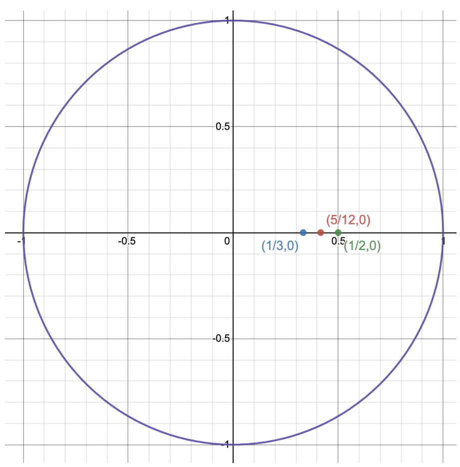
1 Pôles et zéros

1. $$x[n] = \left(\frac{1}{2}\right)^n \epsilon[n] + \left(\frac{1}{3}\right)^n \epsilon[n]$$

   Apply the Z-transform to both sides, using the property of linearity, and the z-transform table we obtain the following:

   $$\begin{aligned} X(z) &= z\left\{\left(\frac{1}{2}\right)^n \epsilon[n]\right\} + z\left\{\left(\frac{1}{2}\right)^n \epsilon[n]\right\} \\ &= \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \frac{1}{1-\left(\frac{1}{3}\right)z^{-1}} \\ &= \frac{2-\frac{5}{6}z^{-1}}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{3}\right)z^{-1}\right)} \end{aligned} \tag{1}$$

   Therefore, $X(z)$ has poles at $z = \frac{1}{2}$ and $z = \frac{1}{3}$ and a zero at $z = \frac{5}{12}$

   The graph above represents the complex plane, with the y-axis as the imaginary axis, and the x-axis as the real axis

2. (a) $$x[n] = \left(\frac{1}{2}\right)^n \epsilon[n] - \left(\frac{1}{3}\right)^n \epsilon[-n-1]$$

   Apply the z-transform to both sides, using the property of linearity of the z-transform (the final ROC of $X(z)$ will be the intersection of the ROCs of its constituent functions)

   $$X(z) = z\left\{\left(\frac{1}{2}\right)^n \epsilon[n]\right\} + z\left\{-\left(\frac{1}{3}\right)^n \epsilon[-n-1]\right\}$$

Using a z-transform table, we obtain the following expression

$$X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \frac{1}{1-\left(\frac{1}{3}\right)z^{-1}} = \frac{2-\frac{5}{6}z^{-1}}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{3}\right)z^{-1}\right)}$$

The poles of found in 1.1 correspond to the ones found here. However, the regions of convergence of the sub-functions $|z| > \frac{1}{2}$ and $|z| < \frac{1}{3}$ do not intersect, therefore, the region of convergence for $X(z)$ is $\emptyset$

(b)

$$x[n] = -\left(\frac{1}{2}\right)^n \epsilon[-n-1] - \left(\frac{1}{3}\right)^n \epsilon[-n-1]$$

Apply the z-transform to both sides, using the property of linearity of the z-transform (the final ROC of $X(z)$ will be the intersection of the ROCs of its constituent functions)

$$X(z) = z\left\{-\left(\frac{1}{2}\right)^n \epsilon[-n-1]\right\} + z\left\{-\left(\frac{1}{3}\right)^n \epsilon[-n-1]\right\}$$

Using a z-transform table, we obtain the following expression

$$X(z) = \frac{2-\frac{5}{6}z^{-1}}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{3}\right)z^{-1}\right)}$$

The poles of found in 1.1 correspond to the ones found here. The regions of convergence of the sub-functions $|z| < \frac{1}{2}$ and $|z| < \frac{1}{3}$ intersect, therefore, the region of convergence for $X(z)$ is $|z| < \frac{1}{3}$

(c)

$$x[n] = -\left(\frac{1}{2}\right)^n \epsilon[-n-1] + \left(\frac{1}{3}\right)^n \epsilon[n]$$

Apply the z-transform to both sides, using the property of linearity of the z-transform (the final ROC of $X(z)$ will be the intersection of the ROCs of its constituent functions)

$$X(z) = z\left\{-\left(\frac{1}{2}\right)^n \epsilon[-n-1]\right\} + z\left\{\left(\frac{1}{3}\right)^n \epsilon[n]\right\}$$

Using a z-transform table, we obtain the following expression

$$X(z) = \frac{2-\frac{5}{6}z^{-1}}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{3}\right)z^{-1}\right)}$$

The poles of found in 1.1 correspond to the ones found here. The regions of convergence of the sub-functions $|z| < \frac{1}{2}$ and $|z| > \frac{1}{3}$ intersect, therefore, the region of convergence for $X(z)$ is $\frac{1}{3} < |z| < \frac{1}{2}$

2 Transformée en z

1.
$$x_1[n] = na^n\epsilon[n]$$
Let $y[n] = a^n\epsilon[n]$, from the z-transform table, we obtain $Y(z) = \frac{1}{1-az^{-1}}$
We also have that the ROC of $Y(z)$ is $|z| > |a|$
$$x_1[n] = ny[n]$$
From the table of z-transforms we also have that given signal $x[n]$ with z-transform $X(z)$ and ROC R,
the signal $nx[n]$ has z-transform $z\{nx[n]\} = -z\frac{dX(z)}{dz}$ and ROC R
$$\begin{aligned} X_1(z) &= -z\frac{dY(z)}{z} \\ &= -z\frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] \\ &= \frac{az}{(z-a)^2} \end{aligned}$$
The ROC of $x_1[n] = ny[n]$ is the same as the ROC of $Y(z)$, namely $|z| > |a|$

2.
$$x_2[n] = n^2a^n\epsilon[n]$$
From the problem 2.1, $x_1[n] = na^n\epsilon[n] \implies Y(z) = \frac{az}{(z-a)^2}$, and that the ROC of $X_1(z)$ is $|z| > |a|$
$$x_2[n] = nx_1[n]$$
From the table of z-transforms we also have that given signal $x[n]$ with z-transform $X(z)$ and ROC R,
the signal $nx[n]$ has z-transform $z\{nx[n]\} = -z\frac{dX(z)}{dz}$ and ROC R
$$\begin{aligned} X_2(z) &= -z\frac{dX_1(z)}{z} \\ &= -z\frac{d}{dz}\left[\frac{az}{(z-a)^2}\right] \\ &= \frac{az(z+a)}{(z-a)^3} \end{aligned}$$
The ROC of $x_2[n] = nx_1[n]$ is the same as the ROC of $X_1(z)$, namely $|z| > |a|$

3.

$$x_3[n] = a^n \cos(\omega_0 n)\epsilon[n]$$

Using the table of z-transforms we obtain the following:

$$\begin{aligned}
X_3(z) &= \sum_{k=-\infty}^{\infty} a^k \cos(\omega_0 k)\epsilon[k]z^{-k} \\
&= \sum_{k=0}^{\infty} a^k \cos(\omega_0 k)z^{-k} \\
&= \frac{1}{2}\sum_{k=0}^{\infty} \left(a^k \left(e^{j\omega_0 k} + e^{-j\omega_0 k}\right)\right) z^{-k} \\
&= \frac{1}{2}\sum_{k=0}^{\infty} a^k e^{j\omega_0 k} z^{-k} + \frac{1}{2}\sum_{k=0}^{\infty} a^k e^{-j\omega_0 k} z^{-k} \\
&= \frac{1}{2}\left(\frac{1}{1 - ae^{j\omega_0}z^{-1}} + \frac{1}{1 - ae^{-j\omega_0}z^{-1}}\right) = ... \\
&= \frac{1 - [a\cos\omega_0]z^{-1}}{1 - [2a\cos\omega_0]z^{-1} + a^2 z^{-2}}
\end{aligned}$$

The ROC of $X_3(z)$ is $|z| > a$

3 Transformée inverses

1.

$$X(z) = \frac{4z^{-1} + 1}{z^{-2} - 3z^{-1} - 10}$$

By partial fraction decomposition, we obtain the following

$$X(z) = \frac{4z^{-1} + 1}{(z^{-1} - 5)(z^{-1} + 2)} = \frac{3}{(z^{-1} - 5)} + \frac{1}{(z^{-1} + 2)} = X_1(z) + X_2(z)$$

Where by the principle of linearity, the ROC of $X(z)$ will be the intersection of the ROCs of $X_1(z)$ and $X_2(z)$.

Assuming that the system is causal, we obtain the following from examining the z-transform table

$$X_1(z) = \frac{3}{(z^{-1} - 5)} = \frac{-\frac{3}{5}}{1 - \frac{z^{-1}}{5}} \implies x_1[n] = -\frac{3}{5}\left(\frac{1}{5}\right)^n \epsilon[n]$$

$$X_2(z) = \frac{1}{(z^{-1} + 2)} = \frac{\frac{1}{2}}{1 + \frac{z^{-1}}{2}} \implies x_2[n] = \frac{1}{2}\left(-\frac{1}{2}\right)^n \epsilon[n]$$

$$X(z) = X_1(z) + X_2(z) \implies x[n] = x_1[n] + x_2[n] = -\frac{3}{5}\left(\frac{1}{5}\right)^n \epsilon[n] + \frac{1}{2}\left(-\frac{1}{2}\right)^n \epsilon[n]$$

$X_1(z)$ has an ROC of $|z| > \frac{1}{5}$ and $X_2(z)$ has an ROC of $|z| > \frac{1}{2}$, therefore, $X(z)$ will have an ROC of $|z| > \frac{1}{2}$

2.

$$\begin{aligned} X(z) &= cos(z^{-1}) \\ &= \frac{1}{2}\left(e^{jz^{-1}} + e^{-jz^{-1}}\right) \end{aligned} \tag{2}$$

Substituing in the taylor series expansion of e^x into the expression above we obtain the following:

$$X(z) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(jz^{-1})^k}{k!} + \sum_{k=0}^{\infty} \frac{(-jz^{-1})^k}{k!}\right) \quad = \sum_{k=0}^{\infty}\left(\frac{(1+(-1)^k)j^k}{2k!}\right) z^{-k} \tag{3}$$

From the definition of the z-transform: $X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$ we obtain the following discrete-time function (where $\epsilon[n]$ is the unit-step function)

$$x[n] = \left(\frac{(1+(-1)^k)j^k}{2k!}\right) \epsilon[n] \tag{4}$$

4 La Fonction de Transfert

1.

$$\begin{aligned} y[n] &= x[n] - \frac{1}{2}x[n-1] + y[n-1] - \frac{1}{3}y[n-2] \\ Y(z) &= X(z) - \frac{1}{2}z^{-1}X(z) + z^{-1}Y(z) - \frac{1}{3}z^{-2}Y(z) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{3}z^{-2}} \end{aligned}$$

Observing the z-transform table,

$$z\left\{(r^n \cos\omega_0 n)\, \epsilon[n]\right\} = \frac{1 - [r\cos\omega_0]z^{-1}}{1 - [2r\cos\omega_0]z^{-1} + r^2 z^{-2}}$$

Therefore, taking $r\cos\omega_0 = \frac{1}{2} \implies r = \frac{\sqrt{3}}{3}$ and $\omega_0 = \frac{\pi}{6}$, we also have that the ROC of $H(z)$ is $|z| > \frac{\sqrt{3}}{3}$

Taking the inverse z-transform of $H(z)$ to determine the impulse response

$$h[n] = \left(\frac{\sqrt{3}}{3}\right)^n \cos\left(\frac{\pi}{6}n\right)\epsilon[n]$$

2. $$y[n] = h[n] * \epsilon[n]$$

   Observing the table of z-transforms,

   $$z\left\{x[n]\right\} = X(z)$$
   $$z\left\{\sum_{k=-\infty}^{n} x[k]\right\} = \frac{1}{1-z^{-1}}X(z)$$

   Using this property of the z-transform, and the property of the z-transform of a convolution

   $$\begin{aligned} Y(z) &= H(z)\left(\frac{1}{1-z^{-1}}\right) \\ y[n] &= \sum_{k=-\infty}^{n} h[k] \\ &= \sum_{k=-\infty}^{n} \left(\frac{\sqrt{3}}{3}\right)^k \cos\left(\frac{\pi}{6}k\right)\epsilon[k] \end{aligned}$$

   The ROC of $y[n]$ in this case is at least $|z| > 1$

3. The system's output from an input of a step function $\epsilon[n]$ is the accumulation of the impulse response. Therefore for LTI systems, an input of $\delta[n]$ will yield the impulse response as an output, and an input of the step function $\epsilon[n]$ will yield the accumulation of said impulse response as an output.