10.32 for $k \in \mathbb{Z}$, show that $2^k - 1$ and $2^k + 1$ are relatively prime.

- If two num are relatively prime, then $gcd(a, b) = 1$
- By definition, we know $K \in \mathbb{N}$, making $2^k - 1$ and $2^k + 1 = 1, 3$ respectively when $k = 1$. We state that $gcd(a, b)$ is also in $\mathbb{N}$. Which means $d \in \mathbb{N}$.
- We know that $gcd(2^k - 1, 2^k + 1) = d$, now we must prove $d = 1$ by direct proof.

$d|2^k - 1$ tells us $2^k - 1 = pd$ for $p \in \mathbb{Z}$
$d|2^k + 1$ tells us $2^k + 1 = qd$ for $q \in \mathbb{Z}$
rearranging $2^k - 1 = pd$, we can get $2^k = pd + 1$
plugging in, we get $pd + 1 + 1 = qd$
moving everything around, we get $2 = d(q - p)$.
This tells us that, to satisfy this equation, we only have 2 possibilities of what d can be $\rightarrow d = 1$, $d = 2$ since $d \in \mathbb{N}$

We can show that $d \neq 2$ by contradiction:
We know that 2^k is an even number, we can write $2^k = 2i$ for $i \in \mathbb{N}$
Both numbers $2^k - 1$ and $2^k + 1$ are odd numbers, we can rewrite it as $2^k + 1 = 2i + 1$ and $2^k - 1 = 2i - 1$
To prove $gcd(2i + 1, 2i - 1) \neq 2$, we contradict it and say $gcd(2i + 1, 2i - 1) = 2$
We can say $2|2i + 1$ and $2|2i - 1$, which means that $2i + 1 = 2z$ and $2i - 1 = 2h$, respective, where $z, h \in \mathbb{Z}$ that satisfy the equation

$$\begin{array}{c|c} 1 = 2z - 2i & -1 = 2h - 2i \\ 1 = 2(z - i) & -1 = 2(h - i) \\ \frac{1}{2} = (z - i) & \frac{-1}{2} = (h - i) \end{array}$$

Oops, the sum of the two integers can never result in a fraction! We have derived a contradiction, proving that $d \neq 2$
This leaves us with $d = 1$, proving that $gcd(2^k - 1, 2^k + 1) = d$, where $d = 1$. Showing that these two numbers are relatively prime.

11.17 A graph G has n vertices

(a) What is the maximum number of edges G can have and not be connected? Prove it.

Consider a graph G with multiple connected components. Let's call the different connected components G' and G'' and so on... Each of the connected components are disconnected from each other. In short, G is a connected graph with multiple connected components.

We prove that there exist a maximum number of edges a graph may have by deriving first a formula.

Assume a connected components G' in G. We can say the number of vertices in this connected component is n'. Considering the question is asking for the maximum number of edges possible, we can consider back to the Handshaking Theorem, and introduce a unique handshake to each of the vertices with all other vertices. With this, we create a complete graph of n' vertices.

Now we know that for a connected component to contain the maximum number of edges possible, it must be a complete graph.

We prove now the number of vertices per component that yield the highest number of edges overall.

In G', we can have at most $n' - 1$ degrees per vertex, assuming $n' \geq 1$. Each vertex can be connected to every other vertex other than itself, then we derive $n'(n'-1)$ for all possible edged. We exclude the duplicates by dividing by 2, leaving us with $e' = \frac{n'(n'-1)}{2}$

In G'', we can use the same logic to derive $e'' = \frac{n''(n''-1)}{2}$

The maximum number of edges we can have in a disconnected graph G is $e = \frac{(n-1)(n-2)}{2}$

Therefore, the following must be true:

$$e' + e'' = e$$
$$\frac{n'(n'-1)}{2} + \frac{n''(n''-1)}{2} = \frac{(n-1)(n-2)}{2}$$

We can simplify $n'' = n - n'$, where

$$\frac{n'(n'-1)}{2} + \frac{(n-n')(n-n'-1)}{2} = \frac{(n-1)(n-2)}{2}$$

expand both sides and simplify

$$\frac{2(n')^2 + n^2 - 2n'n - n}{2} = \frac{n^2 - 2n - n + 2}{2}$$

discard redundant components, leaving us with

$$\begin{aligned} -2n'n + 2(n')^2 &= -2n + 2 \\ 2(-n'n + (n')^2) &= 2(-n + 1) \\ -n'n + (n')^2 &= -n + 1 \end{aligned}$$

We must find values n' that satisfy this statement

$$\begin{aligned} -n'n + (n')^2 + n - 1 &= 0 \\ (n')^2 - n'n + n' - n' + n - 1 &= 0 \\ (n'-1)(n'-n+1) &= 0 \end{aligned}$$

The only values of n' that achieves this equality are $n' = 1$ and $n' = n - 1$, proving that for G to contain the maximum number of edges but remain disconnected. It must contain 2 different components where one must contain 1 vertex and other must contain $n - 1$ vertices ■

(b) What is the minimum number of edges G can have and be connected? Prove it.

For a connected graph G to contain a minimum number of edges between vertices n, it must be interpreted as a single line where each of the endpoints connect to only one single vertex. The formula is $e = n - 1$, for e the minimum edges for n vertices.

We prove by contradition that there exist no smaller number of edges for n number of vertices

Assume $e = n - k$, where $k \in \mathbb{Z}$ and $k > 1$
Assume G is a graph where its endpoints are connected to one other vertex

A connected graph must satisfy the following condition: you may traverse from vertex u to v for any $u, v \in G$.
Assume vertex u, v, $z \in G$, where v is connected to both u and v. An increase in k removes one of the edges in G such that u, $v \in G'$ and $z \in G''$. By our definition of a conncted graph, now there exist no path from z to v or z to u, rendering it a disconnected graph.

We prove by contradiction that there exists no connceted graph with less edges e than $n - 1$, or $e = n - 1$ ■