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Solving an optimization problem employing restraint programming

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Abstract. This paper goes over the theoretical and practical aspects of the resolution of an optimization problem, namely the synthesis of a schedule for a language school, employing restraint programming, more precisely, SICSTUS Prolog's restraint programming capabilities. The paper contains a description of the problem, a detailed analysis of the aproach used to solve it, an explanation of how the solution should be interpreted, a description of obtained results and, finally, conclusions.

Keywords: optimization, restraint programming, PROLOG

1 Fixed-Period Problems: The Sublinear Case

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

$$\begin{aligned} \dot{x} &= JH'(t,x) \\ x(0) &= x(T) \end{aligned}$$

with $H(t,\cdot)$ a convex function of x, going to $+\infty$ when $\|x\| \to \infty$.

1.1 Autonomous Systems

In this section, we will consider the case when the Hamiltonian $H(x)$ is autonomous. For the sake of simplicity, we shall also assume that it is C^1.

We shall first consider the question of nontriviality, within the general framework of (A_∞, B_∞)-subquadratic Hamiltonians. In the second subsection, we shall look into the special case when H is $(0, b_\infty)$-subquadratic, and we shall try to derive additional information.

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$$\gamma := \text{smallest eigenvalue of } \; B_\infty - A_\infty \tag{1}$$

$$\lambda := \text{largest negative eigenvalue of } \; J\frac{d}{dt} + A_\infty \; . \tag{2}$$

Theorem 1 tells us that if $\lambda + \gamma < 0$, the boundary-value problem:

$$\begin{array}{rl} \dot{x} = & JH'(x) \\ x(0) = & x(T) \end{array} \tag{3}$$

has at least one solution $\overline{x}$, which is found by minimizing the dual action functional:

$$\psi(u) = \int_o^T \left[\frac{1}{2} \left(\Lambda_o^{-1} u, u \right) + N^*(-u) \right] dt \tag{4}$$

on the range of Λ, which is a subspace $R(\Lambda)_L^2$ with finite codimension. Here

$$N(x) := H(x) - \frac{1}{2} \left(A_\infty x, x \right) \tag{5}$$

is a convex function, and

$$N(x) \leq \frac{1}{2} \left(\left(B_\infty - A_\infty \right) x, x \right) + c \quad \forall x \ . \tag{6}$$

Proposition 1. *Assume $H'(0) = 0$ and $H(0) = 0$. Set:*

$$\delta := \liminf_{x \to 0} 2N(x) \left\| x \right\|^{-2} \ . \tag{7}$$

If $\gamma < -\lambda < \delta$, the solution $\overline{u}$ is non-zero:

$$\overline{x}(t) \neq 0 \quad \forall t \ . \tag{8}$$

Proof. Condition (7) means that, for every $\delta' > \delta$, there is some $\varepsilon > 0$ such that

$$\|x\| \leq \varepsilon \Rightarrow N(x) \leq \frac{\delta'}{2} \left\| x \right\|^2 \ . \tag{9}$$

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an $\eta > 0$ such that

$$f \left\| x \right\| \leq \eta \Rightarrow N^*(y) \leq \frac{1}{2\delta'} \left\| y \right\|^2 \ . \tag{10}$$

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Since u_1 is a smooth function, we will have $\|hu_1\|_\infty \leq \eta$ for h small enough, and inequality (10) will hold, yielding thereby:

$$\psi(hu_1) \leq \frac{h^2}{2}\frac{1}{\lambda}\|u_1\|_2^2 + \frac{h^2}{2}\frac{1}{\delta'}\|u_1\|^2 \ . \tag{11}$$

If we choose δ' close enough to δ, the quantity $\left(\frac{1}{\lambda} + \frac{1}{\delta'}\right)$ will be negative, and we end up with

$$\psi(hu_1) < 0 \qquad \text{for} \ \ h \neq 0 \ \ \text{small} \ . \tag{12}$$

On the other hand, we check directly that $\psi(0) = 0$. This shows that 0 cannot be a minimizer of ψ, not even a local one. So $\overline{u} \neq 0$ and $\overline{u} \neq \Lambda_o^{-1}(0) = 0$. □

Corollary 1. *Assume H is C^2 and (a_∞, b_∞)-subquadratic at infinity. Let $\xi_1, \ldots, \xi_N$ be the equilibria, that is, the solutions of $H'(\xi) = 0$. Denote by ω_k the smallest eigenvalue of $H''(\xi_k)$, and set:*

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If:

$$\frac{T}{2\pi} b_\infty < -E\left[-\frac{T}{2\pi}a_\infty\right] < \frac{T}{2\pi}\omega \tag{14}$$

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Proof. The spectrum of Λ is $\frac{2\pi}{T}\mathbb{Z} + a_\infty$. The largest negative eigenvalue λ is given by $\frac{2\pi}{T}k_o + a_\infty$, where

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Hence:

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The condition $\gamma < -\lambda < \delta$ now becomes:

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But this index is precisely the index $i_T(\widetilde{x})$ of the T-periodic solution $\widetilde{x}$ over the interval $(0, T)$, as defined in Sect. 2.6. So

$$i_T(\widetilde{x}) = 0 \; . \tag{21}$$

Now if $\widetilde{x}$ has a lower period, T/k say, we would have, by Corollary 31:

$$i_T(\widetilde{x}) = i_{kT/k}(\widetilde{x}) \geq k i_{T/k}(\widetilde{x}) + k - 1 \geq k - 1 \geq 1 \; . \tag{22}$$

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Notes and Comments. The results in this section are a refined version of [1]; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family x_T, $T \in \left(2\pi\omega^{-1}, 2\pi b_{\infty}^{-1}\right)$ of periodic solutions, $x_T(0) = x_T(T)$, with x_T going away to infinity when $T \to 2\pi\omega^{-1}$, which is the period of the linearized system at 0.

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where the Hamiltonian H is $(0, b_\infty)$-subquadratic, and the forcing term is a distribution on the circle:

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Hamiltonian Mechanics2

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Abstract. The abstract should summarize the contents of the paper using at least 70 and at most 150 words. It will be set in 9-point font size and be inset 1.0 cm from the right and left margins. There will be two blank lines before and after the Abstract. . . .

Keywords: graph transformations, convex geometry, lattice computations, convex polygons, triangulations, discrete geometry

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