1. assume n is an integer, give direct and contraposition proofs

   (a) $(n^3 + 5$ is odd$) \to$ (n is even)

      i. direct proof

         - $n^3 + 5$ is odd, $n^3 + 5 = 2k + 1$, $n^3 = 2k - 4$
         - $n = \sqrt[3]{2(k-2)}$
         - a cube root of an even number is always even by definition, statement claimed in p is true

      ii. contraposition proof

         - assume n is not even, n is odd, $n = 2k + 1$
         - $n^3 + 5 = (2k + 1)^3 + 5 => 8k^3 + 12k^2 + 6k + 6$
         - $8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$
         - we have shown that p is even when n is odd, the statement claimed is true

   (b) $(3$ does not divide $n) \to (3$ divides $n^2 + 2)$

      i. direct proof

         - if 3 does not divide n, then $n = 3k + 1$
         - assuming p is true, 3 divides $n^2 + 2$, so $n^2 + 2 = 3k$
         - $n^2 = 3k - 2$, then $n = \sqrt{3k - 2}$
         - plugging in $n = 3k$, we get $n = \sqrt{n - 2}$
         - this is not true for all cases, therefore, this statement is false

      ii. contraposition proof

         - 3 does not divide by $n^2 + 2$
         - therefore, $n^2 + 2 \neq 3k$, for some integer k

- 3 does not divide n, $n \neq 3k$

- $(3k)^2 + 2 \neq 3k$

2. prove by contradiction

   (a) $(x, y) \in \mathbb{Z}^2 \rightarrow x^2 - 4y - 3 \neq 0$

   - assume $x^2 - 4y - 3 = 0$

   - rearranging variables to isolate x, we get $x^2 = 4y + 3$

   - we now know that x^2 is odd, therefore $x^2 = 2k + 1$

   - substitute x^2 in, $2k + 1 - 4y = 3$

   - isolate y, we get $\frac{2k+1-3}{4} = y$, $y = \frac{2k-2}{4}$

   - simplify to get $y = \frac{k-1}{2}$, uh oh, there are integers k that makes y not a positive integer

   - the statement is true due to proof by contradiction

3. prove these if and only if, prove two implications

   (a) prove: 4 divides $n \in \mathbb{Z}$ IF AND ONLY IF $n = 1 + (-1)^k(2k-1)$ for $k \in \mathbb{N}$. (Try $n < 0$, $n = 0$, $n > 0$; k is even/odd.)

   i. p $\rightarrow$ q

      - direct proof: $4w = n \rightarrow n = 1 + (-1)^k(2k-1)$

      - from plugging in, we get $4w = 1 + (-1)^k(2k-1)$

      - if k is even, then $k = 2w$, from plugging in, we get $4w = 1 + (2(2w) - 1)$, equal to $4w = 4w$, from above, we stated that 4 divides n, therefore $n = 4w$

      - if k is odd, then $k = 2w + 1$, from plugging in, we get $4w = 1 + (-1)(2(2w+1) - 1)$, equal to $4w \neq 1 - 4w$, then $n = 1 - 4w$,

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which does not divide 4

- if $k = 0$, then it becomes $4w = 0$, then $n = 0$ as well, which is divisible by 4

- plug in 4w for n in the other equation, then do the three cases where k is even or odd or equal to 0

ii. q $\rightarrow$ p

- direct proof

- assuming that p is true

- assuming $k = 2w$ for even cases, $k = 2w + 1$ for odd cases, $k = 0$ for 0 cases

- even cases of k, $n = 1 + (2(2w) - 1) = 4w$, then $n = 4w$, it does divide by 4

- odd cases of k $n = 1 + (-1)(2(2w + 1) - 1)$, then $n = 1 - 4w$, it does not divide by 4

- 0 case of k $n = 0$, it is divisible by 4

4. determine the type of proof and prove

   (a) If n is odd, then $n^2 - 1$ is divisible by 8.

   - this statement can be proven by a direct proof

   - if n is odd, then $n = 2k + 1$

   - then with $n^2 - 1$, you can substitute n for $2k + 1$

   - becomes $(2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1$

   - at its simplest form, it is $4(k^2 + k) \neq 8k$

   - by direct proof, the statement is false