- **Problem 3.59**

  1. addition: a, b, c, d
  2. subtraction: b, c, d
  3. multiplication: a, b, c, d
  4. division: c, d
  5. exponential: a

- **Problem 4.7**

  1. Assume $n \in \mathbb{Z}$ is true
  2. We then divide it into two cases, where n is odd and n is even
  3. Case 1: $n = 2k + 1$, n is odd $\rightarrow n^2 + n$ is even
     - then $(2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1$
     - then, $4k^2 + 4k + 2k + 2$
     - then, $2(2k^2 + 2k + k + 1)$
     - We prove by direct proof that if n is odd, then $n^2 + n$ is even $\blacksquare$
  4. Case 2: $n = 2k$ is even $\rightarrow n^2 + n$ is even
     - subbing in, we get $(2k)^2 + 2k = 4k^2 + 2k$, which is $2(2k^2 + k)$
     - We prove by direct proof that when n is even, $n^2 + n$ is even $\blacksquare$
  5. We only have two cases of what n could be, which is odd or even, by direct proof, we prove that $n \in \mathbb{Z} \rightarrow n^2 + n$ is true for any positive or negative integer

- **Problem 4.10**

  (k) contrapositive claim: if n is a perfect square, then 3 does not divide $n - 2$
     - Assume n is a perfect square, then $n = k^2$
     - Then, we can say $3w = n - 2$ and $3w = k^2 - 2$
     - Split into odd and even cases:

– Case 1: $k = 2q$, even

$$3w = (2q)^2 - 2$$

$$= 4q^2 - 2$$

$$w = \frac{4q^2}{3} - \frac{2}{3}$$

q is divisible by 3

$$q = 3j$$

$$w = \frac{4 \times 3j \times 3j}{3} - \frac{2}{3}$$

$$= 4j \times 3j - \frac{2}{3}$$

$$w = 12j - \frac{2}{3}$$

q is not divisible by 3

$$q = 3j + 1$$

$$w = \frac{4 \times (3j+1) \times (3j+1)}{3} - \frac{2}{3}$$

$$w = 12j^2 + 8j + \frac{2}{3}$$

q is not divisible by 3 pt 2

$$q = 3j + 2$$

$$w = 12j^2 + 16j - \frac{1}{3}$$

– Case 2: $k = 2q + 1$, odd

$$3w = (2q+1)^2 - 2$$
$$3w = 4q^2 + 4q - 1$$

q is divisible by 3

$$q = 3j$$
$$w = \frac{4q^2}{3} + \frac{4q}{3} - \frac{1}{3}$$
$$w = 12j^2 + 12j - \frac{1}{3}$$

q is not divisible by 3

$$q = 3j + 1$$
$$w = \frac{4(9j^2 + 6j + 1) + 12j + 3}{3}$$
$$w = 12j^2 + 12j + \frac{7}{3}$$

q is not divisible by 3 pt 2

$$q = 3j + 2$$
$$w = \frac{4(3j+2)^2 + 4(3j+2) - 1}{3}$$
$$w = 12j^2 + 28j + \frac{23}{3}$$

For all the following cases, for any $j \in \mathbb{N}$, w will never be a $\mathbb{N}$ due to the fraction, therefore, the following statement is true due to contraposition ■

(l) contrapositive claim: if $p^2 + 1$ is prime, then for $p > 2$ is composite

– Any number p can be written as $k + 2$ or $k + 3$

– if k is even, $k + 2$ is even, $k + 3$ will be even when k is odd

– Case 1: k is even, $k = 2w$ for some integer w

$$\begin{aligned} p^2 + 1 \to (k+2)^2 + 1 &= (2w+2)^2 + 1 \\ &= 4w^2 + 4w + 4w + 4 + 1 \\ &= 4(w^2 + 2w + 1) + 1 \end{aligned}$$

for any value w, it will always be a prime number $\blacksquare$

– Case 2: k is odd, $k = 2w + 1$

$$\begin{aligned} p^2 + 1 \to (k+3)^2 + 1 &= (2w + 1 + 3)2 + 1 \\ &= (2w+4)^2 + 1 \\ &= 4w^2 + 8w + 8w + 16 + 1 \\ &= 4(w^2 + 2w + 2w + 4) + 1 \\ &= 4(w^2 + 4w + 4) + 1 \end{aligned}$$

for any value w, it will be a prime number $\blacksquare$

- **Problem 5.20**

1. Base Case: $n = 1$, $2^0 = 1$, base case proved
2. Induction step: n becomes $n + 1$
3. We can divide it into two cases, where n is even and n is odd
4. Case 1: $n + 1$ is even

$$\begin{aligned} n + 1 &= 2k \\ k &= 2^{w_1} + 2^{w_2} + 2^{w_3} + 2^{w_4} + ... + 2^{w_i} \\ n + 1 &= 2(2^{w_1} + 2^{w_2} + 2^{w_3} + 2^{w_4} + ... + 2^{w_i}) \end{aligned}$$

By induction, we proved that even numbers are created by distinct powers of 2 $\blacksquare$

5. Case 2: $n + 1$ is odd

$$\begin{aligned} n + 1 &= 2k + 2^0 \\ k &= 2^{w_1} + 2^{w_2} + 2^{w_3} + 2^{w_4} + ... + 2^{w_i} \\ n + 1 &= 2(2^{w_1} + 2^{w_2} + 2^{w_3} + 2^{w_4} + ... + 2^{w_i}) + 2^0 \end{aligned}$$

By induction, we proved that odd numbers are created by distinct powers of 2 $\blacksquare$

- **Problem 5.39**

1. Base Cases:

$$
\begin{aligned}
P(n) &= 4a + 5b \\
P(12) &= 4(3) + 5(0) \\
P(13) &= 4(2) + 5(1) \\
P(14) &= 4(1) + 5(2) \\
P(15) &= 4(0) + 5(3)
\end{aligned}
$$

2. We create a variable k, where $k \geq 15$
3. To prove that $P(k)$ is true:

$$
\begin{aligned}
k - 3 &\geq 12 \\
k - 3 &= 4a + 5b \\
\text{Induction Step: } k + 1 &= (k - 3) + 4 \\
&= 4a + 5b + 4 \\
&= 4(a + 1) + 5b
\end{aligned}
$$

4. By leaping induction, we proved that for any $k + 1$, it is simply a variant of $4a + 5b$ ■