***Problem 13.42: To determine if a graph G with 50 vertices is 3-colorable, you test all possible 3-colorings. Your computer checks a million 3-colorings per second. Estimate how long it is going to take, in the worst case.**

Test all 50 vertices with 3 colors:

By sum and product law, we know that the number of subsequent choices aren't affected by the previous, we can conclude that each vertices will have 3 possible choices. All possible 3-colorings are calculated by 3^{50}

Given that the computer checks a million 3-colorings per second, we can calculate:

$$\begin{aligned} T &= \frac{3^{50}}{10^7} \\ &\approx 7.179 \times 10^{16} \end{aligned}$$

This means that all possible 3 colorings can be calculated in about $\boxed{\mathbf{7.179 \times 10^{16}}}$ seconds.

***Problem 13.50. How many 7-digit phone-numbers are non-decreasing (each digit is not less than the previous one.)**

For this problems, we can think of it as choosing 7 random number out of a bucket, each number is placed 7 times, so there is a possible of a single number being chosen 7 times. We can think of it as this 10 containers and we have 7 items to place in 10 different containers. Since we have 10 different containers for the 7 items, we can select k = 7 numbers from r = 10 containers. With this, we can derive the k-subset formula with repetition equal to $\binom{10+7-1}{10-1} = \boxed{\binom{\mathbf{16}}{\mathbf{9}}}$

***Problem 14.15(b-c). Consider the binary strings consisting of 10 bits.**

(b) **How many contain (i) 5 or more consecutive 1's (ii) 5 or more consecutive 0's?**

(i) 5 or more consecutive 1's

For a string of 5 bits, we have 6 possible start locations to begin placing 1's to make sure that we reach 5 consecutive 1's. For the remaining 5 numbers, we have 2 options, represented by:

```
{ {1, 1, 1, 1, 1}, x, x, x, x, x}
x = dont care
```

There are 6×2^5 choices for this subset. A double counting issue such as this arises:

```
{ x {1, 1, 1, 1, 1}, x, x, x, x}
```

where this subset can produce the same string as the one above.
Given that there are 5 different "dont cares", we can restrict each of them, leaving the other 4 free to have all the repeated, forming the equation $2^4 \times 5$
Removing them from our original subset, we get $6 \times 2^5 - 5 \times 2^4 = \boxed{\mathbf{112}}$

(ii) 5 or more consecutive 0's
following the same logic, we can deduct this is the same, at $\boxed{\mathbf{112}}$ possible binary strings.

(c) **How many contain 5 or more consecutive 0's or 5 or more consecutive 1's?** For every binary string that have 5 or more consecutive 1's, just replace those with 0's to achieve 2×112 possibilities. There are two possibilities that are double counted: 1111100000 and 0000011111 = $\boxed{\mathbf{2 \times 112 - 2}}$

***Problem 14.34. Consider all permutations of {1, 2, 3, 4, 5, 6}. A permutation is good if any of the sub-sequences 12, 23, or 56 appear. How many good permutations are there?**

Let us create subsets like such:

```
1. {{1, 2}, 3, 4, 5, 6} subsequence 1,2 appear
2. {1, {2, 3}, 4, 5, 6} subsequence 2,3 appear
3. {1, 2, 3, 4, {5, 6}} subsequence 5,6 appear
4. {{1, 2, 3}, 4, 5, 6} subsequence 1,2 and 2,3 appear
5. {{1, 2}, 3, 4, {5, 6}} subsequence 1,2 and 5,6 appear
6. {1, {2, 3}, 4, {5, 6}} subsequence 2,3 and 5,6 appear
7. {{1, 2, 3}, 4, {5, 6}} all subsequence appear
```

1-3: Total number of subsequences that appears, with duplicated = $3 \times 5!$

4-6: Number that 2 sequences appear = $4!$, total of $3 \times 4!$

7: Number that all sequences appear = $3!$

We can get rid of subset 4-6 by subtracting, but for subset 7, since all sequences appear, we removed it before so we have to add it back.

$$3 \times 5! - 3 \times 4! + 3! = 294$$

There are $\boxed{\mathbf{294}}$ possible permutations that contains 12, 23, or 56.

***Problem 14.63(g). Here are some counting problems on graphs to challenge you.**

(g) **How many Hamiltonian cycles are in $K_{n,n}$? [Hint: a Hamiltonian cycle is a cycle on graph $G = (V, E)$ that starts and ends at vertex $v_0 \epsilon V$, visiting each vertex in set $V - \{v_0\}$ (i.e., all other vertices) exactly once.]**

There are n vertices on each side. If I start from the left side, I can choose n vertices on the right side. After choosing from the right side, I have $n-1$ vertices to choose from the left side, and so on. If I have n = 4, the possibilities sequence will go from $\{4, 3, 3, 2, 2, 1, 1, 1\}$, starting from any vertex. Each vertex has $n \times ((n-1)!)^2$ possibility of hamiltonian cycles. For a bipartite graph $K_{n,n}$, there are $2n$ total vertices. There is a total of $\boxed{\mathbf{2n(n \times ((n-1)!)^2)}}$ possibilities.

***Problem 15.12. Roll a 6-sided die 5 times. What is the probability: (a) some number repeats (b) you get no sixes?**

(a) some number repeats

Total number of combinations when rolling a die 5 times = 6^5

For a number not to repeat, for each roll, the number of possibilities after is one less. To achieve 5 rolls without any repeated the numbers, the product would be $6 \times 5 \times 4 \times 3 \times 2$, or 6!.

To solve this, we can remove the total number of rows without a repeated number from the total combinations, leaving us with $6^5 - 6!$. To get the probability, we can divide it by the total:

$$\frac{6^5 - 6!}{6^5} = 0.907$$

Probabilities of rolling a 6 sided die 5 times and some number repeats = $\boxed{\mathbf{90.7\%}}$

(b) no sixes

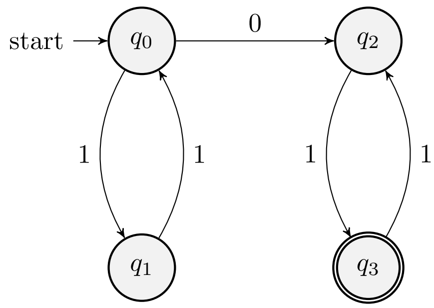
Total number of combinations stay the same at 6^5

6 is eliminated, only 5 possible numbers are left to choose from = 5^5

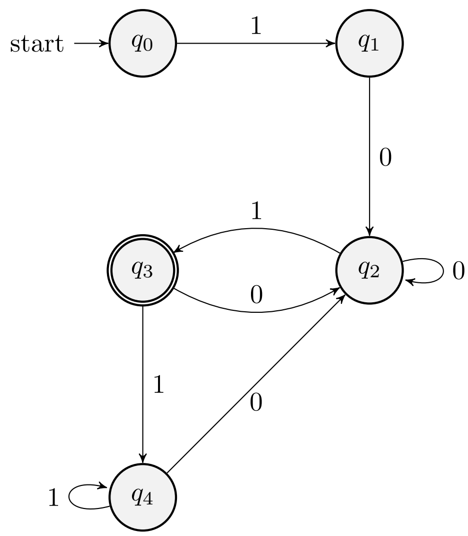
Probabilities of rolling a 6 sided die 5 times and having no 6's = $\frac{5^5}{6^5}$ or $\boxed{\mathbf{40.2\%}}$

***Problem 24.11(f,h,w). Give DFAs for the following languages, a.k.a., computing problems.**

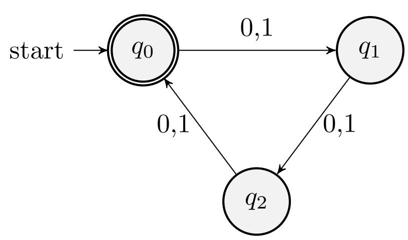
(f) $\mathcal{L} = \{\mathbf{1^{\bullet 2n}01^{\bullet 2k+1}} \mid \mathbf{n, k \geq 0}\}$.



(h) **Strings which begin with 10 and end with 01.**

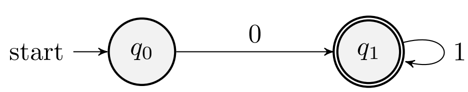


(w) **Strings whose length is divisible by 3.**



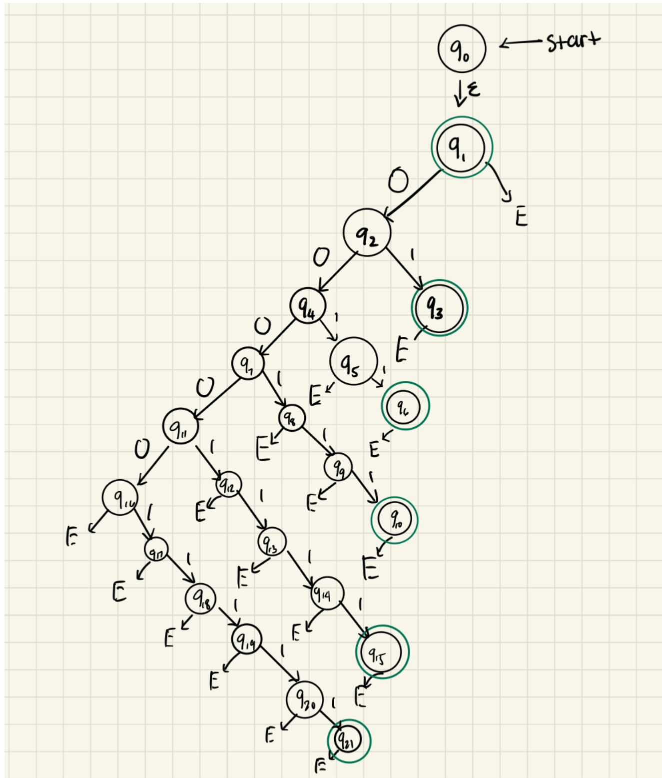
***Problem 25.7 Give a DFA and a CFG for each problem.**

(a) $\mathcal{L} = \{\mathbf{01^{\bullet n}} \mid \mathbf{n \geq 0}\}$



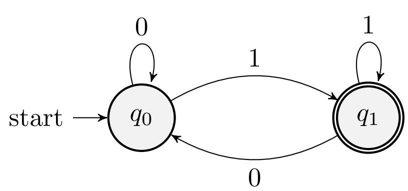
$$\begin{aligned} S &\to \quad 0 \mid ST_1 \\ T_1 &\to \quad 1 \end{aligned}$$

(b) $\mathcal{L} = \{\mathbf{0^{\bullet n} 1^{\bullet n}} \mid \mathbf{0 \leq n \leq 5}\}$



$$S \rightarrow \quad \epsilon \mid 01 \mid 0011 \mid 000111 \mid 00001111 \mid 0000011111$$

(c) $\mathcal{L} = \{$ **strings which end in a 1** $\}$



$$\begin{aligned} S &\rightarrow \quad 1 \mid T_0 S \mid T_1 S \\ T_0 &\rightarrow \quad 0 \\ T_1 &\rightarrow \quad 1 \end{aligned}$$