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Mathe

Problem 7 Functions

a) $f: \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, \dots, 6\}$

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 8-x & \text{if } x \geq 3 \end{cases}$$

i) $f(1)=1$ $f(3)=5$ $f(5)=3$
 $f(2)=4$ $f(4)=4$ $f(6)=2$

The function is not injective as the number 4 has two inputs $f(2)$ and $f(4)$.

ii) The function is not surjective as the number 6 does not have any input.

b) $A := \{2|x-7|; x \in \mathbb{Z} \cap [-2, 2]\}$ and

$B := \{y^2; y \in \mathbb{N} \text{ and } y \leq 5\}$.

$A := x \mapsto 2|x-7| = x; x \in [-2, -7, 0, 7, 2] \rightarrow 2 \cdot |x-7| =$

$A := A(-2)=6$ $A(-7)=4$ $A(0)=2$ $A(7)=0$ $A(2)=2$

$A := \{(-2, 6), (-7, 4), (0, 2), (7, 0), (2, 2)\} \rightarrow \{6, 4, 2, 0\}$

$B := y = \{1, 2, 3, 4, 5\}$ $B := \{1, 4, 9, 16, 25\}$

if $g: A \rightarrow B$ such that g is surjective and not injective

$g := \{(6, 1), (4, 4), (2, 9), (0, 9)\}$

function does not exist as there are too few inputs in A to make it surjective since the cardinality is higher in B than A .
 Must have one input more if to make it

ii) $g_2: A \rightarrow B$ such that g_2 is injective and not surjective

$$g_2: \{(0,1), (2,1), (4,1), (6,1)\}$$

function exists

Problem 4 Induction

$$\sum_{h=1}^n \frac{h}{2^h} = 2 - \frac{n+2}{2^n}$$

Base case

$$\sum_{h=1}^1 \frac{h}{2^h} = 2 - \frac{1+2}{2^1}$$

$$\frac{1}{2} = 2 - \frac{3}{2}$$

$$\frac{1}{2} = \frac{4}{2} - \frac{3}{2}$$

$$\frac{1}{2} = \frac{1}{2} \quad n=1 \text{ is true}$$

Induction Hypothesis ($n \rightarrow n+1$)

$$\sum_{h=1}^n \frac{h}{2^h} = 2 - \frac{n+2}{2^n}$$

$$\sum_{h=1}^{n+1} \frac{h}{2^h} = 2 - \frac{n+3}{2^{n+1}} = \sum_{h=1}^{n+1} \frac{h}{2^h} = \left(\sum_{h=1}^n \frac{h}{2^h} \right) + \frac{n+1}{2^{n+1}}$$

$$= \left(2 - \frac{n+2}{2^n} \right) + \frac{n+1}{2^{n+1}} = \sum_{h=1}^{n+1} \frac{h}{2^h}$$

$$= 2 - \frac{2n+4}{2^{n+1}} + \frac{n+1}{2^{n+1}} = 2 + \frac{-2n-4+n+1}{2^{n+1}} = 2 + \frac{-(n+3)}{2^{n+1}}$$

$$= 2 - \frac{n+3}{2^{n+1}}$$

Thus if the formula holds for n , it also holds for $n+1$. Therefore by mathematical induction, the formula is true for all positive integers n .