Vector Functions and Space Curves

Exr-1 : Find the parametric equation of a tangent line of the circular helix at $t = \dfrac{\pi}{4}$.

$$x = \cos t, \qquad y = \sin t, \qquad z = t$$

Solution:

Let, $\underline{\mathbf{r}}(t) \;=\; x\,\underline{\mathrm{i}} + y\,\underline{\mathrm{j}} + z\,\underline{\mathrm{k}} \;=\; \cos t\,\underline{\mathrm{i}} + \sin t\,\underline{\mathrm{j}} + t\,\underline{\mathrm{k}}$

$$\therefore \underline{\mathbf{r}}'(t) \;=\; -\sin\,\underline{\mathrm{i}} + \cos\,\underline{\mathrm{j}} + 1\,\underline{\mathrm{k}}$$

Now we have,

$$\therefore \underline{\mathbf{r}}(t_o) \;=\; \underline{\mathbf{r}}\left(\frac{\pi}{4}\right) \;=\; \frac{1}{\sqrt{2}}\,\underline{\mathrm{i}} + \frac{1}{\sqrt{2}}\,\underline{\mathrm{j}} + \frac{\pi}{4}\,\underline{\mathrm{k}}$$

$$\therefore \underline{\mathbf{r}}'(t_o) \;=\; \underline{\mathbf{r}}'\left(\frac{\pi}{4}\right) \;=\; -\frac{1}{\sqrt{2}}\,\underline{\mathrm{i}} + \frac{1}{\sqrt{2}}\,\underline{\mathrm{j}} + 1\,\underline{\mathrm{k}}$$

Equation of tangent line is,

$$\begin{aligned}
\underline{\mathbf{r}}(t) \;&=\; \underline{\mathbf{r}}(t_o) + t\,\underline{\mathbf{r}}'(t_o) \\
&=\; \left(\frac{1}{\sqrt{2}}\,\underline{\mathrm{i}} + \frac{1}{\sqrt{2}}\,\underline{\mathrm{j}} + \frac{\pi}{4}\,\underline{\mathrm{k}}\right) + t\left(-\frac{1}{\sqrt{2}}\,\underline{\mathrm{i}} + \frac{1}{\sqrt{2}}\,\underline{\mathrm{j}} + 1\,\underline{\mathrm{k}}\right) \\
&=\; \left(\frac{1}{\sqrt{2}} - \frac{t}{\sqrt{2}}\right)\underline{\mathrm{i}} + \left(\frac{1}{\sqrt{2}} + \frac{t}{\sqrt{2}}\right)\underline{\mathrm{j}} + \left(\frac{\pi}{4} + t\right)\underline{\mathrm{k}}
\end{aligned}$$

The parametric equations are,

$$\underline{\mathrm{x}}(t) \;=\; \frac{1}{\sqrt{2}} - \frac{t}{\sqrt{2}}$$

$$\underline{\mathrm{y}}(t) \;=\; \frac{1}{\sqrt{2}} + \frac{t}{\sqrt{2}}$$

$$\underline{\mathrm{z}}(t) \;=\; \frac{\pi}{4} + t$$

Exr-2 : Find the curvature for the circular helix

$$x = a\cos t, \qquad y = a\sin t, \qquad z = ct$$

Solution:

$$\text{Let, } \underline{\mathbf{r}} = (x,\ y,\ z) = (a\cos t,\ a\sin t,\ ct)$$

$$\therefore \underline{\mathbf{r}}' = (-a\sin t,\ a\cos t,\ c)$$

$$\therefore \underline{\mathbf{r}}'' = (-a\cos t,\ -a\sin t,\ 0)$$

$$\therefore \left\| \underline{\mathbf{r}}' \right\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2} = \sqrt{a^2 + c^2}$$

$$\therefore \underline{\mathbf{r}}' \times \underline{\mathbf{r}}'' = \begin{vmatrix} \underline{\mathrm{i}} & \underline{\mathrm{j}} & \underline{\mathrm{k}} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= (ac\sin t)\,\underline{\mathrm{i}} + (ac\cos t)\,\underline{\mathrm{j}} + (a^2\sin^2 t + a^2\cos^2 t)\,\underline{\mathrm{k}}$$

$$= (ac\sin t)\,\underline{\mathrm{i}} + (ac\cos t)\,\underline{\mathrm{j}} + a^2\,\underline{\mathrm{k}}$$

$$\therefore \left\| \underline{\mathbf{r}}' \times \underline{\mathbf{r}}'' \right\| = \sqrt{a^2c^2\sin^2 t + a^2c^2\cos^2 t + a^4} = a\sqrt{a^2 + c^2}$$

$$\underline{\kappa}(t) = \frac{\left\| \underline{\mathbf{r}}' \times \underline{\mathbf{r}}'' \right\|}{\left\| \underline{\mathbf{r}}' \right\|^3} = \frac{a\sqrt{a^2 + c^2}}{\left(\sqrt{a^2 + c^2}\right)^3} = \frac{a}{a^2 + c^2}$$

Exr-3 : The graph of the ellipse is given by

$$\underline{\mathbf{r}} = 2\cos t\,\underline{\mathrm{i}} + 3\sin t\,\underline{\mathrm{j}} \qquad 0 \le t \le 2\pi$$

Find the curvature of the ellipse at the end point of the major and minor axes.

Solution:

$$\text{Given, } \underline{\mathbf{r}} = (2\cos t)\,\underline{\mathrm{i}} + (3\sin t)\,\underline{\mathrm{j}}$$

$$\underline{\mathbf{r}}' = (-2\sin t)\,\underline{\mathrm{i}} + (3\cos t)\,\underline{\mathrm{j}}$$

$$\underline{\mathbf{r}}'' = (-2\cos t)\,\underline{\mathrm{i}} + (-3\sin t)\,\underline{\mathrm{j}}$$

$$\|\underline{\mathbf{r}}'\| = \sqrt{4\sin^2 t + 9\cos^2 t}$$

$$\therefore \underline{\mathbf{r}}' \times \underline{\mathbf{r}}'' = \begin{vmatrix} \underline{\mathrm{i}} & \underline{\mathrm{j}} & \underline{\mathrm{k}} \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix}$$

$$= i(0) - j(0) + \underline{\mathrm{k}}\left(6\sin^2 t + 6\cos^2 t\right) = 6\underline{\mathrm{k}}$$

$$\therefore \|\underline{\mathbf{r}}' \times \underline{\mathbf{r}}''\| = \sqrt{6^2} = 6$$

$$\underline{\kappa}(t) = \frac{\|\underline{\mathbf{r}}' \times \underline{\mathbf{r}}''\|}{\|\underline{\mathbf{r}}'\|^3} = \frac{6}{\left(\sqrt{4\sin^2 t + 9\cos^2 t}\right)^3} \qquad \cdots\cdots\cdots (1)$$

The end point of the minor axis are $(2, 0)$ and $(-2, 0)$ corresponds to, $t = 0$ and $t = \pi$

Now put $t = 0$ and $t = \pi$ in (1)

$$\Rightarrow \kappa(0) = \frac{6}{(\sqrt{0+9})^3} = \frac{6}{27} = \frac{2}{9} \quad \text{and} \quad \kappa(\pi) = \frac{2}{9}$$

Similarly the end point of the major anis are $(0, 3)$ and $(0, -3)$ corresponds to $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$

Now put $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ in (1),

$$\Rightarrow \kappa\left(\frac{\pi}{2}\right) = \frac{6}{(\sqrt{4+0})^3} = \frac{6}{8} = \frac{3}{4} \quad \text{and} \quad \kappa\left(\frac{3\pi}{2}\right) = \frac{3}{4}$$

Exr-4 : Find the arc length of the circular helix

$$x = a\cos t, \qquad y = a\sin t, \qquad z = ct \qquad \text{where} \quad 0 \le t \le \pi$$

Solution:

Let, $\underline{\mathbf{r}} = (x,\ y,\ z) = (a\cos t,\ a\sin t,\ ct)$

$$\therefore \frac{d\mathbf{r}}{dt} = (-a\sin t,\ a\cos t,\ c)$$

$$\therefore \left\| \frac{d\mathbf{r}}{dt} \right\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2} = \sqrt{a^2 + c^2}$$

$$\therefore L = \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_0^\pi \sqrt{a^2 + c^2}\, dt = \sqrt{a^2 + c^2} \int_0^\pi 1\, dt = \sqrt{a^2 + c^2}\, [\, t \,]_0^\pi = \pi\sqrt{a^2 + c^2}$$

Exr-5 : Find the arc length of the curve

$$\underline{\mathbf{r}}(t) = e^t\cos t\,\underline{\mathrm{i}} + e^t\sin t\,\underline{\mathrm{j}} + e^t\,\underline{\mathrm{k}} \qquad \text{where} \quad 0 \le t \le \frac{\pi}{2}$$

Solution:

Let, $\underline{\mathbf{r}} = (x,\ y,\ z) = (e^t\cos t,\ e^t\sin t,\ e^t)$

$$\therefore \frac{d\mathbf{r}}{dt} = (e^t\cos t - e^t\sin t)\,\underline{\mathrm{i}} + (e^t\sin t + e^t\cos t)\,\underline{\mathrm{j}} + (e^t)\,\underline{\mathrm{k}}$$

$$\therefore \left\| \frac{d\mathbf{r}}{dt} \right\| = \sqrt{e^{2t}\left(\cos^2 t + \sin^2 t - 2\sin t\cos t\right) + e^{2t}\left(\sin^2 t + \cos^2 t + 2\sin t\cos t\right) + e^{2t}}$$

$$= \sqrt{e^{2t}\left(\cos^2 t + \sin^2 t - 2\sin t\cos t + \sin^2 t + \cos^2 t + 2\sin t\cos t + 1\right)}$$

$$= \sqrt{e^{2t}\,(1 + 1 + 1)} = \sqrt{3}e^t$$

$$\therefore L = \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_0^{\frac{\pi}{2}} \sqrt{3}e^t\, dt = \sqrt{3}\int_0^{\frac{\pi}{2}} e^t\, dt = \sqrt{3}\,[\, e^t \,]_0^{\frac{\pi}{2}} = \sqrt{3}\left(e^{\frac{\pi}{2}} - 1\right)$$

Exr-1 : Suppose a particle moves through 3-space and its position vector at time t is

$$\underline{r}(t) = t\,\underline{i} + t^2\,\underline{j} + t^3\,\underline{k}$$

(a) find the scalar tangential and normal components of acceleration at time t

(b) find the scalar tangential and normal components of acceleration at time $t = 1$

(c) find the vector tangential and normal components of acceleration at time $t = 1$.

(d) find the curvature of the path at the point where the particle is located at time $t = 1$

Solution:

(a) Given $\underline{r}(t) = t\,\underline{i} + t^2\,\underline{j} + t^3\,\underline{k}$

$$\therefore \underline{v} = \frac{dr}{dt} = \underline{r}' = 1\,\underline{i} + 2t\,\underline{j} + 3t^2\,\underline{k}$$

$$\therefore \underline{a} = \underline{v}'(t) = 0\,\underline{i} + 2\underline{j} + 6t\,\underline{k}$$

Now,

$$\|\, v \,\| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\therefore \underline{v} \cdot \underline{a} = 0 + 4t + 18t^3$$

$$\therefore a_T = \frac{\underline{v} \cdot \underline{a}}{\|\, v \,\|} = -\frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}$$

$$\therefore \underline{v} \times a = \begin{vmatrix} \underline{i} & \underline{j} & \underline{j} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \underline{i}\left(12t^2 - 6t^2\right) - \underline{j}(6t - 0) + \underline{k}(2 - 0) = 6t^2\,\underline{i} - 6t\,\underline{j} + 2\,\underline{k}$$

$$\therefore a_N = \frac{\|v \times \underline{a}\|}{\|\underline{v}\|} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{1 + +4t^2 + 9t^4}}$$

(b) At $t = 1$, $a_T = \dfrac{4 + 18}{\sqrt{14}} = \dfrac{22}{\sqrt{14}}$

$$a_N = \frac{\sqrt{36 + 36 + 4}}{\sqrt{14}} = \frac{\sqrt{76}}{\sqrt{14}}$$

(c) Since $a_T \underline{T}$ is the vector tangential component of accelaration where $\underline{T} = \dfrac{\underline{v}}{\|\underline{v}\|}$

At $t = 1$, $\underline{T}(1) = \dfrac{v(1)}{\|v(1)\|} = \dfrac{1\,\underline{i} + 2t\,\underline{j} + 3t^2\,\underline{k}}{\sqrt{1+4+9}} = \dfrac{1}{\sqrt{14}}(\underline{i} + 2\dot{J} + 3\underline{k})$

$$\therefore A_T\underline{T}(1) = \frac{22}{\sqrt{14}}\underline{T}(1) = \frac{22}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}}(1\underline{i} + 2\dot{J} + 3\underline{k})$$

$$= \frac{11}{7}(i + 2\underline{i} + 3\underline{k}) = \frac{11}{7}\underline{i} + \frac{22}{7}\underline{j} + \frac{33}{7}\underline{k}$$

Now,

$$\underline{a} = a_T\underline{T} + a_N\underline{N}$$

$$a_N\underline{N} = \underline{a} - a_T\underline{T}$$

At $t = 1$, $a_N N(1) = \underline{a}(1) - a_T T(1)$

$$= (2\underline{j} + 6\underline{k}) - \left(\frac{11}{7}\underline{i} + \frac{22}{7}\underline{j} + \frac{33}{7}\underline{k}\right)$$

$$= -\frac{11}{7}\underline{i} - \frac{8}{7}\underline{j} + \frac{9}{7}\underline{k}$$

(d) At $t = 1$, $\|\underline{v} \times \underline{a}\| = \sqrt{36+36+4} = \sqrt{33}\sqrt{76} = 2\sqrt{19}$

$$\|v\| = \sqrt{1+4+9} = \sqrt{14}$$

$$\therefore \text{curvature } \kappa = \frac{\|\underline{v} \times \underline{a}\|}{\|\underline{v}\|^3} = \frac{2\sqrt{19}}{(\sqrt{14})^3}$$