- **Problem 7.9.** $G^0 = 0$. $G_1 = 1$ and $G_n = 7G_{n-1} - 12G_{n-2}$ for $n > 1$. Compute $G_5$. Show $G_n = 4^n - 3^n$ for $n \geq 0$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $A_n$ | 0 | 1 | 7 | 37 | 175 | 781 |

(i) Prove the base case:

$$G(0) = 4^0 - 3^0 = 0$$

base case is true

(ii) With induction hypothesis: assume $G(n) = 4^n - 3^n$, prove by direct proof that $G(n) = 4^n - 3^n \rightarrow G(n+1) = 4^{n+1} - 3^{n+1}$ for all $n \geq 0$.

$$\begin{aligned} G(n+1) &= 7G(n+1-1) - 12G(n+1-2) \text{ (recursive definition)} \\ &= 7G(n) - 12G(n-1) \\ 7G(n) - 12G(n-1) &= 4^{n+1} - 3^{n+1} \end{aligned}$$

manipulate the LHS

$$\begin{aligned} 7G(n) - 12G(n-1) &= 7(4^n - 3^n) - 12(4^{n-1} - 3^{n-1}) \text{ (induction hypothesis)} \\ &= 7(4^n - 3^n) - \cancel{12}\left(\frac{4^n(3) - 3^n(4)}{\cancel{12}}\right) \\ &= 7(4^n - 3^n) - (3(4^n) - 4(3^n)) \\ &= 7(4^n) - 7(3^n) - 3(4^n) - 4(3^n) \\ &= 4(4^n) - 3(3^n) \\ &= 4^{n+1} - 3^{n+1} \end{aligned}$$

We prove the statement is true for $n \geq 0$ by direct proof ■

- **Problem 7.12(c). (See Problem 7.28 for hints.)** Tinker to guess a formula for each recurrence and prove it. In each case $A_1 = 1$ and for $n > 1$:

(c) $A_n = 10nA_{n-1}/(n-1) + n$

| $n$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $A_n$ | 22 | 333 | 4444 | 55555 |

i. formula found:

$$\frac{10^n - 1}{9} n$$

ii. prove the base case:

$$\begin{aligned} A(2) &= \frac{100 - 1}{9}(2) \\ &= \frac{99}{9}(2) = 22 \end{aligned}$$

base case proven

iii. With induction hypothesis: assume $A(n) = \frac{10^n-1}{9}n$, prove by direct proof that $A(n) = \frac{10^n-1}{9}n \to A(n+1) = \frac{10^{n+1}-1}{9}(n+1)$ for all $n > 0$.

$$A(n) = \frac{10nA(n-1)}{n-1} + n$$
$$A(n+1) = \frac{10(n+1)A(n)}{n} + (n+1) \text{ (recursive definition)}$$

with with RHS

$$\begin{aligned}
A(n+1) &= \frac{10(n+1)A(n)}{n} + (n+1) \text{ (induction hypothesis)} \\
&= \frac{10(n+1)(\frac{10^n-1}{9}\cancel{n})}{\cancel{n}} + (n+1) \\
&= \frac{10(n+1)(10^n-1)}{9} + (n+1) \\
&= \frac{(10n+10)(10^n-1)+9(n+1)}{9} \\
&= \frac{10n(10^n) - 10n + 10(10^n) - 10 + 9n + 9}{9} \\
&= \frac{10n(10^n) - n + 10^{n+1} - 1}{9} \\
&= \frac{10^{n+1}n - n + 10^{n+1} - 1}{9} \\
&= \frac{10^{n+1}-1}{9}(n+1)
\end{aligned}$$

iv. we prove by direct proof that the statement is true for all $n > 1$ ■

- **Problem 7.13(a).** Analyze these very fast growing recursions. [Hint: Take logarithms.] Analyze these very fast growing recursions.

$$\begin{aligned}
&M_1 = 2 \text{ and } M_n = aM_{n-1}^2 \text{ for n} \geq 1 \\
&M_2 = aM_{2-1}^2 = a(M_1)^2 = a(2)^2 = 4a \\
&M_3 = aM_{3-1}^2 = a(M_2)^2 = a(4a)^2 = a(16a^2) = 16a^3 \\
&M_4 = aM_{4-1}^2 = a(M_3)^2 = a(16a^3)^2 = a(256a^6) = 256a^7 \\
&M_n = 2^{2^{n-1}}a^{2^{n-1}-1}
\end{aligned}$$

Can we prove our claim P(n) that $M_n = 2^{2^{n-1}}a^{2^{n-1}-1}$ (thereby removing the recursion)?
Proof. We prove by induction that P(n) is **T** for n $\geq 1$.
Base Case:
P(1) claims $M_1 = 2^{2^{1-1}}a^{2^{1-1}-1} = 2^{2^0}a^{2^0-1} = 2^1a^{1-1} = 2(a^0) = 2(1) = 2$ which is **T** from the recursive definition.
Induction Step:
We prove P(n) $\to$ P(n+1) for all n $\geq 1$ via direct proof.
Assume $M_n = 2^{2^{n-1}}a^{2^{n-1}-1}$; we must prove that $M_{n+1} = 2^{2^{(n+1)-1}}a^{2^{(n+1)-1}-1} = 2^{2^n}a^{2^n-1}$, n $\in \mathbb{N}$.

LHS:

$$\begin{aligned}
M_{n+1} = aM_{n+1-1}^2 &= aM_n^2 \text{ (from recursive definition)} \\
&= a(2^{2^{n-1}} a^{2^{n-1}-1})^2 \text{ (from Induction Hypothesis)} \\
&= a(2^{(\frac{1}{2})(2^n)} a^{(\frac{1}{2})(2^n)-1})^2 \\
&= a(2^{2^n} a^{2^n-2}) \\
&= 2^{2^n} a^{2^n-1} \text{ (after adding exponents of a)}
\end{aligned}$$

By Induction P(n) is **T** for all n $\geq$ 1. ■

- **Problem 7.19(d).** Recall the Fibonacci numbers: $F_1$, $F_2 = 1$; and, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$

  (d) Prove that every third Fibonacci number, $F_{3n}$, is even

  (i) We have to prove that $F_{3n} = 2p$, for some $p \in \mathbb{N}$

  (ii) prove the base case:

$$F_{3n-1} = F_2 \text{ when } n = 1, \ \rightarrow F_2 = 1$$
$$F_{3n-2} = F_1 \text{ when } n = 1, \ \rightarrow F_1 = 1$$

  (iii) since the fibonacci sequence is a sum of the previous two terms, we can make the following assumptions:

By the given formula $F_n = F_{n-1} + F_{n-2}$, we can calculate $F_{3n} = F_{3n-1} + F_{3n-2}$

We know that both $F_{3n-1}$ and $F_{3n-2}$ are sums of even and odd numbers

$$\begin{aligned}
F_{3n-1} &= 2k + (2j + 1) \\
&= 2(k + j) + 1 \\
F_{3n-2} &= (2w + 1) + 2i \\
&= 2(w + i) + 1
\end{aligned}$$

plugging back into the original function, we get:

$$\begin{aligned}
F_{3n} &= [2(k + j) + 1] + [2(w + i) + 1] \\
&= 2(k + j) + 2(w + i) + 2 \\
&= 2(k + j + w + i + 1)
\end{aligned}$$

We prove that the statement is true for all $n > 2$ ■

- **Problem 7.42.** Give pseudocode for a recursive function that computes $3^{2^n}$ on input $n$.

```
out = recur(n)
      if (n == 0) out = 3;
      else out = recur(n-1) * recur(n-1);
```

Proof. We use induction to prove recur(n) $=3^{2^n}$, w/ n $\geq \mathbb{N}_0$.
Base Case:
n = 0: recur(0) $\rightarrow$ out = 3, which is equal to $3^{2^0} = 3^1 = 3 \therefore$ making recur(n) = $3^{2^n}$ **T**.
Induction Step
We prove recur(n) = $3^{2^n} \rightarrow$ recur(n+1) = $3^{2^{n+1}}$ is T for all n $\in \mathbb{N}_0$ via a direct proof.
Assume recur(n) = $3^{2^n}$
LHS:

$$\begin{aligned}
\text{recur(n+1)} &= \text{recur((n+1) - 1) * recur((n+1) -1) (from recursive definition)} \\
&= \text{recur(n) * recur(n)} \\
&= 3^{2^n} * 3^{2^n} \text{ (from Induction Hypothesis)} \\
&= 3^{2(2^n)} \\
&= 3^{2^{n+1}}
\end{aligned}$$

By induction, we have proven recur(n+1) = $3^{2^{n+1}}$ is **T** for all n $\geq 0$. Thus recur(n) = $3^{2^n}$ is **T** for all n $\geq 0$. ■
b.

$$\begin{aligned}
T_0 &= 2 \\
T_1 &= 7 \\
T_2 &= 17 \\
T_3 &= 37 \\
T_n &= 2(T_{n-1}) + 3
\end{aligned}$$

After tinkering, $T_n = 5(2^n) - 3$
Proof. We use induction induction to prove $T_n = 5(2^n) - 3$, for all n $\in \mathbb{N}_0$ via a direct proof.
Base Case:
$T_0 = 5(2^0) - 3 = 2$, which is **T** from the recursive definition.
Induction Step:
We prove $T_n = 5(2^n) - 3 \rightarrow T_{n+1} = 5(2^{n+1}) - 1$ is T for all n $\in \mathbb{N}_0$ via direct proof.
Assume $T_n = 5(2^n) - 3$
LHS:

$$
\begin{aligned}
T_{n+1} &= 2(T_{n+1-1}) + 3 \text{ (from recursive definition)}\\
&= 2(T_n) + 3\\
&= 2(5(2^n) - 3) + 3 \text{ (from induction hypothesis)}\\
&= 5(2^1)(2^n) - 6 + 3\\
&= 5(2^{n+1}) - 3
\end{aligned}
$$

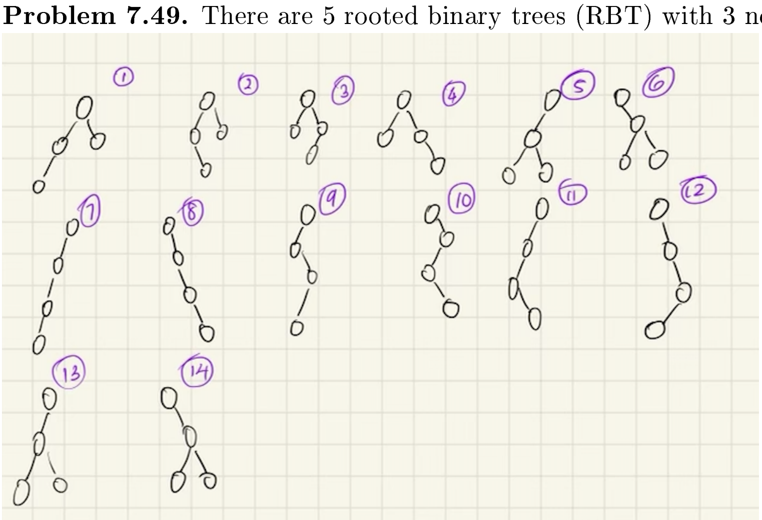
By induction, we have proven $T_{n+1} = 5(2^{n+1}) - 3$ is **T** for all n $\in \mathbb{N}_0$. Thus, $T_n = 5(2^n) - 3$ is **T** for all n $\in \mathbb{N}_0$. ■

- **Problem 7.45(c).** Give recursive definitions for the set $S$ in each of the following cases.

  (c) $S = \{$all strings with the same number of 0's and 1's$\}$ (e.g. 0011,0101,100101).

  1. **[basis]** $\epsilon \in S, 0 \in S, 1 \in S$
  2. **[constructor(i)]** $\epsilon \in S \rightarrow 0 \bullet x \bullet 0 \in S$;
     **[constructor(ii)]** $\epsilon \in S \rightarrow 1 \bullet x \bullet 1 \in S$.

- **Problem 7.49.** There are 5 rooted binary trees (RBT) with 3 nodes. How many have 4 nodes




  - We can make 14 possible rooted binary trees with 4 nodes.

- **Problem 8.12(d).** A set $P$ of parenthesis strings have a recursive definition. The strings in P are balanced, i.e. they have an equal number of open and close parentheses. Let us prove it. To do so, imagine creating the strings $s_1, s_2, s_3$,... of P in some order starting w/ $s_1 = \epsilon$
  $P = \{\epsilon, [], [[]], [][], ...., s_n, ....\}$
  The list has all strings of P in their order of creation. The nth string $s_n$ came from either one previous string $s_i$ or two previous strings $s_i$ and $s_j$ using constructors $s_n = [s_i]$ and $s_n = s_i \times s_j$ respectively. If $s_i$ and $s_j$ are balanced, then so is $s_n$ with one or more open parenthesis and closed parenthesis. Since

the primordial ancestor $s_1 = \epsilon$ balanced, we can prove all the strings in P are balanced.

Q(n): Every string in P is balanced
Proof. We use strong induction to prove Q(n) for n $\geq$ 1
Base Case:
n = 1: $s_1 = \epsilon$ which is clearly balanced, so Q(1) is true.
Induction Step:
We show Q(1)$\wedge$...$\wedge$Q(n) $\rightarrow$ Q(n+1) using direct proof. Assume Q(1),Q(2),...Q(n): $s_1$ ,....., $s_n$ are all balanced. We must show Q(n+1): $s_{n+1}$ is balanced. We have that $s_{n+1} = [s_i]$, where $s_i$ appeared earlier than $s_{n+1}$, so i < n + 1. By the strong induction hypothesis, $s_i$ is balanced so $s_{n+1}$ is balanced because you add one open-parenthesis and close-parenthesis. We also have that $s_{n+1} = s_i s_j$ , where $s_i$ and $s_j$ appeared earlier than $s_{n+1}$. By the strong induction hypothesis, $s_i$ and $s_j$ are balanced. Therefore $s_{n+1}$ is balanced because the previous two results used to produce $s_{n+1}$ are balanced. ■

- **Problem 8.14.** A set $A$ is defined recursively as shown.

  1. $3 \in A$.
  2. $x, y \in A \rightarrow x + y \in A$;
     $x, y \in A \rightarrow x - y \in A$.

$A = \{3, 6, 0, ....s_n, ....\}$
The list has all elements of A in their order of creation. The nth element $s_n$ came from two previous strings $s_i$ and $s_j$ using the constructors $s_n = s_i + s_j$ and $s_n = s_i - s_j$.
Q(n): Every element of A is a multiple of 3
Base Case:
n = 1 where 1 is the first element in A; $s_1 = 3$ which is a multiple of 3, so Q(1) is true.
Induction Step:
We show Q(1)$\wedge$...$\wedge$Q(n) $\rightarrow$ Q(n+1) using direct proof. Assume Q(1), Q(2),...Q(n): $s_1, ...s_n$ are all multiples of 3. We must show Q(n+1) is a multiple of 3. We have that $s_{n+1} = s_i + s_j$, so i, j < n+1. By the strong induction hypothesis, $s_i$ and $s_j$ are both multiples of 3 and come BEFORE $s_{n+1}$. Therefore, $s_{n+1}$ is also a multiple of three because it is the sum (or concatenation) of two multiples of 3. The same can be said about $s_{n+1} = s_i - s_j$, as $s_i$ and $s_j$ are both multiples of three once again, so $s_{n+1}$ is the difference(subtraction) of two multiples of 3, making it itself a multiple of three. By definition, the concatenation/addition and subtraction of multiples of 3 is a multiple of 3. Therefore, $s_{n+1}$ is a multiple of 3 in the case of both constructors. ■

- Prove that every multiple of 3 is in $A$.

  1. We prove by contradiction that every multiple of 3 is in $A$. Consider m, a multiple of 3 that is not in $A$.
  2. **[Case 1]** $k > 0$, $m = 3k$, where $m$ is the largest multiple of 3 NOT in our set $A$.

$$3k = 3 + 3 + 3 + ...$$

We can consider $3(k+1)$, which we know is in our set since it is larger than $m$

$$3(k+1) = 3k+3$$

We know by constructor(ii) that $x - y \in A$, and we know that $3 \in A$ by the basis.

$$\begin{aligned} 3(k) &= x - y, \text{ where } x = 3k+3 \text{ and } y = 3 \\ &= 3k+3-3 \\ &= 3k \text{ woops, we derived a contradiction!} \end{aligned}$$

3. **[Case 2]** $k < 0$, $m = -3k$, where $m$ is the smallest multiple of 3 NOT in our set A.

$$3(-k) = -3-3-3-3-3-3-...$$

We can consider $3(-k-1)$, which we know is in our set since it is smaller than $m$

$$3(-k-1) = -3k-3$$

We know by constructor(i) that $x + y \in A$, and we know that $3 \in A$ by the basis.

$$\begin{aligned} -3k &= x + y, \text{ where } x = -3k-3 \text{ and } y = 3 \\ &= -3k-3+3 \\ &= -3k \text{ woops, we derived a contradiction!} \end{aligned}$$

4. **[Case 3]** $k = 0$, $m = 0$, 0 is not in $A$

We know from the basis that $x = 3$ and $y = 3$
From constructor(ii), we can use $x - y$, where:

$$\begin{aligned} x - y &\in A \\ 3 - 3 &\in A \\ 0 &\in A \text{ woops, we derived a contradiction!} \end{aligned}$$

5. We prove by contradiction for 3 distinct cases of $k$, proving that all multiples of 3 is in $A$ ■