CSCI 2200 FOUNDATIONS OF COMPUTER SCIENCE

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WEDNESDAY RECITATIONS

Start the Problem Set problems as early as possible

Collaborate in groups of up to four students (within your recitation section)

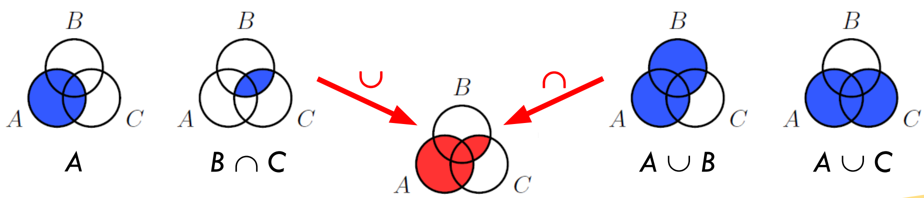
Future Problem Sets will have fewer required problems

Work on the non-required and warm-up problems, too—good exam prep!

PROOFS ABOUT SETS – VENN DIAGRAMS

We can prove equivalences involving sets informally via Venn diagrams…

Venn diagram proof example—prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



$A \cup (B \cap C)$

$(A \cup B) \cap (A \cup C)$

We can also use <u>formal</u> proofs…
(next few slides)

PROOFS ABOUT SETS – FORMAL PROOFS

Formal proof example—prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Can we use the other laws below to prove this…?

1. *Associative:* $(A \cap B) \cap C = A \cap (B \cap C)$; $(A \cup B) \cup C = A \cup (B \cup C)$.
2. *Commutative:* $A \cap B = B \cap A$; $A \cup B = B \cup A$.
3. *Complements:* $\overline{(\overline{A})} = A$; $\overline{A \cap B} = \overline{A} \cup \overline{B}$; $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
4. *Distributive:* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

No, not in this case…

PROOFS ABOUT SETS – FORMAL PROOFS

What do we set out to prove when using the following set operations?

One set is a subset of another set, i.e., $A \subseteq B$…

$x \in A \rightarrow x \in B$

And a proper subset, i.e., $A \subset B$…

$x \in A \rightarrow x \in B$
and $\exists y \in B : y \notin A$

One set is not a subset of another set, i.e., $A \not\subseteq B$…

$\exists x \in A : x \notin B$

Two sets are equal, i.e., $A = B$, which remember is $A \subseteq B$ and $B \subseteq A$…

$x \in A \leftrightarrow x \in B$

PROOFS ABOUT SETS – FORMAL PROOFS

Formal proof example—prove $\underbrace{A \cup (B \cap C)}_{X} = \underbrace{(A \cup B) \cap (A \cup C)}_{Y}$

Proof. We prove the claim by proving each implication (i.e., $X \subseteq Y$ and $Y \subseteq X$).

(i) We use a direct proof for $X \subseteq Y$, or $x \in A \cup (B \cap C) \rightarrow x \in (A \cup B) \cap (A \cup C)$.

Assume element $x \in A \cup (B \cap C)$. There are two cases for x.

Case 1. $x \in A$. Here, since $x \in (A \cup S)$ is **T**, where S is any set, we have $x \in (A \cup B)$ and $x \in (A \cup C)$.

Case 2. $x \notin A$. It must be the case that $x \in (B \cap C)$, i.e., x is in both B and C. Since $x \in B$, we have $x \in (A \cup B)$; since $x \in C$, we have $x \in (A \cup C)$.

In both cases, $x \in (A \cup B) \cap (A \cup C)$ holds **T**.

Complete this proof by showing $Y \subseteq X$…

PROOFS ABOUT SETS – EXAMPLES

Prove the following claims regarding sets using Venn diagrams, then formal proofs…

Claim 1. Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Claim 2. Prove $A \cup \overline{B} = \overline{\overline{A} \cap B}$.

Claim 3. Prove $(\overline{A \cup B}) \cap A = \varnothing$.

Claim 4. Let A be a finite set of size $n \geq 1$. Prove that $|\mathcal{P}(A)| = 2^n$.

Claim 5. Let $A = \{\text{multiples of } 2\}$, $B = \{\text{multiples of } 9\}$, and $C = \{\text{multiples of } 6\}$.

Prove $(A \cap B) \subseteq C$.

Claim 6. Prove $(\overline{A \cap B}) \cap A = A \cap \overline{B}$.

For which of these claims might you try to use an induction proof?

PROOF BY INDUCTION – PROOF TEMPLATE

Given claim $P(n)$, we construct a proof by *induction* to show $P(n)$ holds for all $n \geq n_0$:

Proof. We use induction to prove $\forall n \geq n_0 : P(n)$. [We often set $n_0 = 1$.]

1. Show that $P(n_0)$ is **T**. [Base case.]
2. Show that $P(n) \rightarrow P(n+1)$ for a *general* $n \geq n_0$. [Induction step.]

Direct proof:
Assume $P(n)$ is **T**.
Show $P(n+1)$ is **T**.

induction hypothesis

or

Proof by contraposition:
Assume $P(n+1)$ is **F**.
Show $P(n)$ is **F**.

3. Conclude therefore that $P(n)$ holds for all $n \geq n_0$. ■

WHY DOES INDUCTION WORK?

Using induction, here is how we prove the $\forall n$ part of the claim:

$$\overset{✓}{P(1)} \rightarrow \underbrace{\overset{✓}{P(2)} \rightarrow \overset{✓}{P(3)} \rightarrow \overset{✓}{P(4)} \rightarrow \ldots \rightarrow \overset{✓}{P(k-1)} \rightarrow \overset{✓}{P(k)} \rightarrow \ldots}_{\text{inductive step shows } P(n) \rightarrow P(n+1) \text{ is } \mathbf{T}}$$

base case
$P(n_0)$ is **T**

Prove claim $P(n)$ = "$4^n - 1$ is divisible by 3"

Proof. We use induction to prove $\forall n \geq 1 : P(n)$.

1. $P(1) = 4^1 - 1 = 3$. $P(1)$ is **T**. [Base case.]
2. Prove $P(n) \rightarrow P(n+1)$ for all $n \geq 1$. [Induction step.]

   Proof. We prove the implication using a direct proof.
   (i) Assume that $P(n)$ is **T**, i.e., that $4^n - 1$ is divisible by 3. [Induction hypothesis.]
   (ii) This means that $4^n - 1 = 3k$ for some integer k. Thus, $4^n = 3k + 1$.
   (iii) Observe that $4^{n+1} = 4 \times 4^n$; then, $4^{n+1} = 4(3k + 1) = 12k + 4$.
   Therefore, $4^{n+1} - 1 = 12k + 3 = 3(4k + 1)$, which is a multiple of 3.
   (iv) Since $4^{n+1} - 1$ is a multiple of 3, it follows that $4^{n+1} - 1$ is divisible by 3.
   (v) Therefore, $P(n+1)$ is **T**.
3. By induction, we have proven $P(n)$ for all $n \geq 1$. ■

Prove claim $P(n)$ = "$1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$" using induction

Proof. We use induction to prove $\forall n \geq 1 : P(n)$.

1. **[Base case]** $P(1) = \frac{1}{2}(1)(1 + 1) = 1$. $P(1)$ is **T**.
2. **[Induction step]** We show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$ via a direct proof.

   Assume (*induction hypothesis*) that $P(n)$ is **T**.

   Prove $P(n+1)$: $1 + 2 + \ldots + n + (n+1) = \dfrac{(n+1)(n+2)}{2}$.

   LHS: $1 + 2 + \ldots + n + (n+1) = \underbrace{[1 + 2 + \ldots + n]} + (n+1)$

   plug in the induction hypothesis…

   $$= \frac{n(n+1)}{2} + (n+1) = \frac{1}{2}(n+1)(n+1+1).$$
3. By induction, we have proven $P(n)$ for all $n \geq 1$. ■

Prove claim $S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$ using induction

Proof. We use induction to prove $\forall n \geq 1 : S(n)$.

1. **[Base case]** $S(1) = \frac{1}{6} \times 1 \times 2 \times 3 = 1$. $S(1)$ is **T**.
2. **[Induction step]** We show $S(n) \to S(n+1)$ for all $n \geq 1$ via a direct proof.

   Assume (*induction hypothesis*) that $S(n)$ is **T**.

   Prove $S(n+1)$: $\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$.

   LHS: $\sum_{i=1}^{n+1} i^2 = \underbrace{\sum_{i=1}^{n}}_{\text{plug in the induction hypothesis...}} i^2 + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$

   $= \frac{1}{6}(n+1)(n+2)(2n+3)$.
3. By induction, we have proven $S(n)$ for all $n \geq 1$. ■

PROOF BY INDUCTION — EXAMPLE

Determine what the function below does…

…then prove that the function produces the correct result for all valid inputs

```
def f( n ):
  c = 0
  d = 0
  while ( d != n ):
    c = c + n
    d = d + 1
  return c
```

HINT: define C_{k+1} and D_{k+1} for each iteration of the loop…

PROOF BY INDUCTION

Prove the following claims using induction…

Claim 1. $P(n)$ = "$n^2 - n + 41$ is a prime number." Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 2. $P(n)$ = "$5^n - 1$ is divisible by 4." Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 3. $P(n)$ = "$n \leq 2^n$." Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 4. $P(n)$ = "$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(2n+1)(2n-1)}{3}$."
Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 5. $P(n)$ = "$n^{17} + 9$ and $(n+1)^{17} + 9$ have no factors in common."
Prove $\forall n \geq 1$, $P(n)$ is **T**.

Claim 5 does not hold—tinker with some values but see the textbook for where this fails…

PROOF BY INDUCTION – EXAMPLE

Claim 3. $P(n) = n \leq 2^n$. Prove $\forall n \geq 1$, $P(n)$ is **T**.

Proof. We use induction to prove $\forall n \geq 1 : P(n)$.

1. **[Base case]** For $P(1)$, we have $1 \leq 2^1$, or $1 \leq 2$, which is **T**.
2. **[Induction step]** We show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$ by using a direct proof.

   Assume $n \leq 2^n$ (our *induction hypothesis*). We must prove $n + 1 \leq 2^{n+1}$.

   Since $n \geq 1$, we know $n + 1 \leq n + n$, or $n + 1 \leq 2n$.

   If we can show $2n \leq 2^{n+1}$, then we have shown $n + 1 \leq 2^{n+1}$…

   Multiplying both sides of our induction hypothesis by 2,
   we have $2n \leq 2 \cdot 2^n$, or $2n \leq 2^{n+1}$; from this, $n + 1 \leq 2^{n+1}$ must be **T**.
3. By induction, we have proven $P(n)$ for all $n \geq 1$. ■

THE WELL-ORDERING PRINCIPLE

(we might not know what the minimum element is…)

The *Well-Ordering Principle* is an axiom stating that any non-empty subset of $\mathbb{N}$ has a minimum element

Anything proven by induction can be proven using the Well-Ordering Principle

A proof using this axiom uses contradiction…

…assume there is a counter-example that shows $P(n)$ to be **F**; consider the smallest such counter-example, then show that an even smaller counter-example exists—a contradiction!

A WELL-ORDERING PROOF – EXAMPLE

Go back to our induction proofs and prove using the Well-Ordering Principle…!

Claim 3. $P(n) = n \le 2^n$. Prove $\forall n \ge 1$, $P(n)$ is **T**.

Proof. We prove $\forall n \ge 1 : P(n)$ by contradiction.

Assume there is a counter-example that shows $P(n)$ to be **F**, i.e., $\exists n : n > 2^n$.

Collect all counter-examples into set B.

By the Well-Ordering Principle, set B has minimum element n_*, with $n_* > 2^{n_*}$.

Observe $1 < 2^1$, so $n_* \ge 2$, or $\frac{1}{2} n_* \ge 1$. Next, consider $n_* - 1$:

based on our initial assumption

since $n_* \ge 2$

$$n_* - 1 \ge n_* - \tfrac{1}{2} n_* = \tfrac{1}{2} n_* > \tfrac{1}{2} 2^{n_*} = 2^{n_*-1}$$

Thus, $n_* - 1 > 2^{n_*-1}$, but if $n_* - 1 \in B$, it must be smaller than n_*—a contradiction!

Therefore, we have proven that $\forall n \ge 1 : P(n)$ is **T**. ■

PROOF BY INDUCTION – HARDER EXAMPLE

Given claim $P(n) : n^2 \leq 2^n$. Prove $\forall n \geq 4$, $P(n)$ is **T**.

Why do we start at 4 here…?

Proof. We use induction to prove $\forall n \geq 4 : P(n)$.

1. **[Base case]** For $P(4)$, we have $4^2 \leq 2^4$, or $16 \leq 16$, which is **T**.
2. **[Induction step]** Assume $n^2 \leq 2^n$ (our *induction hypothesis*).

   We must prove $P(n+1)$, i.e., $(n+1)^2 \leq 2^{n+1}$.

   LHS: $(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1$.

The inequality originates from our induction hypothesis…

We are stuck…

…unless we can show $2n + 1 \leq 2^n$—why would that help?

PROOF BY INDUCTION – HARDER EXAMPLE

Given new claim $Q(n) = \overbrace{(n^2 \leq 2^n)}^{P(n)} \wedge (2n + 1 \leq 2^n)$. Prove $\forall n \geq 4$, $Q(n)$ is **T**.

we strengthen the claim, so we assume more…

Proof. We use induction to prove $\forall n \geq 4 : Q(n)$.

1. **[Base case]** For $Q(4)$, we have $4^2 \leq 2^4$ and $2(4) + 1 \leq 16$, both of which are **T**.
2. **[Induction step]** Assume $Q(n)$ is **T**, i.e., (i) $n^2 \leq 2^n$ and (ii) $2n + 1 \leq 2^n$.

   We must prove $Q(n+1)$, i.e., (i) $(n+1)^2 \leq 2^{n+1}$ and (ii) $2(n+1) + 1 \leq 2^{n+1}$.

   LHS (i): $(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \leq 2^n + 2^n = 2^{n+1}$.

   using both assumptions

   LHS (ii): $2(n+1) + 1 = 2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1}$.

   using initial assumption (ii) and the fact that $2 \leq 2^n$

   Both (i) and (ii) together prove $Q(n+1)$ is **T**.
3. By induction, we have proven $Q(n)$ for all $n \geq 4$, which also proves claim $P(n)$. ■

WHY STRENGTHEN THE CLAIM?

When should we strengthen the claim…?

…to get "unstuck" in an induction proof involving an inequality

…to get "unstuck" in an induction proof when our induction hypothesis is not helpful

…to add more to the induction hypothesis—we get to assume more!

PROOF BY INDUCTION – EVEN HARDER EXAMPLES

Prove the following using induction.

Claim 1. $P(n) : n^3 \leq 2^n$. Prove $\forall n \geq 10$, $P(n)$ is **T**. Strengthen P to Q, then Q to R…

Claim 2. $S(n) : 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \leq 2$. Can you prove $\forall n \geq 1$, $S(n)$ is **T**?

Claim 3. $X(n) : 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$. Prove $\forall n \geq 1$, $X(n)$ is **T**.

For Claim 3, once you prove $X(n)$, can you use that to easily prove Claim 2…?

Check out Exercise 6.2…

k-LEAPING INDUCTION — EXAMPLE

Given claim $P(n) : n^3 \le 2^n$, prove $\forall n \ge 10, P(n)$ is **T** using k-leaping induction

Proof. We use 2-leaping induction to prove $\forall n \ge 10 : P(n)$.

1. **[Base cases]** For $P(10)$, we have $10^3 \le 2^{10}$, or $1000 \le 1024$, which is **T**.

   And for $P(11)$, we have $11^3 \le 2^{11}$, or $1331 \le 2048$, which is **T**.

2. **[Induction step]** Assume $n^3 \le 2^n$. We must prove $P(n+2)$, i.e., $(n+2)^3 \le 2^{n+2}$.

   LHS: $(n+2)^3 = n^3 + 6n^2 + 12n + 8 \le n^3 + n \cdot n^2 + n^2 \cdot n + n^3$

   $= 4n^3 \le 4 \cdot 2^n = 2^{n+2}$.

   For n^3 to emerge, if $n \ge 10$, observe that $6 \le n$, $12 \le n^2$, and $8 \le n^3$

   Thus, $P(n) \rightarrow P(n+2)$ is shown to be **T**.

3. By 2-leaping induction, we have proven $P(n)$ for all $n \ge 10$. ■

k-LEAPING INDUCTION — STAMPS EXAMPLE

How can we prove that we can accommodate any postage starting at 24¢ using induction…?

24¢	25¢	26¢	27¢	28¢	29¢	30¢	31¢	…
5¢, 5¢, 7¢, 7¢	5¢, 5¢, 5¢, 5¢, 5¢	5¢, 7¢, 7¢, 7¢	5¢, 5¢, 5¢, 5¢, 7¢	7¢, 7¢, 7¢, 7¢	5¢, 5¢, 5¢, 7¢, 7¢	5¢, 5¢, 5¢, 5¢, 5¢, 5¢	5¢, 5¢, 7¢, 7¢, 7¢	…

The first five solutions are base cases $P(24)$, $P(25)$, $P(26)$, $P(27)$, and $P(28)$

And we must prove that $P(n) \rightarrow P(n+5)$ for all $n \ge 24$

Now can you write a k-leaping induction proof for this problem?

ultra-rare!

WHAT NEXT…?

Do as many problems in the lecture slides as possible—post solutions and questions

Attend as many office hours as you can—we have a lot of weekly coverage

Homework 2 is posted and due by 11:59PM on September 29

Problem Set 3 will be posted by next Monday—due at recitations on September 28

- We are again going to spread out to neighboring rooms: Ricketts 208 and 212
- Note that Ricketts 212 is not available 10:00-10:50AM this week (so for section 01)

Practice! Tinker! Practice! Tinker! Practice! Tinker! Practice! Tinker! Practice!