CSCI 2200 — Foundations of Computer Science (FoCS)
Problem Set 4 (document version 1.1)

Overview

- This problem set is due at your Wednesday, October 12 recitation
- You may work on this problem set in a group of no more than four students; **each of your teammates must be in your recitation section**
- Please start this problem set early and ask questions during office hours and at your recitation section; also ask (and answer) questions on the Discussion Forum
- You can type or hand-write (or both) your solutions to the required graded problems

Problems

These problems are generally good practice problems to work on. Those marked with an asterisk (*) are required and will be reviewed/graded in recitation.

- **Problem 5.62.**
- ***Problem 7.2.**
- ***Problem 7.4(c).**
- ***Problem 7.6.**
- **Problem 7.7.**
- **Problem 7.19.**
- **Problem 7.29.**
- **Problem 7.30.**

(v1.1) Some of the above problems are transcribed below.

- ***Problem 7.2.** Define $f(n)$ for $n \in \mathbb{N}$ by $f(1) = 1$, $f(n) = f(n/2)$ for $n > 1$ even, and $f(n) = f(5n - 9)$ for $n > 1$ odd. Is f a well-defined function? Explain your answer.
- ***Problem 7.4(c).** Guess a formula for A_n and prove it by induction.

  (c) $A_0 = 1$; $A_1 = 2$; $A_n = 2A_{n-1} - A_{n-2} + 2$ for $n \geq 2$. [Hint: Method of differences.]
- ***Problem 7.6.** Define $f(n)$ for $n \in \mathbb{N}$ by $f(1) = 0$ and $f(n) = f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil) + 1$ for $n > 1$. ($\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ round up and down.) (a) Is f a well-defined function? Explain. (b) Tinker, guess a formula for $f(n)$, and prove it.
- **Problem 7.7.** Define $f(n)$ for $n \in \mathbb{N}$ by $f(1) = 1$ and $f(n) = f(n/2) + 1$ for $n > 1$ even, and $f(n) = f(3n + 1)$ for $n > 1$ odd. (a) Compute $f(3)$, $f(5)$, $f(6)$. (b) Is f defined for all $n \in \mathbb{N}$? (See the Collatz conjecture, Problem 1.43.)

- **Problem 7.19.** Recall the Fibonacci numbers: $F_1, F_2 = 1$; and, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.
  - (a) Compute $F_1, \ldots, F_10$ and verify that $F_n < (7/4)^{n-1}$ for $n \geq 1$. Now prove the bound for all $n \geq 1$.
  - (c) Prove that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ for $n \geq 2$.
  - (d) Prove that every third Fibonacci number, F_{3n}, is even.
  - (e) (Sum) Prove: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$ for $n \geq 1$.
  - (f) ...

- **Problem 7.29.** One can use recurrences to define a sequence of strings. Define A_n as follows:

$$
\begin{aligned}
A_0 &= a; \\
A_n &= a \bullet A_{n-1} \bullet ba \qquad \text{for } n \geq 1.
\end{aligned}
$$

  Concatenation, $x \bullet y$, for strings x, y appends y to x, producing the string xy, e.g., $ab \bullet ba = abba$.

  - (a) What are $A_1, \ldots, A_{10}$.
  - (b) Prove $A_n = a^{\bullet n} b (ab)^{\bullet n}$, where $x^{\bullet k}$ is k copies of the string x concatenated together.

- **Problem 7.30.** Give recurrences to generate these string sequences.
  - (a) $A_1 = a$, $A_2 = aa$, $A_3 = aaa$, ..., $A_k = a^{\bullet k}$.
  - (b) $A_1 = abb$, $A_2 = aabb$, $A_3 = aaabb$, ..., $A_k = a^{\bullet k} bb$.
  - (c) $A_1 = ab$, $A_2 = aabb$, $A_3 = aaabbb$, ..., $A_k = a^{\bullet k} b^{\bullet k}$.
  - (d) $A_1 = abb$, $A_2 = aabbbb$, $A_3 = aaabbbbbb$, ..., $A_k = a^{\bullet k} b^{\bullet 2k}$.