Test # 2 Date: Mar 27 2020 Student Number: 20169913

Question # 1

a)
$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 14 \\ -2 & 3 & 8 & 10 \\ 4 & 1 & -2 & 8 \end{array}\right) \xrightarrow[\mapsto R2]{R2-R1} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 14 \\ -4 & 0 & 4 & -4 \\ 4 & 1 & -2 & 8 \end{array}\right]$$

$$\xrightarrow[\mapsto R2]{R2+R3} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 14 \\ 0 & 1 & 2 & 4 \\ 4 & 1 & -2 & 8 \end{array}\right] \xrightarrow[\mapsto R1]{R1-R2} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 10 \\ 0 & 1 & 2 & 4 \\ 4 & 1 & -2 & 8 \end{array}\right] \xrightarrow[\mapsto R1]{R1/2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 4 \\ 4 & 1 & -2 & 8 \end{array}\right]$$

$$\xrightarrow[\mapsto R3]{R3-4R1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & -6 & -12 \end{array}\right] \xrightarrow[\mapsto R3]{R3+3R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\mapsto R1]{R1-R2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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b) The system outlined is that represented by the matrix in part A, so the RREF can be used.

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow$$

x_1, x_2 are dependent
x_3 is independent.
$\therefore x_3 = t$

$x_1 - t = 1 \Rightarrow x_1 = 1 + t$
$x_2 + 2t = 4 \Rightarrow x_2 = 4 - 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

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Question # 2

a) let $\alpha_1, \alpha_2 \in \mathbb{R}$

$(1,0) = \alpha_1(-2,4) + \alpha_2(3,2)$

$= (-2\alpha_1 + 3\alpha_2, 4\alpha_1 + 2\alpha_2)$

$\begin{cases} 1 = -2\alpha_1 + 3\alpha_2 & (1) \\ 0 = 4\alpha_1 + 2\alpha_2 \Rightarrow \alpha_2 = -2\alpha_1 & (2) \end{cases}$

(2) into (1)

$1 = -2\alpha_1 + 3(-2\alpha_1)$ $\qquad \alpha_2 = -2\left(\frac{-1}{8}\right)$

$1 = -2\alpha_1 - 6\alpha_1$ $\qquad = 1/4$

$\alpha_1 = \frac{-1}{8}$

let $\beta_1, \beta_2 \in \mathbb{R}$

$(0,1) = \beta_1(-2,4) + \beta_2(3,2)$

$= (-2\beta_1 + 3\beta_2, 4\beta_1 + 2\beta_2)$

$\begin{cases} 0 = -2\beta_1 + 3\beta_2 \Rightarrow \beta_2 = \frac{2}{3}\beta_1 & (1) \\ 1 = 4\beta_1 + 2\beta_2 & (2) \end{cases}$

(1) into (2) $\qquad \beta_2 = \frac{2}{3}\left(\frac{3}{16}\right)$

$1 = 4\beta_1 + 2\left(\frac{2}{3}\beta_1\right)$ $\qquad = \frac{1}{8}$

$1 = \frac{12}{3}\beta_1 + \frac{4}{3}\beta_1$

$1 = \frac{16}{3}\beta_1 \Rightarrow \beta_1 = \frac{3}{16}$

Thus

$(1,0) = -\frac{1}{8}(-2,4) + \frac{1}{4}(3,2)$

$(0,1) = \frac{3}{16}(-2,4) + \frac{1}{8}(3,2)$

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Question #2 (cont.)

b) $(1,0) = \frac{-1}{8}(-2,4) + \frac{1}{4}(3,2)$

$$L(1,0) = L\left(-\tfrac{1}{8}(-2,4) + \tfrac{1}{4}(3,2)\right)$$ Because L is linear

$$= L\left(-\tfrac{1}{8}(-2,4)\right) + L\left(\tfrac{1}{4}(3,2)\right)$$

$$= -\tfrac{1}{8}L((-2,4)) + \tfrac{1}{4}L((3,2))$$

$$= -\tfrac{1}{8}(-2,8) + \tfrac{1}{4}(-13,28)$$

$$= \left(\tfrac{1}{4}, -1\right) + \left(\tfrac{-13}{4}, 7\right)$$

$$L(0,1) = (-3, 6)$$

$(0,1) = \frac{3}{16}(-2,4) + \frac{1}{8}(3,2)$

$$L(0,1) = L\left(\tfrac{3}{16}(-2,4) + \tfrac{1}{8}(3,2)\right)$$ L is linear

$$= \tfrac{3}{16}L(-2,4) + \tfrac{1}{8}L(3,2)$$

$$= \tfrac{3}{16}(-2,8) + \tfrac{1}{8}(-13,28)$$

$$= \left(\tfrac{-3}{8}, \tfrac{3}{2}\right) + \left(\tfrac{-13}{8}, \tfrac{7}{2}\right)$$

$$L(0,1) = (-2, 5)$$

Standard matrix $= \begin{bmatrix} L(1,0) & L(0,1) \end{bmatrix}$

$$= \begin{bmatrix} -3 & -2 \\ 6 & 5 \end{bmatrix}$$

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c) $$L(1,2) = \begin{bmatrix} -3 & -2 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 1\begin{bmatrix} -3 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

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Question # 3

a) $\{v_1, v_2, \ldots, v_p\}$ is generating for V

$\dim(V) = N$

→ From the key lemma, we know that:

vectors in generating set ≥ lin independent set.

→ We also know that a basis is a linearly independent generating set, and that $\dim(V)$ is the # of vectors in a basis.

→ This implies that:

in generating ≥ # in lin indep = # in basis = $\dim(V)$

$\therefore\ p \geq \dim(V)$ ∴ The statement is true

$\geq N$

↳ This can be illustrated using the standard basis for $\mathbb{R}^3$

$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = (\alpha_1, \alpha_2, \alpha_3)$

↳ $p = N$ $p = N-1$ → $\beta_1(0,1,0) + \beta_3(0,0,1) = (0, \beta_1, \beta_2)$

↳ x axis lost

b) $\{w_1, w_2, \ldots, w_q\}$ is lin indep in V

$\dim(V) = N$

→ According to the key lemma, # in a generating set ≥ # in a lin independent set

→ From A, we know # in generating ≥ $\dim(V)$.

→ This implies that

lin indep ≤ # generating ≥ $\dim(V)$

∴ # lin indep ≤ $\dim(V)$

$q \leq N$ ∴ The statement is true.

↳ This can be graphically proven by considering a 3D grid. As each vector in a lin idep set adds another axis of freedom, after 3 vectors there are no remaining axes.

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Question # 3 (cont)

c) $\dim(V) = N$

A generating set is defined as a set whose span contains all vectors in V.

Therefore:

let $w, v_1, v_2, \ldots, v_p \in V$

$\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$

Basis for W.

$$W = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_N v_N$$

$p = N+1$, added to basis to form generating set.

$$= \underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_N v_N}_{\text{W can be written as a combo of these vectors}} + \alpha_{N+1} v_{N+1}$$

$$W = W + \alpha_{N+1} v_{N+1}$$

$$= W + 0\, v_{N+1}$$

$$W = W \checkmark$$

v_{N+1} can be anything, such that α_{N+1} is zero.

As dimonstrated, a generating set can have more vectors than a basis, as it does not need to be linearly independent.

$\therefore$ The statement is true.

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Question # 4 $W_3 = \{(x,y,z) \in \mathbb{R}^3 : x>0, y>0, z>0\}$

$L: W_3 \to W_3 \Rightarrow L(x,y,z) = (xy, yz, xz)$

a) Addition test

$v_1, v_2 \in W_3$

$$L(v_1+v_2) = L(v_1) + L(v_2)$$
$$= L((x_1,y_1,z_1)) + L(x_2,y_2,z_2)$$
$$= (x_1y_1, y_1z_1, x_1z_1) + (x_2y_2, y_2z_2, x_2z_2)$$
$$= (x_1y_1x_2y_2, y_1z_1y_2z_2, x_1z_1x_2z_2)$$
$$= \underbrace{(x_1x_2y_1y_2, y_1y_2z_1z_2, x_1x_2z_1z_2)}$$
$$= L(x_1x_2, y_1y_2, z_1z_2)$$
$$= L((x_1,y_1,z_1) + (x_2,y_2,z_2)) \checkmark$$

Multiplication test

$v = (x,y,z) \in W_3$, $\alpha \in \mathbb{R}$

$$\alpha L(v) = L(\alpha v)$$
$$= L(\alpha(x,y,z))$$
$$= L(x^\alpha, y^\alpha, z^\alpha)$$
$$= (x^\alpha y^\alpha, y^\alpha z^\alpha, x^\alpha z^\alpha)$$
$$= ((xy)^\alpha, (yz)^\alpha, (xz)^\alpha)$$
$$= \alpha \underbrace{(xy, yz, xz)}$$
$$= \alpha L(x,y,z) \checkmark$$

Both tests passed, L is a linear transformation.

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Question # 4 (cont)

b) let $\bar{0} = (1,1,1) \Rightarrow (x,y,z)+(1,1,1) = (1x, 1y, 1z)$ ✓

$\alpha \cdot (1,1,1) = (1^\alpha, 1^\alpha, 1^\alpha)$
$= (1,1,1)$ ✓

Ker (L) is all vectors where $L(v) = \bar{0}$

let $v = (x,y,z) \in W_3$

$L(x,y,z) = (1,1,1)$

$(xy, yz, xz) = (1,1,1)$

$\Rightarrow \begin{cases} xy = 1 \Rightarrow y = \frac{1}{x} \\ yz = 1 \Rightarrow z = \frac{1}{y} \\ xz = 1 \Rightarrow z = \frac{1}{x} \end{cases}$

$z = \frac{1}{y} = \frac{1}{x}$ only satisfied if $y = x$

$x = \frac{1}{x}$ only satisfied if $x = \pm 1$ — negative solution rejected due to conditions on W_3 that remove negative values

$\therefore x = y = z = 1$

$Ker(L) = \{(1,1,1)\}$
$= \{\bar{0}\}$

c) let $v_1, v_2 \in W_3$

for injectivity: $L(v_1) = L(v_2)$, if and only if $v_1 = v_2$

$L(x_1, y_1, z_1) = L(x_2, y_2, z_2)$

$(x_1y_1, y_1z_1, x_1z_1) = (x_2y_2, y_2z_2, x_2z_2)$

$\Downarrow$

$\begin{cases} x_1y_1 = x_2y_2 \\ y_1z_1 = y_2z_2 \\ x_1z_1 = x_2z_2 \end{cases}$

$y_1 = \frac{x_2y_2}{x_1}$

$y_1 = \frac{y_2z_2}{z_1}$

$\frac{x_2y_2}{x_1} = \frac{y_2z_2}{z_1}$

$z_1x_2\cancel{y_2} = x_1\cancel{y_2}z_2$

$z_1x_2 = x_1z_2 \Rightarrow z_1 = \frac{x_1z_2}{x_2}$

$x_1z_1 = x_2z_2$

$x_1\left(\frac{x_1z_2}{x_2}\right) = x_2z_2$

$x_1^2 = x_2^2$

$\pm\sqrt{x_1^2} = \pm\sqrt{x_2^2}$

$x_1 = \pm x_2$ — W_3 does not have negative values so negative solution rejected

$\boxed{\begin{array}{l} \therefore v_1 = v_2 \\ \therefore L \text{ is injective} \end{array}}$

d) $\dim(ker(L)) = 1$
(see b)

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Question # 5

$L: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

$L(f) = f'' - f$

$\frac{d}{dx} e^{7x} = 7e^{7x}$

$\frac{d^2}{dx^2} e^{7x} = 49e^{7x}$

a) $L(e^{7x}) = \frac{d^2}{dx^2} e^{7x} - e^{7x}$

$= 49e^{7x} - e^{7x} = 48e^{7x}$

b) $L(e^{7x} - 3e^{x}) = \frac{d^2}{dx^2}(e^{7x} - 3e^{x}) - (e^{7x} - 3e^{x})$

$= 49e^{7x} - 3e^{x} - e^{7x} - 3e^{x}$

$= 48e^{7x} - 6e^{x}$

$\frac{d}{dx}(e^{7x} - 3e^{x}) = 7e^{7x} - 3e^{x}$

$\frac{d^2}{dx^2}(7e^{7x} - 3e^{x}) = 49e^{7x} - 3e^{x}$

c) L is not injective, as demonstrated by the functions

$f_0(x) = 0$ and $g(x) = e^{x}$

$L(f_0(x)) = \frac{d^2}{dx^2} 0 - 0$

$= 0 - 0$

$= 0$

$L(g(x)) = \frac{d^2}{dx^2} e^{x} - e^{x}$

$= e^{x} - e^{x}$

$= 0$

Same output, diff. input. ∴ not injective

d) $\text{Dim}(\text{Ker}(L)) > 0$, as demonstrated by the functions

$f_0(x) = 0$ and $f(x) = e^{x}$

$L(f_0(x)) = \frac{d^2}{dx^2} 0 - 0$

$= 0 - 0$

$= 0$

$L(f(x)) = \frac{d^2}{dx^2} e^{x} - e^{x}$

$= e^{x} - e^{x}$

$= 0$

multiple elements in $\text{Ker}(L)$

∴ $\dim(\text{Ker}(L)) > 0$