Fall 2021 - Math 102 - Quiz 2

1. Find the directional derivative of

$$f(x,y)=\begin{cases}\frac{x^3y^2}{2x^6+3y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } x=y=0\end{cases}$$

at the origin whenever it exists.

Let $u=(a,b)$ be a unit vector; that is, $a^2+b^2=1$. Then,

$$\begin{aligned}D_uf(0,0)&=\lim_{h\to 0}\frac{f(ah,bh)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{(ah)^3(bh)^2}{2(ah)^6+3(bh)^4}}{h}\\ &=\lim_{h\to 0}\frac{a^3b^2}{2a^6h^2+3b^4}=\begin{cases}0 & \text{if } ab=0\\ \frac{a^3}{3b^2} & \text{if} ab\neq 0.\end{cases}\end{aligned}$$

2. Find the absolute extrema of $f(x,y)=x^3-y^2$ where $x^2+2y^2\leqslant 3$.

The only critical point is $(0,0)$ obtained from $\nabla f=(3x^2,-2y)$, at which we have $f(0,0)=0$. On the boundary $x^2+2y^2=3$, we use Lagrange, thus we need to solve

$$3x^2=2x\lambda,\quad -2y=4y\lambda,\quad x^2+2y^2=3.$$

If $x=0$, we get $y=\pm\sqrt{3/2}$ and for $y=0$, we get $x=\pm\sqrt{3}$. Assuming $xy\neq 0$ gives

$$3x=2\lambda,\quad \lambda=-1/2,\qquad 3x=-1,\qquad x=-1/3,\quad y=\pm\sqrt{(3-1/9)/2}=\pm\frac{\sqrt{13}}{3}.$$

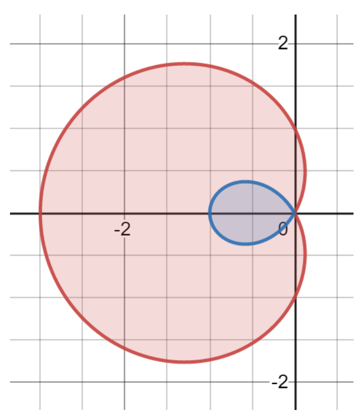
We conclude that

$$f(0,\pm\sqrt{3/2})=-3/2,\quad f(\pm\sqrt{3},0)=\underbrace{\pm 3\sqrt{3}}_{\text{min and max}},\quad f(-1/3,\pm\sqrt{13}/3)=-\frac{1}{27}-\frac{13}{9}=-\frac{40}{27}.$$

3. Determine the area of the larger of the regions bounded by $r=1-2\cos\theta$.

The area of the red region equals

$$\int_{\pi/3}^{2\pi-\pi/3}\int_0^{1-2\cos\theta} r\,dr\,d\theta-\int_{-\pi/3}^{\pi/3}\int_0^{2\cos\theta-1} r\,dr\,d\theta=3\sqrt{3}+\pi.$$



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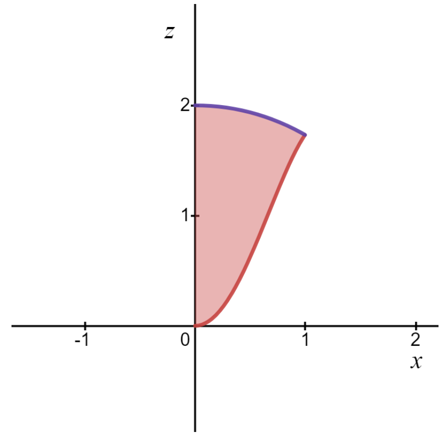
4. Use cylindrical coordinates to set up a triple integral for the volume of the solid which lies above $z = (x^2+y^2)(\sqrt{3}+1-x^2-y^2)$ and below the sphere $x^2+y^2+z^2=4$ for $x^2+y^2 \leqslant 1$.

The volume using cylindrical coordinates equals

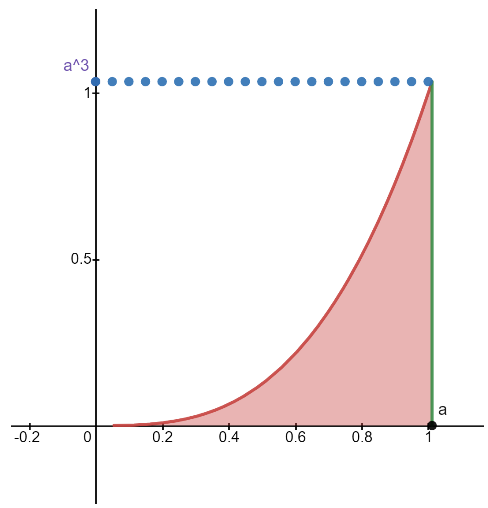
$$\int_0^{2\pi}\int_0^1\int_{r^2(\sqrt{3}+1-r^2)}^{\sqrt{4-r^2}} dz\,r\,dr\,d\theta$$



5. Evaluate $\displaystyle\int_0^{a^3}\int_{\sqrt[3]{y}}^{a} \sin(x^4)\,dx\,dy$, where $a = \sqrt[4]{\pi/3}$.

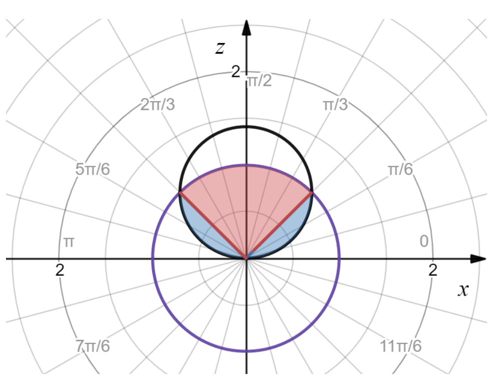
Interchanging the order of integration we get

$$\int_0^a \sin(x^4)\int_0^{x^3} dy\,dx = \int_0^a x^3\sin(x^4)\,dx$$
$$= \frac{1}{4}(1-\cos(\pi/3)) = \frac{1}{8}$$



6. Set up, but do not evaluate, triple integral(s) that gives the volume of the intersection of the *solid* spheres $x^2+y^2+z^2 \leqslant 1$ and $x^2+y^2+z^2 \leqslant \sqrt{2}z$ using the **spherical coordinates**.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



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7. Evaluate $\iint_R x^3y^{-6}e^{x^3/y^3}\,dA$, where R is the region bounded by $y = x^2, x = y^2, 8x = y^2$ and $y = \sqrt{8}x^2$. (Use $u = x/y^2$ and $v = y/x^2$.)

Let $u = x/y^2$ and $v = y/x^2$ so that the corresponding region in uv-coordinates becomes the rectangle $[1/8, 1] \times [1, \sqrt{8}]$. Solving for x and y, we get $x = uy^2 = uv^2x^4$, which gives $x = u^{-1/3}v^{-2/3}$. Similarly, $y = vx^2$ gives $y = v^{-1/3}u^{-2/3}$. Thus,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{3u^{4/3}v^{2/3}} & -\frac{2}{3u^{1/3}v^{5/3}} \\ -\frac{2}{3u^{5/3}v^{1/3}} & -\frac{1}{3u^{2/3}v^{4/3}} \end{vmatrix} = -\frac{1}{3u^2v^2}.$$

Rewriting the integral and noting that $x^3/y^3 = u/v$, we end up with

$$\begin{aligned}\iint_R x^3y^{-6}e^{x^3/y^3}\,dA &= \int_{1/8}^{1}\int_{1}^{\sqrt{8}} u^{-1}v^{-2}v^2u^4e^{u/v}\frac{1}{3u^2v^2}\,dv\,du \\ &= \frac{1}{3}\int_{1/8}^{1}\int_{1}^{\sqrt{8}} e^{u/v}uv^{-2}\,dv\,du = \frac{1}{3}\int_{1/8}^{1}(e^u - e^{u/\sqrt{8}})\,du \\ &= \frac{1}{3}(e - e^{1/8} + \sqrt{8}(e^{1/8\sqrt{8}} - e^{1/\sqrt{8}})).\end{aligned}$$