- **Problem 7.9.** $G^0 = 0$. $G_1 = 1$ and $G_n = 7G_{n-1} - 12G_{n-2}$ for $n > 1$. Compute G_5. Show $G_n = 4^n - 3^n$ for $n \geq 0$.

n	0	1	2	3	4	5
A_n	0	1	7	37	175	781

  (i) prove the base case:

$$G(0) = 4^0 - 3^0 = 0$$

base case is true

  (ii) prove $G(n) = 4^n - 3^n$ for $G(n) = 7G(n-1) - 12G(n-2)$ for $n > 1$

$$4^n - 3^n = 7G(n-1) - 12G(n-2)$$

manipulate the RHS

$$\begin{aligned}
4^n - 3^n &= 7(4^{n-1} - 3^{n-1}) - 12(4^{n-2} - 3^{n-2}) \\
&= 7(\frac{4^n}{4} - \frac{3^n}{3}) - 12(\frac{4^n}{16} - \frac{3^n}{9}) \\
&= 7(\frac{4^n \cdot 3 - 3^n \cdot 4}{12}) - 12(\frac{4^n \cdot 9 - 3^n \cdot 16}{144}) \\
&= 7(\frac{4^n \cdot 3 - 3^n \cdot 4}{12}) - (\frac{4^n \cdot 9 - 3^n \cdot 16}{12}) \\
&= \frac{21(4^n) - 28(3^n) - 9(4^n) + 16(3^n)}{12} \\
&= \frac{12(4^n) - 12(3^n)}{12} \\
&= 4^n - 3^n
\end{aligned}$$

  We prove the statement is true for $n \geq 0$ by direct proof ■

- **Problem 7.12(c). (See Problem 7.28 for hints.)** Tinker to guess a formula for each recurrence and prove it. In each case $A_1 = 1$ and for $n > 1$:

  (c) $A_n = 10nA_{n-1}/(n-1) + n$

n	2	3	4	5
A_n	22	333	4444	55555

    i. formula found:

$$\frac{10^n - 1}{9} n$$

ii. prove the base case:

$$\begin{aligned} A(2) &= \frac{100-1}{9}(2) \\ &= \frac{99}{9}(2) = 22 \end{aligned}$$

base case proven

iii. prove using direct proof

$$10n\frac{A(n-1)}{n-1} + n = \frac{10^n - 1}{9}(n)$$

with with LHS

$$\begin{aligned} 10n\frac{A(n-1)}{n-1} + n &= 10n\frac{\frac{10^{n-1}-1}{9}\cancel{(n-1)}}{\cancel{n-1}} + n \\ &= \cancel{(10)}n(\frac{10^n - 10}{\cancel{10}}\frac{1}{9}) + n \\ &= n\frac{10^n - 10}{9} + n \\ &= n\frac{10^n - 10}{9} + \frac{9n}{9} \\ &= \frac{10^n n - 10n + 9n}{9} \\ &= \frac{10^n n - n}{9} \\ \frac{10^n - 1}{9}n &= \frac{10^n - 1}{9}n \end{aligned}$$

iv. we prove by direct proof that the statement is true for all $n > 1$ ■

- **Problem 7.13(a).** Analyze these very fast growing recursions. [Hint: Take logarithms.]

(a) $M_1 = 2$ and $M_n = aM_{n-1}^2$ for $n > 1$. Guess and prove a formula for M_n. Tinker, tinker.

n	2	3	4	5
A_n	a4	a16	a256	a65536

(i) formula found:

$$M(n) = 2^{2^{n-1}}$$

(ii) base case:

$$\begin{aligned} M(2) &= a(2^{2^1}) \\ &= a(2^2) \\ &= a4 \end{aligned}$$

base case proven

(iii) prove using direct proof

$$
\begin{aligned}
aM(n-1)^2 &= a2^{2^{n-1}} \\
M(n-1)^2 &= 2^{2^{n-1}} \text{ simplify} \\
\log_2(M(n-1)^2) &= \log_2(2^{2^{n-1}}) \text{ log both sides} \\
\log_2(M(n-1)^2) &= 2^{2^{n-1}}
\end{aligned}
$$

work with LHS

$$
\begin{aligned}
\log_2(M(n-1)^2) &= 2\log_2(M(n-1)) \\
&= 2\log_2(2^{2^{(n-1)-1}}) \\
&= 2(2^{n-2}) \\
&= 2^{n-1}
\end{aligned}
$$

(iv) we prove by direct proof that the statement is true for all $n > 1$ ■

- **Problem 7.19(d).** Recall the Fibonacci numbers: F_1, $F_2 = 1$; and, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$

  (d) Prove that every third Fibonacci number, F_{3n}, is even

- **Problem 7.42.** Give pseudocode for a recursive function that computes 3^{2^n} on input n.

Code example:

```
int f(int n):
    if n is 0 return 3
    else return f(n-1) squared
```

Mathematical function:

$$
\begin{aligned}
T_0 &= 3 \\
T_n &= (T_{n-1})^2
\end{aligned}
$$

(a) Prove that your function correctly computes 3^{2^n} for every $n \geq 0$.

n	0	1	2	3	4
T_n	3	9	81	6561	43046721

(i) prove the base case for $n = 1$

$$\begin{aligned} T(n) &= T(n-1)^2 \\ T(1) &= T(0)^2 \\ &= 9 \end{aligned}$$

(ii) prove using a direct proof

$$T(n) = 3^{2^n}$$
$$T(n) = T(n-1)^2$$

$$\begin{aligned} T(n-1)^2 &= 3^{2^n} \\ (3^{2^{n-1}})^2 &= 3^{2^n} \text{ log both sides} \\ \log 3((3^{2^{n-1}})^2) &= \log 3(3^{2^n}) \\ 2\log 3(3^{2^{n-1}}) &= \log 3(3^{2^n}) \\ 2(2^{n-1}) &= 2^n \\ \text{LHS: } 2(2^{n-1}) &= 2^{n-1+1} \\ &= 2^n \end{aligned}$$

(iii) we prove by a direct proof that our function computes 3^{2^n} for every $n \geq 0$ ■

(b) Obtain a recurrence for the runtime T_n. Guess and prove a formula for T_n.

(i) MISSING

- **Problem 7.45(c).** Give recursive definitions for the set S in each of the following cases.

  (c) $S = \{$all strings with the same number of 0's and 1's$\}$ (e.g. 0011,0101,100101).

- **Problem 7.49.**
- **Problem 8.12(d).**
- **Problem 8.14.**