CSCI 2200 FOUNDATIONS OF COMPUTER SCIENCE

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PROOF BY INDUCTION

Given claim $P(n)$, we construct a proof by *induction* to show $P(n)$ holds for all $n \geq n_0$:

Proof. We use induction to prove $\forall n \geq n_0 : P(n)$. [We often set $n_0 = 1$.]

1. Show that $P(n_0)$ is **T**. [Base case.]
2. Show that $P(n) \rightarrow P(n+1)$ for a *general* $n \geq n_0$. [Induction step.]

Direct proof:
Assume $P(n)$ is **T**.
Show $P(n+1)$ is **T**.

or

Proof by contraposition:
Assume $P(n+1)$ is **F**.
Show $P(n)$ is **F**.

3. Conclude therefore that $P(n)$ holds for all $n \geq n_0$. ■

(in LaTeX: `$\hfill\blacksquare$`)

PROOF BY INDUCTION – EXAMPLE

Tinker! What proof techniques might we try here…?

Prove claim $P(n)$ = "$4^n - 1$ is divisible by 3"

Proof. We use induction to prove $\forall n \geq 1 : P(n)$.

1. $P(1) = 4^1 - 1 = 3$. $P(1)$ is **T**. [Base case.]
2. Prove $P(n) \rightarrow P(n+1)$ for all $n \geq 1$. [Induction step.]

   Specifically, if $\underbrace{4^n - 1 \text{ is divisible by } 3}_{P(n)}$, then $\underbrace{4^{n+1} - 1 \text{ is divisible by } 3.}_{P(n+1)}$

Can you prove this implication using a direct proof…?

PROOF BY INDUCTION – EXAMPLE

2. Prove $P(n) \rightarrow P(n+1)$ for all $n \geq 1$. [Induction step.]

   Specifically, if $\underbrace{4^n - 1 \text{ is divisible by } 3}_{P(n)}$, then $\underbrace{4^{n+1} - 1 \text{ is divisible by } 3.}_{P(n+1)}$

Proof. We prove the implication using a direct proof.

(i) Assume that $P(n)$ is **T**, i.e., that $4^n - 1$ is divisible by 3. [Induction hypothesis.]
(ii) This means that $4^n - 1 = 3k$ for some integer k. Thus, $4^n = 3k + 1$.
(iii) Observe that $4^{n+1} = 4 \times 4^n$; then, $4^{n+1} = 4(3k + 1) = 12k + 4$.
Therefore, $4^{n+1} - 1 = 12k + 3 = 3(4k + 1)$, which is a multiple of 3.
(iv) Since $4^{n+1} - 1$ is a multiple of 3, it follows that $4^{n+1} - 1$ is divisible by 3.
(v) Therefore, $P(n+1)$ is **T**.

Are we done…? What have we proven…?

Yes, we are done—we have proven that $4^n - 1$ is divisible by 3 for <u>all</u> $n \geq 1$…

Prove claim $P(n)$ = "$4^n - 1$ is divisible by 3"

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   (i) Assume that $P(n)$ is **T**, i.e., that $4^n - 1$ is divisible by 3. [Induction hypothesis.]
   (ii) This means that $4^n - 1 = 3k$ for some integer k. Thus, $4^n = 3k + 1$.
   (iii) Observe that $4^{n+1} = 4 \times 4^n$; then, $4^{n+1} = 4(3k + 1) = 12k + 4$.
   Therefore, $4^{n+1} - 1 = 12k + 3 = 3(4k + 1)$, which is a multiple of 3.
   (iv) Since $4^{n+1} - 1$ is a multiple of 3, it follows that $4^{n+1} - 1$ is divisible by 3.
   (v) Therefore, $P(n+1)$ is **T**.
3. By induction, we have proven $P(n)$ for all $n \geq 1$. ■

PROOF BY INDUCTION – PROOF TEMPLATE

Given claim $P(n)$, we construct a proof by *induction* to show $P(n)$ holds for all $n \geq n_0$:

Proof. We use induction to prove $\forall n \geq n_0 : P(n)$. [We often set $n_0 = 1$.]

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   Direct proof:
   Assume $P(n)$ is **T**.
   Show $P(n+1)$ is **T**.

   or

   Proof by contraposition:
   Assume $P(n+1)$ is **F**.
   Show $P(n)$ is **F**.
3. Conclude therefore that $P(n)$ holds for all $n \geq n_0$. ■

WHY DOES INDUCTION WORK?

Using induction, here is how we prove the $\forall n$ part of the claim:

$$P(1) \rightarrow \underbrace{P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \ldots \rightarrow P(k-1) \rightarrow P(k) \rightarrow \ldots}_{\text{inductive step shows } P(n) \rightarrow P(n+1) \text{ is } \mathbf{T}}$$

base case
$P(n_0)$ is **T**

What if we do not have P(1) or, more generally, P(n_0)…?

PROOF BY INDUCTION – EXAMPLE

Prove claim $P(n)$ = "$1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$" using induction

Proof. We use induction to prove $\forall n \geq 1 : P(n)$.

1. **[Base case]** $P(1) = ½(1)(1 + 1) = 1$. $P(1)$ is **T**.
2. **[Induction step]** We show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$ via a direct proof.

   Assume (*induction hypothesis*) that $P(n)$ is **T**.

   Prove $P(n+1)$: $1 + 2 + \ldots + n + (n+1) = \dfrac{(n+1)(n+2)}{2}$.

Manipulate the LHS to get it into a form that resembles $P(n)$…

…also look to "plug in" the induction hypothesis…

Prove claim $P(n)$ = "$1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$" using induction

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   Assume (*induction hypothesis*) that $P(n)$ is **T**.

   Prove $P(n+1)$: $1 + 2 + \ldots + n + (n+1) = \dfrac{(n+1)(n+2)}{2}$.

   LHS: $1 + 2 + \ldots + n + (n+1) = \underbrace{[1 + 2 + \ldots + n]} + (n+1)$

   plug in the induction hypothesis…

   $$= \frac{n(n+1)}{2} + (n+1) = \frac{1}{2}\underbrace{(n+1)}_{k}\underbrace{(n+1+1)}_{k+1}.$$

Prove claim $P(n)$ = "$1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$" using induction

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   LHS: $1 + 2 + \ldots + n + (n+1) = \underbrace{[1 + 2 + \ldots + n]} + (n+1)$

   plug in the induction hypothesis…

   $$= \frac{n(n+1)}{2} + (n+1) = \frac{1}{2}\underbrace{(n+1)}_{k}\underbrace{(n+1+1)}_{k+1}.$$

3. By induction, we have proven $P(n)$ for all $n \geq 1$. ■

PROOF BY INDUCTION – EXAMPLE

Prove claim $S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$ using induction

Proof. We use induction to prove $\forall n \geq 1 : S(n)$.

1. **[Base case]** $S(1) = \frac{1}{6} \times 1 \times 2 \times 3 = 1$. $S(1)$ is **T**.
2. **[Induction step]** We show $S(n) \rightarrow S(n+1)$ for all $n \geq 1$ via a direct proof.

   Assume (*induction hypothesis*) that $S(n)$ is **T**.

   Prove $S(n+1)$: $\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$.

Manipulate the LHS to get it into a form that resembles $S(n)$…

…also look to "plug in" the induction hypothesis…

Prove claim $S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$ using induction

Proof. We use induction to prove $\forall n \geq 1 : S(n)$.

1. **[Base case]** $S(1) = \frac{1}{6} \times 1 \times 2 \times 3 = 1$. $S(1)$ is **T**.
2. **[Induction step]** We show $S(n) \rightarrow S(n+1)$ for all $n \geq 1$ via a direct proof.

   Assume (*induction hypothesis*) that $S(n)$ is **T**.

   Prove $S(n+1)$: $\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$.

   LHS: $$\sum_{i=1}^{n+1} i^2 = \underbrace{\sum_{i=1}^{n} i^2}_{\text{plug in the induction hypothesis…}} + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

   $$= \frac{1}{6}(n+1)(n+2)(2n+3).$$

3. By induction, we have proven $S(n)$ for all $n \geq 1$. ■

PROOF BY INDUCTION

Prove the following claims using induction…

Claim 1. $P(n)$ = "$n^2 - n + 41$ is a prime number." Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 2. $P(n)$ = "$5^n - 1$ is divisible by 4." Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 3. $P(n)$ = "$n \leq 2^n$." Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 4. $P(n)$ = "$1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \dfrac{n(2n + 1)(2n - 1)}{3}$."
Prove $\forall n \geq 1$, $P(n)$ is **T**.

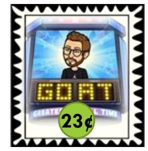
Claim 5. $P(n)$ = "$n^{17} + 9$ and $(n + 1)^{17} + 9$ have no factors in common."
Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 5 does not hold—tinker with some values but see the textbook for where this fails…

PROOF BY INDUCTION – STAMPS EXAMPLE

Assume we have an old stamp-dispensing machine stocked with an infinite number of 5¢ and 7¢ stamps

ultra-rare!



Also assume minimum postage is 19¢

Can we accommodate any postage starting at 19¢—e.g., 19¢, 20¢, 21¢, etc.?

19¢	20¢	21¢	22¢	23¢	24¢	25¢	26¢	…
5¢, 7¢, 7¢	*5¢, 5¢, 5¢, 5¢*	*7¢, 7¢, 7¢*	*5¢, 5¢, 5¢, 7¢*	???	*5¢, 5¢, 7¢, 7¢*	*5¢, 5¢, 5¢, 5¢, 5¢*	*5¢, 7¢, 7¢, 7¢*	…

Can we accommodate any postage starting at 24¢—e.g., 24¢, 25¢, 26¢, etc.?

How can we prove that we can accommodate any postage starting at 24¢ using induction…?

PROOF BY INDUCTION – STAMPS EXAMPLE

How can we prove that we can accommodate any postage starting at 24¢ using induction…?

24¢	25¢	26¢	27¢	28¢	29¢	30¢	31¢	…
5¢, 5¢, 7¢, 7¢	*5¢, 5¢, 5¢, 5¢, 5¢*	*5¢, 7¢, 7¢, 7¢*	*5¢, 5¢, 5¢, 5¢, 7¢*	*7¢, 7¢, 7¢, 7¢*	*5¢, 5¢, 5¢, 7¢, 7¢*	*5¢, 5¢, 5¢, 5¢, 5¢, 5¢*	*5¢, 5¢, 7¢, 7¢, 7¢*	…

Given the first five solutions…

…we can always add another 5¢ stamp to get the next set of five solutions

Can you write an inductive proof for this problem?

WHAT NEXT…?

Problem Set 2 is due at recitations this Wednesday, September 21
- We are going to spread out to neighboring rooms: Ricketts 208 and 212

Homework 2 is posted and due by 11:59PM on September 29

Problem Set 3 will be posted by next Monday—due at recitations on September 28

Email me extra-time accommodations ASAP

Practice! Tinker! Practice! Tinker! Practice! Tinker! Practice! Tinker! Practice!