CSCI 2200
FOUNDATIONS OF COMPUTER SCIENCE

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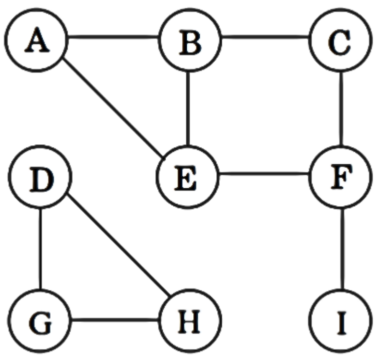
PLANAR GRAPHS

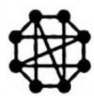
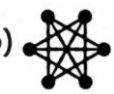
A *planar graph* is one you can draw without crossing any edges



Can you draw graph K_4 as a planar graph…?

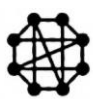
(these three graphs are isomorphic)

Can graphs (a) and (b) be drawn as planar graphs?

(a)  (b) 

PLANAR GRAPHS

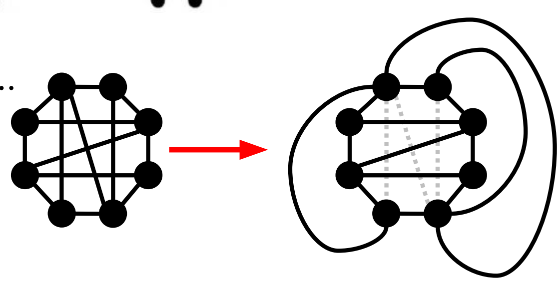
Can graphs (a) and (b) be drawn as planar graphs?

(a)  (b) 

HINT: For graph (a), first determine how many edges need to be changed…

…then connect them without crossing any other edges, if possible

…if not possible, move some vertices around (keep graphs isomorphic)



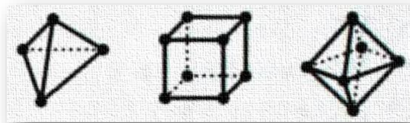
Complete the task for graph (b) — can it be done?

PLANAR GRAPHS – EULER'S INVARIANT

There are exactly five regular polyhedra! What are the other two…?

The *regular polyhedra* are three-dimensional shapes formed from congruent polygon *faces* in which all side lengths and angles are the same…

…attributed to Plato, these are the Platonic solids



Given V vertices, E edges, and F faces, Euler's invariant characteristic claims that

$$F + V - E = 2$$

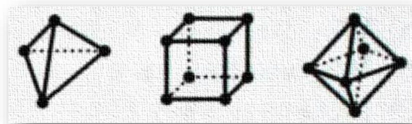
Confirm Euler's invariant holds for the Platonic solids shown…

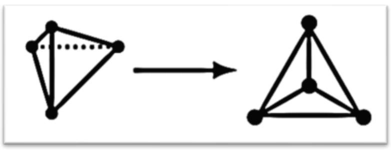
Draw each Platonic solid as a planar graph…

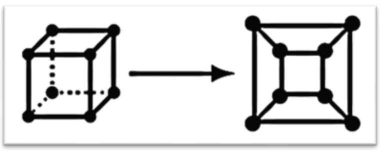
PLANAR GRAPHS – EULER'S INVARIANT

Euler's invariant characteristic claims that $F + V - E = 2$

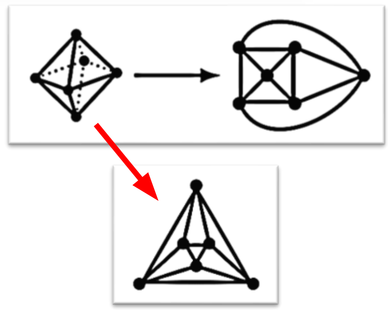
Draw each Platonic solid as a planar graph…







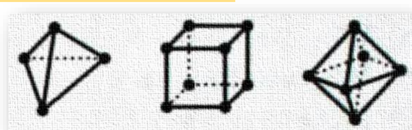
Identify *internal faces* as regions of the planar graph encompassed by a cycle

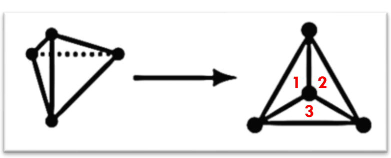


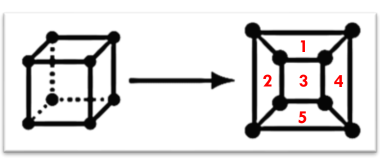
PLANAR GRAPHS – EULER'S INVARIANT

Euler's invariant characteristic claims that $F + V - E = 2$

Draw each Platonic solid as a planar graph…



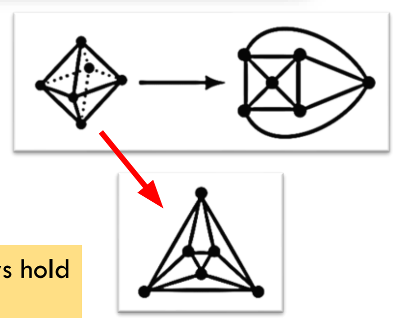




Identify *internal faces* as regions of the planar graph encompassed by a cycle

Add the *external face*, i.e., the unbounded region

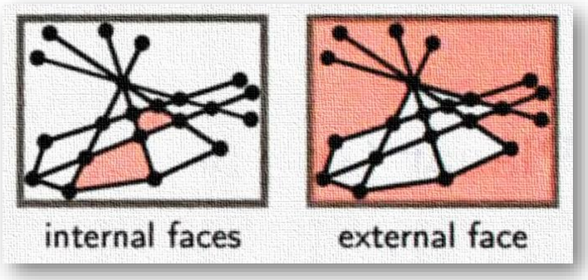
Does Euler's invariant always hold for planar graphs…?



PLANAR GRAPHS – EULER'S INVARIANT

Euler's invariant characteristic claims that $F + V - E = 2$

For any planar graph, assuming it is a connected graph, Euler's invariant holds…



But how can we prove this…?

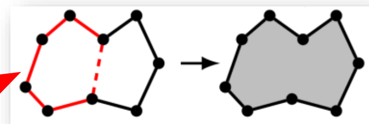
What if the planar graph is a tree…? Since $|E| = |V| - 1$, $F + V - E = 1 + V - (V - 1) = 2$

PLANAR GRAPHS – EULER'S INVARIANT

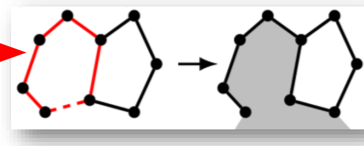
For any connected planar graph, Euler's invariant characteristic claims that $F + V - E = 2$

Proof. We use structural induction to prove the claim.

1. **[Base case]** For any tree T, the claim $F + V - E = 1 + V - (V - 1) = 2$.
2. **[Induction step]** For connected graph G with at least one cycle, removing edge e from any cycle has the following two cases.



Case 1. Edge e separates two internal faces; in this case, the two internal faces merge.



Case 2. Edge e separates an internal face from the external face; in this case, the internal face is lost and the external face grows.

For any connected planar graph, Euler's invariant characteristic claims that $F + V - E = 2$

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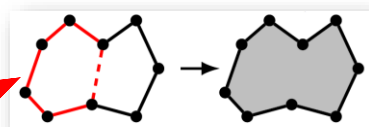

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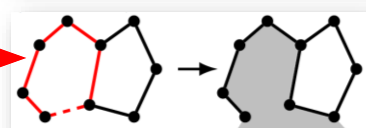


Case 2. Edge e separates an internal face from the external face; in this case, the internal face is lost and the external face grows.

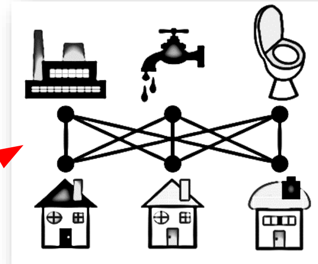
In both cases, E and F are decreased by 1 — and importantly, G is still connected (since paths that used edge e can go the other direction in the now-removed cycle).

Repeatedly removing edges until we have a tree, we have $\Delta F = \Delta E$ (and $\Delta V = 0$). Therefore, $(F + \Delta F) + V - (E + \Delta E) = F + V - E + (\Delta F - \Delta E) = F + V - E = 2$.

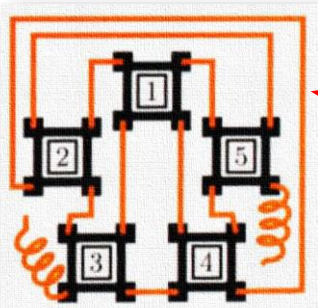
Through structural induction, we have proven the claim. See Exercise 11.7… ■

PLANAR GRAPHS

Applications abound…

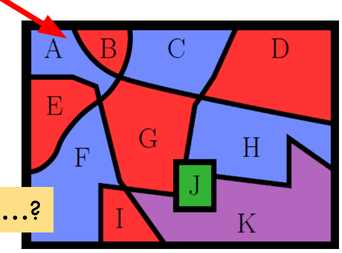


Town planning: connect utilities to homes without crossing any wires or pipes…

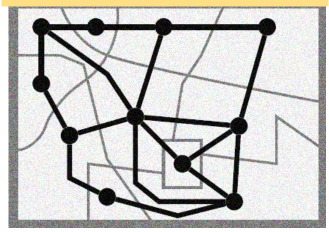


Chip design: components must be connected without crossing any wires…

Map coloring: adjacent regions must have different colors…



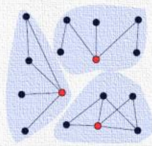
Model as a planar graph!



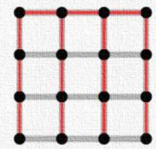
Can you apply Euler's invariant here…?

MANY TYPES OF GRAPH PROBLEMS

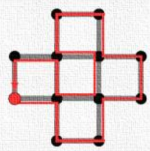
Connected Components. For "viral" marketing, pick one vertex in each *connected component* (e.g. target the "central (red)" vertices). [EASY]



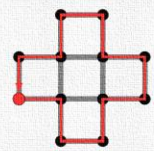
Spanning Tree. In a road grid (gray), to maintain a *minimal* "highway system" that offers high-speed travel we can use a *spanning tree* (red). [EASY]



Euler Cycle. Every winter, Troy typically has a 1-foot snowfall. The snowplow should start at the depot, traverse every road *exactly once* and return to the depot, traversing an *Euler Cycle* (red). [EASY]

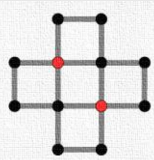


Hamiltonian Cycle. A traveling salesman starts at work and visits every house (vertex) *exactly once*, returning to work. The salesperson follows a *Hamiltonian Cycle*. [HARD]

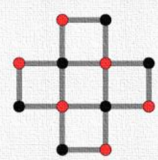


MANY TYPES OF GRAPH PROBLEMS

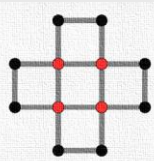
Facility Location (K-center). McDonalds wants to place $K = 2$ restaurants (red) in a road network so that no customer has too drive far to reach their closest McDonalds. [HARD]



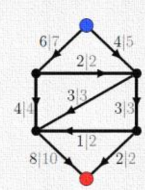
Vertex Cover. Place the minimum number of policemen at intersections so that all roads can be surveiled or "covered". The policeman form a vertex cover. Can you do it with fewer than 6? [HARD]



Dominating Set. Place the fewest hospitals at intersections (vertices) so that every intersection is either at a hospital or one block away from a hospital. The red hospitals are a *dominating set*. [HARD]



Network Flow. A *source*-ISP (blue) sends packets to a *sink*-ISP (red). What is the maximum transmission rate achievable without exceeding the link capacities? We achieved flow rate 10. [EASY]



COUNTING SEQUENCES – SUM RULE

How many binary sequences of length 3 are there?

{ 000, 001, 010, 011, 100, 101, 110, 111 }

There are two types of sequences

$$\{ b_1b_2b_3 \} = \{ b_1b_2\bullet 0 \} \cup \{ b_1b_2\bullet 1 \}$$

Sum Rule. N objects of two types: N_1 of type-1 and N_2 of type-2. Then,
$$N = N_1 + N_2.$$

COUNTING SEQUENCES – SUM RULE

Sum Rule. N objects of two types: N_1 of type-1 and N_2 of type-2. Then,
$$N = N_1 + N_2.$$

$$\begin{aligned}
|\{ b_1b_2b_3 \}| &= |\{ b_1b_2\bullet 0 \}| + |\{ b_1b_2\bullet 1 \}| && \text{(sum rule)} \\
&= |\{ b_1b_2 \}| \times 2 \\
&= (|\{ b_1\bullet 0 \}| + |\{ b_1\bullet 1 \}|) \times 2 && \text{(sum rule)} \\
&= |\{ b_1 \}| \times 2 \times 2 \\
&= 2 \times 2 \times 2 \\
&= 2^3
\end{aligned}$$

COUNTING SEQUENCES – PRODUCT RULE

We can also successively multiply the number of possibilities together to count how many binary sequences of length 3 there are…

$$|\{ b_1 b_2 b_3 \}| = 2 \times 2 \times 2 = 2^3$$

Let N be the number of choices for a sequence

$$x_1 x_2 x_3 \cdots x_{r-1} x_r.$$

Let N_1 be the number of choices for x_1;
Let N_2 be the number of choices for x_2 *after you choose* x_1;
Let N_3 be the number of choices for x_3 *after you choose* $x_1 x_2$;
Let N_4 be the number of choices for x_4 *after you choose* $x_1 x_2 x_3$;
$\vdots$
Let N_r be the number of choices for x_r *after you choose* $x_1 x_2 x_3 \cdots x_{r-1}$.

$$N = N_1 \times N_2 \times N_3 \times N_4 \times \cdots \times N_r.$$

If each x_i is independent, this is easy!
If not, counting can be trickier…

COUNTING SEQUENCES – EXAMPLES

How many possible license plates in New York if we limit the sequence to three uppercase letters followed by four digits (e.g., NYK1973)?

$$n = 26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^3 \times 10^4 = 175{,}760{,}000$$

How many possible passwords of length ℓ exist if the first character is a letter and the subsequent characters are alphanumeric, with $6 \leq \ell \leq 8$?

Define F as the first letter and S^k as k subsequent alphanumeric characters…

$$\begin{aligned} n &= |\{ FS^5 \}| + |\{ FS^6 \}| + |\{ FS^7 \}| && \text{(sum rule)} \\ &= 52 \times 62^5 + 52 \times 62^6 + 52 \times 62^7 && \text{(product rule)} \\ &= 186{,}125{,}210{,}680{,}448 \end{aligned}$$

COUNTING SEQUENCES – EXAMPLES

With 10 runners in a race, how many possible top-3 finishes are there, i.e., how many ways can runners come in first (F), second (S), and third (T)?

$$n = |\{ FST \}| = 10 \times 9 \times 8 = 720$$

Given five students A, B, C, D, and E, we look to form a study group;
the only constraint is that if A does not join, then B will join;
with no other constraints, how many ways can we form this group?

Treat set membership as a binary sequence, i.e., $b_1b_2b_3b_4b_5 = ABCDE$…

…and map the problem to counting how many 5-bit binary numbers begin with 1 or 01

See Problem 13.3…

BUILD-UP COUNTING – EXAMPLE

How many binary sequences of length n contain exactly k ones (with $0 \le k \le n$)…?

$\binom{n}{k}$ = number of n-length binary sequences with exactly k ones

Length 3:
000 001 010 011 100 101 110 111

Length 4: (the rightmost digits are the same in each line)
0000 0001 0010 0011 0100 0101 0110 0111
1000 1001 1010 1011 1100 1101 1110 1111

Length 5:
00000 00001 00010 00011 00100 00101 00110 …

$\binom{n}{k}$ (n \ k)	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6									
7									
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0	1								
1	1	1							
2	1	2	1						
3	1	$\boxed{3}$	$\boxed{3}$	1					
4	1	4	$\boxed{6}$	4	1				
5	1	5	$\boxed{10}$	$\boxed{10}$	5	1			
6	1	6	15	$\boxed{20}$	15	6	1		
7									
8									

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5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7									
8									

$$\{n\text{-sequence with } k \text{ 1's}\} = 0\bullet \underbrace{\{(n-1)\text{-sequence with } k \text{ 1's}\}}_{\binom{n-1}{k}} \cup 1\bullet \underbrace{\{(n-1)\text{-sequence with } (k-1) \text{ 1's}\}}_{\binom{n-1}{k-1}}$$

i.e., $\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$ with base cases: $\binom{n}{0} = 1$; $\binom{n}{n} = 1$.

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$\binom{n}{k}$ = number of n-length binary sequences with exactly k ones

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Length 4: (the rightmost digits are the same in each line)
0000 0001 0010 0011 0100 0101 0110 0111
1000 1001 1010 1011 1100 1101 1110 1111

Length 5:
00000 00001 00010 00011 00100 00101 00110 …

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1	1	1							
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5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Pascal's Triangle

$$\{n\text{-sequence with } k \text{ 1's}\} = 0\bullet \underbrace{\{(n-1)\text{-sequence with } k \text{ 1's}\}}_{\binom{n-1}{k}} \cup 1\bullet \underbrace{\{(n-1)\text{-sequence with } (k-1) \text{ 1's}\}}_{\binom{n-1}{k-1}}$$

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Length 3:
000 001 010 011 100 101 110 111

Length 4: (the rightmost digits are the same in each line)
0000 0001 (
1000 1001 1

Length 5:
00000 00001 00010 00011 00100 00101 00110 …

$\binom{n}{k}$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	$\boxed{3}$	$\boxed{3}$	1					
4	1	4	$\boxed{6}$	4	1				
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Pascal's Triangle

Use the build-up counting method to count the number of subsets of { 1, 2, …, 20 } that do not contain consecutive numbers…

$$\{n\text{-sequence with } k \text{ 1's}\} = 0\bullet \underbrace{\{(n-1)\text{-sequence with } k \text{ 1's}\}}_{\binom{n-1}{k}} \cup 1\bullet \underbrace{\{(n-1)\text{-sequence with } (k-1) \text{ 1's}\}}_{\binom{n-1}{k-1}}$$

i.e., $\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$ with base cases: $\binom{n}{0} = 1$; $\binom{n}{n} = 1$.

BUILD-UP COUNTING — EXAMPLE

Use the build-up counting method to count the number of subsets of $\{ 1, 2, \ldots, 20 \}$ that do not contain consecutive numbers…

Let $F(n)$ be the number of subsets that do not contain consecutive numbers…

(so our aim is to find $F(20)$ here)

…tinker with small values of n

n	subsets	$F(n)$
1	$\varnothing$, $\{1\}$	$F(1) = 2$
2	$\varnothing$, $\{1\}$, $\{2\}$	$F(2) = 3$
3	$\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,3\}$	$F(3) = 5$

What pattern emerges and how can you generalize $F(n)$…?

BUILD-UP COUNTING — EXAMPLE

Use the build-up counting method to count the number of subsets of $\{ 1, 2, \ldots, 20 \}$ that do not contain consecutive numbers…

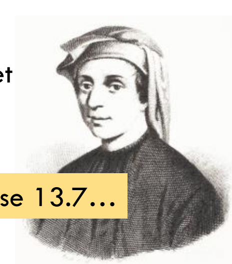
Let $F(n)$ be the number of subsets that do not contain consecutive numbers…

To build set S, consider the nth element — there are two cases:

Case 1. If S contains n, then it cannot contain $n - 1$ and the remaining elements in S are a subset of $\{ 1, 2, \ldots, n - 2 \}$ not containing consecutive numbers — this is $F(n - 2)$

Case 2. If S does not contain n, then the elements in S at this stage are a subset of $\{ 1, 2, \ldots, n - 1 \}$ not containing consecutive numbers — this is $F(n - 1)$

By the sum rule, $F(n) = F(n - 1) + F(n - 2)$ — thus, $F(20) = 17711$



See Exercise 13.7…

WHAT NEXT…?

Exam 2 grading is ongoing

Problem Set 7 has been posted…

- …and is due in your recitations **this week**, Wednesday, November 16

Earned late days should be current in Submitty

Practice! Tinker! Practice! Tinker! Practice! Tinker! Practice! Tinker! Practice!