Spantest: An R Package for Mean-Variance Spanning Tests in Portfolio Analysis

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Abstract Implements a comprehensive suite of portfolio spanning tests, including classical methods by Huberman and Kandel (1987), Gibbons, Ross, and Shanken (1989), Kempf and Memmel (2006), Pesaran and Yamagata (2023), and Gungor and Luger (2016), alongside the recent residual-based subseries tests developed by Ardia and Sessinou (2025). Supports both asset-level and global hypothesis testing with the Cauchy Combination Test (CCT) for robust inference under serial dependence, heteroskedasticity, and high-dimensional regimes where the number of benchmark assets is less than the number of observations.

1 Introduction

Mean-variance spanning (MVS) refers to the hypothesis that a given set of "benchmark" assets already spans the efficient frontier achievable by a larger set of assets. In other words, adding new assets (the "test" assets) to an existing portfolio cannot improve the portfolio's risk-return trade-off if spanning holds (Huberman and Kandel, 1987). If the MVS hypothesis is not rejected, the benchmark assets are mean-variance efficient, and adding the test assets offers no ex ante diversification benefit.

Spanning tests are economically important because they address questions such as: Does an existing portfolio (e.g., a market index) already capture all investment opportunities? Or can diversifying into new asset classes or markets further reduce risk or increase returns? These tests originate in the portfolio efficiency literature (Gibbons et al., 1989) and are closely related to tests of the Capital Asset Pricing Model (CAPM), where one examines whether a market portfolio is efficient (spanning) with respect to individual assets.

From a statistical perspective, mean-variance spanning can be assessed by formulating the portfolio expansion as a set of linear restrictions on the mean and covariance of returns. Early work by Huberman and Kandel (1987) (HK) formalized the spanning hypothesis and proposed a multivariate test for it. Subsequent research developed specialized tests for different aspects of spanning: Gibbons et al. (1989) (GRS) introduced a well-known alpha-spanning test for the efficiency of a given portfolio; Britten-Jones (1999) (BJ) studied the impact of estimation error on efficient portfolios; and more recent studies have extended spanning tests to high-dimensional settings (many assets) and time-varying volatility (Gungor and Luger, 2016).

The `spantest` package brings together these developments, implementing a comprehensive set of mean-variance spanning tests from the literature.

2 Theoretical framework

Consider a benchmark set of K asset returns (e.g., an existing portfolio or set of portfolios) and a test set of N additional asset returns. Let $r_{1,t}$ be the K-dimensional return vector of benchmark assets at time t, and $r_{2,t}$ the N-dimensional return vector of test assets. The MVS hypothesis states that the efficient frontier spanned by $r_{1,t}$ alone is the same as that spanned by the combined $(r_{1,t}, r_{2,t})$. This implies that the new assets r_2 do not improve the minimum achievable variance for a given expected return, nor do they allow a higher expected return for a given variance.

In practical terms, spanning can be characterized by conditions on the means and covariances of returns. A convenient formulation is obtained by regressing each test asset on the benchmarks. For each test asset j, consider the linear model (with constant term α_j and slope vector β_j):

$$r_{2,j,t} = \alpha_j + \beta_j^\top r_{1,t} + \varepsilon_{j,t},$$

where $\mathbb{E}[\varepsilon_{j,t}] = 0$.

If a risk-free asset is available (or returns are measured in excess of the risk-free rate), mean-variance spanning is equivalent to all intercepts being zero, $\alpha_j = 0$ for each test asset (Pesaran and Yamagata, 2023).

This hypothesis is formulated as:

$$H_0^\alpha : \alpha = 0,$$

and is called the *maximum Sharpe ratio spanning test*.

Intuitively, $\alpha_j = 0$ means the new asset j offers no higher expected return than what can be replicated by a combination of the benchmark assets (no abnormal performance).

In the absence of a risk-free asset, full spanning requires that the inclusion of new assets does not improve the *global minimum-variance* (GMV) frontier. Let δ denote the vector of optimal portfolio weights assigned to the test assets in the GMV portfolio formed from both benchmark and test assets. The corresponding null hypothesis is:

$$H_0^{\delta} : \delta = 0,$$

which implies that the test assets receive zero weight in the GMV portfolio and thus provide no variance reduction. This is known as the *global minimum-variance spanning test* (Kan and Zhou, 2012).

Finally, Kan and Zhou (2012) show that the *full MVS* hypothesis can be stated as a joint null:

$$H_0^{\alpha,\delta} : \alpha = 0, \quad \delta = 0,$$

where α is the $N \times 1$ vector of intercepts and δ represents N additional constraints ensuring no variance improvement.

Under $H_0^{\alpha,\delta}$, for every test asset (or any portfolio of them) one can find a portfolio of the K benchmarks that replicates its performance. If either the intercept conditions or the variance conditions fail, the new assets expand the frontier and spanning is violated.

These hypotheses lead to different types of statistical tests. **Maximum Sharpe ratio spanning tests** focus on the $\alpha_j = 0$ restrictions (no excess returns beyond the span of benchmarks). **Global minimum-variance spanning tests** focus on the $\delta = 0$ restrictions (no reduction in minimum variance). **MVS tests** assess the full set of constraints simultaneously.

In all cases, the tests typically involve comparing estimated portfolio moments (means, variances) with and without the new assets, taking into account estimation uncertainty. Early multivariate test procedures for spanning were derived in Huberman and Kandel (1987), who essentially set up the problem as a system of linear equations (restrictions (3a) and (3b) in their notation) and used likelihood-ratio statistics. Modern approaches, while mathematically equivalent in form (often leading to F-statistics or χ^2-statistics under H_0), differ in how they handle finite sample issues, large-dimensional cases, or general distributional assumptions (e.g., non-normal returns).

2.1 Maximum Sharpe Ratio Spanning Test (Mean Efficiency)

Maximum Sharpe ratio spanning tests examine whether adding assets yields any improvement in expected returns for a given level of risk. They test the null hypothesis that all $\alpha_j = 0$, i.e., the benchmark assets already achieve the highest possible Sharpe ratio. A seminal test in this category is the GRS test, which assesses the mean-variance efficiency of a given portfolio.

In the GRS test, one takes a candidate efficient portfolio (e.g., the market index) as the single benchmark and regresses each test asset's excess return on that benchmark. Under H_0^{α} (portfolio is efficient), all intercepts are zero. The GRS statistic has an F-distribution under H_0^{α} and combines the N individual intercept tests into one omnibus test. The `spantest` package implements this test with the `span_grs()` function.

A limitation of the GRS test is that it assumes a moderate number of assets and i.i.d. normally distributed returns. For large cross-sections of assets, the test can suffer from size distortions or low power. Pesaran and Yamagata (2023) address this by proposing two asymptotic Z-tests for alpha spanning in panels with N large relative to T. Their first test (sometimes denoted J_1) is asymptotically standard normal as $N \to \infty$ (with T fixed) under Gaussian errors. Even when the normality assumption is relaxed, a modified test J_2 can be constructed that remains asymptotically normal under certain conditions on cross-sectional dependence.

These PY tests allow one to test spanning in high dimensions (for example, testing if the market index spans thousands of stocks) with controlled type I error. In practice, the `spantest` implementation `span_py()` computes the J-statistics for large-N alpha spanning tests.

It is worth noting that if the benchmark set contains multiple assets (with no risk-free asset), alpha spanning alone is not sufficient for full spanning. Nonetheless, one can still test the $\alpha = 0$ portion separately. Kan and Zhou (2012) term this the mean spanning hypothesis (F1) and derive a corresponding test statistic. In the `spantest` package, the function `span_f1()` provides a test for the intercept restrictions only (analogous to a multivariate t- or F-test on all α_j). This can be useful to isolate whether lack of spanning is coming from mean differences.

Gungor and Luger (2016) (GL) propose a multivariate test for mean-variance efficiency and spanning that accommodates large cross-sections of assets and time-varying covariance structures. Their approach extends classical tests by incorporating dynamic estimation of covariances and allowing for high-dimensional settings where the number of assets may be large relative to the sample size. This is particularly relevant for testing the maximum Sharpe ratio hypothesis, as it directly assesses whether the benchmark portfolio's intercept vector (α) is zero across multiple assets simultaneously, accounting for time-varying risk. The `spantest` package implements this test via the `span_gl_a()` function, enabling practitioners to detect mean spanning violations in complex, realistic market conditions where classical methods may lack power or suffer from size distortions.

In addition to classical tests such as GRS and PY, the recent work by Ardia and Sessinou (2025) (AS) introduces a novel high-dimensional mean-variance spanning test based on a subseries Cauchy Combination Test (SCT) methodology. This approach extends mean-variance spanning tests to settings with many benchmark and test assets where traditional methods suffer from size distortions and power loss due to growing dimensionality and dependence in the data. The SCT aggregates p-values from residual-based subseries regressions across assets, enabling robust inference in the presence of serial correlation, heteroscedasticity, and fat tails without restrictive distributional assumptions.

Their test specifically targets the joint null hypothesis that adding new assets does not improve the maximum Sharpe ratio achievable by the benchmark portfolio, thereby generalizing the classical maximum Sharpe ratio spanning test to high-dimensional regimes where the number of assets may be comparable to or exceed the sample size. Monte Carlo simulations confirm that the SCT maintains proper size and power under realistic market dynamics, outperforming existing tests in complex environments. The `spantest` package includes an implementation of this test via the `span_as()` function, providing a practical and computationally efficient tool for researchers and practitioners assessing portfolio efficiency in large asset universes.

2.2 Variance Spanning Tests (Minimum-Variance Efficiency)

Variance-spanning tests focus on whether adding new assets can achieve a lower global minimum variance. The null hypothesis is that the covariance structure of returns is such that the GMV portfolio of the expanded set has the same variance as the GMV portfolio of the benchmark set. Equivalently, the weight of each test asset in the GMV portfolio is zero under H_0^{δ}. One practical way to test this is to examine the variance reduction obtained by including the new assets.

Kempf and Memmel (2006) (KM) emphasize the importance of the GMV portfolio in investment practice: in many cases the GMV portfolio yields better out-of-sample results than the tangency (maximal Sharpe) portfolio. They provide improved estimates for the GMV weights and variance, which can be used to test variance spanning. Intuitively, if the sample covariance matrix suggests that adding an asset significantly lowers the portfolio variance, one can reject the null of variance spanning. The `spantest` function `span_km()` implements the approach related to the KM test. It specifically evaluates whether the estimated minimum variance is significantly different when the test assets are included.

Another perspective comes from Britten-Jones (1999), who showed that the sampling error in estimates of efficient portfolio weights can be large. This means that even if an expanded portfolio shows a lower variance in-sample, one must account for estimation uncertainty before concluding that variance spanning is violated. BJ derived the distribution of the estimated tangency and GMV portfolio weights, facilitating more accurate inference on spanning. In the `spantest` function `span_bj()`, this is incorporated by adjusting for the estimation error when testing spanning. This yields a more precise p-value for the null of no variance improvement, preventing over-rejection due to noisy estimates.

As with mean-spanning, one can define a pure variance-spanning test that ignores mean returns and only checks the covariance condition. Kan and Zhou (2012)'s formulation of their F2 test includes a separate set of restrictions $\delta = 0$ for variance spanning. The function `span_f2()` in the package performs a test focusing solely on variance spanning (e.g., via a likelihood ratio or F-test on the covariance-related constraints). This can diagnose whether a failure of spanning is due to risk reduction potential even if no significant alpha is found.

As discussed in the previous section on maximum Sharpe ratio spanning tests, Ardia and Sessinou (2025) develop a unified high-dimensional testing framework based on the SCT. This framework also applies naturally to variance spanning hypotheses, assessing whether the inclusion of new assets can significantly reduce the global minimum variance beyond sampling error and estimation uncertainty. By leveraging residual-based subseries p-value aggregation, their SCT-based variance spanning test maintains correct size and power in high-dimensional, heteroskedastic, and dependent data environments where traditional variance spanning tests (e.g., KM or BJ) face limitations. The `span_as()` function in the `spantest` package implements this test, providing a powerful and robust tool to evaluate minimum-variance efficiency in large asset universes.

2.3 Joint Mean-Variance Spanning Tests

Joint tests consider the full mean-variance spanning hypothesis, testing all conditions simultaneously. These tests will reject the null if either expected returns or attainable variance (or both) are improved by the new assets. The original spanning test of Huberman and Kandel (1987) falls in this category: it is a multivariate test (Chi-square or F) for the combined restrictions $\alpha = 0$ and $\delta = 0$.

In practice, one can perform a joint test by combining the statistics for mean and variance restrictions. For example, one approach is to stack the N intercept restrictions and N variance-related restrictions into a single $2N$-dimensional system and compute a Wald statistic; if this statistic exceeds a critical value, spanning is rejected. The `span_hk()` function in `spantest` implements the joint HK test.

Recent advances in joint mean-variance spanning tests have been made by Gungor and Luger (2016), who develop a robust finite-sample testing framework suitable for large cross-sections of assets and time-varying covariances. Their methodology explicitly models estimation error in covariance matrices under conditional heteroskedasticity and maintains correct size and power as the number of assets (N) grows relative to the sample size (T), addressing key limitations of classical approaches. As discussed in the section on maximum Sharpe ratio spanning tests, the GL methodology extends naturally to the full joint null hypothesis $\alpha = 0$ and $\delta = 0$, enabling comprehensive mean-variance spanning assessment in high-dimensional, realistic market environments. The `spantest` package includes implementations of this joint test via the `span_gl_ad()` function.

Finally, Ardia and Sessinou (2025) propose a novel high-dimensional test tailored for the joint mean-variance spanning hypothesis, which simultaneously evaluates whether adding new assets improves either expected returns or achievable variance efficiency. Their approach leverage the SCT methodology, aggregating residual-based subseries p-values to deliver a robust test statistic valid under complex serial and cross-sectional dependence, heteroskedasticity, and heavy tails. Unlike classical methods limited to small asset sets, this test remains reliable when both benchmark and test asset dimensions are large, making it highly suitable for modern portfolio applications involving extensive asset universes. The spantest package's `span_as()` function provides p-values for this joint null alongside the marginal alpha and delta hypotheses. Readers are referred to the preceding sections for detailed discussion of the test's treatment of the individual mean or variance spanning components. This joint test represents a significant advancement for portfolio efficiency evaluation, offering practitioners a rigorous and computationally efficient tool for high-dimensional empirical analysis

In summary, the `spantest` package provides a suite of functions covering all the above categories of tests. Table 1 summarizes the tests, their null hypotheses, their restrictions and their references.

2.4 Summary of Spanning Tests

Table @ref(tab:spanning_tests) summarizes the main spanning tests implemented or planned in the `spantest` package. Each test is characterized by its null hypothesis and sample size or dimension requirements.

Name	Reference	Hypothesis	Restrictions
span_hk	@HubermanKandel1987	$\alpha = 0$ and $\delta = 0$	$T - K - N \geq 1$
span_gl_ad	@GungorLuger2016	$\alpha = 0$ and $\delta = 0$	None
span_as	@ArdiaSessinou2025	$\alpha = 0$ and $\delta = 0$	None
span_gl_a	@GungorLuger2016	$\alpha = 0$	None
span_grs	@GRS1989	$\alpha = 0$	$T - K - N \geq 1$
span_f1	@KanZhou2012	$\alpha = 0$	$T - K - N \geq 1$
span_bj	@BrittenJones1999	$\alpha = 0$	$T - K - N \geq 1$
span_py	@PesaranYamagata2023	$\alpha = 0$	$T - K - N \geq 1$
span_as	@ArdiaSessinou2025	$\alpha = 0$	None
span_km	@KempfMemmel2006	$\delta = 0$	$T - K - N \geq 1$
span_f2	@KanZhou2012	$\delta = 0$	$T - K - N \geq 1$
span_as	@ArdiaSessinou2025	$\delta = 0$	None

T represents the number of observations, K is the number of benchmark assets, and N is the number of test assets. Aside from `span_as`, each function returns a list containing the p-value, the test statistic, and the hypothesis that is being tested for easy interpretation. The `span_as` function returns a list of each test p-value (depending on the hypothesis being tested since it covers $H_0^{\alpha,\delta}$, H_0^{α}, and H_0^{δ}). To validate the implementation of the spanning tests and provide guidance on their practical use, we conduct an extensive simulation study.

This analysis evaluates the empirical power and size properties of each test across a range of data-generating processes and dimensions. The resulting power curves, available in the package's GitHub repository, serve as a reference to help users select appropriate tests under varying empirical conditions.

3 Empirical illustration

To demonstrate the practical use of the `spantest` package, we conduct a comprehensive empirical analysis based on a monthly rolling-window framework, using one year of daily returns per window. This approach is inspired from the methodology employed by Gungor and Luger (2016). Our setup mimics the real-world challenge faced by a U.S.-based investor who is already exposed to North American markets (S&P 500 and Dow Jones) and seeks to assess the incremental diversification value of adding European equities (DAX and CAC 40).

We begin by downloading daily adjusted prices from Yahoo Finance using quantmod, and we compute arithmetic daily returns, ensuring all missing values are appropriately filled forward. The North American indices are treated as the benchmark assets ($R1$), while the European indices serve as the test assets ($R2$). For each rolling window, we apply the full suite of spanning tests implemented in `spantest`.

We report the p-values over time of the HK test in figure 1.

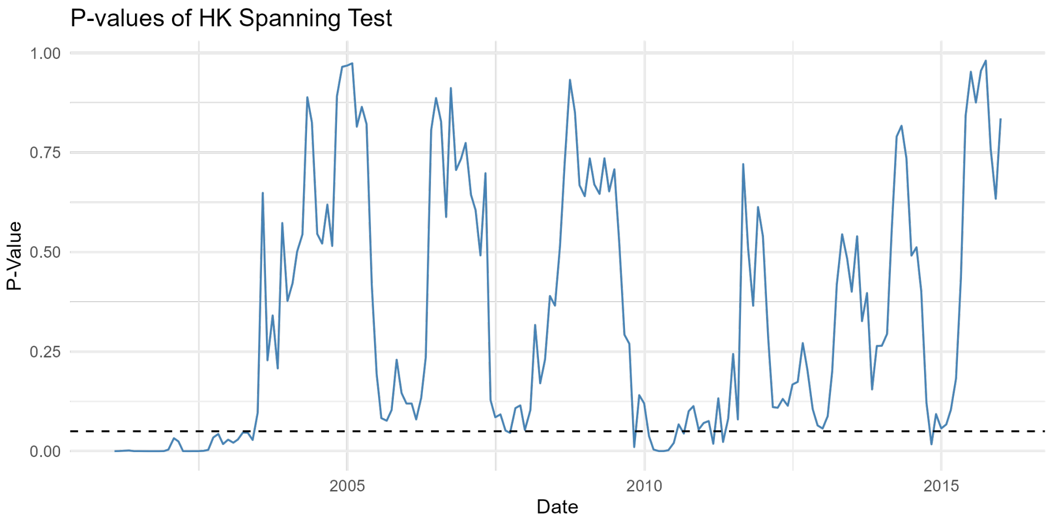


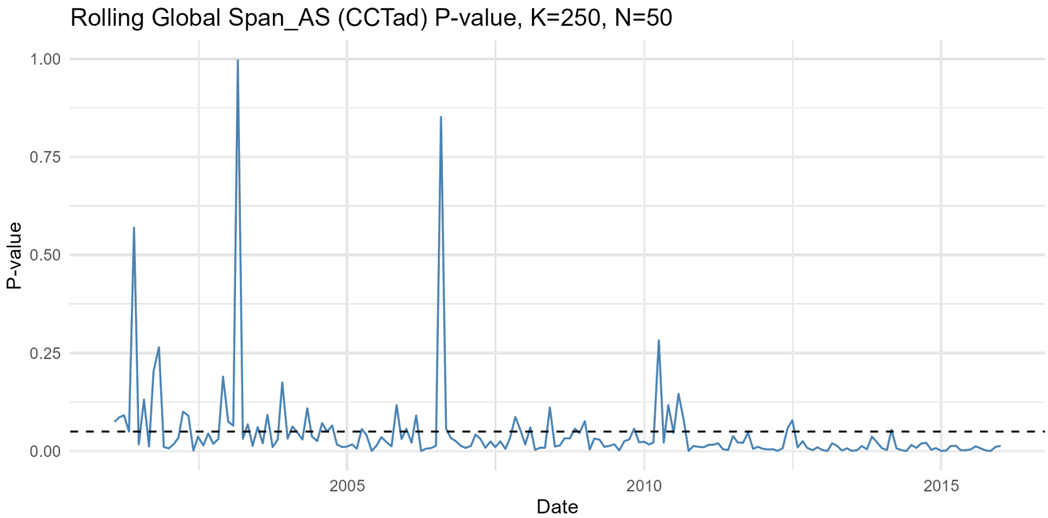
Figure 1 shows the monthly p-values of the HK spanning test computed over a rolling one-year window of daily returns, using the S&P 500 and Dow Jones indices as the benchmark portfolio and the DAX and CAC 40 as test assets. The graph reveals the time-varying nature of the spanning relationship between North American and European equity markets.

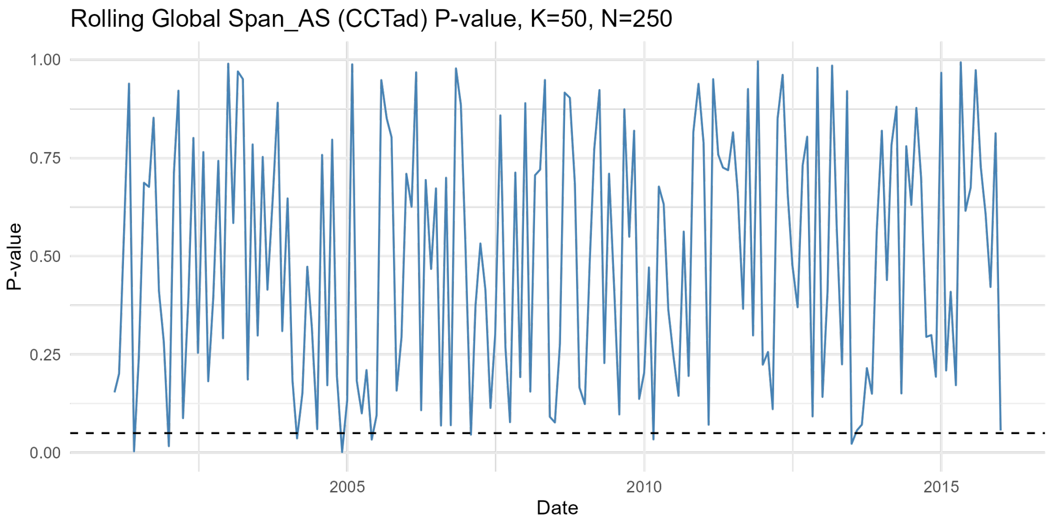
Periods when the p-values are close to or above 1 indicate no evidence against the null hypothesis, suggesting the benchmark portfolio fully spans the test assets, i.e., adding European equities does not improve the mean-variance efficient frontier. Conversely, dips toward zero signal statistically significant rejection of spanning, meaning the European indices contribute diversification benefits and improve portfolio efficiency during those times.

This graph shows pronounced clustering of high p-values near 1, indicating that for extended periods the benchmark dominates the test assets in terms of mean-variance efficiency. The fluctuations throughout the 2000s and early 2010s coincide with known market events and structural changes, such as the dot-com bubble burst, the 2008 financial crisis, and the Eurozone debt crisis, reflecting evolving correlation and integration dynamics.

Overall, this visualization emphasizes the importance of dynamically assessing spanning relations in global portfolio construction, as the benefits of international diversification appear strongly time-dependent.

The previous example is done on a low dimensional setting (2 benchmark assets, and 2 test assets). We provide another empirical illustration using the `span_as` function, designed to be robust under higher dimensional settings. The p-values are reported in figure 2.





We perform a monthly rolling-window analysis similar to the previous example, but now using 250 randomly sampled stocks from the S&P 500 and the full EuroStoxx index constituents. Daily returns are computed as arithmetic differences of forward-filled adjusted prices, and windows span one year of daily observations.

The two plots illustrate rolling p-values from the high-dimensional `span_as()` test applied in reversed roles between two portfolios: one with a large number of benchmark assets (S&P500 constituents: $K = 250$) and a small test set (Eurostoxx constituents: $N = 50$), and the other with a small benchmark ($K = 50$) and a large test set ($N = 250$).

The first plot corresponds to the scenario where the large portfolio of 250 S&P 500 stocks serves as the benchmark (K), and the smaller Eurostoxx 50 portfolio is the test set (N). Here, the p-values frequently fall below the 5% significance threshold, indicating repeated rejection of the spanning null hypothesis. This suggests that the large benchmark set does not fully span the test assets, meaning the additional Eurostoxx stocks provide incremental diversification benefits not captured by the large S&P benchmark alone.

The second plot flips the roles: the Eurostoxx 50 stocks are now the benchmark (K) and the 250 S&P 500 stocks form the test set (N). In this configuration, the p-values are mostly above the 5% level, implying fewer rejections of the spanning hypothesis. This indicates that the smaller Eurostoxx portfolio sufficiently spans the larger S&P 500 test set, so adding the larger pool of S&P stocks does not significantly improve mean-variance efficiency relative to the Eurostoxx benchmark.

This asymmetry may be caused by the fact that the smaller Eurostoxx 50 index represents a more concentrated and possibly more diversified set in terms of risk factors relevant to the joint market, whereas the larger S&P 500 portfolio contains many stocks with overlapping risk exposures. Consequently, the large S&P portfolio cannot fully capture the specific risk-return characteristics of the Eurostoxx 50, while the reverse is less true. Moreover, the random selection might play a role too if the stocks selected are closer in terms of risk-return dynamics, thus containing overlapping characteristics.

In summary, these results emphasize the importance of benchmark selection and the dimensionality of the spanning problem: a larger benchmark does not necessarily guarantee better spanning, and a smaller, carefully selected benchmark may capture most of the investment opportunity set. The `span_as()` test, by accommodating high-dimensional settings and dependence, provides a powerful tool to detect such nuanced relationships.

Together, these results underscore the value of using robust, high-dimensional spanning tests when evaluating investment decisions in realistic, evolving market conditions. While classical tests provide a baseline, they may miss local violations of spanning due to their assumptions of i.i.d. returns and constant covariances. The `spantest` package equips analysts with a diverse set of tools that reveal more nuanced dynamics in asset return structures—particularly useful in modern portfolio construction and international allocation decisions.

The package's consistent interface (matrix of benchmark returns, matrix of test returns) makes it straightforward to try different tests on the same data.

4 Conclusion

Mean-variance spanning tests provide a rigorous tool to evaluate portfolio efficiency and the merits of new investment opportunities. The `spantest` package assembles a broad collection of spanning tests from the literature – from classical F-tests of portfolio efficiency (Gibbons et al., 1989) to modern high-dimensional tests [Gungor and Luger (2016); Ardia & Sessinou 2025] – under a unified framework.

We have discussed the intuition and hypotheses behind each type of test (alpha spanning, variance spanning, joint spanning) and demonstrated how to perform a spanning test using real market data. These tests enable researchers and practitioners to answer the following question: "Will this new asset enhance my portfolio's risk-return profile?" in a statistically sound way.

By citing the foundational works (Gibbons et al., 1989; Britten-Jones, 1999; Gungor and Luger, 2016) and offering reproducible examples, this vignette has outlined both the theoretical background and practical usage of mean-variance spanning tests. The reader is encouraged to explore the `spantest` documentation for details on each test's assumptions and to apply these tests to their own portfolio data for better-informed investment decisions.

Extension to Mean-Variance Efficiency Testing: While our examples focus on asset-based spanning, these tests can also be adapted to assess mean-variance efficiency by replacing benchmark asset returns with portfolio excess returns and test assets with factor returns (e.g., Fama-French factors), as illustrated by Gungor and Luger (2016). This expands the utility of the `spantest` package to rigorous factor model validation and performance evaluation beyond asset allocation.

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