Exercise Problems II

1. Use polar coordinates to evaluate the double integral $I = \iint_D \frac{x}{\sqrt{x^2+y^2}} dA$, where

$$D = \{(x,y) : 1 \leqslant x^2 + 4y^2, x^2 + y^2 \leqslant 4, x/\sqrt{3} \leqslant y \leqslant x\sqrt{3}\}.$$

The ellipse $x^2 + 4y^2 = 1$ corresponds to $r = 1/\sqrt{\cos^2\theta + 4\sin^2\theta}$. Thus, we have

$$\begin{aligned}
I &= \int_{\pi/6}^{\pi/3} \int_{\frac{1}{\sqrt{\cos^2\theta+4\sin^2\theta}}}^{2} r\cos\theta \, dr\, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{\pi/3} \cos\theta \left(r^2 \Big|_{\frac{1}{\sqrt{\cos^2\theta+4\sin^2\theta}}}^{2} \right) d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3} \cos\theta \left(4 - \frac{1}{\cos^2\theta + 4\sin^2\theta} \right) d\theta \\
&= 2\sin\theta \Big|_{\pi/6}^{\pi/3} - \frac{1}{2}\int_{\pi/6}^{\pi/3} \frac{\cos\theta}{1+3\sin^2\theta} d\theta \\
&= \sqrt{3} - 1 - \frac{1}{2\sqrt{3}} \arctan(\sqrt{3}\sin\theta)\Big|_{\pi/6}^{\pi/3} \\
&= \sqrt{3} - 1 - \frac{1}{2\sqrt{3}}\big(\arctan(3/2) - \arctan(\sqrt{3}/2)\big).
\end{aligned}$$

2. Explain whether the following argument holds water.

The limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} = 0$ since using polar coordinates we see that $r \to 0$ as $(x,y) \to (0,0)$ and we get

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r(\cos\theta + \sin\theta)} = \lim_{r\to 0} \frac{r\cos\theta\sin\theta}{\cos\theta + \sin\theta} = 0.$$

This argument is false since it is assumed that θ is independent of r, which is not true. For example, if one takes $r\sin\theta = y = x^3 - x = r^3\cos^3\theta - r\cos\theta$, we get

$$\sin\theta = r^2\cos^3\theta - \cos\theta,$$

and the limit above becomes

$$\lim_{r\to 0} \frac{r\cos\theta\sin\theta}{r^2\cos^3\theta}$$

which clearly doesn't exist.

3. Evaluate $\lim_{(x,y)\to(0,0)} \frac{8^{x^2}16^{y^2} - 1}{3x^2 + 4y^2}$.

Using $x = \frac{1}{\sqrt{3}} r\cos\theta$ anad $y = \frac{1}{2} r\sin\theta$, we get $r^2 = 3x^2 + 4y^2 \to 0$ and thus

$$\lim_{(x,y)\to(0,0)} \frac{8^{x^2}16^{y^2} - 1}{3x^2 + 4y^2} = \lim_{r\to 0} \frac{2^{r^2} - 1}{r^2} = \frac{d2^r}{dr}\Big|_{r=0} = \ln 2.$$

4. Find $\frac{\partial f}{\partial x}(1,-1)$, where

$$f(x,y) = e^{x^2(y+1)^{2/3}} x \ln(x^{1/7} + y^3 + 2y^2 + 5y + 4)(x + 1 - \cos(y+1))^{-1/3}.$$

$$\begin{aligned}\frac{\partial f}{\partial x}(1,-1) &= \lim_{h\to 0} \frac{f(1+h,-1) - f(1,-1)}{h} \\ &= \lim_{h\to 0} \frac{f(1+h,-1)}{h} = \lim_{h\to 0} \frac{g(1+h) - g(1)}{h} = g'(1),\end{aligned}$$

where

$$g(x) = \frac{1}{7}x^{2/3}\ln x, \qquad g'(x) = x^{-1/3}(\frac{2}{21}\ln x + \frac{1}{7}), \qquad g'(1) = \frac{1}{7}.$$

5. Evaluate $\displaystyle\iint_{|x|^{1/5}+|y|^{1/2}\leqslant 1} \frac{2x^3 - x^2y}{x^2 + y^4 + 2} dA.$

Since the region is symmetric and $f(-x,-y) = -f(x,y)$, we get zero.

6. Find the absolute extrema of $f(x,y) = 4x^4 - 12x^2y + 9y^2 - 5$.

Since

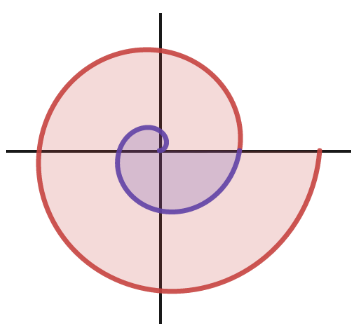
$$f = (2x^2 - 3y)^2 - 5$$

it has no max, and -5 abs. min.

7. (a) Find the area of the region $r \leqslant \theta$ for $0 \leqslant \theta \leqslant 4\pi$.

The area is

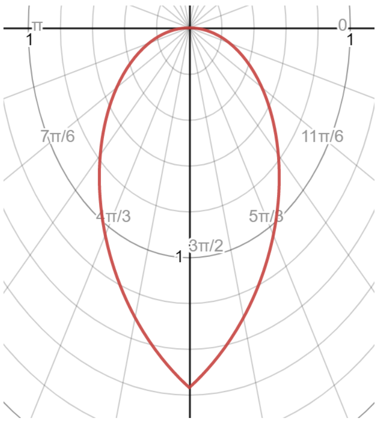
$$\int_{2\pi}^{4\pi}\int_0^{\theta} r\,dr\,d\theta = \frac{1}{2}\int_{2\pi}^{4\pi}\theta^2 d\theta = \frac{1}{6}((4\pi)^3 - (2\pi)^3) = \frac{28}{3}\pi^3.$$



(b) Find the volume under $z = \sqrt{x^2+y^2}$ above the region $r = \theta - \pi$ for $\theta \in [\pi/2, 3\pi/2]$.

The volume is

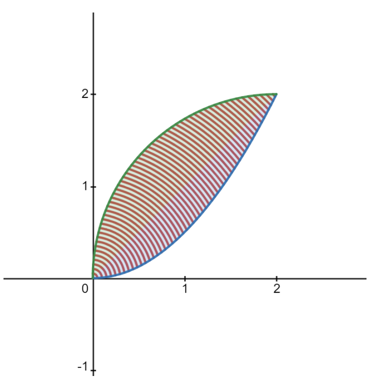
$$\int_{\pi/2}^{3\pi/2}\int_0^{|\theta-\pi|} r^2 dr d\theta = \frac{1}{3}\int_{\pi/2}^{3\pi/2} |\theta-\pi|^3 d\theta = \frac{2}{3}\int_{\pi}^{3\pi/2} (\theta-\pi)^3 d\theta$$
$$= \frac{1}{6}(\theta-\pi)^4\Big|_{\pi}^{3\pi/2} = \frac{\pi^4}{96}.$$



8. Set up double integral in Cartesian coordinates using both $dxdy$ and $dydx$ for the area of the region inside the circle $(x-2)^2+y^2=4$ and above $2y=x^2$. Then, do the same using polar coordinates in two ways using $d\theta dr$ and then $dr d\theta$.

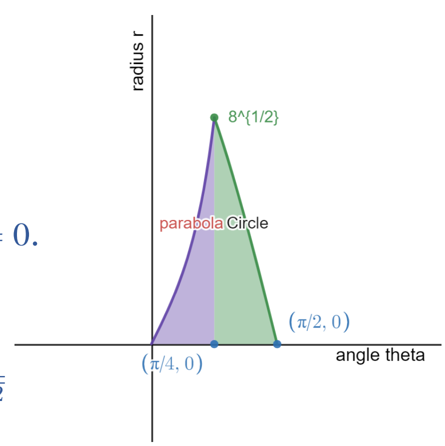
Cartesian coordinates:

$$\int_0^2\int_{x^2/2}^{\sqrt{4-(x-2)^2}} dydx = \text{ Area } = \int_0^2\int_{2-\sqrt{4-y^2}}^{\sqrt{2y}} dxdy$$



Polar coordinates: $2y=x^2$ gives $r=2\sin\theta\sec^2\theta$ and the circle equals $r=4\cos\theta$. Thus, the area is

$$\int_0^{\pi/4}\int_0^{2\sin\theta\sec^2\theta} r dr d\theta + \int_{\pi/4}^{\pi/2}\int_0^{4\cos\theta} r dr d\theta$$



Using the above equation we also get

$$1-\sin^2\theta=\cos^2\theta = 2\sin\theta/r \Longrightarrow \sin^2\theta + 2\sin\theta/r - 1 = 0.$$

Solving the quadratic gives

$$\sin\theta = \frac{-2/r+\sqrt{4/r^2+4}}{2} = \frac{-1+\sqrt{1+r^2}}{r} = \frac{r}{1+\sqrt{1+r^2}}$$

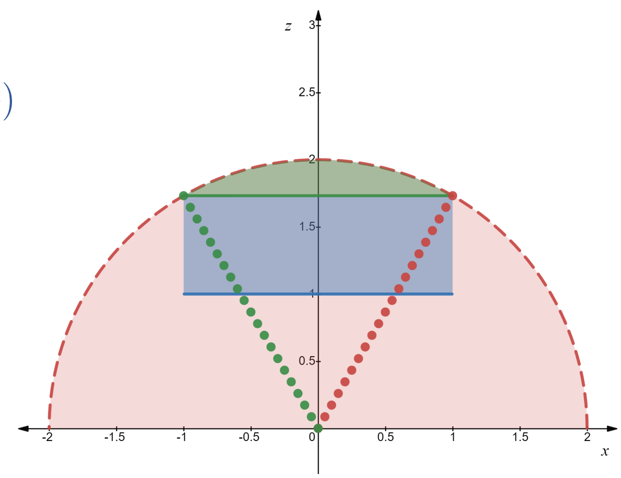
Similarly, circle gives $\theta = \arccos(r/4)$. Using these two angles, we get another expression for the area

$$\int_0^{\sqrt{8}} \int_{\arcsin(r/(1+\sqrt{1+r^2}))}^{\arccos(r/4)} d\theta r dr.$$

9. Find the volume of the solid obtained by removing the cube of volume 8 centered at $(0, 0, \sqrt{3})$ from the solid hemi-sphere $0 \leqslant z \leqslant \sqrt{4 - x^2 - y^2}$.

The volume is obtained by subtracting the volume of half the cube (that is, 4) and the volume V of the top part of the sphere lying above the lower half of the cube from the volume of the hemisphere, which is $\frac{2}{3}\pi 2^3$. To find the volume of the top part of the sphere we use the spherical coordinates to get

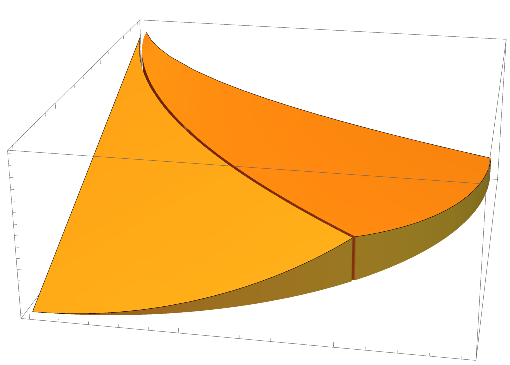
$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{\pi/6} \int_{\sqrt{3}\sec\phi}^{2} \rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/6} (8\sin\phi - 3\sqrt{3}\sin\phi\sec^3\phi) \\
&= \frac{1}{3} \int_0^{2\pi} \left(-8\cos\phi - \frac{3}{2}\sqrt{3}\sec^2\phi\right)\Big|_0^{\pi/6} d\theta \\
&= \frac{2\pi}{3}\left(8 - \frac{9\sqrt{3}}{2}\right)
\end{aligned}$$



10. Set up a triple integral that gives the volume of the solid determined by the conditions $x, y \geqslant 0, x^2 + y^2 \leqslant 2$ and $0 \leqslant z \leqslant \min\{1 - \sqrt{y}, 1 - x\}$.

Note first that $\min\{1-\sqrt{y}, 1-x\} = 1-\sqrt{y}$ provided $y \geqslant x^2$ and this parabola intersects the boundary circle of the given disc at $(1, 1)$. Hence the volume is

$$\int_0^1 \int_{\sqrt{y}}^{\sqrt{2-y^2}} \int_0^{1-x} dz dx dy + \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} \int_0^{1-\sqrt{y}} dz dy dx$$

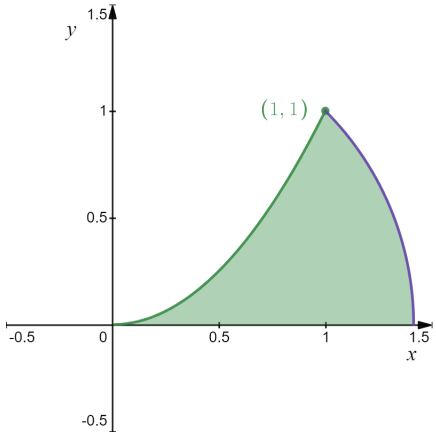


Exercise Problems II

11. Find the area of the region in the first quadrant below $y = x^2$ and inside the circle $x^2 + y^2 = 2$ using polar coordinates.

Parabola is given by $r = \sin\theta \sec^2\theta$. Hence, the area is

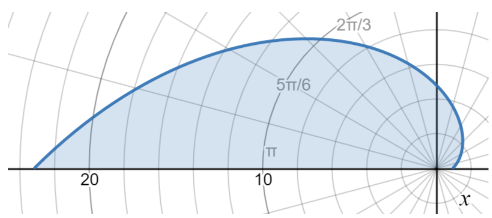
$$\int_0^{\pi/4} \int_{\sin\theta\sec^2\theta}^{\sqrt{2}} r dr d\theta$$



12. (a) Express the area of the region lying below $r = e^\theta$ for $\theta \in [0, \pi]$ and above the x- axis as a double integral using polar coordinates in the order

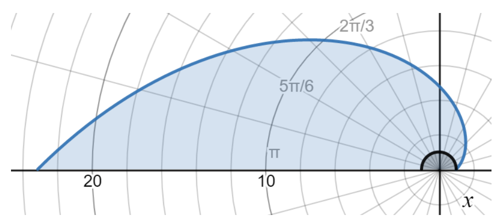
i. $dr d\theta$

$$\int_0^{\pi} \int_0^{e^\theta} r dr d\theta$$



ii. $d\theta dr$

$$\int_0^1 \int_0^{\pi} r d\theta dr + \int_1^{e^\pi} \int_{\ln r}^{\pi} r d\theta dr.$$



(b) Use one of the above to evaluate the area.

Using part (a), we get the area equal to

$$\int_0^{\pi} \frac{1}{2} e^{2\theta} d\theta = \frac{1}{4}(e^{2\pi} - 1).$$

13. Find all $\alpha \in \mathbb{R}$ such that

$$\lim_{s \to 0^+} \iint\limits_{s \leqslant x^2+y^2 \leqslant 1} \frac{1}{(x^2+y^2)^\alpha} dA$$

exists.

Using polar coords, we get

$$\lim_{s\to 0^+} \iint\limits_{s\leqslant x^2+y^2\leqslant 1} \frac{1}{(x^2+y^2)^\alpha} dA = \lim_{s\to 0^+} \int_0^{2\pi}\int_s^1 r^{1-2\alpha} dr d\theta = \lim_{s\to 0^+} = 2\pi \lim_{s\to 0^+} \begin{cases} \frac{1-s^{2-2\alpha}}{2-2\alpha} & \text{if } \alpha \neq 1 \\ -\log s & \text{if } \alpha = 1. \end{cases}$$

Thus, we see that the limit exists provided that $\alpha < 1$.