Traitement de signal
Analyse Spectrale : Fourier, temps-fréquence, temps-échelle

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Reminders on analog signals

- Integrable signals

$$s \in L_1(\mathbb{R}) \longleftrightarrow \int_{-\infty}^{\infty} |s(t)| dt < \infty$$

- Finite energy signals

$$s \in L_2(\mathbb{R}) \longleftrightarrow ||s||^2 = \int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$$

- Scalar product

$$\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} s_1(t) s_2^*(t) dt$$

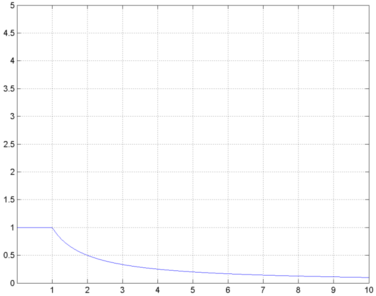
- Remark

$$L_1(\mathbb{R}) \not\subset L_2(\mathbb{R}) \quad \text{and} \quad L_2(\mathbb{R}) \not\subset L_1(\mathbb{R})$$

Reminders - Counter-example 1

- Signal in $L_2(\mathbb{R})$ but not in $L_1(\mathbb{R})$

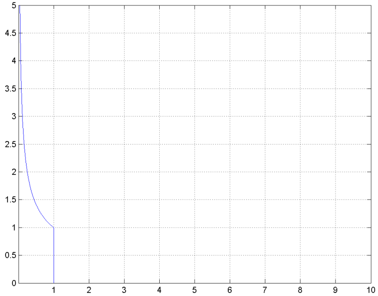
$$x = \begin{cases} 1 & \text{if } |t| < 1 \\ \frac{1}{|t|} & \text{otherwise.} \end{cases}$$



Reminders - Counter-example 1

- Signal $L_1(\mathbb{R})$ but not in $L_2(\mathbb{R})$

$$x = \begin{cases} \frac{1}{\sqrt{t}} & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



History

A few names in the history:

- Joseph Fourier: 1807 (heat equation) [1768-1830]
- Gauß: 1805 (interpolation of orbits of celestial bodies)
- Runge: 1903
- Danielson-Lanczos: 1942
- Good: 1958
- Cooley-Tuckey: 1965 (FFT)
- Frigo-Johnson: 1998 (Fastest Fourier Transform in the West)

Source: Gauß and the history of the Fourier transform. Heideman, Johnson, Burrus, IEEE ASSP Magazine, 1984

History

Lettre de C. Jacobi à A. Legendre, 1830

> *M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde.*

Continuous Fourier transform

- Definition

$$s(t) \xrightarrow{\mathrm{FT}} S(f) = \int_{-\infty}^{\infty} s(t) e^{-\imath 2\pi f t} dt$$

  function of the (dual) variable frequency $f \in \mathbb{R}$
- Existence 1: if $s \in L_1(\mathbb{R})$, then
  - S(f) is continuous and bounded
  - $\lim_{|f|\to\infty} S(f) = 0$
- Existence 2:

$$s \in L_2(\mathbb{R}), \text{ iff } S \in L_2(\mathbb{R})$$

- Inversion:

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{\imath 2\pi f t} df$$

  for almost every t

Fourier transform - properties 1

- Linearity

$$s_1(t) \xrightarrow{\mathrm{FT}} S_1(f), \quad s_2(t) \xrightarrow{\mathrm{FT}} S_2(f)$$

then

$$\forall(\lambda, \mu) \in \mathbb{C}^2 \quad \lambda s_1(t) + \mu s_2(t) \xrightarrow{\mathrm{FT}} \lambda S_1(f) + \mu S_2(f)$$

- Delay/translation

$$\forall b \in \mathbb{R}, \quad s(t-b) \xrightarrow{\mathrm{FT}} e^{-\imath 2\pi b f} S(f)$$

delay/translation invariance in amplitude spectrum

- Modulation:

$$\forall \nu \in \mathbb{R}, \quad e^{\imath 2\pi \nu t} s(t) \xrightarrow{\mathrm{FT}} S(f-\nu)$$

Fourier transform - properties 2

- Scale change

$$\forall \alpha \in \mathbb{R}^*, \quad s(\alpha t) \xrightarrow{\text{FT}} \frac{1}{|\alpha|} S(\frac{f}{\alpha})$$

  turns dilatation onto contraction
- Time inverse: a special case

$$s(-t) \xrightarrow{\text{FT}} S(-f)$$

- Conjugation

$$s^*(t) \xrightarrow{\text{FT}} S^*(-f)$$

- Hermitian symmetry: if s is real, then:
  - $\mathrm{Re}(S(f))$ even, $\mathrm{Im}(S(f))$ odd
  - $|S(f)|$ even, $\arg S(f)$ odd $\pmod{2\pi}$

Fourier transform - properties 3

- Convolution

$$(s_1 * s_2)(t) = \int_{-\infty}^{\infty} s_1(u) s_2(t-u) du = (s_2 * s_1)(t)$$

- Conditions
  - $s_1 \in L_1(\mathbb{R}), s_2 \in L_1(\mathbb{R}) \Rightarrow s_1 * s_2 \in L_1(\mathbb{R})$
  - $s_1 \in L_2(\mathbb{R}), s_2 \in L_2(\mathbb{R}) \Rightarrow s_1 * s_2 \in L_2(\mathbb{R})$

$$(s_1 * s_2)(t) \xrightarrow{\text{FT}} S_1(f) S_2(f)$$

- Parseval-Plancherel equalities: if $s_1, s_2 \in L_1(\mathbb{R})$:
  - $\langle s_1, s_2 \rangle = \langle S_1, S_2 \rangle$
  - $\|s\|^2 = \|S\|^2$

Fourier transform - examples

- $s(t) = e^{-at}u(t), \Re(a) > 0$:
  $S(f) = \frac{1}{a+2\imath\pi f}$
- $s(t) = e^{-at}\cos(2\pi f_0 t)u(t)$:
  $S(f) = \frac{1}{2}\left[\frac{1}{a+2\pi\imath(f-f_0)} + \frac{1}{a+2\pi\imath(f+f_0)}\right]$
- $s(t) = e^{-a|t|}$:
  $S(f) = \frac{2|a|}{a^2+(2\pi f)^2}$

Digital signals - sampling

- For a signal $s(t)$, regularly sampled:

$$s[k] = s(kT)$$

  with T: sampling period; $f_s = 1/T$: sampling frequency

- Sampling theorem: if $S(f) = 0$ for $|f| \geq B$ and $f_s \geq 2B$, the the signal is sampled without information loss (theoretically), with Shannon-Nyquist formula:

$$s(t) = \sum_{k=-\infty}^{\infty} s[k] \mathrm{sinc}\left(\frac{\pi(t-kT)}{T}\right), \mathrm{sinc}(t) = \begin{cases} \frac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

- cardinal theorem of interpolation theory: (Cauchy)-Whittaker, 1915; Nyquist, 1928; Kotelnikov, 1933; Whittaker, 1935; Raabe, 1938; Gabor, 1946; Shannon, 1948; Someya, 1949
- Note: Balian-Low theorem for time-frequency analysis

Digital signals - sampling

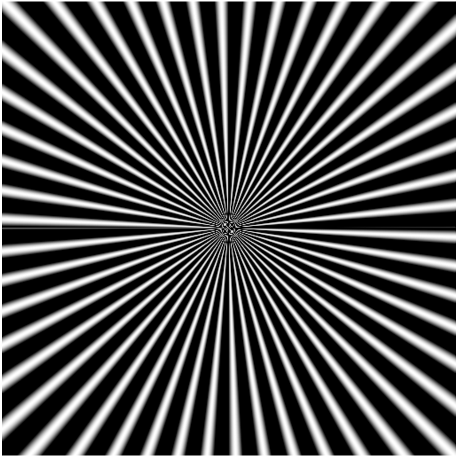


Figure 1: Example of aliasing

Digital signals - sampling

- Comments
  - sufficient but non-necessary condition
  - two-part theorem: (1) sampling, (2) reconstruction
  - caution: jitter, amplitude quantization, noise
  - slow convergence of the sinc, instability
  - signals cannot be time-limited and frequency-bounded
  - extensions to band-limited signals exist (iterative Papoulis-Gerchberg); optimally $2(f_u - f_l)$. Aliases in $\pm$ freq. content do not overlap for integer k:

$$\frac{2f_u}{k} \leq f_s \leq \frac{2f_l}{k-1}$$

  - alternatives: non-regularly sampled data (Lomb-Scargle periodogram); sparse/finite-rate-of-innovation signals (no more than n events per unit of time): compressive sensing, sparse sampling
- Beware of weak analogies between continuous/discrete

Digital signals

- Absolutely convergent sequences

$$(s[k])_{k\in\mathbb{Z}} \in l^1(\mathbb{R}) \Leftrightarrow \sum_{k=-\infty}^{\infty} |s[k]| < \infty$$

- Square-summable sequences

$$(s[k])_{k\in\mathbb{Z}} \in l^2(\mathbb{R}) \Leftrightarrow \sum_{k=-\infty}^{\infty} |s[k]|^2 < \infty$$

- Remark

$$l^1(\mathbb{R}) \subset l^2(\mathbb{R})$$

Discrete-Time Fourier transform (DTFT)

- Definition 1: z-transform

$$(s[k])_{k\in\mathbb{Z}} \xrightarrow{\text{ZT}} S(z) = \sum_{k=-\infty}^{\infty} s[k]z^{-k}$$

- Definition 2: Discrete-Time Fourier transform

$$(s[k])_{k\in\mathbb{Z}} \xrightarrow{\text{DTFT}} S(f) = \sum_{k=-\infty}^{\infty} s[k]z^{-k}, \quad z = e^{\imath 2\pi f}$$

Discrete-Time Fourier transform (DTFT)

- Normalized frequency

$$f = T f_{\text{phys}}$$

- Periodicity

$$f = 1 \ : S(f+1) = S(f)$$

- Existence

$$(s[k])_{k \in \mathbb{Z}} \in l^2(\mathbb{R})$$

- Special case: if $(s[k])_{k \in \mathbb{Z}} \in l^1(\mathbb{R})$, then $S(f)$ is continuous and bounded

DTFT - properties 1

- Linearity

$$\lambda s_1[k] + \mu s_2[k] \xrightarrow{\text{DTFT}} \lambda S_1(f) + \mu S_2(f)$$

- Integer delay/translation

$$s[k-n] \xrightarrow{\text{DTFT}} e^{-\imath 2\pi n f} S(f)$$

- Modulation

$$e^{\imath 2\pi\nu} s[k] \xrightarrow{\text{DTFT}} S(f-\nu)$$

- Time inversion

$$s[-k] \xrightarrow{\text{DTFT}} S(-f)$$

DTFT - properties 2

- Conjugaison

$$s^*[k] \xrightarrow{\text{DTFT}} S^*(-f)$$

- Parseval-Plancherel equalities

$$\sum_{k=-\infty}^{\infty} s_1[k]s_2^*[k] = \int_{-1/2}^{1/2} S_1(f)S_2^*(f)df$$

$$\sum_{k=-\infty}^{\infty} |s[k]|^2 = \int_{-1/2}^{1/2} |S(f)|^2 df$$

DTFT - properties 3

- Convolution

$$(s_1 * s_2)[k] = \sum_{l=-\infty}^{\infty} s_1[l] s_2[k-l] \xrightarrow{\text{DTFT}} S_1(f) S_2(f)$$

- Inversion

$$s[k] = \int_{-1/2}^{1/2} S(f) e^{\imath 2\pi f k} df$$

Fourier series

- If s is periodic (and continuous), $s(t+2\pi) = s(t)$ define:

$$a_k = \frac{1}{\pi} \int s(t) \cos(kt) dt$$

$$b_k = \frac{1}{\pi} \int s(t) \sin(kt) dt$$

then the infinite Fourier series is:

$$S(k) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_n \cos(kt) + b_n \sin(kt)]$$

Discrete Fourier transform

- Definition

$$(s[k])_{0\leq k\leq K-1} \xrightarrow{\mathrm{DTF_K}} \hat{s}[p]_{0\leq p\leq K-1}$$

with

$$\hat{s}[p]_{0\leq p\leq K-1} = \sum_{k=0}^{K-1} s[k]e^{-\imath 2\pi \frac{kp}{K}}$$

- Link to the DTFT : if $s[k] = 0$ pour $k < 0$ et $k \geq K$, then

$$\hat{s}[p] = S\left(\frac{p}{K}\right)$$

i.e. K-sample sampling of DTFT $S(f)$ on $[0, 1]$

- Inversion

$$s[k] = \frac{1}{K}\sum_{p=0}^{K-1} \hat{s}[p]e^{\imath 2\pi \frac{kp}{K}}$$

All Fourier transforms unite: Pontryagin duality

Transform	Original domain	Transform domain
Fourier transform	$\mathbb{R}$	$\mathbb{R}$
Discrete-time Fourier transform (DTFT)	$\mathbb{Z}$	T
Fourier series	T	$\mathbb{Z}$
Discrete Fourier transform (DFT)	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$

All Fourier transforms unite: Pontryagin duality

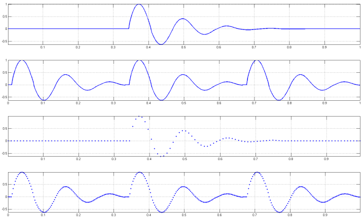


Figure 2: Signals on the different domains

Fast Fourier transform - FFT

- Fast algorithms exist (since Gauss)

$$\mathrm{FFT_K} \Rightarrow \text{complexity of } O(K\log_2(K)) \text{ operations}$$

- Cyclic or periodic convolution: Let $(s_1[k])_{0\leq k\leq K-1}$ and $(s_2[k])_{0\leq k\leq K-1}$

$$(s_1 \circledast s_2)[k] \stackrel{\mathrm{DTF_K}}{\longrightarrow} \hat{s}_1[p]\cdot\hat{s}_2[p]$$

where $(s_1 \circledast s_2)[k]$ represents the K-point convolution of the periodized sequences:

$$(s_1 \circledast s_2)[k] = \sum_{l=0}^{k} s_1[l]s_2[k-l] + \sum_{l=k+1}^{K-1} s_1[l]s_2[K+k-l]$$

Reminders - Key message

- Nature of the data and the transform
  - Continuous and discrete natures ARE different
  - Generally stuff works
  - Intuition may be misleading (ex.: FFT on 8-sample signals, non proper windows)
  - Sometimes special care is needed: re-interpolation, pre-processing to avoid edge effects, instabilities, outliers

Side dishes - Hilbert transform

- linear operator (Cauchy kernel)
- important tool in signal processing

$$Hf(t) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{+\infty}\frac{f(s)}{t-s}ds$$

$$\mathcal{F}(Hf(t))(\omega) = (-\imath\,\text{sign}(\omega)).\mathcal{F}(f)(\omega)$$

- analytic signal: $f(t) + \imath Hf(t)$
- examples of Hilbert pairs:
  - $\cos(t) \to \sin(t) \quad (e^{\imath t})$
  - $\frac{1}{1+t^2} \to \frac{t}{1+t^2}$
- useful for envelope extraction, time-frequency processing

Windows

- Several uses
  - apodization, tapering (edges, jumps)
  - "stationarizing"
  - spectral estimation, filter design

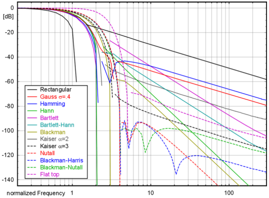


Figure 3: Origin: Wikipedia

- Many (parametric) designs:
  - Rect., Bartlett, Hann, Hamming, Kayser, Chebychev, Blackman-Harris,...
  - Gen. cosine: $a_0 - a_1 \cos(\frac{2\pi n}{M-1}) + a_2 \cos(\frac{4\pi n}{M-1}) - a_3 \cos(\frac{6\pi n}{M-1})$

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Reminders: set averages

- $s(n)$: discrete time random process (stationary stochastic process)
- expectation:
$$\mu_s(n) = E\{s(n)\}$$
- variance:
$$\sigma_s^2(n) = E\{|s(n) - \mu_s(n)|^2\}$$
- autocovariance:

$$c_s(k,l) = E\{(s(k) - \mu_s(k))(s(l) - \mu_s(l))^*\}$$

Reminders: power spectral density

For an autocorrelation ergodic process:

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} s(n+k)s(n) = r_{ss}(k)$$

- if $s(n)$ is known for every n, power spectrum estimation
- caveat 1: samples are not unlimited $[0,\ldots,N-1]$, sometimes small
- caveat 2: corruption (noise, interfering signals)

Recast the problem: from the biased estimator of the ACF

$$\hat{r}_{ss}(k) = \sum_{n=0}^{N-1-k} s(n+k)s(n)$$

estimate power spectrum (periodogram)

$$\hat{P}_x(e^{\imath\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_{ss}(k)e^{\imath k\omega}$$

Time and frequency resolution

- Energy

$$E = \int |s(t)|^2 dt = \int |S(f)|^2 df$$

- Time or frequency location

$$\bar{t} = 1/E \int t|s(t)|^2 dt \qquad \overline{f} = 1/E \int f|S(f)|^2 df$$

- Energy dispersion

$$\Delta t = \sqrt{1/E \int (t - \bar{t})^2 |s(t)|^2 dt}$$

$$\Delta f = \sqrt{1/E \int (f - \overline{f})^2 |S(f)|^2 df}$$

Heisenberg-Gabor inequality

- Theorem (Weyl, 1931)
  If $s(t), ts(t), s'(t) \in \mathrm{L}^2$ then

$$\|s(t)\|^2 \leq 2\|ts(t)\|\|s'(t)\|$$

- Equality:
  Iff $s(t)$ is a modulated Gaussian/Gabor elementary function:

$$s'(t)/s(t) \propto t$$

$$s(t) = C \exp[-\alpha(t - t_m)^2 + \imath 2\pi\nu_m(t - t_m)]$$

- Proof
  Integration by part + Cauchy-Schwarz

Uncertainty principle (UP)

- Time-frequency UP
  For finite-energy every signal $s(t)$, with Δt and Δf finite:
  $$\Delta t \Delta f \geq \frac{1}{4\pi}$$
  with equality for the modulated Gaussian only
- Principles
  $$\|s'(t)\|^2 = |\imath 2\pi|^2 \|fS(f)\|^2$$
- Observations
  - Fourier (continuous) fundamental limit: arbitrary "location" cannot be attained both in time and frequency
  - have to choose between time and frequency locations
  - Gaussians are "the best"

Uncertainty principle (UP) for project management

Applies to other domains

Figure 4: Dilbert

Uncertainty principle - time

- One may write

$$s(t) = \int s(u)\delta(t-u)du$$

  - $\delta(t)$ is neutral w.r.t. convolution
  - interpreted as a decomposition of $s(t)$ onto a basis of shifted $\delta(t-u)$: $\Delta t = 0$ at
  - FT of basis functions: $e^{-\imath 2\pi f t}$: $\Delta f = \infty$

UP: as a limit of $0 \times \infty$

Uncertainty principle - frequency

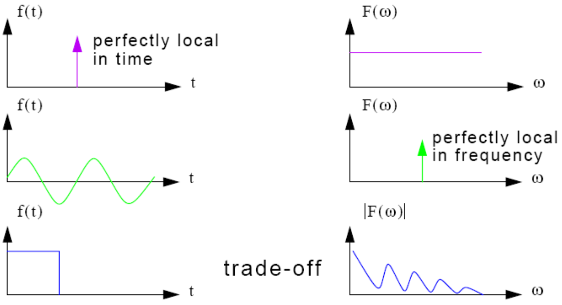
- One may write

$$s(t) = \int S(f) e^{\imath 2\pi f t} df$$

  - interpreted as a decomposition on pure waves $e^{\imath 2\pi f t}$: $\Delta t = \infty$
  - FT of basis functions: $\delta(f - t)$: $\Delta f = 0$

UP: as a limit of $\infty \times 0$

Uncertainty principle - illustration



Basis formalism interpretation

- Orthonormality

$$\langle e^{\imath 2\pi f t}, e^{\imath 2\pi g t} \rangle = \delta(f-g)$$

$$\langle \delta(f), \delta(g) \rangle = \delta(f-g)$$

- Scalar product

$$S(f) = \langle s(t), e^{\imath 2\pi f t} \rangle$$

$$s(t) = \langle s(t), \delta(f) \rangle$$

- Matching of a signal with a vector, a basis function (pure wave, Dirac)
  - Synthesis: continuous sum of orthogonal projection onto basis functions
  - Relative interest of the two bases? Other bases? (Walsh-Hadamard, DCT, eigenbase)
  - How to cope with mixed resolution?

Sliding window Fourier transform

- Principles
  Fourier analysis on time-space slices of the continuous $s(t)$ with a sliding window $h(t-\tau)$
- Short-term/short-time Fourier transform (STFT)

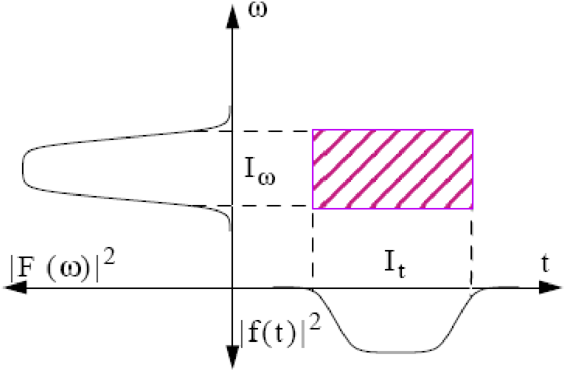
$$S_s(\tau, f; h) = \int s(t)h^*(t-\tau)e^{-\imath 2\pi f t}dt$$

- Wider domain of applications than FT
  - depend on h
  - FT as a peculiar instance (valid for other transforms: not a new tool, only a more versatile "leatherman"-like multi-tool)

Sliding window Fourier transform

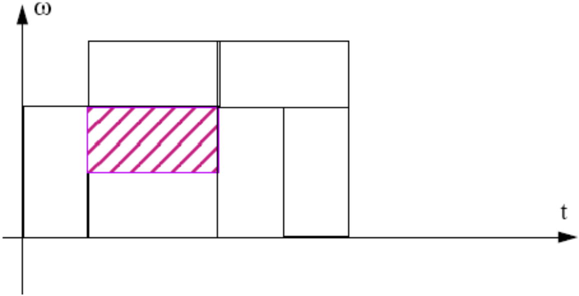
Figure 5: Leatherman wave black

Sliding window Fourier transform - illustration



Sliding window Fourier transform - time-freq. completude

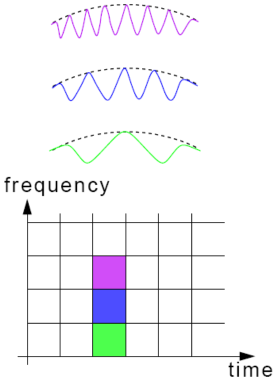
Notion of a "complete" descrition (*i.e.* somehow invertible)



Sliding window Fourier transform - windows

- Related to frequency analysis
  Depend on the window choice h (shape, length)
- Continuous time windows
  - rectangular: poor frequency resolution
  - gaussian: best time-frequency trade-off? (Gabor, 1946)
- Discrete time windows
  - τ discretized (jumps vs. redundancy)
  - different criteria: side lobes, *equiripple*, apodizing; Bartlett, Hamming, Hann, Blackmann-Harris, Blackmann-Nutall, Kaiser, Chebychev, Bessel, Generalized raised cosine, Lánczos, Flat-top,...

Sliding window Fourier transform - paving



Sliding window Fourier transform - reconstruction

- Simple analogy
  - synthesis: what two numbers add to result 3
  - $a + b = 3$: infinite number of solutions, *e.g.* $2945.75 + (-2942.75)$ but irrelevant
  - assume they are integers?
  - assume they are positive? $1 + 2 = 3$ or $2 + 1 = 3$
  - aim: increase interpretability, information compaction $(0 + 3 = 3)$, reduce overshoot

Sliding window Fourier transform - reconstruction

- Redundant transformation!
- Inversion

$$s(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S_s(u, \xi; h) g(t-u) e^{\imath 2\pi t u} du d\xi,$$

provided that

$$\int_{-\infty}^{+\infty} g(t) h^*(t) dt = 1.$$

(perfect reconstruction, no information loss)

  - special case: admissible normalized window h

$$g(t) = h(t)$$

    but not the only solution (truncated $\sin$)
  - a bit more involved for discrete time, less if only approximate

Sliding window Fourier transform - spectrogram

- Definition

$$|S_s(\tau, f; h)|^2$$

- The spectrogram is a (bilinear) time-frequency distribution

$$E = \iint |S_s(\tau, f; h)|^2 d\tau df$$

for normalized admissible window h

- Parseval formula

$$\langle s_1, s_2 \rangle = \iint S_{s_1}(\tau, f; h) S_{s_2}(\tau, f; h) d\tau df$$

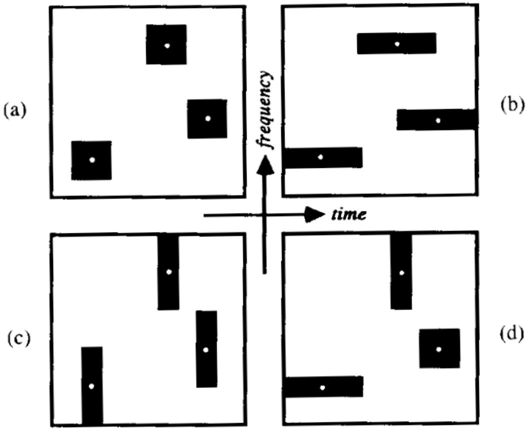
Sliding window Fourier transform - monoresolution

- Reason: basis functions

$$h(t-\tau)e^{\imath 2\pi f t}$$

  all possess similar resolution
- Examples:
  - $s(t) = \delta(t-t_0) \longrightarrow |S_s(\tau, f; h)|^2 = |h(t_0-\tau)|^2$
  - $s(t) = e^{\imath 2\pi f_0 t} \longrightarrow |S_s(\tau, f; h)|^2 = |H(f_0-f)|^2$
- Uses
  - for long range oscillatory signals, long windows are necessary
  - for short range transient, short windows needed
  - possibility to use several in parallel
  - incentive to use several ones simultaneously

Sliding window Fourier transform - illustration



Other time frequency distributions

- Quadratic or bilinear distributions
  - Wigner-Ville and avatars (smoothed, pseudo-, reweighted)
  - Cohen class (WV, Rihaczek, Born-Jordan, Choi-Williams)
  - property based: covariance, unitarity, inst. freq. & group delay, localization (for specific signals), support preservation, positivity, stability, interferences
  - Bertrand class, fractional Fourier transforms, linear canonical transformation (4 param.: FT, fFT, Laplace, Gauss-Weierstrass, Segal-Bargmann, Fresnel transforms)
  - generally not applied in more than 1-D

Wavelets and multiresolution

- New transformation group:

$$\text{affine} = \text{translation} + \text{dilation}$$

- Basis functions:

$$h_{\tau,a}(t) = \frac{1}{\sqrt{a}} h\left(\frac{t-\tau}{a}\right)$$

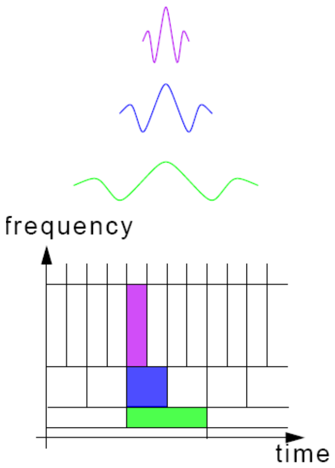
  - $a > 1$: dilation
  - $a < 1$: contraction
  - $1/\sqrt{a}$: energy normalization
  - group invariant measure:

$$d(t,a) = \frac{dtda}{a^2}$$

  - multiresolution (vs. monoresolution)

$$h_{\tau,a}(t) \xrightarrow{\text{FT}} \sqrt{a} H(af) e^{-\imath 2\pi f \tau}$$

Wavelets and multiresolution - illustration



Continuous wavelets transform (CWT)

- Definition

$$C_s(\tau, a) = \int s(t) h^*_{\tau,a}(t) dt$$

- Vector interpretation

$$C_s(\tau, a) = \langle s(t), h_{\tau,a}(t) \rangle$$

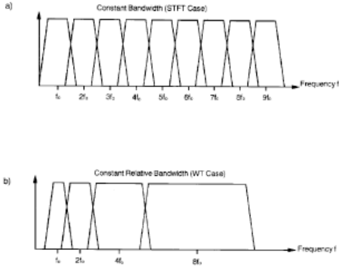
Projection onto time-scale atoms (vs. time-frequency)

- Redundant transform!
- Parseval

$$C_s(\tau, a) = \langle X(f), H_{\tau,a}(f) \rangle$$

related to constant-Q filter banks

STFT vs. CWT



a)
Constant Bandwidth (STFT Case)
Frequency f
b)
Constant Relative Bandwidth (WT Case)
Frequency f

STFT vs. CWT

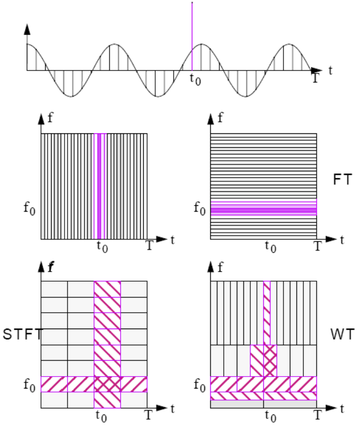
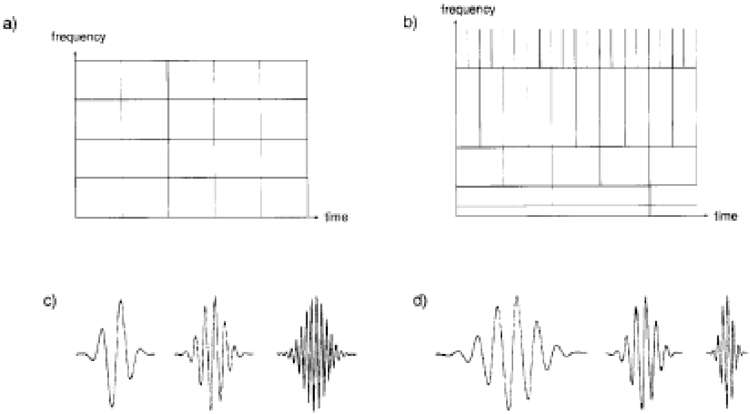


Figure 6: Dirac and sin

STFT vs. CWT



Continuous wavelets transform (CWT)

- Wavelet existence: admissibility criterion

$$0 < A_h = \int_0^{+\infty} \frac{\widehat{G}^*(\nu)H(\nu)}{v} dv = \int_{-\infty}^{0} \frac{\widehat{G}^*(\nu)H(\nu)}{v} dv < \infty$$

  generally normalized to 1
- Induces band-pass property:
  - necessary condition: $|H(0)| = 0$, or zero-average shape
  - amplitude spectrum neglectable w.r.t. $|v|$ at infinity
- examples: Morlet-Gabor (non. adm.) and "mexican hat"

$$h(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} e^{-\imath 2\pi f_0 t} \qquad h(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \left(1 - \frac{t^2}{\sigma^2}\right) e^{-\frac{t^2}{2\sigma^2}}$$

Continuous wavelets transform (CWT)

- Inversion

$$s(t) = \iint C_s(u,a) g_{u,a}(t) \frac{duda}{a^2}$$

- Scalogram

$$|C_s(t,a)|^2$$

- Energy conversation

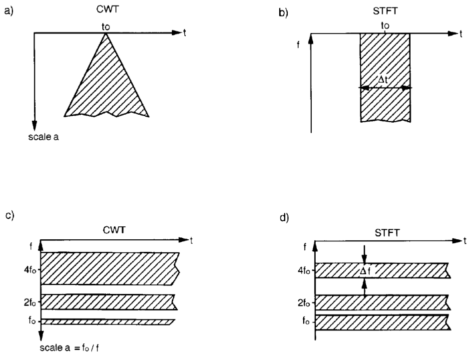
$$E = \iint |C_s(t,a)|^2 \frac{dtda}{a^2}$$

- Parseval

$$\langle s_1, s_2 \rangle = \iint C_{s_1}(t,a) C^*_{s_2}(t,a) \frac{dtda}{a^2}$$

STFT vs. CWT

Principles: impulse and 3 sines



Demo?

Discrete Fourier transform

- Definition

$$(s[k])_{0\leq k\leq K-1} \xrightarrow{\mathrm{DTF_K}} \hat{s}[p]_{0\leq p\leq K-1}$$

with

$$\hat{s}[p]_{0\leq p\leq K-1} = \sum_{k=0}^{K-1} s[k]e^{-\imath 2\pi\frac{kp}{K}}$$

- Link to the DTFT : if $s[k] = 0$ pour $k < 0$ et $k \geq K$, then

$$\hat{s}[p] = S\left(\frac{p}{K}\right)$$

i.e. K-sample sampling of DTFT $S(f)$ on $[0, 1]$

- Inversion

$$s[k] = \frac{1}{K}\sum_{p=0}^{K-1} \hat{s}[p]e^{\imath 2\pi\frac{kp}{K}}$$

Question: can we discretize time-frequency/time-scale?

- In time-frequency (STFT):

$$S_s(\tau, f; h) = \int s(t)h^*(t-\tau)e^{-\imath 2\pi f t}$$

regular sampling is possible

$$s_{m,n} = S_s(nt_0, mf_0), \quad (m,n) \in \mathbb{Z}$$

- In time-scale (CWT):

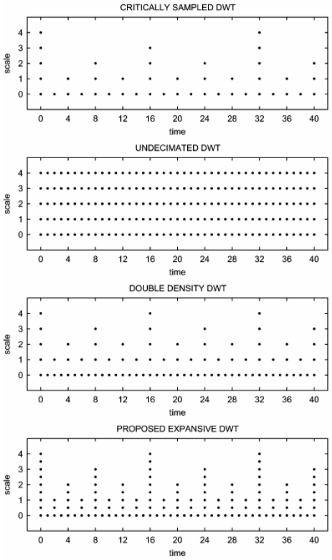
$$C_s(\tau, a) = \int s(t)\psi^*_{\tau,a}(t)dt, \quad \psi_{\tau,a}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-\tau}{a}\right)$$

can one sample with

$$c_{j,k} = C_s(kb_0a_0^j, a_0^j), \quad (j,k) \in \mathbb{Z}$$

and be able to recover $s(t)$?

Typical discretization/sampling patterns



Question: how may this representation be sampled?

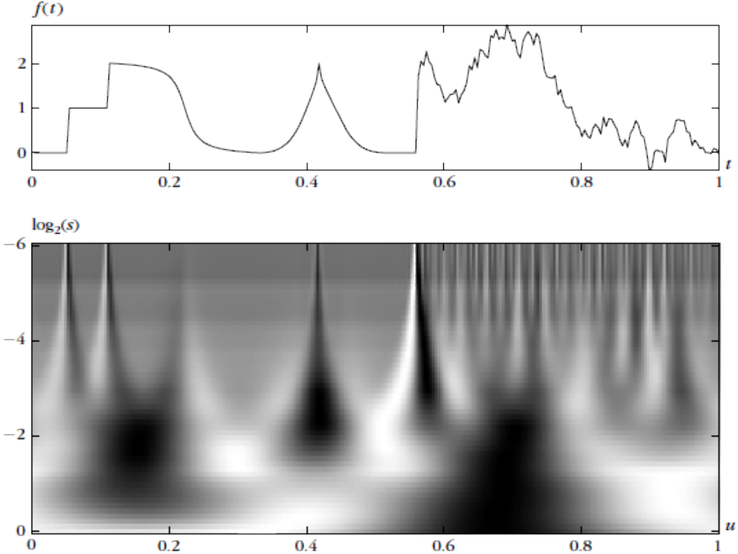
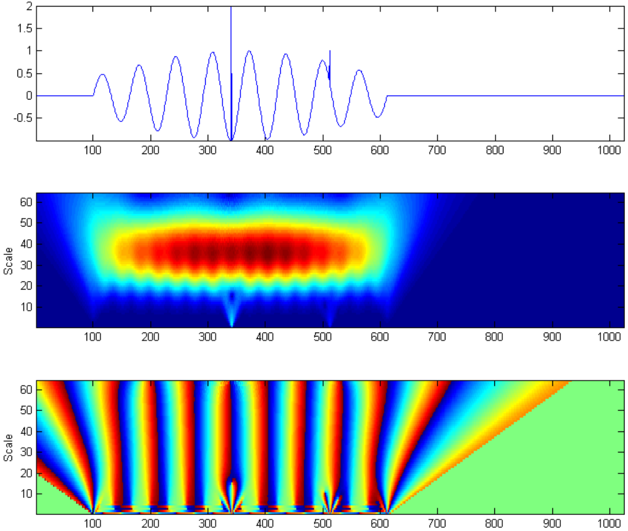
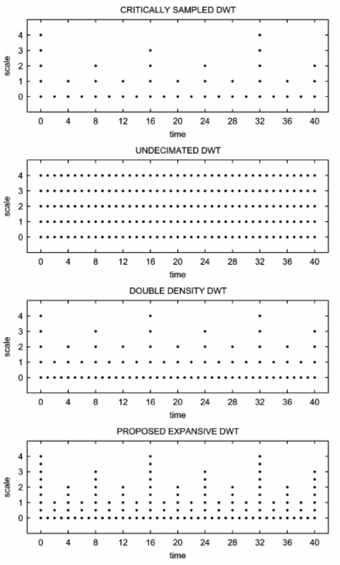


Figure 7: Real wavelet transform (Mexican hat)

Question: how may this representation be sampled?



Potential discretizations



Link with frames

- Definition: a sequence $e_n, n \in \mathbb{N}$ in Hilbert space H is a frame iff there exists $A > 0$ et $B < +\infty$ so that, $\forall s \in H$

$$A\|s\|^2 \leq \sum_n |\langle s, e_n \rangle|^2 \leq B\|s\|^2$$

  with describe s in a stable way
- Operator $F : s \to \langle s, e_n \rangle$ is continuous (infimum), left invertible (supremum)
- One may construct the sequence $\tilde{e}_n = (F^*F)^{-1}e_n$ (with F^* adjoint operator of F)
- $\tilde{e}_k$ is also a frame, called dual frame

Link with frames

- F: operator in a Hilbert space
- If F is linear, continuous (or bounded), then there exists a unique continuous operator, the adjoint such that:

$$\forall x, y \in H, \langle Fx, y\rangle = \langle x, F^{\dagger} y\rangle$$

Link with frames

- Reconstruction:

$$s = \sum_n \langle s, e_n \rangle \tilde{e}_n$$

- With wavelets $\tilde{e}_n = \psi_{j,k}$
- Iterative reconstruction:
  1. $s_0(t) = 0$
  2. For $n \geq 1$

$$s_n(t) = \frac{2}{A+B} \sum_{j,k} c_{j,k} \psi_{j,k}(t) + R_n(t)$$

$$R_n(t) = s_{n-1}(t) - \frac{2}{A+B} \sum_{j,k} \langle s_{n-1}(t), \psi_{j,k} \rangle \psi_{j,k}$$

- Convergence:

$$\|s - s_n\| \leq \frac{B}{A} \left(\frac{B-A}{A+B} \right)^{n+1} \|s\|$$

Link with frames

- Result 1 (Daubechies, 1984): there exists a wavelet frame if

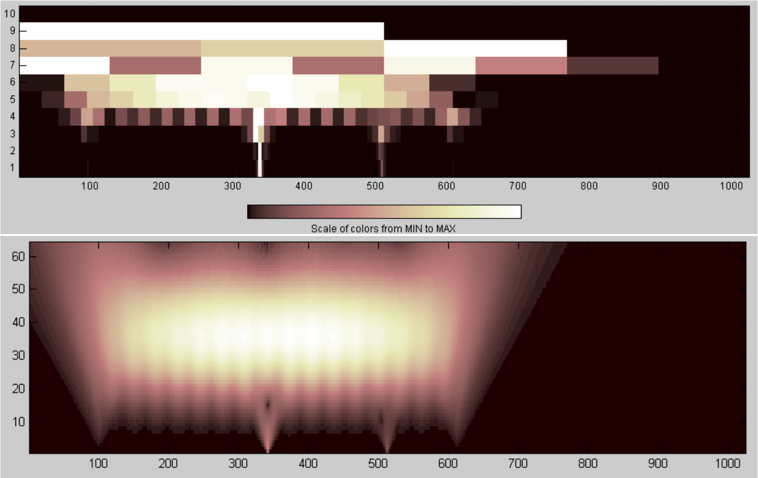
$$a_0 b_0 < C,$$

  C depends on wavelet ψ. A frame is generally redundant
-
- Result 2 (Meyer, 1985): there exist an orthonormal basis for a specific ψ (non trivial, Meyer wavelet) and

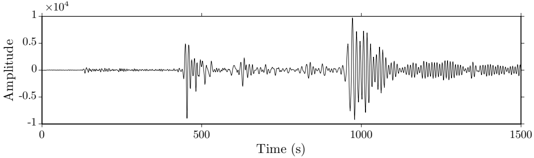
$$a_0 = 2 \qquad b_0 = 1$$

- Result 3 (Mallat, Meyer, 1986): major link between multiscale analysis (D. Marr, 1982) and filter banks

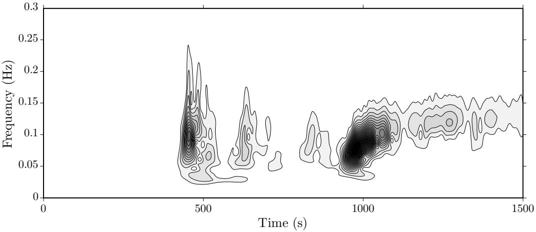
Illustration 2



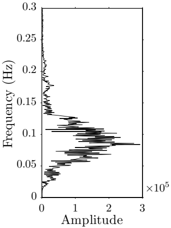
Time-Frequency Analysis



(a) Seismic profile



(b) Wavelet transform modulus



(c) Spectrum

Figure 9: Earthquake occurred in April 2007 at New Zealand (epicentral distance 87.1°, depth 53.7 km) recorded at Chile by a Geoscope station.

Short Time Fourier Transform

- Definition:

$$S_u(\tau,\xi) = \langle u, g_{\tau,\xi}(t)\rangle = \int_{-\infty}^{\infty} u(t) g^*_{\tau,\xi}(t)dt$$

- Family of functions:

$$g_{\tau,\xi}(t) = g(t-\tau)e^{i\xi t}$$

- Operators on $g(t)$:
  - $\tau \quad \Longrightarrow \quad$ Translation operator.
  - $e^{i\xi t} \quad \Longrightarrow \quad$ Modulation operator.
- The signal spread (a.k.a. resolution) in the time-frequency plane is independent of τ and ξ:

$$\sigma_t^2 = \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \qquad \sigma_\omega^2 = \int_{-\infty}^{\infty} \omega^2 |\hat{g}(\omega)|^2 d\omega$$

Short Time Fourier Transform

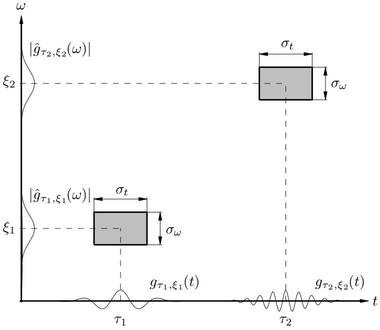


Figure 10: STFT Resolution

Time-Frequency Resolution

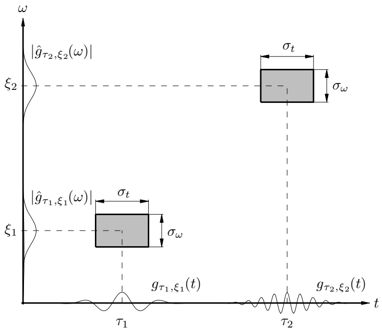


Figure 11: Linear resolution vs. logarithmic resolution

From STFT to CWT

For the gaussian window:

$$g(t) = \pi^{-1/4}\sigma^{-1/2}e^{-\frac{t^2}{2\sigma^2}}$$

the STFT family of function is:

$$g_{\tau,\xi}(t) = \pi^{-1/4}\sigma^{-1/2}e^{-\frac{(t-\tau)^2}{2\sigma^2}}e^{i\xi t}$$

- 1) To have log. resolution we can change the length of $g(t)$ with the frequency,

$$g'_{\tau,\xi}(t) = \pi^{-1/4}(\sigma/\xi)^{-1/2}e^{-\frac{(t-\tau)^2\xi^2}{2\sigma^2}}e^{i\xi t}$$

- 2) To achieve separable operators, we must add and extra phase term:

$$g''_{\tau,\xi}(t) = \pi^{-1/4}(\sigma/\xi)^{-1/2}e^{-\frac{(t-\tau)^2\xi^2}{2\sigma^2}}e^{i\xi(t-\tau)}$$

From STFT to CWT

- 3) Defining $\lambda = \omega_0/\xi$ (scale) we have a wavelet **like** family of functions:

$$g''_{\tau,\lambda}(t) = \pi^{-1/4} \left(\frac{\omega_0}{\sigma\lambda}\right)^{-1/2} e^{-\frac{1}{2}\left(\frac{(t-\tau)\omega_0}{\lambda\sigma}\right)^2} e^{i\omega_0\left(\frac{t-\tau}{\lambda}\right)}$$

When $\sigma/\omega_0 = 1$, $g''_{\tau,\lambda}(t)$ is the Morlet wavelet:

$$g''_{\tau,\lambda}(t) = \frac{1}{\sqrt{\lambda}}\psi\left(\frac{t-\tau}{\lambda}\right) \qquad \text{with} \qquad \psi(t) = \pi^{1/4} e^{-t^2/2} e^{i\omega_0 t}$$

From STFT to CWT

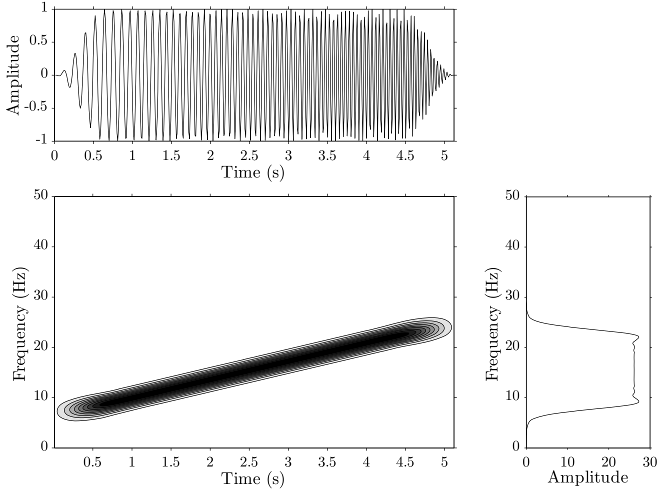
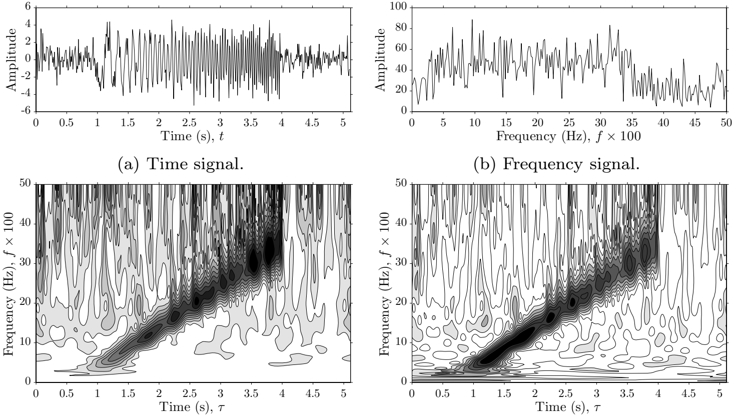


Figure 12: Fourier and short time Fourier transforms of a constant amplitude chirp signal.

From STFT to CWT



(a) Time signal.

(b) Frequency signal.

(c) Amplitude normalized.

(d) Energy normalized (CWT).

Figure 13: Fourier and continuous wavelet transforms of a constant amplitude chirp signal among white Gaussian noise.

From STFT to CWT

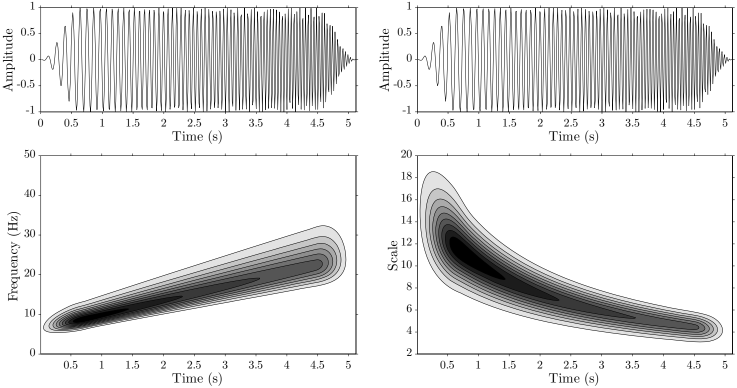


Figure 14: Continuous wavelet transform of a constant amplitude chirp signal.

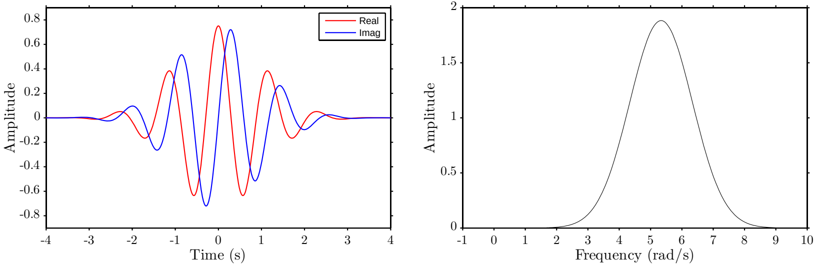
CWT: Morlet wavelet

Approximate Mother wavelet (not **fully** zero-mean!)

$$\psi(t) = \pi^{1/4} e^{-t^2/2} e^{i\omega_0 t}$$

Exact Morlet wavelet

$$\psi(t) = \pi^{-1/4} \left(e^{i\omega_0 t} - e^{-\omega_0^2/2} \right) e^{-t^2/2}$$

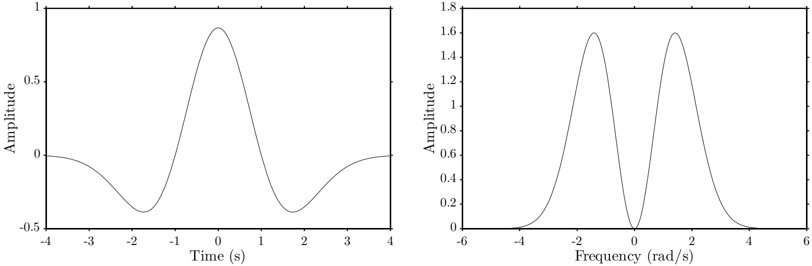


Note that $\|\psi(t)\| = 1$.

CWT: Mexican hat wavelet

Mother wavelet:

$$\psi(t) = \frac{2}{\sqrt{3\sigma}}\pi^{-1/4}\left(1 - \frac{t^2}{\sigma^2}\right)e^{-t^2/2\sigma^2}$$



CWT resolution

Energy center:

- Time

$$\tau = \frac{1}{\left\|\psi(t)\right\|^2} \int_{-\infty}^{\infty} t \left|\psi(t)\right|^2 dt$$

- Frequency

$$\eta = \frac{1}{2\pi \left\|\psi(t)\right\|^2} \int_{-\infty}^{\infty} \omega \left|\hat{\psi}(\omega)\right|^2 d\omega$$

Energy dispersion:

- Time

$$\sigma_t^2 = \frac{1}{\left\|\psi(t)\right\|^2} \int_{-\infty}^{\infty} (t-\tau)^2 \left|\psi(t)\right|^2 dt$$

- Frequency

$$\sigma_\omega^2 = \frac{1}{2\pi \left\|\psi(t)\right\|^2} \int_{-\infty}^{\infty} (\omega-\eta)^2 \left|\hat{\psi}(\omega)\right|^2 d\omega$$

CWT resolution

Time and frequency resolution with scale:

- Time resolution

$$\sigma_t^2(\lambda) = \frac{1}{\left\|\psi(t)\right\|^2}\int_{-\infty}^{\infty}(t-\tau)^2\left|\frac{1}{\sqrt{\lambda}}\psi\left(\frac{t-\tau}{\lambda}\right)\right|^2 dt = \lambda^2\sigma_t^2$$

- Frequency resolution

$$\sigma_\omega^2(\lambda) = \frac{1}{2\pi\left\|\psi(t)\right\|^2}\int_{-\infty}^{\infty}\left(\omega-\frac{\eta}{\lambda}\right)^2\left|\sqrt{\lambda}\hat{\psi}(\lambda\omega)e^{-i\omega\tau}\right|^2 d\omega = \frac{\sigma_\omega^2}{\lambda^2}$$

CWT inverses

Due to the CWT redundancy, there exist an infinite number of inverses.

- Single integral formula

$$u(t) = \frac{1}{C_{\psi_1,\delta}} \int_{-\infty}^{\infty} \frac{Wu(t,\lambda)}{|\lambda|^{3/2}} d\lambda$$

with

$$C_{\psi_1,\delta} = \int_{-\infty}^{\infty} \frac{\hat{\psi}_1^*(\omega)}{|\omega|} d\omega$$

Admissibility condition:

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}_1(\omega)|}{|\omega|} d\omega < \infty$$

CWT inverses

- Least-square approach

$$u(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Wu(\tau, \lambda) \psi_{\tau,\lambda}(t) \frac{d\lambda \, d\tau}{\lambda^2}$$

with

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

For continuously differentiable functions, the admissibility condition reduces to:

$$\int_{-\infty}^{\infty} \psi(t) dt = \hat{\psi}(0) = 0$$

CWT inverses

- Generalization

$$u(t) = \frac{1}{C_{\psi_1,\psi_2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Wu(\tau,\lambda)\psi_{2_{\tau,\lambda}}(t) \frac{d\lambda\, d\tau}{\lambda^2}$$

with

$$C_{\psi_1,\psi_2} = \int_{-\infty}^{\infty} \frac{\hat{\psi}_1^*(\omega)\hat{\psi}_2(\omega)}{|\omega|} d\omega$$

Admissibility condition:

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}_1(\omega)||\hat{\psi}_2(\omega)|}{|\omega|} d\omega < \infty$$

CWT implemetation

Fair sampling: Constant information per sample.

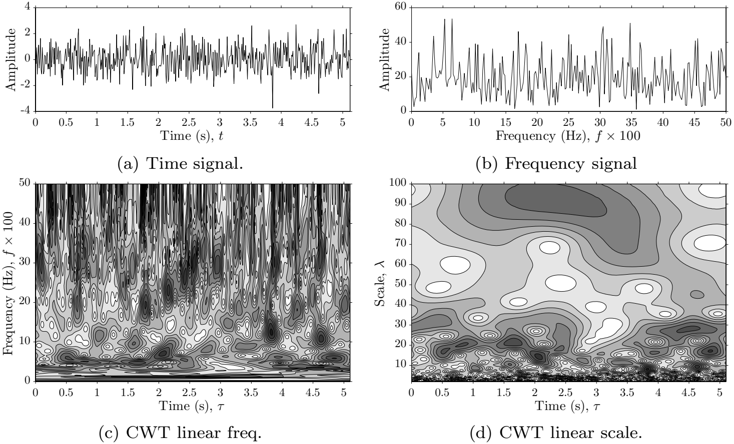


Figure 15: CWT of white Gaussian noise.

CWT implemetation

Continuous wavelet family

$$\psi_{\tau,\lambda}(t) = \frac{1}{\sqrt{|\lambda|}}\psi\left(\frac{t-\tau}{\lambda}\right)$$

Sampling:

- Scale $\lambda = a_0^j$
- Time $t = nT$, $\tau = rb_0a_0^j$
- Continuous wavelet family sampling

$$\psi_{r,j}[n] = a_0^{-j/2}\psi\left(\frac{nT - rb_0a_0^j}{a_0^j}\right)$$

- Discrete sequences: $a_0 = 2$, $j \to j + v/V$, $V \in \mathbb{N}$ $v \in [0, V-1]$

$$\psi_{r,j}^{v}[n] = \frac{1}{\sqrt{2^{j+v/V}}}\psi\left(\frac{nT - r2^jb_0}{2^{j+v/V}}\right)$$

Frame of Wavelets: Main ideas

- Stable reconstruction sii $A > 0$ y $B < \infty$

$$A \left\| u \right\|^2 \leq \sum_{r,j} \left| \langle u, \psi_{r,j} \rangle \right|^2 \leq B \left\| u \right\|^2$$

- Exact rec.: Different family functions in analysis and synthesis,

$$u[n] = \sum_r \sum_j W u[r,j] \widetilde{\psi}_{r,j}[n]$$

- Approximation (first iteration)

$$u[n] \simeq \frac{2}{A+B} \sum_r \sum_j W u[r,j] \psi_{r,j}[n]$$

with an error of $R\, u[n]$,

$$\|R\| \leq \frac{B-A}{B+A} = \frac{B/A - 1}{B/A + 1}$$

Frame bounds of the Morlet wavelet

	$V=2$			$V=3$		
b_0	A	B	B/A	A	B	B/A
0.5	6.019	7.820	1.299	10.295	10.467	1.017
1.0	3.009	3.910	1.230	5.147	5.234	1.017
1.5	1.944	2.669	1.373	3.366	3.555	1.056
2.0	1.173	2.287	1.950	2.188	3.002	1.372

	$V=4$		
b_0	A	B	B/A
0.5	13.837	13.846	1.0006
1.0	6.918	6.923	1.0008
1.5	4.540	4.688	1.032
2.0	3.013	3.910	1.297

Frame bounds of the Mexican hat wavelet

	$V=1$			$V=2$		
b_0	A	B	B/A	A	B	B/A
0.25	13.097	14.183	1.083	27.273	27.278	1.0002
0.5	6.546	7.092	1.083	13.673	13.639	1.0002
1.0	3.223	3.596	1.116	6.768	6.870	1.015
1.5	0.325	4.221	12.986	2.609	6.483	2.485

	$V=3$			$V=4$		
b_0	A	B	B/A	A	B	B/A
0.25	40.914	40.194	1.0000	54.552	54.552	1.0000
0.5	20.457	20.457	1.0000	27.276	27.276	1.0000
1.0	10.178	10.279	1.010	13.586	12.690	1.007
1.5	4.659	9.009	1.947	6.594	11.590	1.758

Main advantage and drawbacks

Advantage

- Flexibility.
- Controlled redundancy.

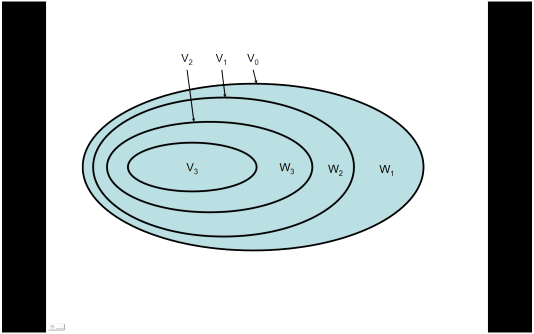
Drawbacks

- No perfect reconstruction in practice.
- Non-redundant decompositions are difficult to achieve.

Multiresolution analysis (MRA)

- Definition: a multiresolution analysis in $L^2(\mathbb{R})$ is a sequence of linear vector subspaces (closed) of $L^2(\mathbb{R})$ satisfying axioms:
  1. $(V_j)_{j\in\mathbb{Z}}$ is nested (embedded):
  $$\forall j \in \mathbb{Z}, \quad V_{j+1} \subset V_j$$
  2. Limits
  $$\lim_{j\to+\infty}(V_j) = \{0\}$$
  $$\lim_{j\to-\infty}(V_j) \text{ dense in } L^2(\mathbb{R})$$
  3. A function $s(t) \in V_j$ iff its dilated $s(t/2) \in V_{j+1}$
  4. There exists (hypothesis) $\phi(t) \in L^2(\mathbb{R})$, called scaling function or father wavelet such that $\{\phi(t-k), k \in \mathbb{Z}\}$ is an orthonormal basis of V_0

Multiresolution analysis (MRA) - nested subspaces



MRA - interpretation

- A MRA of $s(t)$ consists in computing orthogonal projections of $s(t)$ onto subspaces V_j
- Inclusion relation entails that the projection onto V_{j+1} is a less faithful approximation of the signal than the projection onto V_j. In terms of causality, the approximation onto V_j contains all information required to compute the approximation V_{j+1}
- When $j \to +\infty$, we loose all information, when $j \to -\infty$, we asymptotically approximate the original signal
- j indexes a level of resolution: $s(t) \in V_0$ iff $s(t/2^j) \in V_j$
- V_j is translation invariant with a shift multiple of 2^j:

$$\forall k \in \mathbb{Z}, \quad s(t) \in V_j \Rightarrow s(t - k2^j) \in V_j$$

- Thus, $\{2^{-j/2}\phi(t/2^j - k), k \in \mathbb{Z}\}$ form an orthonormal basis of V_j

MRA - interpretation

- What is the difference between to subsequent approximations (in V_{j-1} and V_j): a "detail", containted in a subspace W_j, orthogonal to V_j in V_{j-1} (direct sum)

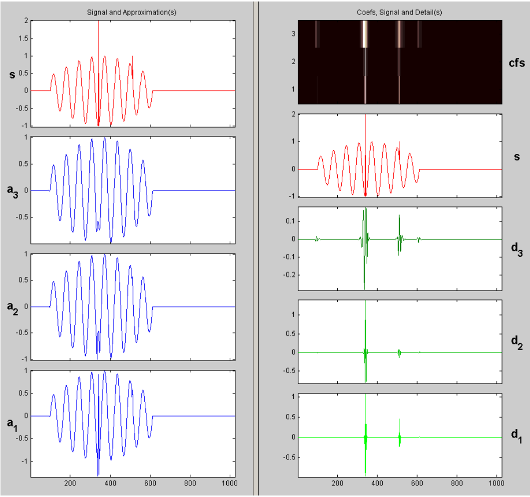
$$V_{j-1} = V_j \oplus W_j$$

- Projections on $s(t)$

$$P_{V_{j-1}} s(t) = P_{V_j} s(t) + P_{W_j} s(t)$$

- Mother wavelet: there exists $\psi(t)$ such that $\{2^{-j/2}\psi(t/2^j - k), (j,k) \in \mathbb{Z}^2\}$ forms an orthonormal basis of $L^2(\mathbb{R})$

MRA - illustration



Wavelets and filter banks

- Discrete wavelets may be computed on discrete signals by a bank of filters (Mallat)
- Due to the inclusion relation, every function of V_1 is a linear combination of functions $\phi(t-l)$ in V_0. Therefore there exists coefficients $h_0[l]$ such that (two-scale equation)

$$\frac{1}{\sqrt{2}}\phi(t/2) = \sum_{l=-\infty}^{\infty} h_0[l]\phi(t-l)$$

- $\phi(t-l), l \in \mathbb{Z}$ is orthogonal by definition, so:

$$h_0[l] = \langle \frac{1}{\sqrt{2}}\phi(t/2), \phi(t-l)\rangle.$$

In other words, they are defined by the father wavelet (if it exists!)

Wavelets and filter banks

- Two-scale equation between scales j and $j+1$

$$\frac{1}{2^{\frac{j+1}{2}}}\phi\left(\frac{t}{2^{j+1}}-k\right)=\sum_{l=-\infty}^{\infty}h_0[l-2k]\frac{1}{2^{j/2}}\phi\left(\frac{t}{2^j}-l\right)$$

- Half filter-bank: h_0 is interpreted as the impulse response of a filter
- We have $H_0(0)=\sqrt{2}$ et $|H_0(f)|^2+|H_0(f+1/2)|^2=2$
- To the limit: $\Phi(f)=\prod_{p=1}^{\infty}\left(\frac{1}{\sqrt{2}}H_0(\frac{f}{2^p})\right)$

Wavelets and filter banks

- Onto W_j

$$\frac{1}{2^{\frac{j+1}{2}}}\psi\left(\frac{t}{2^{j+1}}-k\right)=\sum_{l=-\infty}^{\infty}h_1[l-2k]\frac{1}{2^{j/2}}\phi\left(\frac{t}{2^j}-l\right)$$

- Another half filter banks: h_1 is interpreted as the impulse response of a filter
- On a $|H_1(f)|^2+|H_1(f+1/2)|^2=2$,
  $H_0(f)H_1^*(f)+H_0(f+1/2)H_1^*(f+1/2)=0$, puis
  $|H_0(f)|^2+|H_1(f)|^2=2$
- As $H_0(0)=\sqrt{2}$, $H_1(0)=0$
- Potential low-pass (h_0) and high-pass (h_1) filters

Filter bank: interpolation by 2

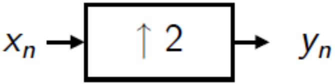
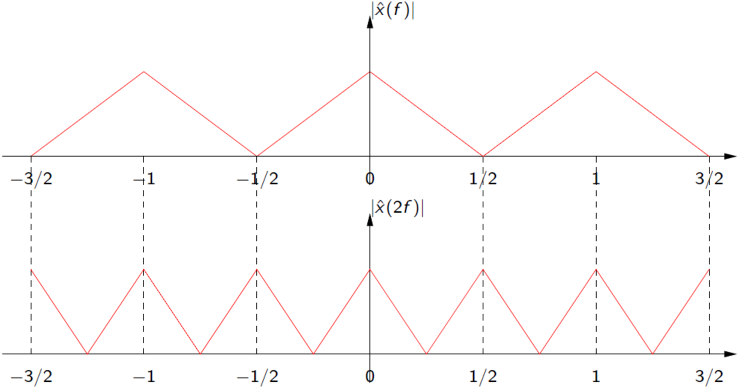


Figure 16: Interpolation

$$\begin{cases} y_{2n} = x_n \\ y_{2n+1} = 0 \end{cases}$$

Filter bank: interpolation by 2 - frequency



Filter bank: decimation by 2

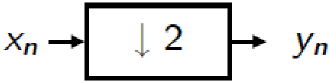
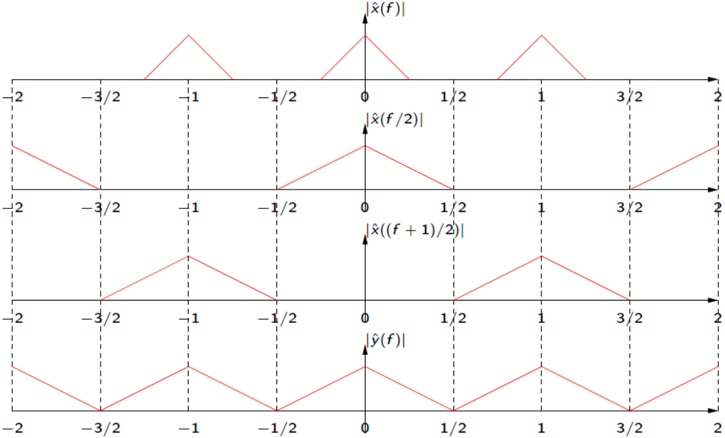


Figure 17: Decimation

$$\left\{ y_n = x_{2n} \right.$$

Filter bank: decimation by 2 - frequency



Mallat fast algorithm

- Project $s(t)$ onto basis $\{2^{-j/2}\phi(t/2^j - k), k \in \mathbb{Z}\}$ and $\{2^{-j/2}\psi(t/2^j - k), k \in \mathbb{Z}\}$ of V_j and W_j:

$$P_{V_j} s(t) = \sum_{k=-\infty}^{\infty} a_j[k] \frac{1}{2^{j/2}} \phi(t/2^j - k)$$

$$P_{W_j} s(t) = \sum_{k=-\infty}^{\infty} c_j[k] \frac{1}{2^{j/2}} \psi(t/2^j - k)$$

  avec

$$a_j[k] = \langle s(t), \frac{1}{2^{j/2}} \phi(t/2^j - k) \rangle$$

$$c_j[k] = \langle s(t), \frac{1}{2^{j/2}} \psi(t/2^j - k) \rangle$$

- Shall we compute all scalar product? No!

Mallat fast algorithm

- Thanks to the two-scale equation:

$$a_j[k] = \sum_{l=-\infty}^{\infty} a_{j-1}[l] h_0^*[l-2k], \qquad c_j[k] = \sum_{l=-\infty}^{\infty} a_{j-1}[l] h_1^*[l-2k]$$

- Filters $\tilde{h}_0[l] = h_0^*[-l]$, $\tilde{h}_1[l] = h_1^*[-l]$ followed by decimation:

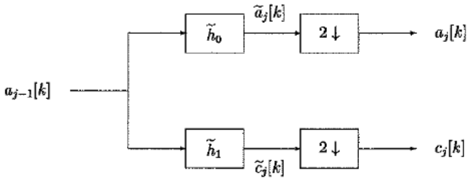
$$\tilde{a}_j[k] = \sum_{l=-\infty}^{\infty} a_{j-1}[l] h_0^*[l-k], \qquad \tilde{a}_j[2k] = a_j[k]$$

$$\tilde{c}_j[k] = \sum_{l=-\infty}^{\infty} a_{j-1}[l] h_1^*[l-k], \qquad \tilde{c}_j[2k] = a_j[k]$$

Mallat fast algorithm - illustration

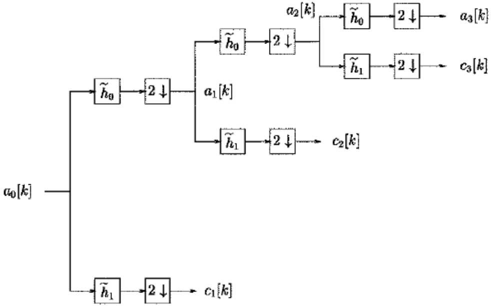
$$\tilde{a}_j[k] = \sum_{l=-\infty}^{\infty} a_{j-1}[l] h_0^*[l-k], \qquad \tilde{a}_j[2k] = a_j[k]$$

$$\tilde{c}_j[k] = \sum_{l=-\infty}^{\infty} a_{j-1}[l] h_1^*[l-k], \qquad \tilde{c}_j[2k] = a_j[k]$$



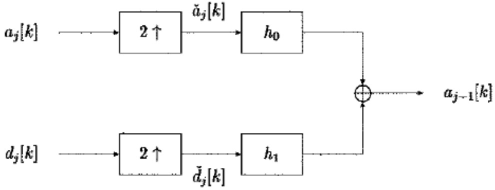
Mallat fast algorithm - cascade

- Complexity proportional to the signal's length for short FIR filters ($O(n)$)

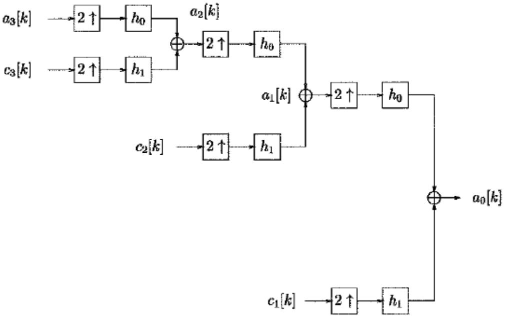


Mallat fast algorithm - synthesis

$$a_{j-1}[k] = \sum_{l=-\infty}^{\infty} a_j[l] h_0^*[k-2l] + \sum_{l=-\infty}^{\infty} c_j[l] h_1^*[k-2l]$$



Algorithme de Mallat - synthesis



Wavelet families - Shannon-Nyquist

- Goal: produce a sequence of coarser approximations of $s(t)$
  - idea: progressively reduce the band-width
  - approximation spaces:

$$V_j = \{s(t) \in \mathrm{L}^2(\mathbb{R}) \mid S(f) = 0 \quad \text{pour} \quad |f| \geq 2^{-j-1}\}$$

  - related to the sampling theorem:

$$s(t) = \sum_{k=-\infty}^{\infty} s(2^j k) \,\mathrm{sinc}\left(\pi\left(\frac{t}{2^j} - k\right)\right)$$

  - the father wavelet is: $\phi(t) = \mathrm{sinc}(\pi t)$

Wavelet families - Shannon-Nyquist

- Two-scale equation in the frequency domain

$$\Phi(f) = \frac{1}{\sqrt{2}}\, H_0\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

- Low-pass filter:
  - ideal
  - cut-off frequency (reduced): $f_s = 1/4$
  - frequency response at $\sqrt{2}$ in the pass-band
- Mother wavelet

$$\psi(t) = \operatorname{sinc}\left(\frac{\pi t}{2}\right)\cos\left(\frac{3\pi t}{2}\right)$$

Wavelet families - Shannon-Nyquist

- Characteristics
  - well-localized in frequency
  - poorly localized in time
  - IIR filters

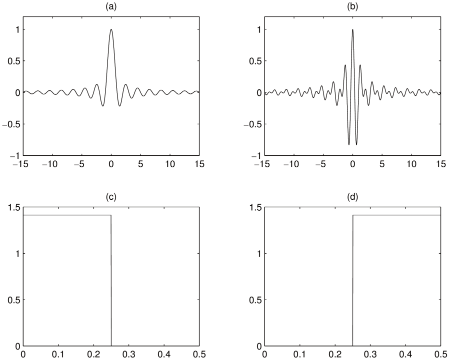


Figure 18: Functions: (a) $\phi(t)$ (b) $\psi(t)$ (c) $|H_0(f)|$ (d) $|H_1(f)|$

Wavelet families - Haar

Dual (time-frequency) approach to Shannon-Nyquist:

$$V_j = \left\{ s(t) \in \mathrm{L}^2(\mathbb{R}) \;\middle|\; \forall k \in \mathbb{Z},\ s(t) \text{ constant pour } t \in [2^j k, 2^j(k+1)[\ \right\}$$

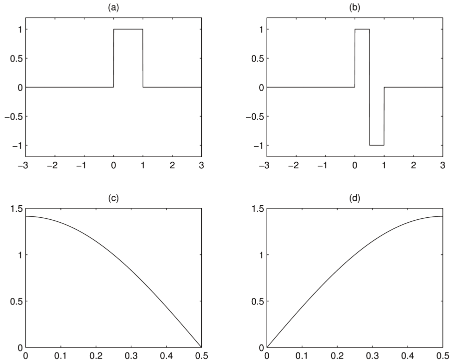


Figure 19: Functions: (a) $\phi(t)$ (b) $\psi(t)$ (c) $|H_0(f)|$ (d) $|H_1(f)|$

Wavelet families - Haar

- Characteristics
  - $\{2^{-j/2}\, r_{[2^j k, 2^j(k+1)[}(t),\ k \in \mathbb{Z}\}$ is an orthonormmal bais of V_j
  - scale function/father wavelet: $\phi(t) = r_{[0,1[}(t)$
  - low-pass filter

$$h_0[k] = \begin{cases} 1/\sqrt{2} & \text{if} \quad k \in \{0,1\}, \\ 0 & \text{otherwise} \end{cases}$$

  - mother wavelet: $\psi(t) = \phi(2t) - \phi(2t-1)$
  - high-pass filter

$$h_1[k] = \begin{cases} 1/\sqrt{2} & \text{if} \quad k = 0, \\ -1/\sqrt{2} & \text{if} \quad k = 1, \\ 0 & \text{otherwise} \end{cases}$$

  - 2-tap filter
  - poor frequency localization

Wavelet families - Splines

- Goal: better trade-off in time-frequency localization
- Idea: signal approximations on each interval by N-degree polynomials (which junction derivability)
- Standard tools for approximation in functional analysis
- Associated filters:
  - infinite impulse response (IIR)
  - linear phase
- Corresponding wavelets: Battle-Lemarié
  - exponential decay
  - $N+1$ vanishing moments: $\int \psi(t) t^n dt = 0$, $n \leq N+1$

Wavelet families - Splines (Franklin)

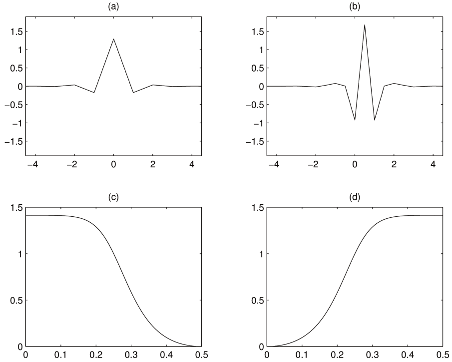


Figure 20: Franklin MRA ($N = 1$): (a) $\phi(t)$ (b) $\psi(t)$ (c) $|H_0(f)|$ (d) $|H_1(f)|$

Wavelet families - Splines (cubic)

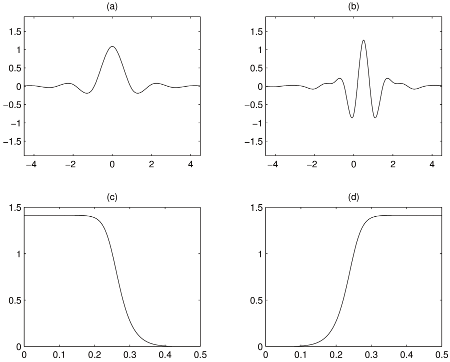


Figure 21: Cubic spline MRA ($N = 1$): (a) $\phi(t)$ (b) $\psi(t)$ (c) $|H_0(f)|$ (d) $|H_1(f)|$

Wavelet families - Daubechies

- Goal:
  - compact support wavelets with FIR filters
- Characteristics:
  - minimal length filters $(2N)$ for N vanishing moments
  - asymmetric (non-linear phase)

Wavelet families - Daubechies

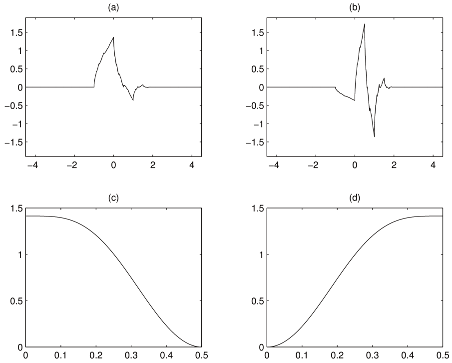


Figure 22: Daubechies $(N = 2)$: (a) $\phi(t)$ (b) $\psi(t)$ (c) $|H_0(f)|$ (d) $|H_1(f)|$

Wavelet families - Daubechies

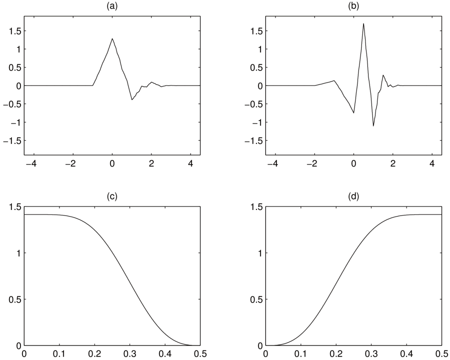


Figure 23: Daubechies ($N = 3$): (a) $\phi(t)$ (b) $\psi(t)$ (c) $|H_0(f)|$ (d) $|H_1(f)|$

Wavelet families - orthogonal others

- Symmlets:
  - more symmetric shape
- Coiflets:
  - scale function has vanishing moments
- Wavelets on the interval:
  - better adapted to short signals (cf. DFT)
  - wavelet bases are modified at the end on the support interval (cf. DFT)
  - limits edge effects

Wavelet families - biorthogonal

- Main issue:
  - FIR filters, real, linear phase, orthogonal (symmetric): only Haar (not smooth)
  - may relax orthogonality
- Consequence: two wavelets, at analysis AND synthesis

$$\left\langle \frac{1}{2^{j/2}}\psi_1\left(\frac{t}{2^j}-k\right), \frac{1}{2^{j'/2}}\psi_2^*\left(\frac{t}{2^{j'}}-k'\right)\right\rangle = \begin{cases} 1 & \text{if } j=j' \text{ and } k=k', \\ 0 & \text{otherwise.}\end{cases}$$

- Signal decomposition onto to bases

$$\begin{aligned} s(t) &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left\langle s(\tau), \frac{1}{2^{j/2}}\psi_1\left(\frac{\tau}{2^j}-k\right)\right\rangle \frac{1}{2^{j/2}}\psi_2\left(\frac{t}{2^j}-k\right) \\ &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left\langle s(\tau), \frac{1}{2^{j/2}}\psi_2\left(\frac{\tau}{2^j}-k\right)\right\rangle \frac{1}{2^{j/2}}\psi_1\left(\frac{t}{2^j}-k\right) \end{aligned}$$

Wavelet families - biorthogonal

- Decomposition algorithm:
  - same as in the orthogonal case
- Differences with orthogonal case:
  - Relation $\widetilde{h}_0[\ell] = h_0^*[-\ell]$ et $\widetilde{h}_1[\ell] = h_1^*[-\ell]$ not satisfied anymore
- Perfect reconstruction conditions:

$$
\begin{aligned}
H_0(f)\widetilde{H}_0(f) + H_1(f)\widetilde{H}_1(f) = 2,\\
H_0(f+1/2)\widetilde{H}_0(f) + H_1(f+1/2)\widetilde{H}_1(f) = 0.
\end{aligned}
$$

Wavelet families - biorthogonal

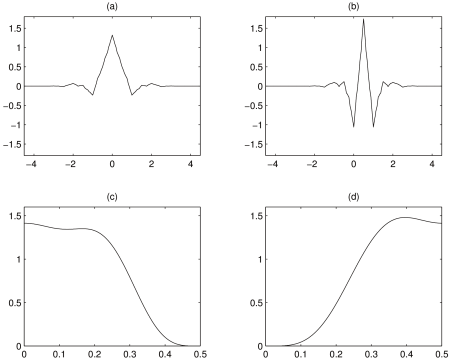


Figure 24: Biorthogonal 9/7 (analysis): (a) $\phi(t)$ (b) $\psi(t)$ (c) $|H_0(f)|$ (d) $|H_1(f)|$

Wavelet families - biorthogonal

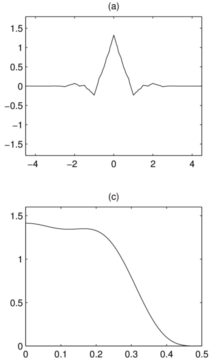


Figure 25: Biorthogonal 9/7 (synthesis): (a) $\phi(t)$ (b) $\psi(t)$ (c) $|H_0(f)|$ (d) $|H_1(f)|$

Wavelet families - multidimensional separable extensions

- Vector spaces tensor products $\phi(t)$ scaling function for MRA $(V_j)_{j\in\mathbb{Z}}$
- An orthonormal basis of V_j^2 is:

$$\left\{\frac{1}{2^j}\phi\left(\frac{x}{2^j}-n\right)\phi\left(\frac{y}{2^j}-m\right),\ (n,m)\in\mathbb{Z}^2\right\}$$

- Supplementary subspaces:

$$V_{j-1}^2 = V_j^2 \oplus W_j^{\mathrm{H}} \oplus W_j^{\mathrm{V}} \oplus W_j^{\mathrm{D}}$$

- Mother wavelets

$$\begin{aligned}\psi_{\mathrm{H}}(x,y) &= \phi(x)\psi(y),\\ \psi_{\mathrm{V}}(x,y) &= \psi(x)\phi(y),\\ \psi_{\mathrm{D}}(x,y) &= \psi(x)\psi(y)\end{aligned}$$

Wavelet families - multidimensional separable extensions

- Detail subspaces orthonormal bases

$$W_j^{\mathrm{H}}: \quad \left\{ \frac{1}{2^j}\psi_{\mathrm{H}}\left(\frac{x}{2^j}-n, \frac{y}{2^j}-m\right),\ (n,m)\in\mathbb{Z}^2 \right\}$$

$$W_j^{\mathrm{V}}: \quad \left\{ \frac{1}{2^j}\psi_{\mathrm{V}}\left(\frac{x}{2^j}-n, \frac{y}{2^j}-m\right),\ (n,m)\in\mathbb{Z}^2 \right\}$$

$$W_j^{\mathrm{D}}: \quad \left\{ \frac{1}{2^j}\psi_{\mathrm{D}}\left(\frac{x}{2^j}-n, \frac{y}{2^j}-m\right),\ (n,m)\in\mathbb{Z}^2 \right\}$$

- Approximation of an image at resolution 2^{-j}:

$$\mathrm{proj}_{V_j^2} s(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_j[n,m]\,\frac{1}{2^j}\phi\left(\frac{x}{2^j}-n\right)\phi\left(\frac{y}{2^j}-m\right)$$

where

$$\begin{aligned} a_j[n,m] &= \left\langle\left\langle s(x,y), \frac{1}{2^j}\phi\left(\frac{x}{2^j}-n\right)\phi\left(\frac{y}{2^j}-m\right)\right\rangle\right\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x,y)\,\frac{1}{2^j}\phi^*\left(\frac{x}{2^j}-n\right)\phi^*\left(\frac{y}{2^j}-m\right)\,dx\,dy \end{aligned}$$

Wavelet families - multidimensional separable extensions

- Detail subspaces orthonormal bases

$$W_j^{\mathrm{H}}: \quad \left\{ \frac{1}{2^j} \psi_{\mathrm{H}} \left(\frac{x}{2^j} - n, \frac{y}{2^j} - m \right),\ (n,m) \in \mathbb{Z}^2 \right\}$$

$$W_j^{\mathrm{V}}: \quad \left\{ \frac{1}{2^j} \psi_{\mathrm{V}} \left(\frac{x}{2^j} - n, \frac{y}{2^j} - m \right),\ (n,m) \in \mathbb{Z}^2 \right\}$$

$$W_j^{\mathrm{D}}: \quad \left\{ \frac{1}{2^j} \psi_{\mathrm{D}} \left(\frac{x}{2^j} - n, \frac{y}{2^j} - m \right),\ (n,m) \in \mathbb{Z}^2 \right\}$$

Wavelet families - multidimensional separable extensions

- Approximation of an image at resolution 2^{-j}:

$$\mathrm{proj}_{V_j^2} s(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_j[n,m] \frac{1}{2^j} \phi\left(\frac{x}{2^j} - n\right) \phi\left(\frac{y}{2^j} - m\right)$$

where

$$\begin{aligned} a_j[n,m] &= \left\langle \left\langle s(x,y), \frac{1}{2^j} \phi\left(\frac{x}{2^j} - n\right) \phi\left(\frac{y}{2^j} - m\right) \right\rangle \right\rangle \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(x,y) \frac{1}{2^j} \phi^*\left(\frac{x}{2^j} - n\right) \phi^*\left(\frac{y}{2^j} - m\right) dxdy \end{aligned}$$

- `Demo_Wavelet_Image.m`

Denoising and compression

- Principles:
  - take advantage of wavelet approximation properties
  - describe singularities with a few big coefficients
- Idea:
  - keep those big coefficient
  - more generally, sort coefficients by size (disregarding "frequency or scale"
  - threshold

Denoising - settings

We observe a signal (image) z such that

$$z = x + n$$

where n is an additive gaussian white noise with variance σ^2.
After a wavelet decomposition:

$$w_j^z = w_j^x + w_j^n$$

where w_j^x are wavelet coefficients of x. Objective: estimate x by $\hat{x}$ with only observations z.

Denoising - thresholding

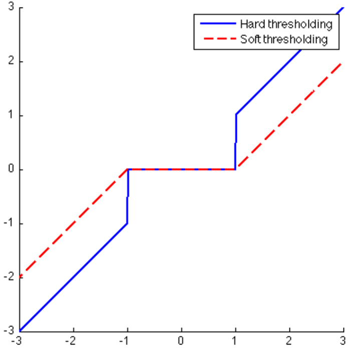


Figure 26: Two-standard thresholding functions

Denoising - thresholding

- Hard thresholding (blue line) :

$$w_j^{\hat{x}}(n) = \begin{cases} w_j^z(n) & \text{if} |w_j^z(n)| > \chi \\ 0 & \text{otherwise} \end{cases}$$

- Soft thresholding (red line) :

$$w_j^{\hat{x}}(n) = sign(w_j^z(n)) \max\{|w_j^z(n)| - \chi, 0\}.$$

where χ is a threshold value, real and positive.
Approximation coefficients are generally not thresholded

Denoising - threshold choice

- Universal threshold: $\chi = \sigma\sqrt{2\ln(N)}$
- Sureshrink : based on Stein's principle
- Generalized threshold functions
- Scalar or block-wise
- Knowledge based

where χ is a threshold value, real and positive.