CSCI 2200 FOUNDATIONS OF COMPUTER SCIENCE

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PROOF – EVEN (AND ODD) SQUARES

If we square an even integer, what do we get…?

And for odd integers…?

n	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	± 10	± 11	$\ldots$
n^2	$\boxed{\mathbf{0}}$	1	$\boxed{\mathbf{4}}$	9	$\boxed{\mathbf{16}}$	25	$\boxed{\mathbf{36}}$	49	$\boxed{\mathbf{64}}$	81	$\boxed{\mathbf{100}}$	121	$\ldots$

Conjecture:
Even squares come from even numbers—and even numbers have even squares

How do I convince you this is true? We need proof!

PROOF – EVEN (AND ODD) SQUARES

Conjecture:
Even squares come from even numbers—and even numbers have even squares

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n^2	$\boxed{\mathbf{0}}$	1	$\boxed{4}$	9	$\boxed{\mathbf{16}}$	25	$\boxed{\mathbf{36}}$	49	$\boxed{\mathbf{64}}$	81	$\boxed{\mathbf{100}}$	121	$\ldots$

Proof: We use an exhaustive (case-by-case) proof

(a) n is even $\rightarrow$ $n = 2k$ $(k \in \mathbb{Z})$ $\rightarrow$ $n^2 = 2(2k^2)$ $\rightarrow$ n^2 is even (divisible by 2)

(b) n is odd $\rightarrow$ $n = 2k + 1$ $(k \in \mathbb{Z})$ $\rightarrow$ $n^2 = 2(2k^2 + 2k) + 1$ $\rightarrow$ n^2 is odd

Here, n must be even or odd (*exhaustive*)—also, n is *general*, i.e., no restrictions on n

PROOF – EVEN (AND ODD) SQUARES

Theorem:
Every even square comes from an even number—and every even number has an even square

n	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	±10	±11	$\ldots$
n^2	$\boxed{\mathbf{0}}$	1	$\boxed{4}$	9	$\boxed{\mathbf{16}}$	25	$\boxed{\mathbf{36}}$	49	$\boxed{\mathbf{64}}$	81	$\boxed{\mathbf{100}}$	121	$\ldots$

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A PROOF MUST CONVINCE

A proof strings together multiple "truths" to *convince* the reader of something *new*

Last class, our proof that $\sqrt{2}$ is irrational strung together several truths:

(a) The definition of $\mathbb{Q}$, the set of rational numbers

(b) The Well-Ordering Principle (which remember is an *axiom*)

(c) Simple algebra to manipulate both sides of equalities

(d) Our theorem that states that squares are even

ALGORITHM FOR MAKING AND PROVING A CLAIM

Step 1: Precisely state the right thing to prove

Use existing and known definitions, sets, axioms, theorems, etc.

Be precise—apply your creativity and imagination through tinkering

Step 2: Prove the claim

This will take practice and your toolbox of proof techniques (which will grow)

Step 3: Check the proof for correctness

Proofread the proof—no creativity is needed to look a proof in the eye and determine if it is correct, i.e., are you convinced?

MAKING PRECISE STATEMENTS

Sentences formed using English are often filled with assumptions and ambiguities

Which of these statements are true…? Which are false…?

(a) You can have cake or ice-cream

(b) If pigs can fly then you get an A

(c) Every person has a story

(d) $2 + 2 = 4$

(e) $1 + 1 = 10$

MAKING PRECISE STATEMENTS

Sentences formed using English are often filled with assumptions and ambiguities

Which of these statements are true…? Which are false…?

(a) You can have cake or ice-cream

(b) If pigs can fly then you get an A

(c) Every person has a story

(d) $2 + 2 = 4$

(e) $1 + 1 = 10$

These statements are not precise…
…and therefore are not useful claims to be proven

Before seeking proof, we must precisely state the correct logical meaning of the claim

PROPOSITIONS

A *proposition* is a statement that precisely states a claim without ambiguity

A proposition is true (**T**) or false (**F**), though it may be difficult or even impossible to assign a truth value to a proposition

Identify propositions using lowercase letters $p, q, r, s, t, \dots$

p: pigs can fly q: you got an A

r: every map can be colored with four colors such that no adjacent regions are the same color

s: 4^2 is even

t: within a circle, any triangle with points on the circle's circumference and with the circle's diameter as a side is a right triangle

COMPOUND PROPOSITIONS

Use logical connectors to form *compound propositions* from simple (atomic?) propositions

Connector	Symbol	Example
NOT	$\neg p$	IT IS NOT THE CASE THAT (pigs can fly)
AND	$p \wedge q$	(pigs can fly) AND (you got an A)
OR	$p \vee q$	(pigs can fly) OR (you got an A) OR both!
IF…THEN…	$p \rightarrow q$	IF (pigs can fly) THEN (you got an A)

NEGATION (NOT)

Connector	Symbol
NOT	$\neg p$
AND	$p \wedge q$
OR	$p \vee q$
IF…THEN…	$p \rightarrow q$

The negation $\neg p$ is **T** when p is **F**, and the negation $\neg p$ is **F** when p is **T**

The proposition "pigs can fly" is **F…**

…therefore, IT IS NOT THE CASE THAT "pigs can fly" is **T**

The proposition "4^2 is even" is **T…**

…therefore, IT IS NOT THE CASE THAT "4^2 is even" is **F**

CONJUNCTION (AND)

Connector	Symbol
NOT	$\neg p$
AND	$p \wedge q$
OR	$p \vee q$
IF…THEN…	$p \rightarrow q$

Both p and q must be **T** for $p \wedge q$ to be **T**; otherwise, the claim $p \wedge q$ is **F**

The proposition "pigs can fly" is **F…**

…therefore, "pigs can fly" $\wedge$ "you got an A" is **F**

In this case, q does not matter and we do not know whether "you got an A" is **T**

DISJUNCTION (OR)

Connector	Symbol
NOT	$\neg p$
AND	$p \wedge q$
OR	$p \vee q$
IF…THEN…	$p \rightarrow q$

Both p and q must be **F** for $p \vee q$ to be **F**; otherwise, the claim $p \vee q$ is **T**

The proposition "pigs can fly" is **F**…

…therefore, "pigs can fly" $\vee$ "you got an A" is **T** or **F**, i.e., q <u>does</u> matter

Disjunction always allows for both p and q to be **T**…

…you can "have your cake" and "eat it, too"

IMPLICATION (IF…THEN…)

Connector	Symbol
NOT	$\neg p$
AND	$p \wedge q$
OR	$p \vee q$
IF…THEN…	$p \rightarrow q$

With implication, $p \rightarrow q$ is **F** only when p is **T** and q is **F**; otherwise, $p \rightarrow q$ is **T**

Suppose the proposition "pigs can fly" $\rightarrow$ "you got an A" is **T**…

…since pigs cannot fly, does this mean you cannot get an A?

The theorem "n^2 is even" $\rightarrow$ "n is even" is **T**…

…if n^2 is even, can we deduce that $n \neq 5$?

Given propositions p: "it rained last night" and q: "the grass is wet"…

…what does $p \rightarrow q$ tell us?

Connector	Symbol
NOT	$\neg p$
AND	$p \wedge q$
OR	$p \vee q$
IF…THEN…	$p \rightarrow q$

IMPLICATION (IF…THEN…)

p : it rained last night

q : the grass is wet

$p \rightarrow q$: IF "it rained last night" THEN "the grass is wet"

We learn from the morning news that it rained last night…

$p \rightarrow q$ is **T** : IF "it rained last night" THEN "the grass is wet"

p is **T** : "it rained last night"

$\therefore$ q is **T** : "the grass is wet"

For a **true** implication $p \rightarrow q$, when p is **T,** we can conclude that q is also **T**

Connector	Symbol
NOT	$\neg p$
AND	$p \wedge q$
OR	$p \vee q$
IF…THEN…	$p \rightarrow q$

IMPLICATION (IF…THEN…)

p : it rained last night

q : the grass is wet

$p \rightarrow q$: IF "it rained last night" THEN "the grass is wet"

Walking to class, we observe that the grass is wet…

$p \rightarrow q$ is **T** : IF "it rained last night" THEN "the grass is wet"

q is **T** : "the grass is wet"

$\therefore$ p is **T** or **F** : we do not know if "it rained last night"

For a **true** implication $p \rightarrow q$, when q is **T,** we **cannot** deduce whether p is **T** or **F**

IMPLICATION (IF…THEN…)

Connector	Symbol
NOT	$\neg p$
AND	$p \wedge q$
OR	$p \vee q$
IF…THEN…	$p \rightarrow q$

p : it rained last night

q : the grass is wet

$p \rightarrow q$: IF "it rained last night" THEN "the grass is wet"

We learn from the morning news that it did not rain last night…

$p \rightarrow q$ is **T** : IF "it rained last night" THEN "the grass is wet"

p is **F** : IT IS NOT THE CASE THAT "it rained last night"

$\therefore$ q is **T** or **F** : we do not know if "the grass is wet"

For a **true** implication $p \rightarrow q$, when p is **F**, we **cannot** deduce whether q is **T** or **F**

IMPLICATION (IF…THEN…)

Connector	Symbol
NOT	$\neg p$
AND	$p \wedge q$
OR	$p \vee q$
IF…THEN…	$p \rightarrow q$

p : it rained last night

q : the grass is wet

$p \rightarrow q$: IF "it rained last night" THEN "the grass is wet"

*What if we learn that it rained last night but we observe the grass to be dry…? …then we have a falsifying scenario and the implication $p \rightarrow q$ is **F***

Walking to class, we observe that the grass is not wet…

$p \rightarrow q$ is **T** : IF "it rained last night" THEN "the grass is wet"

q is **F** : IT IS NOT THE CASE THAT "the grass is wet"

$\therefore$ p is **F** : IT IS NOT THE CASE THAT "it rained last night"

For a **true** implication $p \rightarrow q$, when q is **F**, we can conclude that p is **F**

TRUTH TABLES

We can summarize our logical connectors using a truth table

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$
F	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**
T	**T**	**F**	**T**	**T**	**T**

Use these logical connectors to form compound propositions

Construct a truth table for $(p \vee q) \rightarrow r$, then highlight the rows for which $(p \vee q) \rightarrow r$ is **T**

TRUTH TABLES – AN EXAMPLE

Construct a truth table for $(p \vee q) \rightarrow r$, then highlight the rows for which $(p \vee q) \rightarrow r$ is **T**

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$
F	**F**	**F**	**F**	**T**
F	**F**	**T**	**F**	**T**
F	**T**	**F**	**T**	**F**
F	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**
T	**F**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**
T	**T**	**T**	**T**	**T**

TRUTH TABLES – AN EXAMPLE

Construct a truth table for $(p \vee q) \rightarrow r$, then highlight the rows for which $(p \vee q) \rightarrow r$ is **T**

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$
F	**F**	**F**	**F**	**T**
F	**F**	**T**	**F**	**T**
F	T	F	T	F
F	**T**	**T**	**T**	**T**
T	F	F	T	F
T	**F**	**T**	**T**	**T**
T	T	F	T	F
T	**T**	**T**	**T**	**T**

Interpret each row given propositions

p : "you are hungry"

q : "you are thirsty"

r : "you go to the dining hall"

TRUTH TABLES – AN EXAMPLE

Construct a truth table for $(p \vee q) \rightarrow r$, then highlight the rows for which $(p \vee q) \rightarrow r$ is **T**

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$
F	**F**	**F**	**F**	**T**
F	**F**	**T**	**F**	**T**
F	T	F	T	F
F	**T**	**T**	**T**	**T**
T	F	F	T	F
T	**F**	**T**	**T**	**T**
T	T	F	T	F
T	**T**	**T**	**T**	**T**

Interpret each row given propositions

p : "you are hungry"

q : "you are thirsty"

r : "you go to the dining hall"

TRUTH TABLES – AN EXAMPLE

Construct a truth table for $(p \lor q) \to r$, then highlight the rows for which $(p \lor q) \to r$ is **T**

p	q	r	$p \lor q$	$(p \lor q) \to r$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

Interpret each row given propositions

p : "you are hungry"

q : "you are thirsty"

r : "you go to the dining hall"

TRUTH TABLES – AN EXAMPLE

Construct a truth table for $(p \lor q) \to r$, then highlight the rows for which $(p \lor q) \to r$ is **T**

p	q	r	$p \lor q$	$(p \lor q) \to r$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

Interpret each row given propositions

p : "you are hungry"

q : "you are thirsty"

r : "you go to the dining hall"

PROVING AN IMPLICATION

Note that we only have proven that the implication is true!

Can we prove the **implication** that if n^2 is even, n is even (for $n \in \mathbb{Z}$)?

$p:$ n^2 is even

$q:$ n is even

$p \to q:$ if n^2 is even, n is even

p	q	$p \to q$
F	**F**	**T**
F	**T**	**T**
T	**F**	**F**
T	**T**	**T**

Can we prove that the **F** row **cannot** occur?

In this row, q is **F**, so n is odd, i.e., $n = 2k + 1$

From this, $n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$, which means n^2 is odd, i.e., p must be **F**

The highlighted row **cannot** occur—therefore, $p \to q$ is always **T**

PROVING EQUIVALENCE

We can prove the equivalence of different statements by comparing their truth tables…

…or we can apply the rules shown to the right for manipulating logical connectors

Which approach is better…?

1. *Associative:* $(p \wedge q) \wedge r \overset{\text{eqv}}{\equiv} p \wedge (q \wedge r);$
   $(p \vee q) \vee r \overset{\text{eqv}}{\equiv} p \vee (q \vee r).$
2. *Commutative:* $p \wedge q \overset{\text{eqv}}{\equiv} q \wedge p;$
   $p \vee q \overset{\text{eqv}}{\equiv} q \vee p.$
3. *Negations:* $\neg(\neg p) \overset{\text{eqv}}{\equiv} p;$
   $\neg(p \wedge q) \overset{\text{eqv}}{\equiv} \neg p \vee \neg q;$
   $\neg(p \vee q) \overset{\text{eqv}}{\equiv} \neg p \wedge \neg q.$
4. *Distributive:* $p \vee (q \wedge r) \overset{\text{eqv}}{\equiv} (p \vee q) \wedge (p \vee r);$
   $p \wedge (q \vee r) \overset{\text{eqv}}{\equiv} (p \wedge q) \vee (p \wedge r).$
5. *Implication:* $p \to q \overset{\text{eqv}}{\equiv} \neg q \to \neg p;$
   $p \to q \overset{\text{eqv}}{\equiv} \neg p \vee q.$

A (TRICKY) PROBLEM TO WORK ON…

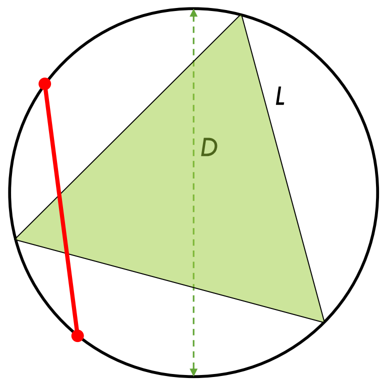
Can you solve this problem… …by Tuesday's lecture?

Take any circle with diameter *D* and inscribe within it an equilateral triangle with side length *L*

Select a *chord* at random…

Recall that a chord is a line segment with its two endpoints on the circumference of the circle

What is the probability that the length of the chord is greater than *L*…?



WHAT NEXT…?

Homework 1 is due by 11:59PM on Thursday, September 15 (in Submitty)

Recitation next week (September 14) is a Q&A session to work on the homework

If you want to switch sections, email me…

…the "add" deadline is Monday, September 12

Get the textbook!

Email me extra-time accommodations ASAP