Selling Training Data

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- monopolist screening: one data seller, one data buyer, a statistical decision problem (hypothesis testing)
- data buyer is endowed with **private dataset** and seeks to purchase additional dataset to improve the test
- data seller versions the data and designs the associated tariff to screen the buyers with different private datasets
- our question: what is the **optimal data selling mechanism**?

Timeline

1. the seller posts a mechanism $\mathcal{M} = \{\mathcal{E}, t\}$
   1. a collection of experiments $\mathcal{E}$
   2. associated tariff $t : \mathcal{E} \to \mathbb{R}_+$
2. the buyer (with private experiment) chooses an experiment $E \in \mathcal{E}$ and pays price $t(E)$
3. the true state ω is realized
4. the buyer receive **two signal realizations** to update his belief, one from her **private experiment**, another from **the experiment E he purchased**, and she chooses an action a to maximize her expected utility
5. payoffs are realized

Statistical Decision Making

- two states : ω_1 (null hypothesis), ω_2 (alternative hypothesis), prior: $\mu_0 = (\frac{1}{2}, \frac{1}{2})$
- hypothesis test: binary action $\{a_1, a_2\}$ and payoff $u(a_i, \omega_j) = 1_{i=j}$ (correct identification)
- private experiment: two signals s_1' (acceptance), s_2' (rejection)
- private type: (α, β), Type I error $\alpha = \Pr(s_2'|\omega_1)\mu(\omega_1)$, Type II error $\beta = \Pr(s_1'|\omega_2)\mu(\omega_2)$

  Remark: buyer with high-quality private dataset is low type. ($\alpha + \beta$ is low)

	E'	s_1'	s_2'	Statistical Errors
Null hypothesis →	ω_1	π_1'	$\boxed{1-\pi_1'}$	→ Type I Error $\alpha = \Pr(\mu_2)(1-\mu_2)$
Alternative hypothesis →	ω_2	$\boxed{1-\pi_2'}$	π_2'	→ Type II Error $\beta = \Pr(\mu_1)(1-\mu_1)$

Private Data $(1 \geq \pi_1', \pi_2' \geq 0.5)$

Supplementary Data

E is obedient for type (α, β) if every signal $s_k = (a_{k_1}, a_{k_2})$ is obeyed for (α, β), i.e.

$$a_{k_j} \in \arg\max_{a_{j'} \in A} \mathrm{E}[u_{ij'} | s_k, s'_j] \text{ for all } s_k \text{ and } j = 1, 2.$$

Lemma

The outcome of every menu $\mathcal{M}$ can be attained by a direct and straight mechanism $\mathcal{M} = \{\mathcal{E}_\Theta, t\}$, where each type $\theta = (\alpha, \beta)$ buys obedient E_θ from $\mathcal{E}_\Theta$, and pays $t : \mathcal{E}_\Theta \to \mathbb{R}_+$.

E	(a_1, a_1)	(a_1, a_2)	(a_2, a_1)	(a_2, a_2)
ω_1	π_{11}	π_{12}	π_{13}	π_{14}
ω_2	π_{21}	π_{22}	π_{23}	π_{24}

Table: Straight Experiment

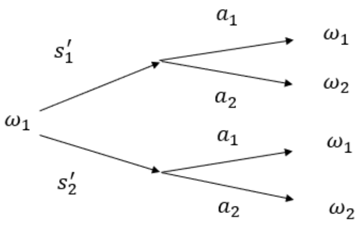


Figure: Data Merging

- Type-wise reduction data structure (π_1, π_2): reduce Type i error by a ratio π_i

  π_i: probability inducing **Type i** error from identifying ω_{-i} in ω_i

Lemma

The revenues can always be weakly improved by replacing a direct and straight mechanism $\mathcal{M} = \{\mathcal{E}_\Theta, t\}$ with an alternative direct and straight mechanism $\mathcal{M} = \{\mathcal{E}'_\Theta, t'\}$, where $E'_\theta \in \mathcal{E}'_\Theta$ is Type-wise reduction for all θ.

- obedience constraint: $\pi_1\alpha + \pi_2\beta \leq \min\{\frac{1}{2}\pi_1, \frac{1}{2}\pi_2\}$

E	(a_1, a_1)	(a_1, a_2)	(a_2, a_1)	(a_2, a_2)
ω_1	$1 - \pi_1$	π_1	0	0
ω_2	0	π_2	0	$1 - \pi_2$

Table: Type-wise Reduction Experiment

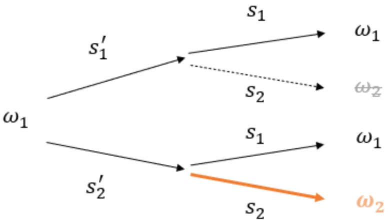


Figure: Reducing Type I error

revelation principle for $\theta = (\alpha, \beta)$:

1. direct mechanism $\mathcal{M} = \{E_\theta, t_\theta\}_{\theta\in\Theta}$: IC + IR
2. straight information design with Type-wise reduction E_θ $(\pi_1(\theta), \pi_2(\theta))$: Ob-edience

value of experiment (π_1, π_2) for (α, β): incremental probability of correct identification

$$V(E,\theta) = \underbrace{\alpha + \beta}_{\text{initial overall error}} - \min \underbrace{\{\alpha\pi_1 + \beta\pi_2, , \frac{1}{2}\pi_1, \frac{1}{2}\pi_2\}}_{\text{new overall error}}$$

Designer's Problem

$$\max_{\mathcal{M}} \int_\Theta t_\theta dF(\theta)$$

$$\alpha\pi_1(\theta) + \beta\pi_2(\theta) \leq \min\{\tfrac{1}{2}\pi_1(\theta), \tfrac{1}{2}\pi_2(\theta)\}$$

$$\alpha + \beta - \alpha\pi_1(\theta) - \beta\pi_2(\theta) - t_\theta \geq 0, \ \forall\theta \in \Theta$$

$$\alpha + \beta - \alpha\pi_1(\theta) - \beta\pi_2(\theta) - t_\theta \geq \alpha + \beta - \underbrace{\min\{\alpha\pi_1(\theta') + \beta\pi_2(\theta'), \frac{1}{2}\pi_1(\theta'), \frac{1}{2}\pi_2(\theta')\}}_{\textbf{two-step deviation}} - t_{\theta'}, \forall\theta, \theta' \in \Theta$$

Key Attributes of Data Goods

- data goods: sell statistical error (specific multi-dimensional goods)
- interdependence between different Types of error imposes rigidity on the menu structure:
  1. obedience, $\alpha\pi_1 + \beta\pi_2 \leq \min\{\frac{1}{2}\pi_1, \frac{1}{2}\pi_2\}$, constrains the allocation of statistical error
  2. double-deviation, $\min\{\alpha\pi_1 + \beta\pi_2, \frac{1}{2}\pi_1, \frac{1}{2}\pi_2\}$, weakens the differentiation
- inclusion, exclusion and extraction principles + allocation rigidity shape the bundling policy
- trade-off: extraction of low type surplus v.s. reduce information rent
- the seller can exploit the horizontal difference to neutralize the vertical difference, through subtly designing the lower-tiered dataset to nullify the impact of private datase

Data Goods and Other Goods

flexibility: physical goods $<$ information goods $<$ data goods $<$ bundling of physical goods

1. physical goods: posted-price, no-haggling
2. information goods: position and informativeness (separately "multi-dimensional" goods)
   1. position: the Type of error (either Type I or II) - $(\alpha, 0)$ or $(0, \beta)$
   2. informativeness: the probability of corresponding Type error - α or β
   3. allocation: reducing corresponding Type error- $(\pi_1, \pi_2) = (\pi_1, 1)$ when $(\alpha, 0)$, or $(1, \pi_2)$ when $(0, \beta)$

   the design and price of information $\Longleftrightarrow$ one-dimensional allocation (differentiated informativeness) $+$ one-dimensional preference with incongruent order
3. data goods: allocate different Types of error (specific multi-dimesional goods)
4. multi-dimensional goods: optimal bundling policy tends to be complex and infinite

Literature Review

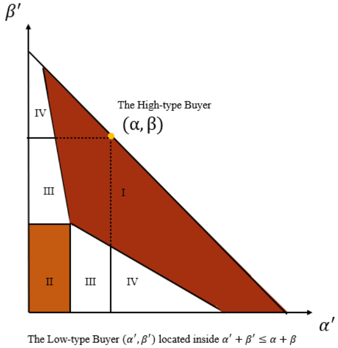
1. Information Design as Screening Tools: Admati and Pfleiderer (1986), Admati and Pfleiderer (1990), Babaioff et al. (2012), Bergemann et al. (2018), Yang (2022), Segura-Rodriguez (2022), Bonatti et al. (2023), Bonatti et al. (2024), Rodriguez Olivera (2024)
2. Multi-dimensional Screening: Adams and Yellen (1976), McAfee et al. (1989), Armstrong and Rochet (1999), Manelli and Vincent (2007), Hart and Reny (2015), Daskalakis et al. (2017), Carroll (2017), Haghpanah and Hartline (2021); Yang (2022), Deb and Roesler (2023)

Main Results: Binary Situation

roadmap of main results:

1. binary type (low (α', β') and high (α, β), $\alpha' + \beta' \leq \alpha + \beta$, uniform distribution): four polices
2. continuous type space: two-tiered partial grand bundling scheme

Lemma: sell fully informative $\overline{E}$ to type-H & type-L experiment E should be obedient for type-H



Region	Data Menu	Selling Policy
I	$(\overline{E}, \overline{E})$	Inclusive Grand Bundling
II	$(\overline{E}, \phi)$	Exclusive Grand Bundling

grand bundling: sell $\overline{E}$ with $(\pi_1, \pi_2) = (0, 0)$

I: low rent, high type-L surplus - including type-L

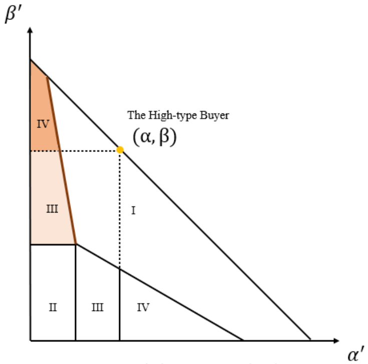
II: high rent, low type-L surplus - excluding type-L boundary line: inclusion/exclusion of type-L (rent extraction v.s. surplus extraction)

Region	Data Menu	Selling Policy	Binding Constraints
III	$(\overline{E}, E^*)$	Nested Bundling	(IR-L),(IC-H),(Ob-H)

$(\alpha, \beta) > (\alpha', \beta') \Rightarrow \pi_1\alpha + \pi_2\beta > \pi_1\alpha' + \pi_2\beta'$, given (π_1, π_2)

$\Rightarrow$ information rent > 0 & higher tendency to make another Type error ((Ob-H) is binding)

E^*: **only** reduce some Type error by a **constant ratio** (e.g. when $\frac{\beta}{2} < \beta' < \beta$ and $\alpha' < \frac{\alpha}{2}$)



The Low-type Buyer (α', β') located inside $\alpha' + \beta' \le \alpha + \beta$

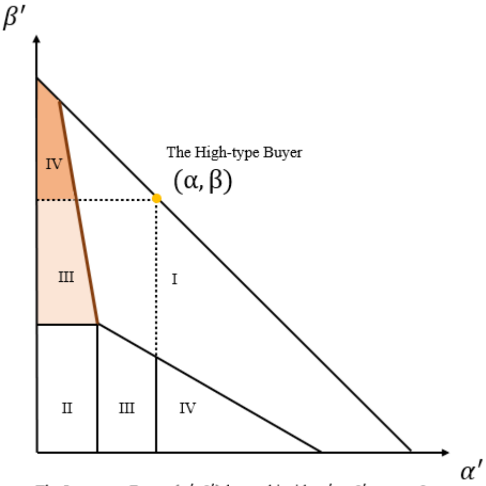
exploitation of data structure:

$1 \cdot \alpha + \pi_2^*\beta = \frac{1}{2}\pi_2^*$

	(s_1, s_1)	(s_1, s_2)	(s_2, s_2)
ω_1	0	1	0
ω_2	0	π_2^*	$1 - \pi_2^*$

two benefits for including β': relatively low rent and high surplus

Region	Data Menu	Selling Policy	Binding Constraints
IV	$(\overline{E}, E_{(\alpha,\beta)})$	Partial Grand Bundling	(IR-L),(IC-H),(Ob),(IR-H)



The Low-type Buyer (α', β') located inside $\alpha' + \beta' \leq \alpha + \beta$

$E_{(\alpha,\beta)}$: reduce both Types of error

(i) exploitation of data structure:
$\alpha\pi_1^* + \beta\pi_2^* = \frac{1}{2}\pi_i^*$

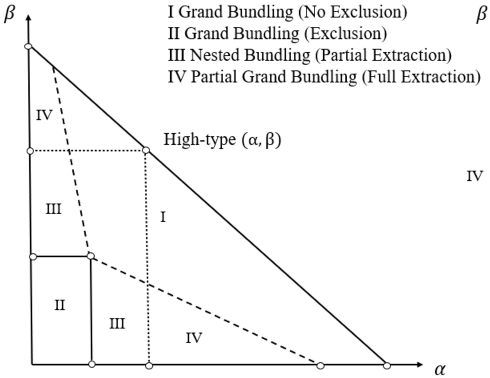
(ii) no information rent:
$\alpha\pi_1^* + \beta\pi_2^* = \alpha'\pi_1^* + \beta'\pi_2^*$

	(s_1, s_1)	(s_1, s_2)	(s_2, s_2)
ω_1	$1 - \pi_1^*(\alpha', \beta')$	$\pi_1^*(\alpha', \beta')$	0
ω_2	0	$\pi_2^*(\alpha', \beta')$	$1 - \pi_2^*(\alpha', \beta')$

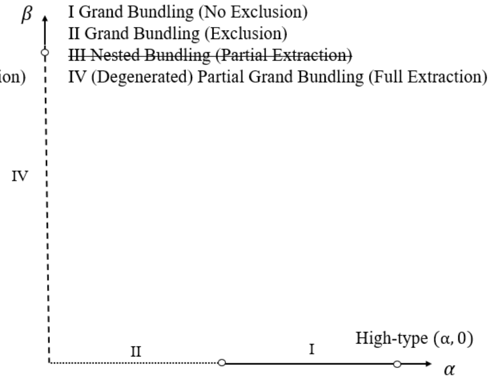
Selling Data v.s. Selling Information (Bergemann et al.2018)

private type in Bergemann et al(2018): private signal before contracting/interim belief

the buyer commits either Type I error or Type II error - private type is $(\alpha, 0)$ or $(0, \beta)$



The Low-type Buyer (α', β') located inside $\alpha' + \beta' \leq \alpha + \beta$



The Low-type Buyer $(\alpha', 0)$ or $(0, \beta')$ located inside $\alpha' \leq \alpha$ or $\beta' \leq \alpha$

Continuous Type Space

assumption

The statistical error of buyer's private data is characterized by a linear relationship: for the private type (α, β), it holds that $k\alpha + \beta = m$, with $m \in [0, \frac{1}{2})$ and $k \in [0, 2m]$.

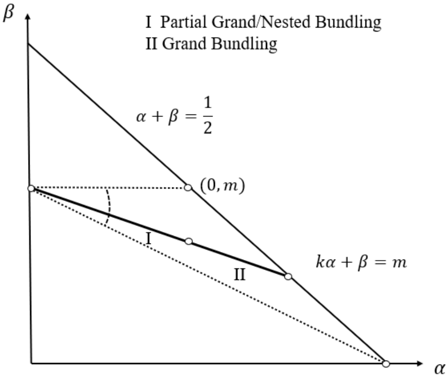
three characteristics: the coexistence of both horizontal and vertical differences, the obedience constraints and the possibilities of double deviation.

private type can be represented as $(\alpha, m - k\alpha)$, where $\alpha \in \mathcal{A} = [\underline{\alpha}, \overline{\alpha}] = [0, \frac{\frac{1}{2} - m}{1 - k}]$ draws from distribution F with a continuous, strictly positive density f

optimal mechanism

The optimal selling mechanism is two-tiered pricing:

1. $(E_\alpha, t_\alpha) = (\overline{E}, \overline{t})$ for $\alpha \in [\alpha^*, \overline{\alpha}]$
2. $(E_\alpha, t_\alpha) = (E^*, t^*)$ for $\alpha \in [\underline{\alpha}, \alpha^*)$, where $\pi_1^* = 1 - k(1 - \frac{\alpha^*}{\overline{\alpha}})$, $\pi_2^* = \frac{\alpha^*}{\overline{\alpha}}$
3. $\alpha^* \in \arg\max_\alpha \alpha \left((1-k)\overline{\alpha} - \frac{1}{2}F(\alpha)\right)$



$k\alpha + \beta = m, m \in \left[0, \frac{1}{2}\right], k \in [0,2m]$

exploit the horizontal differences to neutralize the vertical difference and include the low type

improve E along the neutralization line $\frac{1-\pi_1}{1-\pi_2} = k$ and extract all the additional value

$$V(E^*, \alpha) = \alpha + (m - k\alpha) - \alpha\pi_1^* - (m - k\alpha)\pi_2^* = m(1-\pi_2^*) + \alpha[(1-k) - (\pi_1^* - k\pi_2^*)] = m(1-\pi_2^*)$$

such operation can be continued until it hits the obedient boundary (rigidity)

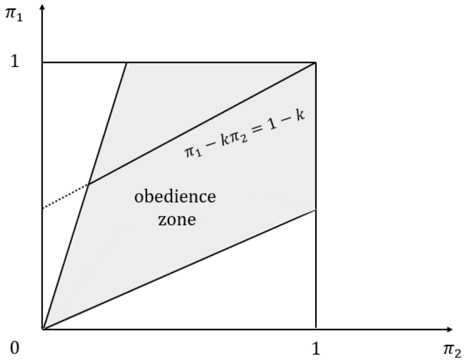


Figure: neutralization line

Proof Sketch

Denote $V(E,\alpha) = \max\{V_r(E,\alpha), V_n(E,\alpha)\}$, where

$$V_r(E,\alpha) = \alpha(1-\pi_1) + (m-k\alpha)(1-\pi_2), \quad V_n(E,\alpha) = \alpha + (m-k\alpha) - \min\{\tfrac{1}{2}\pi_1, \tfrac{1}{2}\pi_2\}$$

two properties of the value functions:

Property 1. (“same difference”)

$$V_n(E',\alpha') - V_n(E,\alpha') = V_n(E',\alpha) - V_n(E,\alpha), \forall E, E', \alpha, \alpha'.$$

Property 2. (“increasing difference”)

$V_r(E',\alpha') - V_r(E,\alpha') \geq V_r(E',\alpha) - V_r(E,\alpha), \forall \alpha' > \alpha$ if and only if $\pi_1' - k\pi_2' \leq \pi_1 - k\pi_2$, where inequality binds if and only if $\pi_1' - k\pi_2' = \pi_1 - k\pi_2$.

denote $\lambda(\alpha) : \mathcal{A} \to \mathcal{A}$ the type of buyers who are exactly indifferent to following seller's recommendation or not, when merging his own private dataset.

$$\text{(i)}\lambda(\alpha) = \frac{(\frac{1}{2} - m)\pi_2(\alpha)}{\pi_1(\alpha) - k\pi_2(\alpha)} \text{ if } \pi_1(\alpha) \neq 0; \text{ (ii) } \lambda(\alpha) = \bar{\alpha} \text{ otherwise.}$$

Lemma (Characterization of Obedience Zone)

Optimal menu (E_α, t_α) satisfies

1. $\pi_2(\alpha)/\pi_1(\alpha) \leq 1$
2. *There exists a threshold α^* such that*
   1. *for any $\alpha < \alpha$, $\alpha < \lambda(\alpha)$ and there exists some $\alpha' > \lambda(\alpha)$ such that $IC[\alpha' \to \alpha]$ binds;*
   2. *$E_\alpha = \bar{E}$ if and only if $\alpha \geq \alpha^*$.*

a class of perturbations $\{(-k\Delta\pi, -\Delta\pi : \Delta\pi \geq 0)\}$ on supplementary datasets, which does not change the difference in evaluating the dataset between V_r, but enlarge it between V_r and V_n

exploit such perturbation of informativeness improvement to the maximal degree

$\Rightarrow$ either double-deviation IC, or Ob is binding define $\gamma(\alpha)$ some type who is indifferent between choosing $E_{\gamma(\alpha)}$ and conducting double deviation by choosing E_α.

$$\gamma(\alpha) = \begin{cases} \alpha & \text{if } \alpha = \lambda(\alpha) \\ \tilde{\alpha} \in \{\alpha' > \lambda(\alpha) : \text{IC}[\alpha' \to \alpha] \text{ is binding}\} & \text{if } \alpha < \lambda(\alpha) \end{cases}$$

Lemma (Properties of λ and γ)

In optimal menu,

1. $\lambda(\alpha) \leq \lambda(\hat{\alpha}) \leq \gamma(\alpha)$ *for* $\hat{\alpha} \in [\alpha, \lambda(\alpha)]$.
2. $\pi(\alpha) := \pi_1(\alpha) - k\pi_2(\alpha)$ *is non-increasing for* $\alpha \in [0, \bar{\alpha}]$*;*

two key observations:

the supplementary dataset amplifies the quality gap of baseline/private datasets.

the private dataset narrows the quality gap of supplementary datasets.

Lemma (Equivalent Transformation of Constraints)

In the optimal mechanism, the IC, IR and Ob conditions are equivalent to

1. $\frac{1}{2}\pi_2(\alpha) + t_\alpha = t^*$ for all $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, where t^* is the associated tariff for $\bar{E}$;
2. $V(E_\alpha, \alpha) = \int_{\underline{\alpha}}^{\alpha} (1 - k - \pi(t))dt + V(E_{\underline{\alpha}}, \underline{\alpha})$
3. $\pi(\alpha) : [\underline{\alpha}, \bar{\alpha}] \to [0, 1-k]$ is non-increasing;
4. $\mathrm{IR}[\hat{\theta}]$ holds for some $\hat{\alpha} = \inf\{\alpha | \pi(\alpha) \leq 1 - k\}$.

condition 1: the price difference between any pair of supplementary datasets in the menu should exactly measure their difference in Type II error reduction.

seller's optimization problem can be transformed as

$$\max_{\pi} \int_{\underline{\alpha}}^{\overline{\alpha}} \frac{-1}{1-2m} \left[\int_{\alpha}^{\bar{\alpha}} (1 - F(t) - tf(t))dt + 2m\alpha \right] d\pi(\alpha)$$

$$s.t. \begin{cases} \pi : [\underline{\alpha}, \bar{\alpha}] \to [0, 1-k] \text{ is non-increasing} \\ \pi(\bar{\alpha}) = 0 \end{cases}$$

a classic one-dimensional screening problem