > \alpha^*$. The designer can strictly increase her revenue by replacing the menu of α^* to the fully informative one, because $t_{\alpha^*} < \bar{t}$, without violating other conditions.

C.2 Proof of Lemma 5

If $\alpha \in [\alpha^*, \overline{\alpha}]$, all conclusions trivially hold. Now we discuss that $\alpha \in [\underline{\alpha}, \alpha^*)$ where obedience is not binding a.e, i.e. $\lambda(\alpha) > \alpha$.

A Local Monotonicity Property of $\pi(\alpha)$

We first prove that $\pi(\alpha) \geq \pi(\hat{\alpha})$, for $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$. E_α is obedient for $\hat{\alpha}$. Since $\hat{\alpha} > \alpha$, $E_{\hat{\alpha}}$ is obedient for α.

IC$[\alpha \to \hat{\alpha}]$

$$-\alpha\pi_1(\alpha) - (m-k\alpha)\pi_2(\alpha) - t_\alpha \geq -\alpha\pi_1(\hat{\alpha}) - (m-k\alpha)\pi_2(\hat{\alpha}) - t_{\hat{\alpha}}$$
$$t_{\hat{\alpha}} - t_\alpha \geq \alpha[(\pi_1(\alpha) - k\pi_2(\alpha)) - (\pi_1(\hat{\alpha}) - k\pi_2(\hat{\alpha}))] + m[\pi_2(\alpha) - \pi_2(\hat{\alpha})]$$

IC[$\hat{\alpha} \to \alpha$]

$$-\hat{\alpha}\pi_1(\hat{\alpha}) - (m - k\hat{\alpha})\pi_2(\hat{\alpha}) - t_{\hat{\alpha}} \geq -\hat{\alpha}\pi_1(\alpha) - (m - k\hat{\alpha})\pi_2(\alpha) - t_\alpha$$
$$\hat{\alpha}[(\pi_1(\alpha) - k\pi_2(\alpha)) - (\pi_1(\hat{\alpha}) - k\pi_2(\hat{\alpha}))] + m[\pi_2(\alpha) - \pi_2(\hat{\alpha})] \geq t_{\hat{\alpha}} - t_\alpha$$

By IC[$\hat{\alpha} \to \alpha$] and IC[$\alpha \to \hat{\alpha}$], we derive that

$$(\hat{\alpha} - \alpha)[\pi(\alpha) - \pi(\hat{\alpha})] \geq 0.$$

Therefore $\pi(\alpha) \geq \pi(\hat{\alpha})$.

The First Part of Lemma 5

Part 1 $\lambda(\hat{\alpha}) \leq \gamma(\alpha)$

For $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$, if $\lambda(\hat{\alpha}) > \gamma(\alpha) > \hat{\alpha}$, then $E_{\hat{\alpha}}$ is obedient for $\gamma(\alpha)$, we have

$$\gamma(\hat{\alpha}) > \lambda(\hat{\alpha}) > \gamma(\alpha) > \lambda(\alpha) \geq \hat{\alpha} > \alpha$$

Therefore, both the obedience of α and $\hat{\alpha}$ are not binding while the IC[$\gamma(\alpha) \to \alpha$] and IC[$\gamma(\hat{\alpha}) \to \hat{\alpha}$] are binding.

IC[$\gamma(\alpha) \to \alpha$] binding and IC[$\gamma(\alpha) \to \hat{\alpha}$]

$$V(E_\alpha, \gamma(\alpha)) - t_\alpha \geq V(E_{\hat{\alpha}}, \gamma(\alpha)) - t_{\hat{\alpha}}$$
$$-\tfrac{1}{2}\pi_2(\alpha) - t_\alpha \geq -\gamma(\alpha)\pi_1(\hat{\alpha}) - (m - k\gamma(\alpha))\pi_2(\hat{\alpha}) - t_{\hat{\alpha}}$$
$$t_{\hat{\alpha}} - t_\alpha \geq \tfrac{1}{2}\pi_2(\alpha) - \gamma(\alpha)\pi_1(\hat{\alpha}) - (m - k\gamma(\alpha))\pi_2(\hat{\alpha})$$

IC[$\gamma(\hat{\alpha}) \to \hat{\alpha}$] is binding and IC[$\gamma(\hat{\alpha}) \to \alpha$] (Moreover, E_α is not obedient for $\gamma(\hat{\alpha})$)

$$V(E_{\hat{\alpha}}, \gamma(\hat{\alpha})) - t_{\hat{\alpha}} \geq V(E_\alpha, \gamma(\hat{\alpha})) - t_\alpha$$
$$-\tfrac{1}{2}\pi_2(\hat{\alpha}) - t_{\hat{\alpha}} \geq -\tfrac{1}{2}\pi_2(\alpha) - t_\alpha$$
$$\tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}) \geq t_{\hat{\alpha}} - t_\alpha$$

Combining the two equations above, we have

$$\tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}) \geq \tfrac{1}{2}\pi_2(\alpha) - \gamma(\alpha)\pi_1(\hat{\alpha}) - (m - k\gamma(\alpha))\pi_2(\hat{\alpha})$$
$$\gamma(\alpha)\pi(\hat{\alpha}) - (\tfrac{1}{2} - m)\pi_2(\hat{\alpha}) \geq 0$$
$$\gamma(\alpha) \geq \lambda(\hat{\alpha})$$

which contradicts that $\gamma(\alpha) < \lambda(\hat{\alpha})$.

Therefore $\lambda(\hat{\alpha}) \in [\hat{\alpha}, \gamma(\alpha)]$, or $E_{\hat{\alpha}}$ is not obedient for $\gamma(\alpha)$.

Part 2 $\lambda(\hat{\alpha}) \geq \lambda(\alpha)$

For $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$, when $k > 0$ or $\pi_2(\alpha) < 1$ holds, if there exists $\alpha < \hat{\alpha} \leq \lambda(\alpha)$, $\lambda(\hat{\alpha}) < \lambda(\alpha)$, consider the following mutual IC.

IC[$\gamma(\alpha) \to \alpha$] binding and IC[$\gamma(\alpha) \to \hat{\alpha}$] ($E_{\hat{\alpha}}$ is not obedient for $\gamma(\alpha)$ since $\lambda(\hat{\alpha}) < \gamma(\alpha)$)

$$
\begin{aligned}
V(E_{\alpha}, \gamma(\alpha)) - t_{\alpha} &\geq V(E_{\hat{\alpha}}, \gamma(\alpha)) - t_{\hat{\alpha}} \\
-\tfrac{1}{2}\pi_2(\alpha) - t_{\alpha} &\geq -\tfrac{1}{2}\pi_2(\hat{\alpha}) - t_{\hat{\alpha}} \\
t_{\hat{\alpha}} - t_{\alpha} &\geq \tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha})
\end{aligned}
$$

IC[$\hat{\alpha} \to \alpha$]

$$
\hat{\alpha}[\pi(\alpha) - \pi(\hat{\alpha})] + m[\pi_2(\alpha) - \pi_2(\hat{\alpha})] \geq t_{\hat{\alpha}} - t_{\alpha}
$$

Combining the two inequalities, we have

$$
\begin{aligned}
\hat{\alpha}[\pi(\alpha) - \pi(\hat{\alpha})] + m[\pi_2(\alpha) - \pi_2(\hat{\alpha})] &\geq \tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}) \\
\hat{\alpha}\,[\pi(\alpha) - \pi(\hat{\alpha})] &\geq (\tfrac{1}{2} - m)\,[\pi_2(\alpha) - \pi_2(\hat{\alpha})] \\
\hat{\alpha}[\pi(\alpha) - \pi(\hat{\alpha})] &\geq \lambda(\alpha)\pi(\alpha) - \lambda(\hat{\alpha})\pi(\hat{\alpha}) \\
[\hat{\alpha} - \lambda(\hat{\alpha})][\pi(\alpha) - \pi(\hat{\alpha})] + [\lambda(\hat{\alpha}) - \lambda(\alpha)]\pi(\alpha) &\geq 0
\end{aligned}
$$

By $\hat{\alpha} < \lambda(\hat{\alpha})$, $\lambda(\hat{\alpha}) < \lambda(\alpha)$ and $\pi(\alpha) \geq \pi(\hat{\alpha})$ since $\alpha < \hat{\alpha}$, we have

$$
0 > [\hat{\alpha} - \lambda(\hat{\alpha})][\pi(\alpha) - \pi(\hat{\alpha})] + [\lambda(\hat{\alpha}) - \lambda(\alpha)]\pi(\alpha) \geq 0,
$$

which is impossible.

Therefore, $\lambda(\hat{\alpha}) \geq \lambda(\alpha)$ for $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$. This completes the proof of statement 1.

The Second Part of Lemma 5

Considering $\alpha < \hat{\alpha} < \hat{\alpha}' < \lambda(\alpha)$, by the **Local Monotonicity Property of** $\pi(\alpha)$, we learn that $\pi(\alpha) \geq \pi(\hat{\alpha}), \forall \hat{\alpha} \in [\alpha, \lambda(\alpha)]$. Reuse this result on $\hat{\alpha}$ to get $\pi(\hat{\alpha}) \geq \pi(\hat{\alpha}')$.

Now, we show that for any $\alpha < \alpha^*$, there exists $\alpha' < \alpha$ such that $\lambda(\alpha') \geq \alpha$.

Otherwise, there exists $\alpha < \alpha^*$, for any $\alpha' < \alpha$, $\lambda(\alpha') < \alpha$. Then there exists $\alpha_n \to \alpha$ such that the limit menu $(\lim_{\alpha_n \to \alpha} E(\alpha_n), \lim_{\alpha_n \to \alpha} t_{\alpha_n}) := (E^l_{\alpha}, t^l_{\alpha})$ exists. A transform from $(E(\alpha), t_{\alpha})$ to $(E^l_{\alpha}, t^l_{\alpha})$ in buyer α's menu maintaining his net utility will never break any IC constraints, as the adjusted IC[$\alpha'' \to \alpha$] holds as the limit of the original IC[$\alpha'' \to \alpha_n$] for any $\alpha'' \in \mathcal{A}$. However, by $\alpha > \lambda(\alpha_n) > \alpha_n$, as $\alpha_n \to \alpha$, we learn that $\lambda(E^l_\alpha) = \lim_{\alpha_n \to \alpha} \lambda(\alpha_n) = \alpha$ where $\lambda(E^l_\alpha)$ defines as the value of λ function in adjusted menu for α. Thus, in the whole adjusted menu, $\lambda(\alpha) = \alpha$ for a $\alpha < \alpha^*$, which contradicts to Lemma 4.

Thus, intervals $[\alpha, \lambda(\alpha)]$ is transitive on $[\underline{\alpha}, \alpha^*)$. By the monotonicity on intervals $[\alpha, \lambda(\alpha)]$, we can get the global monotonicity on interval $[\underline{\alpha}, \alpha^*)$. Since $\lambda(\alpha) = \bar{\alpha}$ and $\pi(\alpha) = 0$ for $\alpha \in [\alpha^*, \bar{\alpha}]$, we then get the global monotonicity on interval $[\underline{\alpha}, \bar{\alpha}]$.

C.3 Optimality of Tiered Pricing Mechanism

We can characterize derive a sharper prediction about the structure of the optimal mechanism. A mechanism is called tiered pricing mechanism if it implements the policy where the type space $\mathcal{A}$ is partitioned into intervals or singleton $\{I_d\}_{d \in \mathcal{D}}$, and in every partition set, the all types share the same menu, i.e. $(E_\alpha, t_\alpha) = (\mathrm{E}_{\alpha'}, t_{\alpha'})$ for all $\alpha, \alpha' \in I_d$.

Lemma 8 (Structure of the Optimal Mechanism)**.** *1. $\lambda(\alpha) : \mathcal{A} \to \mathcal{A}$ is non-decreasing and thereby can be decomposed into $\lambda(\alpha) = \sum_{d \in \mathcal{D}} c_d \mathbb{I}_{\alpha \geq \alpha_d}$ with partition index set $\mathcal{D}$.*

2. The optimal mechanism is a tiered pricing mechanism with the partition set $\mathcal{D}$ above.

Proof. **The First Part of Lemma 8**

Considering $\alpha < \hat{\alpha} < \hat{\alpha}' < \lambda(\alpha)$, by the result of **Part 2** in the **the first part of lemma 5**, we learn that $\lambda(\hat{\alpha}) < \lambda(\hat{\alpha}')$ as $\hat{\alpha} < \hat{\alpha}' < \lambda(\hat{\alpha})$ by reusing that result on $\hat{\alpha}$. $[\alpha, \lambda(\alpha)]$ is transitive on $[\underline{\alpha}, \alpha^*)$. By the same argument in the proof of **the second part of lemma 5**, we then get the global monotonicity on interval $[\underline{\alpha}, \bar{\alpha}]$.

The Second Part of Lemma 8

It is equivalent to show that, if $\lambda(\alpha) = \lambda(\alpha')$ for some $\alpha' > \alpha$, then for all $\hat{\alpha} \in [\alpha, \alpha']$, $\hat{\alpha}$ share the same menu with α, i.e. $(\pi_1(\hat{\alpha}), \pi_2(\hat{\alpha}), t_{\hat{\alpha}}) = (\pi_1(\alpha), \pi_2(\alpha), t_\alpha)$.

By $\lambda(\alpha) = \lambda(\alpha')$ and $\lambda(\hat{\alpha})$ is non-decreasing for $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$ by lemma **??**, $\lambda(\hat{\alpha}) = \lambda(\alpha) = \lambda(\alpha') < \gamma(\alpha)$ for $\hat{\alpha} \in [\alpha, \alpha']$. We know that both E_α and $E_{\hat{\alpha}}$ are obedient for both α and $\hat{\alpha}$, and both not obedient for $\gamma(\alpha)$.

IC$[\gamma(\alpha) \to \alpha]$ binding and IC$[\gamma(\alpha) \to \hat{\alpha}]$

$$V(E_\alpha, \gamma(\alpha)) - t_\alpha \geq V(E_{\hat{\alpha}}, \gamma(\alpha)) - t_{\hat{\alpha}}$$
$$t_{\hat{\alpha}} - t_\alpha \geq \tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha})$$

IC$[\hat{\alpha} \to \alpha]$

$$V(E_{\hat{\alpha}}, \hat{\alpha}) - t_{\hat{\alpha}} \geq V(E_\alpha, \hat{\alpha}) - t_\alpha$$
$$\hat{\alpha}[\pi(\alpha) - \pi(\hat{\alpha})] + m[\pi_2(\alpha) - \pi_2(\hat{\alpha})] \geq t_{\hat{\alpha}} - t_\alpha$$

Combining the two inequalities, we have

$$[\hat{\alpha} - \lambda(\hat{\alpha})][\pi(\alpha) - \pi(\hat{\alpha})] + [\lambda(\hat{\alpha}) - \lambda(\alpha)]\pi(\alpha) \geq 0$$
$$[\hat{\alpha} - \lambda(\hat{\alpha})][\pi(\alpha) - \pi(\hat{\alpha})] \geq 0$$

So we have $\pi(\hat{\alpha}) = \pi(\alpha)$ for all $\hat{\alpha}$. By $\lambda(\hat{\alpha}) = \lambda(\alpha)$, we have $\pi_1(\hat{\alpha}) = \pi_1(\alpha)$ and then $\pi_2(\hat{\alpha}) = \pi_2(\alpha)$. Revisiting the above IC conditions, we can further derive that $t_{\hat{\alpha}} = t_\alpha$.

□

C.4 Proof of Lemma 6

Denote $V(\alpha) = V(E_\alpha, \alpha) - t_\alpha$ as the net value of type α.

Necessity

With IC, IR and obedience,

Statement 1:

We first prove that for all $\hat{\alpha} \in [\alpha, \lambda(\alpha))$,

$$\tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}) = t_{\hat{\alpha}} - t_\alpha$$

By $\gamma(\hat{\alpha}) \geq \lambda(\hat{\alpha}) > \lambda(\alpha) > \hat{\alpha}$, IC$[\gamma(\hat{\alpha}) \to \alpha]$ is binding, combining with IC$[\gamma(\hat{\alpha}) \to \hat{\alpha}]$:

$$V(E_{\hat{\alpha}}, \gamma(\hat{\alpha})) - t_{\hat{\alpha}} \geq V(E_\alpha, \gamma(\hat{\alpha})) - t_\alpha$$
$$-\tfrac{1}{2}\pi_2(\hat{\alpha}) - t_{\hat{\alpha}} \geq -\tfrac{1}{2}\pi_2(\alpha) - t_\alpha$$
$$\tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}) \geq t_{\hat{\alpha}} - t_\alpha$$

IC$[\gamma(\alpha) \to \alpha]$ binding and IC$[\gamma(\alpha) \to \hat{\alpha}]$

$$\tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}) \leq t_{\hat{\alpha}} - t_\alpha$$

Therefore, for all $\hat{\alpha} \in [\alpha, \lambda(\alpha))$,

$$\tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}) = t_{\hat{\alpha}} - t_\alpha$$

Now we prove that for all $\alpha, \hat{\alpha} \in \mathcal{A}$, $\alpha < \hat{\alpha}$,

$$\tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}) = t_{\hat{\alpha}} - t_\alpha$$

By the proof above, we can also get that for any $\alpha \le \hat{\alpha}' < \lambda(\alpha)$,

$$\tfrac{1}{2}\pi_2(\alpha) - \tfrac{1}{2}\pi_2(\hat{\alpha}') = t_{\hat{\alpha}'} - t_\alpha$$

Therefore, for all $\alpha \le \hat{\alpha} < \hat{\alpha}' < \lambda(\alpha)$,

$$\tfrac{1}{2}\pi_2(\hat{\alpha}') - \tfrac{1}{2}\pi_2(\hat{\alpha}) = t_{\hat{\alpha}} - t_{\hat{\alpha}'}$$

By that $\lambda(\alpha) > \alpha$ a.e. in $\mathcal{A}$, this relation can be transitive across different $[\alpha, \lambda(\alpha))$, $\alpha < \alpha^*$. Therefore, $\frac{1}{2}\pi_2(\alpha) + t_\alpha$ is a constant when $\alpha < \alpha^*$.

Now, consider an arbitrary $\alpha < \alpha^*$, IC$[\alpha^* \to \alpha]$ tells that $-t_{\alpha^*} \ge -\frac{1}{2}\pi_2(\alpha) - t_\alpha$ while IC$[\gamma(\alpha) \to \alpha]$ binding combining with IC$[\gamma(\alpha) \to \alpha^*]$ tells that $-\frac{1}{2}\pi_2(\alpha) - t_\alpha \ge -t_{\alpha^*}$. Thus, for all $\alpha \in \mathcal{A}$, $\frac{1}{2}\pi_2(\alpha) + t_\alpha = \bar{t}$.

Statement 2: With statement 1, we can reduce the net value function into one dimension.

$$\begin{aligned}
V(\alpha) &= \alpha + (m - k\alpha) - \alpha\pi_1(\alpha) - (m - k\alpha)\pi_2(\alpha) - t_\alpha \\
&= (1-k)\alpha + m - m\pi_2(\alpha) - \alpha\pi(\alpha) - t_\alpha \\
&= (1-k)\alpha + m - 2m(\bar{t} - t_\alpha) - \alpha\pi(\alpha) - t_\alpha \\
&= \alpha(1 - k - \pi(\alpha)) - (1-2m)t_\alpha + m(1 - 2\bar{t})
\end{aligned}$$

By the optimal structure of the menu, for any $\alpha \in [\underline{\alpha}, \alpha^*)$, there always exist ϵ, $E_{\alpha'}$ is always obedient for α, $\alpha' \in (\alpha - \epsilon, \alpha + \epsilon)$. $V(\pi_2, \alpha)$ is differentiable and absolutely continuous for on $\alpha \in [\underline{\alpha}, \alpha^*)$[31]. By the envelope theorem (Milgrom and Segal (2002), Sinander (2022)), for all such α,

$$V(\alpha) = V(\underline{\alpha}) + \textstyle\int_{\underline{\alpha}}^{\alpha} V_t(\pi, t)dt = V(\underline{\alpha}) + \int_{\underline{\alpha}}^{\alpha} (1 - k - \pi(t))dt$$

Statement 3 and 4: Statement 3 is from lemma **??**; Statement 4 trivialy holds when all IR conditions hold.

Sufficiency

Construct $t_\alpha = \frac{\alpha}{1-2m}(1 - k - \pi(\alpha)) + t_{\underline{\alpha}} - \frac{\int_{\underline{\alpha}}^{\alpha}(1-k-\pi(t))dt}{1-2m}$.

Incentive Compatibility

i.one-step deviation IC

IC$[\alpha \to \alpha']$ where $E_{\alpha'}$ is obedient for α

[31] Here only for α^*, the differentiability of $V(\alpha)$ may not hold when $\lambda(\alpha) = \alpha^*$ for $\alpha \in (\alpha - \epsilon, \alpha)$ for some ϵ. With Therefore, we omit the tedious description of this situation which does not impair our conclusion.

$$
\begin{aligned}
& V\left(E_{\alpha}, \alpha\right)-t_{\alpha}-V\left(E_{\alpha'}, \alpha\right)+t_{\alpha'} \\
= & -\alpha \pi_1(\alpha)-(m-k\alpha) \pi_2(\alpha)-t_{\alpha}+\alpha \pi_1(\alpha')+(m-k\alpha) \pi_2(\alpha')+t_{\alpha'} \\
= & -\alpha \pi(\alpha)-m \pi_2(\alpha)-t_{\alpha}+\alpha \pi(\alpha')+m \pi_2(\alpha')-t_{\alpha'} \\
= & \ \alpha[\pi(\alpha')-\pi(\alpha)]+(1-2m)(t_{\alpha'}-t_{\alpha}) \\
= & \ \alpha[\pi(\alpha')-\pi(\alpha)]+[\alpha'(1-k-\pi(\alpha'))-\alpha(1-k-\pi(\alpha))+\int_{\alpha'}^{\alpha}(1-k-\pi(t))dt] \\
= & \ (\alpha-\alpha')\pi(\alpha')-\int_{\alpha'}^{\alpha}\pi(t)dt \\
\geqslant & \ 0
\end{aligned}
$$

ii.double deviation IC

IC$[\alpha \rightarrow \alpha']$ where $E_{\alpha'}$ is not obedient for α

$$
\begin{aligned}
V\left(E_{\alpha}, \alpha\right)-t_{\alpha}-V\left(E_{\alpha'}, \alpha\right)+t_{\alpha'} & =-\alpha \pi(\alpha)-2m(\bar{t}-t_{\alpha})-t_{\alpha}+\frac{1}{2}\pi_2(\alpha')+t_{\alpha'} \\
& =-\alpha\pi(\alpha)-(1-2m)t_{\alpha}+(1-2m)\bar{t} \\
& =(1-2m)(\bar{t}-t_{\alpha})-\alpha\pi(\alpha) \\
& =(\frac{1}{2}-m)\pi_2(\alpha)-\alpha\pi(\alpha) \\
& =\pi(\alpha)[\lambda(\alpha)-\alpha] \\
& \geqslant 0.
\end{aligned}
$$

Individual Rationality

The IR constraints can be written as $V(\alpha) \geq 0$. From statement 2 and 3, we know that $V(\alpha) = \int_0^{\alpha}(1-k-\pi(t))dt + V(\underline{\alpha})$, which is increasing with respect to α if $\pi(\alpha) \leq 1-k, \forall\alpha$; otherwise, $V(\alpha)$ get its minimum at $\hat{\alpha} = \inf\{\alpha|\pi(\alpha) \leq 1-k\}$. As a result, statement 4 guarantees that all IR holds.

Obedience

For α

$$
\begin{aligned}
(\frac{1}{2}-m+k\alpha)\pi_2(\alpha) & =(1-2m)[\bar{t}-t_{\alpha}]+k\alpha\pi_2(\alpha) \\
& =[\alpha^*(1-k-\pi(\alpha^*))-\alpha(1-k-\pi(\alpha))-\int_{\alpha}^{\alpha^*}(1-k-\pi(t))dt]+k\alpha\pi_2(\alpha) \\
& =\alpha\pi(\alpha)+\int_{\alpha}^{\alpha^*}\pi(t)dt+k\alpha\pi_1(\alpha) \\
& \geqslant \alpha[\pi(\alpha)+k\pi_2(\alpha)] \\
& =\alpha\pi_1(\alpha)
\end{aligned}
$$

So far we have proved the sufficiency.

C.5 Proof of Theorem 2

Case 1. First, we consider the case where $\pi(\underline{\alpha}) \leq 1-k$. Then the optimal question is,

$$\max_{\pi(\alpha),\pi_2(\alpha),t_\alpha} \int_{\underline{\alpha}}^{\overline{\alpha}} t_\alpha dF(\alpha)$$

s.t.

$$\begin{aligned}
&\pi(\alpha) : \mathcal{A} \to [0, 1-k] \text{ is non-increasing} \\
&t_\alpha = \tfrac{\alpha}{1-2m}(1-k-\pi(\alpha)) + t_{\underline{\alpha}} - \tfrac{\int_{\underline{\alpha}}^{\alpha}(1-k-\pi(t))dt}{1-2m} \\
&t_\alpha + \tfrac{1}{2}\pi_2(\alpha) = \bar{t} \\
&m - m\pi_2(\underline{\alpha}) - t_{\underline{\alpha}} \geq 0
\end{aligned}$$

And [32]

$$\begin{aligned}
&\int_{\underline{\alpha}}^{\overline{\alpha}} t_\alpha dF(\alpha) \\
=&\int_{\underline{\alpha}}^{\overline{\alpha}} \left(\frac{\alpha}{1-2m}(1-k-\pi(\alpha)) + t_{\underline{\alpha}} - \frac{\int_{\underline{\alpha}}^{\alpha}(1-k-\pi(t))dt}{1-2m} \right) dF(\alpha) \\
=&\frac{1}{1-2m}\int_{\underline{\alpha}}^{\overline{\alpha}} \left(\frac{1-F(\alpha)}{f(\alpha)} - \alpha \right) \pi(\alpha) dF(\alpha) + t_{\underline{\alpha}} \\
=&\frac{1}{1-2m}\int_{\underline{\alpha}}^{\overline{\alpha}} \left[\int_{\alpha}^{\overline{\alpha}} (1-F(t)-tf(t))dt \right] d\pi(\alpha) + t_{\underline{\alpha}}
\end{aligned}$$

It is also easy to verify that $m = m\pi_2(\underline{\alpha}) + t_{\underline{\alpha}}$. Therefore, given the existence of α^* and the formulation $t_\alpha = \frac{\alpha}{1-2m}(1-k-\pi(\alpha)) + t_{\underline{\alpha}} - \frac{\int_{\underline{\alpha}}^{\alpha}(1-k-\pi(t))dt}{1-2m}$, we have

$$\begin{aligned}
&\bar{t} = t_{\underline{\alpha}} + \tfrac{\int_{\underline{\alpha}}^{\bar{\alpha}} \pi(t)dt}{1-2m} \\
&\pi_2(\underline{\alpha}) = \tfrac{2\int_{\underline{\alpha}}^{\bar{\alpha}} \pi(t)dt}{1-2m} \\
&t_{\underline{\alpha}} = m[1 - \tfrac{2\int_{\underline{\alpha}}^{\bar{\alpha}} \pi(t)dt}{1-2m}]
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{1-2m}\int_{\underline{\alpha}}^{\overline{\alpha}} \left[\int_{\alpha}^{\overline{\alpha}} (1-F(t)-tf(t))dt \right] d\pi(\alpha) + t_{\underline{\alpha}} \\
=&\frac{1}{1-2m}\int_{\underline{\alpha}}^{\overline{\alpha}} \left[\int_{\alpha}^{\overline{\alpha}} (1-F(t)-tf(t))dt \right] d\pi(\alpha) + m[1 - \frac{2\int_{\underline{\alpha}}^{\bar{\alpha}} \pi(t)dt}{1-2m}] \\
=&\int_{\underline{\alpha}}^{\overline{\alpha}} \{ \frac{1}{1-2m} \left[\int_{\alpha}^{\overline{\alpha}} (1-F(t)-tf(t))dt + 2m\alpha \right] \} d\pi(\alpha) + m
\end{aligned}$$

[32] Here we use the fact that $\int_a^b g(\alpha)x(\alpha)d\alpha = \int_{\underline{\alpha}}^{\bar{\alpha}} \mathbf{1}_{\{\alpha \leq b\}} \left(\int_{\max\{a,\alpha\}}^{b} g(\tau)d\tau \right) dx(\alpha)$.

To fully transform the problem into a question about determining function π, consider the following relations:

$$\pi_2(\alpha) = \frac{2}{1-2m}[\int_{\alpha}^{\bar{\alpha}} \pi(t)dt + \alpha\pi(\alpha)] = \frac{2}{1-2m}\int_{\alpha}^{\bar{\alpha}}(-t)d\pi(t)$$
$$\pi_1(\alpha) = \pi(\alpha) + k\pi_2(\alpha) = \int_{\alpha}^{\bar{\alpha}}[-1 - \frac{2kt}{1-2m}]d\pi(t)$$

It follows that $0 \leq \pi_2(\alpha) \leq \pi_1(\alpha)$ and both of them are non-increasing with respect to α. Thus, it only requires $\pi_1(\underline{\alpha}) \leq 1$ to make $\pi_1(\alpha)$ and $\pi_2(\alpha)$ valid, i.e.

$$\int_{\underline{\alpha}}^{\bar{\alpha}}[-1 - \frac{2kt}{1-2m}]d\pi(t) \leq 1.$$

Notice that under $\pi(\overline{\alpha}) = 0$, $\pi(\underline{\alpha}) \leq 1-k$ is equivalent to $\int_{\underline{\alpha}}^{\bar{\alpha}}(-1)d\pi(t) \leq 1-k$. Given this inequality, we can learn that $\int_{\underline{\alpha}}^{\bar{\alpha}}[-1 - \frac{2kt}{1-2m}]d\pi(t) \leq \int_{\underline{\alpha}}^{\bar{\alpha}}(-1)(1+k)d\pi(t) \leq (1-k)(1+k) < 1$ holds, thus the constraint $\int_{\underline{\alpha}}^{\bar{\alpha}}[-1 - \frac{2kt}{1-2m}]d\pi(t) \leq 1$ is never binding.

If $k = 1$,[33] then selling $\overline{E}$ to all types is optimal.

If $k < 1$, we can rewrite the optimization problem as

$$\max_{\tilde{\pi}(\alpha)} \int_{\underline{\alpha}}^{\overline{\alpha}} \Phi(\alpha)d\tilde{\pi}(\alpha)$$

s.t.

$$\begin{aligned}&\tilde{\pi}(\alpha) : \mathcal{A} \to [0,1] \text{ is non-increasing,}\\ &\tilde{\pi}(\bar{\alpha}) = 0\end{aligned}$$

where $\Phi(\alpha) = \frac{-1}{2\bar{\alpha}}\left[\int_{\alpha}^{\overline{\alpha}}(1-F(t)-tf(t))dt + 2m\alpha\right]$, $\tilde{\pi}(\alpha) = \frac{\pi(\alpha)}{1-k}$.

Ignoring the constraint $\tilde{\pi}(\bar{\alpha}) = 0$ and applying the general extension of Carathéodory's theorem in Kang (2023), it follows that there exists an optimal allocation rule that is one of the extreme points of the set of non-increasing functions ranging from $[0,1]$, where $\operatorname{im}\tilde{\pi}(\cdot) \subseteq \{0,1\}$, also satisfying $\tilde{\pi}(\bar{\alpha}) = 0$.[34] With the previous conclusions, we know that $\pi(\alpha) = 1-k$ on $\alpha \in [\underline{\alpha}, \alpha^*)$ while $\pi(\alpha) = 0$ on $\alpha \in [\alpha^*, \overline{\alpha}]$ for some α^*. By the relations between $\pi(\alpha)$ with $\pi_1(\alpha)$ and $\pi_2(\alpha)$, we can get that in the two-tier optimal menu, $\pi_2(\alpha) = \frac{\alpha^*}{\bar{\alpha}}$ and $\pi_1(\alpha) = \frac{\frac{1}{2}-m+k\alpha^*}{\bar{\alpha}} = 1-k(1-\frac{\alpha^*}{\bar{\alpha}})$. on $\alpha \in [\alpha^*, \overline{\alpha}]$ for some α^* for $0 \leq \alpha < \alpha^*$. Meanwhile, the obedience of α^* is binding,[35] as $\frac{\pi_1(\alpha)}{\pi_2(\alpha)} = \frac{\frac{1}{2}-m+k\alpha^*}{\alpha^*}$ for $0 \leq \alpha < \alpha^*$.

[33] Under our assumption, $k = 1$ occurs if and only if $m = \frac{1}{2}$. But this outcome can be extended to any situation where $\alpha + \beta = Constant$ for all types.

[34] See more details in our appendix D or Kang (2023).

[35] In the optimal menu, α^* chooses the menu with fully informative experiment. Here the binding obedience of α^* means that the obedience constraints for arbitrary sequence in $[0, \alpha^*)$ converging to α^* implies the obedience of α^* for the partially informative menu is binding.

Then, $t_{\underline{\alpha}} = m[1 - \frac{2\alpha^*\pi^*}{1-2m}] = m(1 - \frac{\alpha^*}{\bar{\alpha}})$ and $\bar{t} = m + \alpha^*\pi^* = m + (1-k)\alpha^*$. The designer determines $\alpha^* \in [\underline{\alpha}, \bar{\alpha}]$ to maximize $t_{\underline{\alpha}} F(\alpha^*) + \bar{t}(1 - F(\alpha^*))$, i.e.

$$\begin{aligned} & m(1 - \tfrac{\alpha^*}{\bar{\alpha}})F(\alpha^*) + (m + (1-k)\alpha^*)(1 - F(\alpha^*)) \\ = \; & m + \tfrac{\alpha^*}{\bar{\alpha}}[(1-k)\bar{\alpha} - \tfrac{1}{2}F(\alpha^*)] \end{aligned}$$

Therefore the optimal mechanism is that

$$\begin{gathered} \alpha^* \in \arg\max_\alpha \; \alpha\left[(1-k)\overline{\alpha} - \tfrac{1}{2}F(\alpha)\right] \\ \pi_2(\underline{\alpha}) = \tfrac{\alpha^*}{\bar{\alpha}} \text{ and } \pi_1(\underline{\alpha}) = 1 - k(1 - \tfrac{\alpha^*}{\bar{\alpha}}) \end{gathered}$$

Case 2. Now suppose in the optimal menu $\pi(\alpha)$, $\hat{\alpha} = \inf\{\alpha | \pi(\alpha) \leq 1-k\} > \underline{\alpha}$. In this case, IR$[\hat{\alpha}]$ is binding, i.e.

$$m - m\pi_2(\underline{\alpha}) - t_{\underline{\alpha}} + \int_{\underline{\alpha}}^{\hat{\alpha}} (1 - k - \pi(\alpha))dt = 0$$

Considering $\pi_2(\alpha) = \frac{2}{1-2m}[\int_\alpha^{\bar{\alpha}} \pi(t)dt + \alpha\pi(\alpha)]$, substitute $\pi_2(\underline{\alpha}) = \frac{2}{1-2m}\int_{\underline{\alpha}}^{\bar{\alpha}} \pi(t)dt$ to get

$$t_{\underline{\alpha}} = m - \frac{2m}{1-2m}\int_{\underline{\alpha}}^{\bar{\alpha}} \pi(t)dt - \int_{\underline{\alpha}}^{\hat{\alpha}} (1 - k - \pi(\alpha))dt$$

Then, the assigned transfer to α induced by the designed $\pi(\alpha)$ is given by

$$\begin{aligned} t_\alpha &= \frac{\int_{\underline{\alpha}}^{\alpha} \pi(t)dt - \alpha\pi(\alpha)}{1-2m} + m - \frac{2m}{1-2m}\int_{\underline{\alpha}}^{\bar{\alpha}} \pi(t)dt - \int_{\underline{\alpha}}^{\hat{\alpha}} (1 - k - \pi(\alpha))dt \\ &= \frac{-1}{1-2m}\int_\alpha^{\hat{\alpha}} \pi(t)dt - \frac{2m}{1-2m}\int_\alpha^{\bar{\alpha}} \pi(t)dt - \frac{\alpha\pi(\alpha)}{1-2m} + (1-k)\hat{\alpha} + m \end{aligned}$$

Define $\hat{\pi}(\alpha)$ as

$$\hat{\pi}(\alpha) = \begin{cases} \pi(\alpha) & \text{if } \alpha \in \{\alpha | \pi(\alpha) \leq 1-k\} \\ 1-k & \text{otherwise} \end{cases}$$

Then, the assigned transfer $\hat{t}_\alpha$ to α induced by the designed $\hat{\pi}(\alpha)$ satisfied $t_\alpha = \hat{t}_\alpha$ for $\alpha \in \{\alpha | \pi(\alpha) \leq 1-k\}$ and $t_\alpha \leq \hat{t}_\alpha$ otherwise. Since $\hat{\alpha} > \underline{\alpha}$, then $\hat{\pi}$ induces a strictly better menu than π, a contradiction.

D Appendix: an Infinite-dimensional Extension of Carathéodory's Theorem

In the proof of theorem 2, we transform the optimization into maximizing a linear functional subject to a non-increasing function. Here we refer to an infinite-dimensional extension of Carathéodory's theorem found in Kang (2023), and the characterization of extreme points of randomized allocation function.

D.1 Extreme Points of $\Pi = \{\pi | \pi : \Theta \to [0,1],\ \pi$ is non-increasing$\}$

Denote $\mathcal{P} = \{\pi | \pi : \Theta \to [0,1],\ \pi \text{ is non-increasing, and } \operatorname{im}\pi \subseteq \{0,1\}\}$.

Proof of $\mathcal{P} \subseteq \mathrm{ex}\Pi$

Suppose that $\pi \in \mathcal{P}$, and $\pi = \alpha\pi_1 + (1-\alpha)\pi_2$ for $\pi_1, \pi_2 \in \Pi$ and $\alpha \in (0,1)$. Then $\alpha\pi_1(\theta) + (1-\alpha)\pi_2(\theta) = \pi(\theta) \in \{0,1\}$ for almost every $\theta \in [\underline{\theta}, \overline{\theta}]$, which implies that $\pi_1 = \pi_2 = \pi$ for almost everywhere on Θ. Hence $\pi \in \mathrm{ex}\,\Pi$.

Proof of $\mathrm{ex}\Pi \subseteq \mathcal{P}$

For any $\pi \in \Pi$ satisfying $m(\{\theta | x(\theta) \notin \{0,1\}\}) > 0$, where m is the Lebesgue measure. Define $\pi_1, \pi_2 : \Theta \to [0,1]$ by $\pi_1 = \pi^2$ and $\pi_2 = 2\pi - \pi^2$; by construction, π_1, $\pi_2 \in \Pi$ and $\pi = (\pi_1 + \pi_2)/2$. Note that $\pi_1 \neq \pi_2$. Therefore $\pi = (\pi_1 + \pi_2)/2$ where $\pi_1, \pi_2 \in \Pi$ are distinct; hence $\pi \notin \mathrm{ex}\Pi$.

Therefore $\mathcal{P} = \mathrm{ex}\Pi$.

D.2 An Infinite-dimensional Extension of Carathéodory Theorem

Theorem 3. *Let K be a convex, compact set in a locally convex Hausdorff space, and let $l : K \to \mathcal{R}^m$ be a continuous affine function such that $\sum \subseteq \operatorname{im} l$ is a closed and convex set. Suppose that $l^{-1}(\sum)$ is nonempty and and that $\Omega : K \to \mathcal{R}$ is a continuous convex function. Then there exists $z^* \in l^{-1}(\sum)$ such that $\Omega(z^*) = \max_{z \in l^{-1}(\sum)} \Omega(z)$ and*

$$z^* = \textstyle\sum_{i=1}^{m+1} \alpha_i z_i, \text{ where } \sum_{i=1}^{m+1} \alpha_i = 1, \text{ and for all } i,\ \alpha_i \geq 0,\ z_i \in \mathrm{ex}K$$

Proof. See in Bauer (1958) and Szapiel (1975). □

Kang (2023) shows that these three conditions (convexity, compact in the L_1 topology, and the existence of the optimal mechanism) are satisfied in a mechanism design with transferable utility setting. Therefore we can apply Carathéodory Theorem to our problem.

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