CSCI 2200 FOUNDATIONS OF COMPUTER SCIENCE

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EXAM 1 LOGISTICS...

We will review homework and problem set solutions in lecture on October 4 and in recitations on October 5

Exam 1 is on October 5 in our 6:00-7:50PM testblock in West Hall Auditorium

- If you have extra-time accommodations, I will email you further details regarding when/where
- **Please bring your RPI ID**

You can also bring one double-sided or two single-sided 8½"×11" cribsheets

- Feel free to collaborate on creating your cribsheets

Exam 1 will be graded out of 50 points

- We will have 18 multiple choice questions and three short answer questions
- Each multiple choice question will be worth 2 points—no partial credit—for a total of 36 points
- The three short answer questions will be worth 4 points, 5 points, and 5 points

Exam 1 covers everything through our September 30 lecture, i.e., through Chapter 7

To study for the exam, review both required and practice/warm-up problems (and solutions)...

STRONG INDUCTION

Also note that the representation of n as a product of primes is unique (aside from ordering)

Consider $P(n)$, the *Fundamental Theorem of Arithmetic*, which states that for all $n \geq 2$, we can write n as the product of one or more prime numbers

$12180 = 2 \times 2 \times 3 \times 5 \times 7 \times 29$ $\qquad$ $2022 = 2 \times 3 \times 337$

$P(n) \rightarrow P(n+1)$?

How do we prove $P(n)$ using induction…?

Tinker: what is the *prime factorization* of 2021? $\qquad$ $2021 = 43 \times 47$

How can we strengthen our claim?

STRONG INDUCTION

Consider $P(n)$, the *Fundamental Theorem of Arithmetic*, which states that for all $n \geq 2$, we can write n as the product of one or more prime numbers

$12180 = 2 \times 2 \times 3 \times 5 \times 7 \times 29$ $\qquad$ $2022 = 2 \times 3 \times 337$

How do we prove $P(n)$ using induction…?

Smaller values do help, e.g., $12180 = 60 \times 203$, or $P(60) \wedge P(203) \rightarrow P(12180)$

From this, $P(4) \wedge P(15) \rightarrow P(60)$ and $P(7) \wedge P(29) \rightarrow P(203)$—and so on!

Our much stronger claim is $Q(n)$: 2, 3, …, n are <u>all</u> products of prime numbers

STRONG INDUCTION

Consider $P(n)$, the *Fundamental Theorem of Arithmetic*, which states that for all $n \geq 2$, we can write n as the product of one or more prime numbers

Claim $Q(n)$ is $P(2) \wedge P(3) \wedge \ldots \wedge P(n)$, i.e., $2, 3, \ldots, n$ are all products of prime numbers

Proof. We prove by induction that $Q(n)$ is **T** for $n \geq 2$.

1. **[Base case]** $Q(2) = P(2)$, i.e., 2 is a product of prime numbers 2×1.
2. **[Induction step]** We show $Q(n) \rightarrow Q(n+1)$ for all $n \geq 2$ via a direct proof.

   Assume $Q(n)$ is **T**: each of $2, 3, \ldots, n$ is a product of prime numbers.
   We must prove $Q(n+1)$ is **T**: each of $2, 3, \ldots, n, n+1$ is a product of primes.

   By our induction hypothesis $Q(n)$, observe that $2, 3, \ldots, n$ are products of primes.

   Therefore, we need only prove that $n+1$ is a product of primes…

Consider $P(n)$, the *Fundamental Theorem of Arithmetic*, which states that for all $n \geq 2$, we can write n as the product of one or more prime numbers

Claim $Q(n)$ is $P(2) \wedge P(3) \wedge \ldots \wedge P(n)$, i.e., $2, 3, \ldots, n$ are all products of prime numbers

Proof. We prove by induction that $Q(n)$ is **T** for $n \geq 2$.

1. **[Base case]** $Q(2) = P(2)$, i.e., 2 is a product of prime numbers, which is **T**.
2. **[Induction step]** We show $Q(n) \rightarrow Q(n+1)$ for all $n \geq 2$ via a direct proof.

   Assume $Q(n)$ is **T**: each of $2, 3, \ldots, n$ is a product of prime numbers.
   We must prove $Q(n+1)$ is **T**: each of $2, 3, \ldots, n, n+1$ is a product of primes.

   By our induction hypothesis, $Q(n)$, observe that $2, 3, \ldots, n$ are products of primes.

   Therefore, we need only prove that $n+1$ is a product of primes—two cases:

   Case 1. $n+1$ is prime. In this case, nothing more to prove.

   Case 2. $n+1$ is not prime, so $n+1 = k\ell$, where $2 \leq k, \ell \leq n$. From our induction hypothesis, both $P(k)$ and $P(\ell)$ are **T**, which shows k and ℓ to be products of primes.

   Therefore, $n+1 = k\ell$ is a product of primes and $Q(n+1)$ is shown to be **T**.
3. By induction, $Q(n)$ is **T** for all $n \geq 2$. ■

STRONG INDUCTION

In strong induction, we strengthen claim $P(n)$ to include all n down to the base case…

To prove $P(n)$ for all $n \geq n_0$, use strong induction to prove stronger claim $Q(n)$

$$Q(n) : \text{each of } P(n_0), P(n_0+1), P(n_0+2), \ldots, P(n) \text{ are } \mathbf{T}$$

Compare ordinary induction with strong induction (assume $n_0 = 1$):

	Ordinary induction	Strong induction
Base case	Prove $P(1)$	Prove $Q(1) = P(1)$
Induction step	Assume $P(n)$ is **T** Prove $P(n+1)$	Assume $Q(n) = P(1) \wedge P(2) \wedge \ldots \wedge P(n)$ is **T** Prove $P(n+1)$

Strong induction is always easier because you get to assume more!

LEAPING STRONG INDUCTION

Suppose $P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(n) \rightarrow P(n+4)$ and base case $P(1)$ is true…

…is $P(n)$ true for all $n \geq 1$?

If so, explain why.

If not, what additional base cases do you need?

SOLUTION: No, only $P(1)$ and therefore $P(5)$ are true! Here, $P(1) \rightarrow P(5)$.

To prove $P(n)$ is true for <u>all</u> $n \geq 1$, we need base cases $P(1)$, $P(2)$, $P(3)$, and $P(4)$. Why…?

Given these four base cases, $P(1) \rightarrow P(5)$; $P(1) \wedge P(2) \rightarrow P(6)$; $P(1) \wedge P(2) \wedge P(3) \rightarrow P(7)$; $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \rightarrow P(8)$; $P(1) \wedge \ldots \wedge P(5) \rightarrow P(9)$; $P(1) \wedge \ldots \wedge P(6) \rightarrow P(10)$; etc.

L-TILE LAND PROBLEM

Given an unlimited supply of L-shaped tiles, can we tile a $2^n \times 2^n$ square patio, ignoring only one center tile?

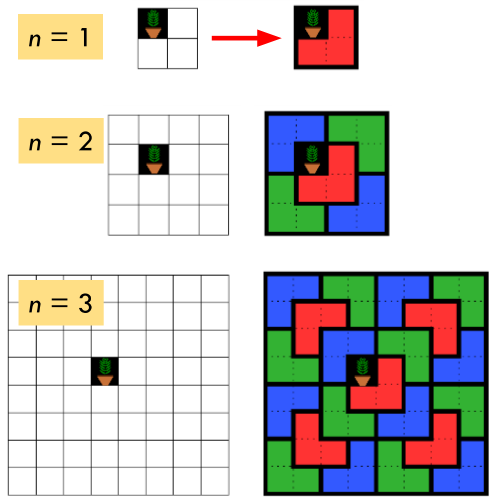
L-tiles come in red, green, and blue

Claim $P(n)$: Any $2^n \times 2^n$ grid with $n \geq 1$ minus a center square can be L-tiled

Prove $P(n)$ using induction…

…but what is $P(n + 1)$?

A separate problem is whether we can ensure no pair of adjacent tiles has the same color…!

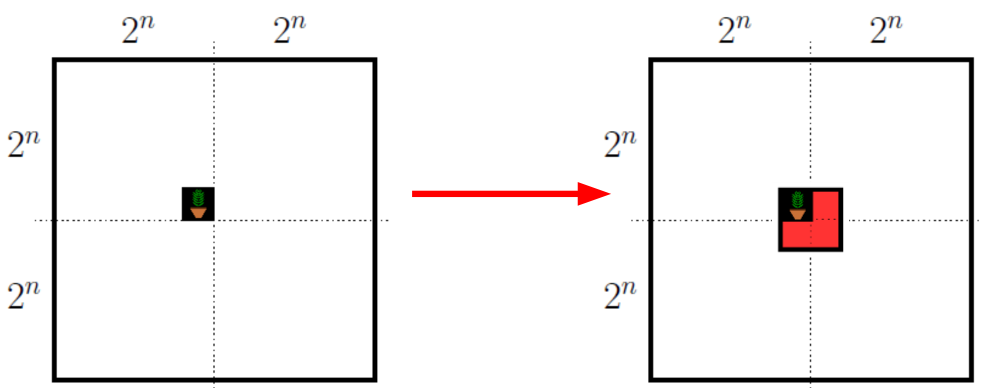


L-TILE LAND PROBLEM – STRUCTURAL INDUCTION

Claim $P(n)$: Any $2^n \times 2^n$ grid with $n \geq 1$ minus a center square can be L-tiled

For $P(n + 1)$, we construct a $2^{n+1} \times 2^{n+1}$ grid using four $2^n \times 2^n$ grids…

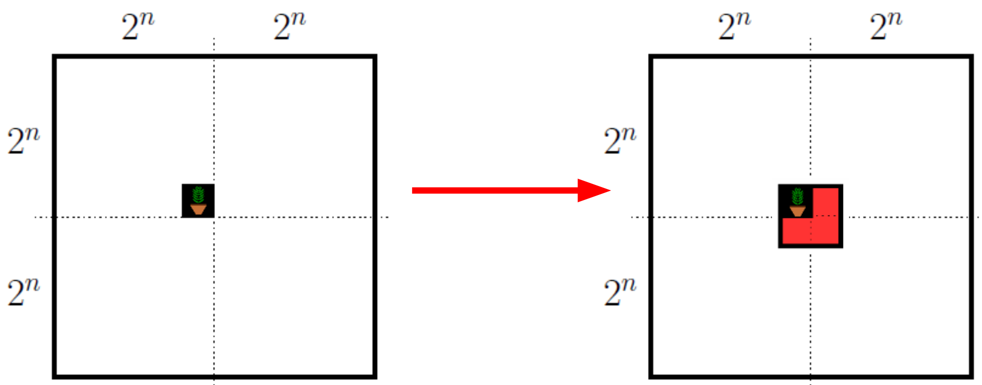
…now each $2^n \times 2^n$ sub-grid has exactly one missing square!



Claim $P(n)$: Any $2^n \times 2^n$ grid with $n \geq 1$ minus a center square can be L-tiled

For $P(n+1)$, we construct a $2^{n+1} \times 2^{n+1}$ grid using four $2^n \times 2^n$ grids…

…now each $2^n \times 2^n$ sub-grid has exactly one missing square!



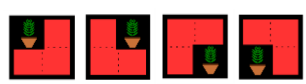
We can only use $P(n)$ for <u>center</u> squares—ugh, this approach has missing <u>corner</u> squares…

How can we strengthen claim $P(n)$ to also include grids with missing corner squares?

L-TILE LAND PROBLEM – STRENGTHENED…

Claim $Q(n)$: (i) Any $2^n \times 2^n$ grid with $n \geq 1$ minus a <u>center</u> square can be L-tiled; and
(ii) Any $2^n \times 2^n$ grid with $n \geq 1$ minus a <u>corner</u> square can be L-tiled

Proof. We prove claim $Q(n)$ for all $n \geq 1$ by induction.

1. **[Base case]** $Q(1)$ holds for center and corner squares: 
2. **[Induction step]** We prove $Q(n) \rightarrow Q(n+1)$ for $n \geq 1$ using a direct proof.

   Assume $Q(n)$: (i) Any $2^n \times 2^n$ grid minus a center square can be L-tiled;
   and (ii) Any $2^n \times 2^n$ grid minus a corner square can be L-tiled.

   Prove $Q(n+1)$: (i) Any $2^{n+1} \times 2^{n+1}$ grid minus a center square can be L-tiled;
   and (ii) Any $2^{n+1} \times 2^{n+1}$ grid minus a corner square can be L-tiled.

   (i) For a $2^{n+1} \times 2^{n+1}$ grid missing a center square, …

   (ii) For a $2^{n+1} \times 2^{n+1}$ grid missing a corner square, …

See the rest of the proof on page 73…

WHAT NEXT...?

Attend as many office hours as you can—we have a lot of weekly coverage

Homework 2 is posted and due by 11:59PM on September 29

- Problem 7.4(c) is no longer required—it is now a warm-up problem…

Problem Set 3 is due at recitations on September 28—tomorrow!

- We are again going to spread out to neighboring rooms: Ricketts 208 and 212
- Note that Ricketts 212 is not available 10:00-10:50AM this week (so for section 01)

Practice! Tinker! Practice! Tinker! Practice! Tinker! Practice! Tinker! Practice!