CSCI 2200 — Foundations of Computer Science (FoCS)
Homework 4 (document version 1.0)

Overview

- This homework is due by 11:59PM on Thursday, November 3
- You may work on this homework in a group of no more than four students; unlike recitation problem sets, **your teammates may be in any section**
- You may use at most **three** late days on this assignment
- Please start this homework early and ask questions during office hours; also ask (and answer) questions on the Discussion Forum
- Please be concise in your answers; even if your solution is correct, if it is not well-presented, you may still lose points
- You can type or hand-write (or both) your solutions to the required graded problems below; **all work must be organized in one PDF that lists all teammate names**
- You are strongly encouraged to use LaTeX, in particular for mathematical symbols; see references in Course Materials
- **EARNING LATE DAYS:** for each homework that you complete using LaTeX (including any tables, graphs, etc., i.e., no hand-written anything), you earn one additional late day; you can draw graphs and other diagrams in another application and include them as image files
- To earn a late day, you must submit your LaTeX files (i.e., `*.tex`) along with your one required PDF file—please name the PDF file `hw4.pdf`
- Also note that the earned late day can be used retroactively, even back to the first homework assignment!

Warm-up exercises

The problems below are good practice problems to work on. Do not submit these as part of your homework submission. **These are ungraded problems.**

- **Problem 9.20.**
- **Problem 9.23(a).**
- **Problem 9.28.**
- **Problem 10.8.**
- **Problem 10.10.**
- **Problem 10.11.**
- **Problem 10.15(a-b).**
- **Problem 10.22(b-c,e).**
- **Problem 10.29.**
- **Problem 11.3.**
- **Problem 11.5.**
- **Problem 11.10.**
- **Problem 11.13.**
- **Problem 11.15.**

Graded problems

The problems below are required and will be graded.

- ***Problem 9.23(b).**
- ***Problem 10.13.**
- ***Problem 10.18(a-b).**
- ***Problem 10.22(a,d).**
- ***Problem 10.32.**
- ***Problem 11.11.**
- ***Problem 11.17.**
- ***Problem 11.27.**

Some of the above problems (graded and ungraded) are transcribed in the pages that follow.

Graded problems are noted with an asterisk (*).

If any typos exist below, please use the textbook description.

- **Problem 9.20.** Prove or disprove:

  (a) $\dfrac{n^3+2n}{n^2+1} \in \Theta(n)$ (b) $(n+1)! \in \Theta(n!)$ (c) $n^{1/n} \in \Theta(1)$ (d) $(n!)^{1/n} \in \Theta(n)$

- **Problem 9.23(a).** Prove by contradiction: (a) $n^3 \notin O(n^2)$

- ***Problem 9.23(b).** Prove by contradiction: (b) $2^n \notin O(3^n)$

- **Problem 9.28.** For recurrence $f(0) = 1$; $f(n) = nf(n-1)$, compare $f(n)$ with (a) 2^n (b) n^n.

- **Problem 10.8.** What natural numbers are relatively prime to 2, 3, and 6?

- **Problem 10.10.** For any $m, n, x \in \mathbb{Z}$, prove that $\gcd(m, n) = \gcd(m, n - mx)$.

- **Problem 10.11.** Use Euclid's algorithm and the remainders generated to solve these problems.

  (a) Compute $\gcd(1200, 2250)$ and find $x, y \in \mathbb{Z}$ for which $\gcd(1200, 2250) = 1200 \cdot x + 2250 \cdot y$.

  (b) Find x, y as in (a) but with the additional requirement that $x \leq 0$ and $y \geq 0$.

- ***Problem 10.13.** Let $d = \gcd(m, n)$, where $m, n > 0$. Bezout gives $d = mx + ny$, where $x, y \in \mathbb{Z}$. Prove or disprove:

  (a) It is always possible to choose: (i) $x > 0$ (ii) $x < 0$.

  (b) It is possible to find another $x, y \in \mathbb{Z}$ for which $0 < mx + ny < d$.

  (c) It is always possible to find $a, b \in \mathbb{Z}$ for which $ax + by = 1$.

- **Problem 10.15(a-b).** Prove.

  (a) If a divides bc and $\gcd(a, b) = 1$ then a divides c.

  (b) For any prime p, if $p | a_1 a_2 \ldots a_n$ then p divides one of the a_i.

- ***Problem 10.18(a-b).** The Fibonacci numbers are: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

  (a) Prove that $\gcd(F_n, F_{n+1}) = 1$. (Consecutive Fibonacci numbers are relatively prime.)

  (b) Prove that for $n \geq 1$, $F_m | F_{mn}$.

- ***Problem 10.22(a,d).** You may find Bezout's identity useful for answering these questions.

  (a) Prove that consecutive integers n and $n + 1$ are relatively prime.

  (d) For $k \in \mathbb{Z}$, prove that $2k + 1$ and $9k + 4$ are relatively prime.

- **Problem 10.22(b-c,e).** You may find Bezout's identity useful for answering these questions.
  - (b) For which positive n are the pair n and $n+2$ relatively prime? Prove your answer.
  - (c) Let p be a prime. For which positive n are the pair n and $n+p$ relatively prime? Prove your answer. [Hint: If n is not a multiple of p then $\gcd(n,p)=1$.]
  - (e) As a function of $k \in \mathbb{Z}$, compute $\gcd(2k-1, 9k+4)$.

- **Problem 10.29.** Solve each measuring problem, or explain why it can't be done. (You have unlimited water.)
  - (a) Using 6- and 15-gallon jugs, measure (i) 3 gallons (ii) 4 gallons (iii) 5 gallons.
  - (b) Using 5- and 11-gallon jugs, measure (i) 6 gallons (ii) 7 gallons.

- ***Problem 10.32.** For $k \in \mathbb{N}$, show that $2^k - 1$ and $2^k + 1$ are relatively prime.

- **Problem 11.3.** Give the degree sequences of K_{n+1}, $K_{n,n}$, L_n, C_n, S_{n+1}, and W_{n+1}.

- **Problem 11.5.** A graph is regular if every vertex has the same degree. Which of these graphs are regular?

  (a) K_6 (b) $K_{4,5}$ (c) $K_{5,5}$ (d) L_6 (e) S_6 (f) W_4 (g) W_5

- **Problem 11.10.** Give graphs with these degree distributions, or explain why you can't. Verify $2|E| = \sum_{i=1}^{n} \delta_i$.

(a) $[5,3,3,2,1]$	(d) $[3,3,3,3,3]$	(g) $[4,4,4,4,4]$	(j) $[3,3,3,2,2]$
(b) $[3,2,1,1,1]$	(e) $[3,3,3,3,3,3]$	(h) $[4,4,3,2,1]$	(k) $[3,3,3,3,2]$
(c) $[3,3,2,1]$	(f) $[3,3,2,2,2]$	(i) $[4,3,3,2,2]$	(l) $[5,3,2,2,2]$

- ***Problem 11.11.** In a graph only the two vertices u, v have odd degree. Prove there is a path from u to v.

- **Problem 11.13.** Compute the number of edges in the following graphs:

  (a) K_n (b) $K_{n,\ell}$ (c) W_n

- **Problem 11.15.** A graph is r-regular if every vertex has the same degree r. Show:

  (a) If r is even and $n > r$, there is an r-regular graph with n vertices. (Tinker!)

  (b) If r is odd and n is odd, there is no r-regular graph with n vertices.

  (c) If r is odd and $n > r$ is even, there is an r-regular graph with n vertices.

  (d) An r-regular graph with $4k$ vertices must have an even number of edges.

- ***Problem 11.17.** A graph G has n vertices.

  (a) What is the maximum number of edges G can have and not be connected? Prove it.

  (b) What is the minimum number of edges G can have and be connected? Prove it.

- ***Problem 11.27.** Every vertex degree in a graph is at least 2. Prove that there is at least one cycle.