On the Existence of Fully Informative Experiment in Optimal Menu [*]

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Lemma 1. *Given information structure E, $V(E,\theta)$ is continuous with respect to θ.*

Lemma 2. *The net utility $V(E(\theta),\theta)-t(\theta)$ in a feasible menu (satisfied IC and IR conditions), is lower semi-continuous with respect to θ.*

Proof. We prove it via contradiction.

If the net utility in the optimal menu is not lower semi-continuous. Then at some $\theta \in \Theta$, there exists $\delta > 0$ and for any $\epsilon > 0$, there always exists θ' such that $d(\theta,\theta') < \epsilon$ and $[V(E(\theta'),\theta')-t(\theta')]-[V(E(\theta),\theta)-t(\theta)] \leq -\delta$, i.e.

$$t(\theta)-t(\theta') \leq V(E(\theta),\theta)-V(E(\theta'),\theta')-\delta \tag{1}$$

Now considering the IC conditions that θ' is unwilling to imitiate θ, i.e.

$$t(\theta)-t(\theta') \geq V(E(\theta),\theta')-V(E(\theta'),\theta') \tag{2}$$

With inequality (1) and (2), we can derive that

$$V(E(\theta),\theta)-V(E(\theta),\theta') \geq \delta$$

which contradicts with the Lemma 1. □

Proof. Suppose $\bar{E}$ is not in the original menu $\mathcal{M} = \{E(\theta), t(\theta)\}$.

For every type θ, θ', denote $DS(\theta,\theta') = V(E(\theta),\theta)-t(\theta)+t(\theta')-V(E(\theta'),\theta)$ and $DA(\theta,\theta') = (V(\bar{E},\theta)-V(E(\theta'),\theta))-(V(\bar{E},\theta')-V(E(\theta'),\theta'))$.

Case 1: If there exists a buyer θ' such that for any $\theta \neq \theta'$, $DS(\theta,\theta') \geq DA(\theta,\theta') > 0$ or $DA(\theta,\theta') \leq 0$ holds, then we can replace the experiment of θ' as $\bar{E}$ and charge a strictly higher

[*] All notations are directly from Bergemann et al. (2018).

price $\hat{t}(\theta') = t(\theta') + V(\bar{E}, \theta') - V(E(\theta'), \theta')$. Obviously, we only need to verify the IC conditions that other types $\theta \neq \theta'$ are unwilling to imitiate θ'.

$$[V(E(\theta), \theta) - t(\theta)] - [V(\bar{E}, \theta) - \hat{t}(\theta')] = [V(E(\theta), \theta) - t(\theta) + t(\theta') - V(E(\theta'), \theta)] + [V(\bar{E}, \theta') + V(E(\theta'), \theta) - V(\bar{E}, \theta) - V(E(\theta'), \theta')] = DS(\theta, \theta') - DA(\theta, \theta') \geq 0.$$

Case 2: If for any buyer θ', there exists a θ such that $DA(\theta, \theta') > DS(\theta, \theta') \geq 0$, denote $f(\theta) \triangleq V(E(\theta), \theta) - t(\theta) - V(\bar{E}, \theta)$, then we can derive that

$$\begin{aligned}
&f(\theta) - f(\theta') \\
=&[V(E(\theta), \theta) - t(\theta) - V(\bar{E}, \theta)] - [V(E(\theta'), \theta') - t(\theta') - V(\bar{E}, \theta')] \\
=&[V(E(\theta), \theta) - t(\theta) - V(E(\theta'), \theta) + t(\theta')] \\
&- [V(\bar{E}, \theta) - V(\bar{E}, \theta') + V(E(\theta'), \theta') - V(E(\theta'), \theta)] \\
=&DS(\theta, \theta') - DA(\theta, \theta') < 0
\end{aligned}$$

,which means for any θ', there exists a $\theta \neq \theta'$ having a strictly lower value of $f(\theta)$ than that of θ'.

However, by Lemma 1 and Lemma 2, we know that $f(\theta)$ is a lower semi-continuous function over $\theta \in \Theta$. Since Θ as a simplex space is compact, by Weierstrass Theorem, there exists a θ', for any $\theta \in \Theta$, $f(\theta) \geq f(\theta')$, a contradiction.

Thus for every optimal menu not containing $\bar{E}$, we feasiblely adjust it to a new menu containing $\bar{E}$ which strictly increases the seller's revenue. $\square$

References

Bergemann, Dirk, Alessandro Bonatti, and Alex Smolin, "The Design and Price of Information," *American Economic Review*, January 2018, *108* (1), 1–48.