

- **Problem 7.9.**  $G^0 = 0$ .  $G_1 = 1$  and  $G_n = 7G_{n-1} - 12G_{n-2}$  for  $n > 1$ . Compute  $G_5$ . Show  $G_n = 4^n - 3^n$  for  $n \geq 0$ .

|       |   |   |   |    |     |     |
|-------|---|---|---|----|-----|-----|
| $n$   | 0 | 1 | 2 | 3  | 4   | 5   |
| $A_n$ | 0 | 1 | 7 | 37 | 175 | 781 |

- (i) prove the base case:

$$G(0) = 4^0 - 3^0 = 0$$

base case is true

- (ii) prove  $G(n) = 4^n - 3^n$  for  $G(n) = 7G(n-1) - 12G(n-2)$  for  $n > 1$

$$4^n - 3^n = 7G(n-1) - 12G(n-2)$$

manipulate the RHS

$$\begin{aligned}
 4^n - 3^n &= 7(4^{n-1} - 3^{n-1}) - 12(4^{n-2} - 3^{n-2}) \\
 &= 7\left(\frac{4^n}{4} - \frac{3^n}{3}\right) - 12\left(\frac{4^n}{16} - \frac{3^n}{9}\right) \\
 &= 7\left(\frac{4^n \cdot 3 - 3^n \cdot 4}{12}\right) - 12\left(\frac{4^n \cdot 9 - 3^n \cdot 16}{144}\right) \\
 &= 7\left(\frac{4^n \cdot 3 - 3^n \cdot 4}{12}\right) - \left(\frac{4^n \cdot 9 - 3^n \cdot 16}{12}\right) \\
 &= \frac{21(4^n) - 28(3^n) - 9(4^n) + 16(3^n)}{12} \\
 &= \frac{12(4^n) - 12(3^n)}{12} \\
 &= 4^n - 3^n
 \end{aligned}$$

We prove the statement is true for  $n \geq 0$  by direct proof ■

- **Problem 7.12(c).** (See Problem 7.28 for hints.) Tinker to guess a formula for each recurrence and prove it. In each case  $A_1 = 1$  and for  $n > 1$ :

- (c)  $A_n = 10nA_{n-1}/(n-1) + n$

|       |    |     |      |       |
|-------|----|-----|------|-------|
| $n$   | 2  | 3   | 4    | 5     |
| $A_n$ | 22 | 333 | 4444 | 55555 |

- i. formula found:

$$\frac{10^n - 1}{9}n$$

ii. prove the base case:

$$\begin{aligned} A(2) &= \frac{100-1}{9}(2) \\ &= \frac{99}{9}(2) = 22 \end{aligned}$$

base case proven

iii. prove using direct proof

$$10n \frac{A(n-1)}{n-1} + n = \frac{10^n - 1}{9}(n)$$

with with LHS

$$\begin{aligned} 10n \frac{A(n-1)}{n-1} + n &= 10n \frac{\frac{10^{n-1}-1}{9} \cancel{(n-1)}}{\cancel{n-1}} + n \\ &= \cancel{(10)} n \left( \frac{10^n - 10}{9} \right) + n \\ &= n \frac{10^n - 10}{9} + n \\ &= n \frac{10^n - 10}{9} + \frac{9n}{9} \\ &= \frac{10^n n - 10n + 9n}{9} \\ &= \frac{10^n n - n}{9} \\ \frac{10^n - 1}{9} n &= \frac{10^n - 1}{9} n \end{aligned}$$

iv. we prove by direct proof that the statement is true for all  $n > 1$  ■

• **Problem 7.13(a).** Analyze these very fast growing recursions. [Hint: Take logarithms.]

(a)  $M_1 = 2$  and  $M_n = aM_{n-1}^2$  for  $n > 1$ . Guess and prove a formula for  $M_n$ . Tinker, tinker.

|       |    |     |      |        |
|-------|----|-----|------|--------|
| $n$   | 2  | 3   | 4    | 5      |
| $A_n$ | a4 | a16 | a256 | a65536 |

(i) formula found:

$$M(n) = 2^{2^{n-1}}$$

(ii) base case:

$$\begin{aligned} M(2) &= a(2^{2^1}) \\ &= a(2^2) \\ &= a4 \end{aligned}$$

base case proven

(iii) prove using direct proof

$$aM(n-1)^2 = a2^{2^{n-1}}$$

$$M(n-1)^2 = 2^{2^{n-1}} \text{ simplify}$$

$$\log_2(M(n-1)^2) = \log_2(2^{2^{n-1}}) \text{ log both sides}$$

$$\log_2(M(n-1)^2) = 2^{n-1}$$

work with LHS

$$\begin{aligned} \log_2(M(n-1)^2) &= 2 \log_2(M(n-1)) \\ &= 2 \log_2(2^{(n-1)-1}) \\ &= 2(2^{n-2}) \\ &= 2^{n-1} \end{aligned}$$

(iv) we prove by direct proof that the statement is true for all  $n > 1$  ■

- **Problem 7.19(d).** Recall the Fibonacci numbers:  $F_1, F_2 = 1$ ; and,  $F_n = F_{n-1} + F_{n-2}$  for  $n > 2$

(d) Prove that every third Fibonacci number,  $F_{3n}$ , is even

(i) We have to prove that  $F_{3n} = 2p$ , for some  $p \in \mathbb{N}$

(ii) prove the base case:

$$F_{3n-1} = F_2 \text{ when } n = 1, \rightarrow F_2 = 1$$

$$F_{3n-2} = F_1 \text{ when } n = 1, \rightarrow F_1 = 1$$

(iii) since the fibonacci sequence is a sum of the previous two terms, we can make the following assumptions:

By the given formula  $F_n = F_{n-1} + F_{n-2}$ , we can calculate  $F_{3n} = F_{3n-1} + F_{3n-2}$

We know that both  $F_{3n-1}$  and  $F_{3n-2}$  are sums of even and odd numbers

$$F_{3n-1} = 2k + (2j + 1)$$

$$= 2(k + j) + 1$$

$$F_{3n-2} = (2w + 1) + 2i$$

$$= 2(w + i) + 1$$

plugging back into the original function, we get:

$$\begin{aligned} F_{3n} &= [2(k+j) + 1] + [2(w+i) + 1] \\ &= 2(k+j) + 2(w+i) + 2 \\ &= 2(k+j+w+i+1) \end{aligned}$$

We prove that the statement is true for all  $n > 2$  ■

- **Problem 7.42.** Give pseudocode for a recursive function that computes  $3^{2^n}$  on input  $n$ .

Code example:

```
int f(int n):
    if n is 0 return 3
    else return f(n-1) squared
```

Mathematical function:

$$\begin{aligned} T_0 &= 3 \\ T_n &= (T_{n-1})^2 \end{aligned}$$

- (a) Prove that your function correctly computes  $3^{2^n}$  for every  $n \geq 0$ .

|       |   |   |    |      |          |
|-------|---|---|----|------|----------|
| $n$   | 0 | 1 | 2  | 3    | 4        |
| $T_n$ | 3 | 9 | 81 | 6561 | 43046721 |

- (i) prove the base case for  $n = 1$

$$\begin{aligned} T(n) &= T(n-1)^2 \\ T(1) &= T(0)^2 \\ &= 9 \end{aligned}$$

- (ii) prove using a direct proof

$$\begin{aligned} T(n) &= 3^{2^n} \\ T(n) &= T(n-1)^2 \end{aligned}$$

$$\begin{aligned}
T(n-1)^2 &= 3^{2^n} \\
(3^{2^{n-1}})^2 &= 3^{2^n} \text{ log both sides} \\
\log 3((3^{2^{n-1}})^2) &= \log 3(3^{2^n}) \\
2 \log 3(3^{2^{n-1}}) &= \log 3(3^{2^n}) \\
2(2^{n-1}) &= 2^n \\
\text{LHS: } 2(2^{n-1}) &= 2^{n-1+1} \\
&= 2^n
\end{aligned}$$

(iii) we prove by a direct proof that our function computes  $3^{2^n}$  for every  $n \geq 0$  ■

(b) Obtain a recurrence for the runtime  $T_n$ . Guess and prove a formula for  $T_n$ .

(i) runtime  $T_n$

- assume squaring a number is passed onto a function such as:

```

int square(int n) {
    return n * n;
}
// two steps in total
// (1) multiplication
// (2) return

```

- $T_0 = 2$ , when  $n$  is 0  $\rightarrow$  (test, return)
- $T_1 = 6$ , when  $n$  is 1  $\rightarrow$  (test, multiplication(2), set, and  $T_0$ )
- $T_2 = 10$ , when  $n$  is 2  $\rightarrow$  (test, multiplication(2), set, and  $T_1$ )
- $T_n = T_{n-1} + 4$  for  $n \geq 2$

(ii) derived formula:  $T_n = 4n + 2$

base case:  $n = 1$

$$\begin{aligned}
T(1) &= 4(1) + 2 \\
&= 6
\end{aligned}$$

prove by direct proof

$$\begin{aligned}
T(n) &= T(n-1) + 4 \\
T(n) &= 4n + 2
\end{aligned}$$

Setting both equations equal, we get

$$\begin{aligned}
T(n-1) + 4 &= 4n + 2 \\
\text{LHS} \rightarrow T(n-1) + 4 &= 4(n-1+2+4) \\
&= 4n\cancel{-4} + 2\cancel{-4} \\
&= 4n + 2
\end{aligned}$$

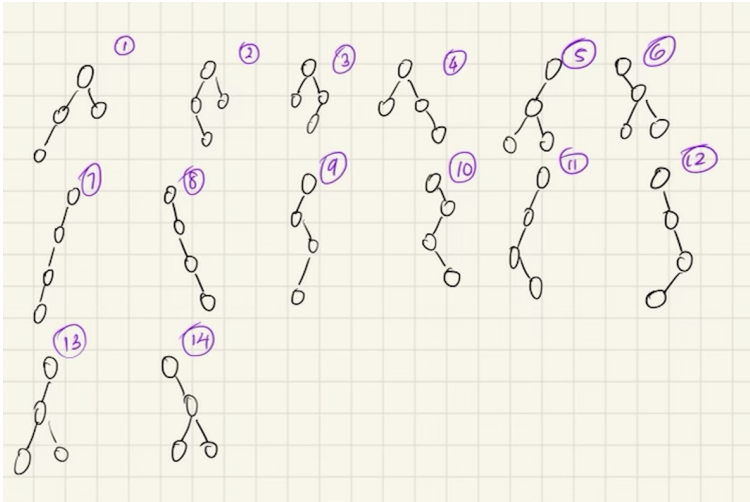
(iii) we prove by direct proof that our formula  $T_n$  accurately calculates the runtime  $T_n$  for  $n \geq 1$  ■

• **Problem 7.45(c).** Give recursive definitions for the set  $S$  in each of the following cases.

(c)  $S = \{\text{all strings with the same number of 0's and 1's}\}$  (e.g. 0011,0101,100101).

1. **[basis]**  $\epsilon \in S, 0 \in S, 1 \in S$
2. **[constructor(i)]**  $\epsilon \in S \rightarrow 0 \bullet x \bullet 0 \in S$ ;  
**[constructor(ii)]**  $\epsilon \in S \rightarrow 1 \bullet x \bullet 1 \in S$ .

• **Problem 7.49.** There are 5 rooted binary trees (RBT) with 3 nodes. How many have 4 nodes



• We can make 14 possible rooted binary trees with 4 nodes.

• **Problem 8.12(d).** A set  $P$  of parenthesis strings have a recursive definition.

1.  $\epsilon \in P$
2.  $x \in P \rightarrow [x] \in P$   
 $x, y \in P \rightarrow xy \in P$

(d) Prove by structural induction that every string  $P$  is balanced.

- i. **[Base case]** When  $n = 1$  and  $x_1 = \epsilon$ , it is clearly balanced,  $P(1)$  is true
- ii. **[Induction step]** show

• **Problem 8.14.** A set  $A$  is defined recursively as shown.

1.  $3 \in A$ .
2.  $x, y \in A \rightarrow x + y \in A$ ;  
 $x, y \in A \rightarrow x - y \in A$ .

(a) Prove that every element of  $A$  is a multiple of 3.

1. Prove by structural induction that every element in  $A$  is a multiple of 3.
2. **[Base case]** for  $P(0)$ , we have both:

$$3 + 3 \in A = 6$$

$$3 - 3 \in A = 0$$

both are multiples of 3

3. **[Induction step]** suppose  $x, y \in A$  and both  $x$  and  $y$  are multiples of 3

$$x = 3k$$

$$y = 3k$$

the constructor rules allow us to create the following formula:

$$x + y \in A$$

$$3k + 3w \in A$$

$$3(k + w) \in A$$

Adding two numbers that are multiples of 3 will always result in a number that is a multiple of 3

4. By structural induction, we conclude that every member of  $A$  is a multiple of 3 ■

(b) Prove that every multiple of 3 is in  $A$ .