Ordinal Centrality

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Introduction

1 Introduction

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3 Centrality in Network Games

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Centrality Matters in Network Analysis

- centrality: the description of the position of an agent in the network
- different centrality captures different facets of a person's centrality, power, prestige, or influence
- A Taxonomy from (Jackson et.al, 2022):
  1. which information they make use of about nodes' positions
  2. how that information is weighted as a function of distance from the node in question
- centrality may also relate to action in equilibrium

Canonical Centrality

- Canonical Centrality
  1. Katz-Bonacich centrality
  2. eigenvector centrality
  3. degree centrality
  4. betweenness centrality
  5. diffusion centrality
- We can classify these centrality into two different groups:
  1. local centrality captures the node's position from his neighbors
  2. global centrality captures the node's position in the whole network

Ordinal Centrality

- Intuitively, one person with **more neighbors and more important neighbors** will (certainly) be more importantly than another one in many facets of ”influence”
- This motivates the definition of **recursive monotonicity** , based on which we construct two new (ordinal) centrality measures and relate this two measures with actions in equilibrium
  1. Strong centrality: all ordinal centralities rank vertex i over $j \Rightarrow j \preceq_{strong} i$
  2. Weak centrality: extended strong centrality motivated by comparison between linked nodes, which is necessarily the retraction of degree centrality and can be a total order in a special structure called overlapping hierarchies
- recursive monotonicity $\Rightarrow$ ordinal centrality $\Rightarrow$ strong centrality $(\Rightarrow)$ weak centrality

Two Key Findings

- ordinal centralities $\Leftrightarrow$ equilibria in network games of strategic complements
  $\Rightarrow$ strong centrality: **common comparisons across all equilibria** in all network games of strategic complements.
- The order induced by either the minimal or the maximal equilibrium is always **an extension of weak centrality**

Centrality

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Recursive Monotonicity

- undirected graph $G = (V, E)$, neighbor set of i: G_i
- binary relation $i \succsim j$

  preorder $\succsim$: reflexive and transitive, extension and retraction $\succsim \subseteq \succsim'$
- $S \subseteq V \succsim$ -dominates $S' \subseteq V$ ($S \succsim S'$): there exists an injective function $f : S' \to S$ such that $f(i) \succsim i$ for all $i \in S'$

Recursive Monotonicity

The binary relation $\succsim$ satisfies recursive monotonicity if $i \succsim j$ whenever $G_i \succsim G_j$ An ordinal centrality $\succsim$ on $G = (V, E)$ is a preorder on V that satisfies recursive monotonicity

- vertex i is more central than vertex j if i has more connections to more central nodes than j, represented by $c : V \to R$ if $i \succsim j$ iff $c(i) \geq c(j)$

Several Useful Properties

1. invariant extension: if $G_i \succsim G_j$, then we have $G_i \succsim' G_j$ according to any extension $\succsim \subseteq \succsim'$
2. reflexivity: $G_i \succsim G_i$
3. closed under intersections: $\succsim \cap \succsim'$
4. subset means less important: if $G_j \subseteq G_i$, then we have $i \succsim j$

Examples

- prestige centralities, or positive-feedback centralities:
    1. degree centrality: $c(i) = |G_i|$
    2. Eigenvector centrality: $c(i) = (1/\lambda)\Sigma_{k \in G_i} c(k)$
    3. Katz-Bonacich centrality:$c(i) = |G_i| + \delta \sum_{k \in G_i} c(k)$
- Generally: $c(i) = h(\sum_{k \in G_i} c(k))$ for any increasing $h : R_+ \to R_+$
- not capture the measures that depend on global network structure, such a node's role as an intermediary (betweenness centrality and closeness centrality)

Failures: Graphical Illustration

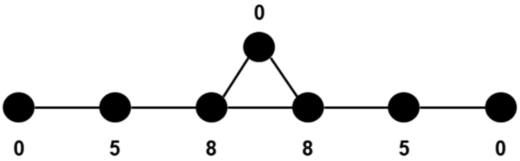

FIG. 1.—Betweenness centrality is not recursively monotone.

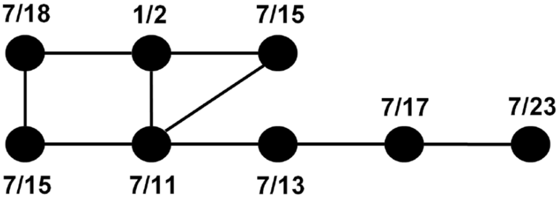

FIG. 2.—Closeness centrality is not recursively monotone.

Strong Centrality

Strong Centrality

$i \succsim_S j$, iff if $i \succsim j$ for every ordinal centrality on G

- strong centrality is the intersection of all ordinal centralities

Algorithm for Strong Centrality

1. Set $\succsim_0$ as the binary relation in which $i \succsim_0 i$ for each $i \in V$, with no other comparisons.
2. Define $\succsim_n$ by $i \succsim_n j$ if and only if $G_i \succsim_{n-1} G_j$

- Strong centrality makes few comparisons:
  1. the intersection of all ordinal centrality: robustness vs completeness
  2. failure to compare seemingly natural pairs

Few Comparisons: Graphical Illustration

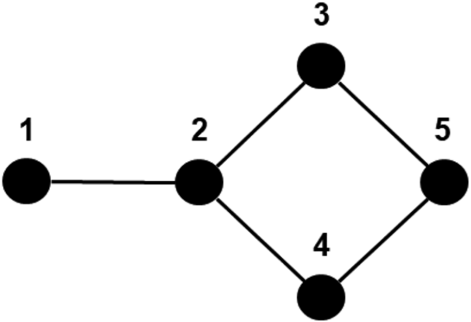

FIG. 3.—Strong centrality is not total.

- we want: $2 \succsim 3 \sim 4 \succsim 5$
- strong centrality $3(/4) \succsim_S 1$ and $2 \succsim_S 5$
- strong centrality often cannot compare immediate neighbors

Few Comparisons: Graphical Illustration

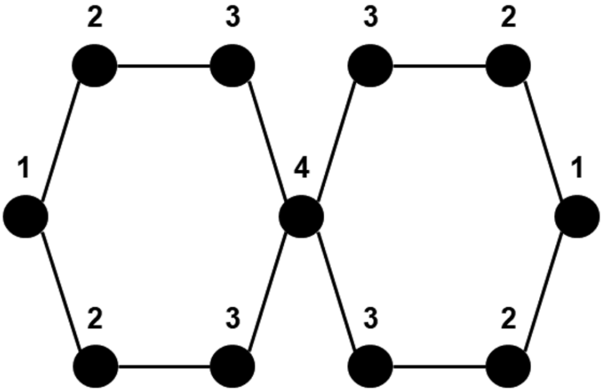

FIG. 4.—No nodes are comparable according to $\succsim_s$.

- No comparisons due to two totally reverse ordinal rankings!

Weak Monotoncity

Strict Ordinal Centrality

An ordinal centrality $\succsim$ on $G = (V, E)$ is strict if $\bar{G}_i \succsim \bar{G}_j$ whenever $i \succsim j$.

- self-inclusive neighborhood: $\bar{G}_i = G_i \cup i$

Weak Monotoncity

$i \succsim_W j$ iff there exists a strict ordinal centrality $\succsim$ such that $i \succsim j$

- a strict ordinal centrality can justify itself through recursive monotonicity

Lemma for Computation Algorithm

every reflexive binary relation $\succsim$ has a unique minimal extension to an ordinal centrality $\succsim \subseteq \succsim_0$. If $\succsim$ satisfies $i \succsim j$ only if $G_i \succsim G_j$, then this extension is strict.

Key Properties of Weak Centrality

Key Properties of Weak Centrality

1. Weak centrality $\succsim_W$ is a strict ordinal centrality.
2. Weak centrality is the maximal strict retraction of degree centrality $\succsim_{deg}$

- isomorphic vertices are always equivalent in the weak-centrality order

Algorithm

1. Set $\succsim_0 = \succsim_{deg}$
2. Define $\succsim_n$ by $i \succsim_n j$ if and only if $\bar{G}_i \succsim_{n-1} \bar{G}_j$

Examples

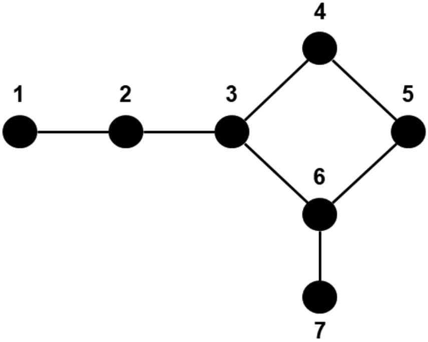

FIG. 5.—Square with two asymmetric spokes.

- strong centrality: $3 \succsim_S 1, 3 \succsim_S 5 \succsim_S 7$ and $6 \succsim_S 4$
- allow self-inclusive set: $2 \succsim 1$ and $6 \succsim 7$
- several rankings; we can select iteratedly by degree centrality algorithm

When WC Is Total: Overlapping Hierarchies

- a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_K\}$
- the subset $S \subseteq V$ dominates $S' \subseteq V$ with respect to $\mathcal{P}$: for each $k = 1, 2, \ldots, K$, we have $\left| S \cap \left(\bigcup_{\ell=k}^{K} V_\ell \right) \right| \geq \left| S' \cap \left(\bigcup_{\ell=k}^{K} V_\ell \right) \right|$

  the subset S contains at least as many vertices in partition element k or higher for every k

Overlapping Hierarchy

$G = (V, E)$ is an overlapping hierarchy if we can find a partition $\mathcal{P}$ such that $\bar{G}_i$ dominates $\bar{G}_j$ with respect to $\mathcal{P}$ whenever i is in a higher partition element than j

Total Order

Given a graph $G = (V, E)$, weak centrality is a total order on V if and only if G is an overlapping hierarchy.

Overlapping Hierarchies: Graphical Illustration

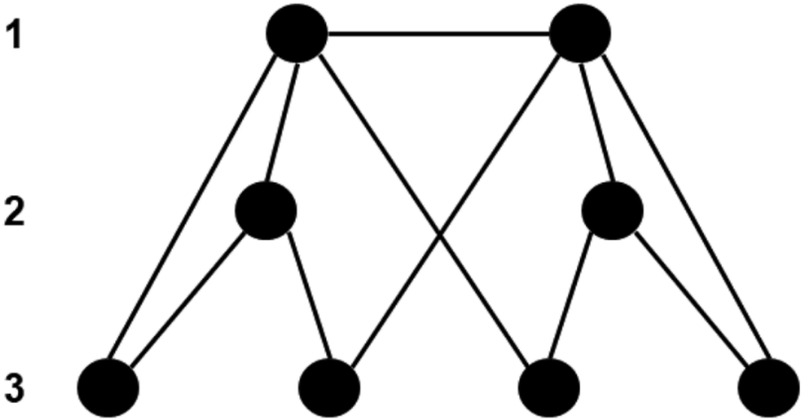

FIG. 6.—An overlapping hierarchy.

Weighted and Directed Graphs

- revision: $G = \left(V, \{w_{ij}\}_{i,j\in V}\right)$, $G_i = \{j : w_{ij} > 0\}$, $\bar{G}_i = G_i \cup \{i\}$
- i $\succsim$ -dominates that of j $\Leftrightarrow$ an injective function $f : G_j \to G_i(\bar{G}_j \to \bar{G}_i)$ such that $f(k) \succsim k$ and $w_{if(k)} \geq w_{jk}$ for every $k \in G_j$

  we take $\omega_{ii} = \omega_{ij} = \omega_{jj}$ (the transitivity issue)

  both neighbors and link strength matter
- strong/weak centrality still holds
- strict ordinal centrality: $\bar{G}_i \succsim \bar{G}_j$ whenever $i \succsim j$ and there is no third vertex k such that $i \gtrnsim k \gtrnsim j$

Algorithm

1. Step 0. Set $\succsim_0 = \succsim_{\deg}$
2. Step $n \geq 1$. Define $\succsim_n$ by $i \succsim_n j$ if and only if there exists a chain $i = i_0 \succsim_{n-1} i_1 \succsim_{n-1} \ldots \succsim_{n-1} i_K = j$ such that $\bar{G}_{i_k} \succsim_{n-1} \bar{G}_{i_{k+1}}$ for each $k = 0, 1, 2, \ldots, K-1$

Centrality in Network Games

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Setup

- the vector of edge weights: $\mathbf{w}_i = (\omega_{i1}, \ldots, \omega_{ii-1}, \omega_{ii+1}, \ldots, \omega_{iN})$
- A symmetric network game of strategic complements is a quadruple $\Gamma = \{(N, G, S, u)\}$ in which:
  1. the set of players in the game is N ;
  2. the players are linked according to the weighted graph $G = \left(N, \{w_{ij}\}_{i,j \in N}\right)$
  3. every player has the same action space S , a complete lattice with partial order $\succcurlyeq$
  4. players share a common payoff function $u : S \times S^{N-1} \times \mathbb{R}_+^{N-1} \to \mathbb{R}$; if the strategy profile is $\mathbf{s} = (s_i, \mathbf{S}_{-i})$, then player i earns utility $u\left(s_i, (\mathbf{s}_{-i}, \mathbf{w}_{-i})\right)$

Setup

Assumptions of Payoff Function

1. invariant to permutations of the indices in $(\mathbf{s}_{-i}, \mathbf{w})$, and if $\omega_j = 0$, then u is invariant to $\mathcal{S}_j = 0$
2. continuous in s according to the order topology along every chain in S
3. supermodular in s, and it has strictly increasing differences in s and $(\mathbf{s}_{-i}, \mathbf{w})$

1. Payoffs depend on network connections.
2. The continuity assumption ensures that the best-response correspondence is always nonempty
3. Supermodularity and increasing differences capture the essence of strategic complements

Strong Centrality and Equilibrium Actions

- A strategy profile $s \in S^N$ represents the centrality $\succsim$ on G if we have $i \succsim j$i □ j if and only if $s_i \succsim s_j$

Strong Centrality and Equilibrium Actions

1. Every pure-strategy Nash equilibrium σ of τ represents an ordinal centrality on G
2. Given any ordinal centrality $\succsim$ on G, there exists a network game of strategic complements τ with a pure-strategy Nash equilibrium σ that represents $\succsim$.

- an equivalence between ordinal centralities and the equilibria of network games
- corollary: ranking action $\Leftrightarrow$ ordinal centrality

Weak Centrality and Equilibrium Actions

- $\bar{\sigma}$ and $\underline{\sigma}$ are the highest and lowest Nash equilibria of τ

Weak Centrality and Equilibrium Actions

1. Given any weighted graph (V, ω_{ij}), the maximal and minimal equilibira of corresponding game represents $\succsim_W$
2. If $i \succsim_W j$, then $\bar{\sigma}_i \succsim \bar{\sigma}_j$ and $\underline{\sigma_i} \succsim \underline{\sigma_j}$

- Intuition: iterative procedure to compute these equilibira preserves any comparison that is supported through an infinite chain of justification

Examples and CSA

- a technology adoption/norm transition issue: at least k neighbors choose active action $\Rightarrow$ the $k-th$ one in weak centrality is the threshold in the maximal equilibrium

  finitely many different behaviors (or products or technologies)
- BCZ(2006): Katz-Bonacich centrality determines equilibrium actions
- Comparative Static Analysis: considering two graphs with the same node set and different (comparable) weighted links, we can deduce the node in a denser graph is more central than in another one by considering the union of them

Discussions and Conclusion

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Discussions

1. Recursive Antimonotonicity and Strategic Substitutes
2. Positional Dominance
3. Relationship to Wolitzky(2013)

Conclusions

- This paper proposes the notion of recursive monotonicity from an intuitive criterion(measure) of a node's significance: the number of neighbors and the significance of neighbors
- Based on this, this paper highlight the common structure of centrality measures and propose a robust ordinal centrality called strong centrality, which is the intersection of all ordinal centrality
- Motivated by the limits of strong centrality (coarse comparisons), this paper proposes another centrality called weak centrality, which is an extraction of degree centrality and can be iteratedly computed with perserving comparisons

Conclusions

- Action and Equilibrium in Strategic Complementary Network Games:
  1. action $\Leftrightarrow$ Ordinal Centrality
  2. Strong centrality: robust among all equilibria
  3. Weak centrality: minimal and maximal equilibria