Green's, Gauss's, Stoke's Theorem

Green's Theorem

Exr. State Green's theorem for a plane. Verify Green's theorem in the xy plane for $\oint_C \left(xy + x^2\right) dx + xy^2\, dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Green's Theorem: Let C be a simple closed curve in the xy plane such that a line parallel to either axis cuts C in at most two points. Let $M(x, y)$, $N(x, y)$, $\frac{\partial N}{\partial x}, \frac{\partial M}{\partial y}$ be continuous function of x and y inside and on C and R be the region inside C then, $\oint_c M(x, y)\, dx + N(x, y)\, dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

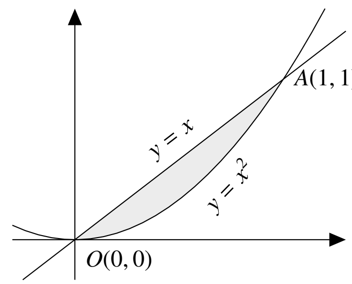
Solution:

$$L.H.S = \oint_C \left(xy + x^2\right) dx + xy^2\, dy$$

$$= \int_{\substack{OA \\ y=x^2 \\ dy=2x\,dx}} \left(x^3 + x^2\right) dx + x^5 2x\, dx + \int_{\substack{AO \\ y=x \\ dy=dx}} \left(x^2 + x^2\right) dx + x^3\, dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} + 2\cdot\frac{x^7}{7}\right]_0^1 + \left[2\frac{x^3}{3} + \frac{x^4}{4}\right]_1^0$$

$$= \frac{1}{4} + \frac{1}{3} + \frac{2}{7} - \frac{2}{3} - \frac{1}{4} = -\frac{1}{21}$$



Given, $y = x$ and $y = x^2$

Then, $x = x^2$

$\Rightarrow x - x^2 = 0$

$\Rightarrow x(1 - x) = 0$

$\therefore x = 0, 1$

$\therefore y = 0, 1$

$$R.H.S = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} \left(y^2 - x\right) dy\, dx \quad = \int_0^1 \left[\frac{y^3}{3} - xy\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^3}{3} - x^2 - \frac{x^6}{3} + x^3\right) dx \quad = \int_0^1 \left(\frac{4}{3}x^3 - x^2 - \frac{x^6}{3}\right) dx$$

$$= \left[\frac{4}{3}\frac{x^4}{4} - \frac{x^3}{3} - \frac{1}{3}\frac{x^7}{7}\right]_0^1 \quad = \frac{1}{3} - \frac{1}{3} - \frac{1}{21} - 0 \quad = -\frac{1}{21}$$

Here,

$M(x, y) = xy + x^2$

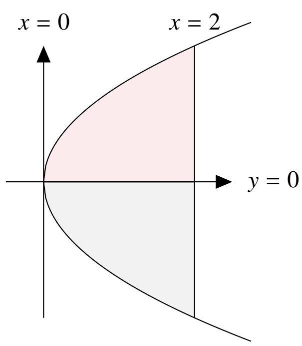
$N(x, y) = xy^2$

$\therefore \frac{\partial N}{\partial x} = y^2$

$\therefore \frac{\partial M}{\partial y} = x$

Since, $L.H.S = R.H.S$. Hence, Green's Theorem has been verified.

19(b) Using Green's theorem evaluate $\oint_C (x^2 - 2xy)\,dx + (x^2y + 3)\,dy$, where C is the closed curve of the region bounded by the $y^2 = 8x$ and $x = 2$.



$M(x, y) = x^2 - 2xy \qquad N(x, y) = xy^2 + 3$

$\therefore \dfrac{\partial N}{\partial x} = 2xy \qquad \therefore \dfrac{\partial M}{\partial y} = -2x$

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)\,dx\,dy \quad = 2 \int_{x=0}^{2} \int_{y=0}^{\sqrt{8x}} (2xy + 2x)\,dx\,dy$$

$$= 2 \int_{x=0}^{2} 2x \left[\frac{y^2}{2} + y\right]_0^{\sqrt{8x}} dx \quad = 2 \int_{x=0}^{2} 2x \left(\frac{8x}{2} + \sqrt{8x}\right) dx$$

$$= 2 \int_{x=0}^{2} \left(8x^2 + 2\sqrt{8}x^{\frac{3}{2}}\right) dx \quad = 2\left[8\frac{x^3}{3} + 2\sqrt{8}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$= 2\left(\frac{64}{3} + \frac{64}{5}\right) \quad = \frac{1024}{15}$$

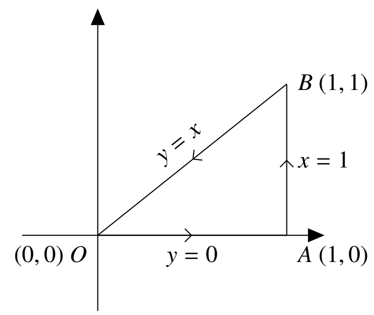
15(a) Using Green's theorem to evaluate $\oint_c x^2y\,dx + x^2\,dy$ where C is the boundary described counter clockwise of the C triangle with vertices $(0,0),\ (1,0),\ (1,1)$.

Solution:

Here,

$M = x^2y \qquad N = x^2$

$\dfrac{\partial M}{\partial Y} = x^2 \qquad \dfrac{\partial N}{\partial x} = 2x$



$$\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x} \left(2x - x^2\right) dy\,dx = \int_0^1 \left[2xy - x^2y\right]_0^x dx$$

$$= \int_0^1 \left(2x^2 - x^3\right) dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

Jacobian

6(a) Define Jacobian of two variables. If $x = \rho\sin\theta\cos\phi,\ y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\theta$ then show that $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\sin\theta$

Jacobian of two variables:

If u and v are functions of two independent variable x and y, then the determinant is known as Jacobian of u and v with respect to x and y

$$J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Solution:

By definition of Jacobian:

$$|J| = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix}$$

Given that,

$x = \rho\sin\theta\cos\phi, \quad y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\theta$

$$\frac{\partial x}{\partial\rho} = \sin\theta\cos\phi \qquad \frac{\partial x}{\partial\theta} = \rho\cos\theta\cos\phi \qquad \frac{\partial x}{\partial\phi} = -\rho\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial\rho} = \sin\theta\sin\phi \qquad \frac{\partial y}{\partial\theta} = \rho\cos\theta\sin\phi \qquad \frac{\partial y}{\partial\phi} = \rho\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial\rho} = \cos\theta \qquad \frac{\partial z}{\partial\theta} = -\rho\sin\theta \qquad \frac{\partial z}{\partial\phi} = 0$$

$$= \cos\theta\left[\rho^2\sin\theta\cos\theta\cos^2\phi + \rho^2\sin\theta\cos\theta\sin^2\phi\right] + \rho\sin\theta\left[\rho\sin^2\theta\cos^2\phi + \rho\sin^2\theta\sin^2\phi\right] + 0$$

$$= \cos\theta\left[\rho^2\sin\theta\cos\theta\left(\cos^2\phi + \sin^2\phi\right)\right] + \rho\sin\theta\left[\rho\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$

$$= \rho^2\sin\theta\cos^2\theta + \rho^2\sin^3\theta = \rho^2\sin\theta\left(\cos^2\theta + \sin^2\theta\right) = \rho^2\sin\theta$$

Jacobian of n variables:

If $u_1,\ u_2, \ldots u_n$ are n functions of n variables $x_1,\ x_2,\ \ldots x_n$.

Then the determinant,

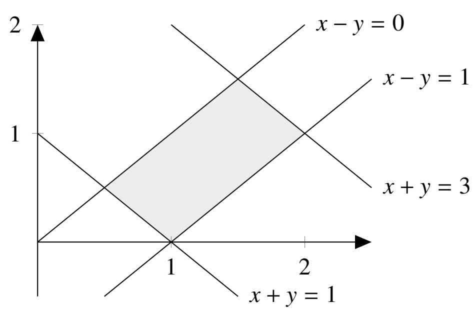
$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1}, & \dfrac{\partial u_1}{\partial x_2}, & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1}, & \dfrac{\partial u_2}{\partial x_2}, & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \dfrac{\partial u_3}{\partial x_1}, & \dfrac{\partial u_3}{\partial x_2}, & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1,\ u_2,\ \ldots u_n$ with respect to $x_1,\ x_2\ \ldots x_n$ and denoted by $J(u_1, u_2, \ldots u_n)$ or $\dfrac{\partial(u_1, u_2 \ldots u_n)}{\partial(x_1, x_2, \cdots x_n)}$

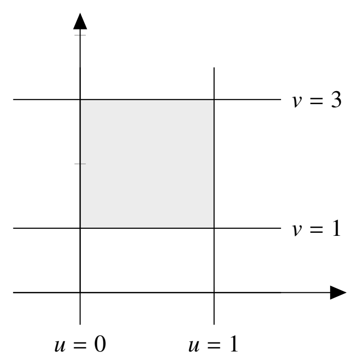
14. Evaluate the $\displaystyle\iint_R \frac{x-y}{x+y}\, dA$ where R is the region enclosed by the lines $x-y=0,\ x-y=1,\ x+y=1,\ x+y=3.$

Solution:

Let, $u = x - y$ and $v = x + y$ then xy plane corresponds to the uv plane. $u = 0,\ u = 1,\ v = 1,\ v = 3$



xy plane



uv plane

$$\therefore \iint_R \frac{x-y}{x+y} dA = \int_{u=0}^{1}\int_{v=1}^{3} \frac{u}{v}\,|J|\; dv\, du = \int_{u=0}^{1}\int_{v=1}^{3} \frac{u}{v}\frac{1}{2}\, dv\, du$$

$$= \frac{1}{2}\int_{u=0}^{1}\int_{v=1}^{3} \frac{1}{v}\, dv \cdot u\, du = \frac{1}{2}\int_{u=0}^{1} [\ln v]_1^3\, u\, du = \frac{1}{2}\int_0^1 (\ln 3 - \ln 1) u\, du$$

$$= \frac{1}{2}(\ln 3 - 0)\left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}\ln 3\left(\frac{1}{2} - 0\right) = \frac{1}{4}\ln 3$$

$$u = x - y \qquad v = x + y$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 1$$

$$|J'| = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2$$

$$|J| = \frac{1}{|J'|} = \frac{1}{2}$$

16(b) Evaluate the $\iint_R e^{xy}\,dA$ where R is the region enclosed by the lines $y = \frac{1}{2}x,\ \ y = x$ and the hyperbolas $y = \frac{1}{x}$ and $y = \frac{2}{x}$.

Solution:

$$y = \frac{1}{2}x \quad \Rightarrow \frac{y}{x} = \frac{1}{2}$$

$$y = x \quad \Rightarrow \frac{y}{x} = 1$$

Also,

$$y = \frac{1}{x} \quad \Rightarrow xy = 1$$

$$y = \frac{2}{x} \quad \Rightarrow xy = 2$$

Let $u = \frac{y}{x}$ and $v = xy$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2} \qquad \frac{\partial u}{\partial y} = \frac{1}{x}$$

$$\frac{\partial v}{\partial x} = y \qquad \frac{\partial v}{\partial y} = x$$

$$|J'| = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix}$$

$$= -\frac{y}{x} - \frac{y}{x} = -\frac{2y}{x} = -2u$$

$$\therefore |J| = \frac{1}{|J'|} = \frac{1}{-2u} \qquad [\because |J| \cdot |J'| = 1\,]$$

Now,

$$\iint_R e^{xy}\,dA = \int_{u=\frac{1}{2}}^{1}\int_{v=1}^{2} e^v\,|J|\,dv\,du = \int_{u=\frac{1}{2}}^{1}\int_{v=1}^{2} e^v\,\frac{-1}{2u}\,dv\,du = \int_{u=\frac{1}{2}}^{1} [e^v]_1^2\,\frac{-1}{2u}\,du$$

$$= -\frac{1}{2}\int_{\frac{1}{2}}^{1}\left(e^2 - e^1\right)\frac{1}{u}\,du = -\frac{1}{2}\left(e^2 - e\right)[\ln u]_{\frac{1}{2}}^{1}$$

$$= -\frac{1}{2}\left(e^2 - e\right)\left(\ln 1 - \ln\frac{1}{2}\right) = -\frac{1}{2}\left(e^2 - e\right)(\ln 1 - (\ln 1 - \ln 2))$$

$$= -\frac{1}{2}\left(e^2 - e\right)\ln 2$$

Gauss Divergence Theorem

Gauss Theorem: The surface integral of the normal component of a continuous diffentiable vector $\underline{F}$ taken over a closed surface S is equal to the integral of the divergence of $\underline{F}$ taken over the volume V enclosed by the surface.

$$\text{Mathematically} \iint_S (\underline{F}\cdot\underline{n})\, dS = \iiint_V (\underline{\nabla}\cdot\underline{F})\, dV$$

where n is the positive normal to s.

16(a) Use Divergence theorem to evaluate $\iint_S \overline{F}.\,\overline{n}\, dS$ where $\underline{F} = 4x\underline{i} - 2y^2\underline{j} + z^2\underline{k}$ and S is the surface bounded by the S region $x^2 + y^2 = 4,\ \ z = 0$ and $z = 3$.

Solution:

Here,

$$\underline{\nabla}\cdot\underline{F} = \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}\left(2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) \quad = 4 - 4y + 2z$$

Here,

$x^2 + y^2 = 4$

$\Rightarrow y = \pm\sqrt{4 - x^2}$

Also,

$x^2 = 4$

$\Rightarrow x = \pm 2$

$$\therefore \text{R.H.S} = \iiint_v \underline{\nabla}\cdot\underline{F}\, dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} 4 - 4y + 2z\ dz\,dy\,dx \quad = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + 2\frac{z^2}{2}\right]_0^3 dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9 - 0)\ dy\,dx \quad = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\ dy\,dx$$

$$= \int_{-2}^{2}\left[21y - 12\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \quad = \int_{-2}^{2}\left[21y - 6y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[21\left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) - 6\left(\left(\sqrt{4-x^2}\right)^2 - \left(-\sqrt{4-x^2}\right)^2\right)\right] dx$$

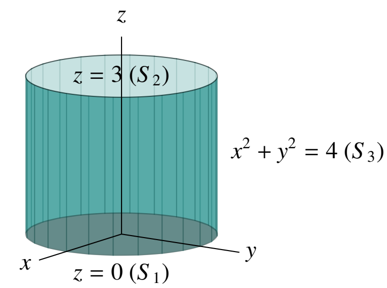
$$= \int_{-2}^{2}\left[21\left(2\sqrt{4-x^2}\right) - 6\left((4-x^2) - (4-x^2)\right)\right] dx$$

$$= \int_{-2}^{2}\left[21\left(2\sqrt{4-x^2}\right) - 6\left(4 - x^2 - 4 + x^2\right)\right] dx \quad = 42\int_{-2}^{2}\sqrt{4-x^2}\ dx$$

$$= 42\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2} \qquad \left[\because \int\sqrt{a^2-x^2} = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C\right]$$

$$= 42\left[0 + 2\sin^{-1}(1) - 0 - 2\sin^{-1}(-1)\right] \quad = 42\cdot\left[2\cdot\frac{\pi}{2} + 2\cdot\frac{\pi}{2}\right] \quad = 84\pi$$

$$\therefore \text{L.H.S} = \iint_S \underline{F}.\,\underline{n}\,dS = \iint_{\substack{S1\\(z=0)}} \underline{F}.\,\underline{n}\,dS_1 + \iint_{\substack{S2\\(z=3)}} \underline{F}.\,\underline{n}\,dS_2 + \iint_{\substack{S3\\(x^2+y^2=4)}} \underline{F}.\,\underline{n}\,dS_2$$



On S_1 : $z = 0 \quad n = \underline{k} \qquad \therefore \underline{F}\cdot\underline{n} = z^2 = 0 \qquad \therefore \iint_{S1} \underline{F}.\,\underline{n}\,dS_1 = 0$

On S_2 :

$z = 3 \quad n = \underline{k} \qquad \therefore \underline{F}\cdot\underline{n} = z^2 = 3^2 = 9$

$$\therefore \iint_{S2} \underline{F}.\,\underline{n}\,dS_2 = \iint_{S2} 9\,dS_2 = 9 \times 4\pi = 36\pi$$

Here, $x^2 + y^2 = 4 \qquad \therefore r = 2$

Area of $S_2 = \pi r^2 = \pi(2)^2 = 4\pi$

On S_3 : $x^2 + y^2 - 4 = 0$

$$\therefore \underline{F}.\,\underline{n} = \left(4x\underline{i} - 2y^2\underline{j} + z^2\underline{k}\right)\cdot\left(\frac{x}{2}\underline{i} + \frac{y}{2}\underline{j}\right)$$
$$= 2x^2 - y^3 + 0$$

$$\therefore \iint_{S3} \underline{F}.\,\underline{n}\,dS_3 = \iint_{S3} \left(2x^2 - y^3\right)\,dS_3$$

$$\underline{\nabla}(x^2 + y^2 - 4) = 2x\underline{i} + 2y\underline{j}$$

$$\therefore \underline{n} = \frac{2x\underline{i} + 2y\underline{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{2x\underline{i} + 2y\underline{j}}{\sqrt{4\,(x^2 + y^2)}}$$
$$= \frac{2x\underline{i} + 2y\underline{j}}{\sqrt{16}} = \frac{x}{2}\underline{i} + \frac{y}{2}\underline{j}$$

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{3} 2(2\cos\theta)^2 - (2\sin\theta)^3\ \ 2\,d\theta\,dz$$
$$= 2\int_0^{2\pi} [z]_0^3 \left(8\cos^2\theta - 8\sin^3\theta\right) d\theta$$
$$= 2\int_0^{2\pi} 3\left(8\cos^2\theta - 8\sin^3\theta\right) d\theta$$
$$= 6\int_0^{2\pi} 4\cdot 2\cos^2\theta\,d\theta \ - 6\int_0^{2\pi} 2\cdot 4\sin^3\theta\,d\theta$$
$$= 24\int_0^{2\pi} (1 + \cos 2\theta)d\theta - 12\int_0^{2\pi} (3\sin\theta - \sin 3\theta)d\theta$$
$$= 24\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} - 12\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{2\pi}$$
$$= 24[2\pi + 0 - 0] - 12\left[-3.1 + \frac{1}{3} + 3 - \frac{1}{3}\right] = 48\pi$$

Let,

$x = r\cos\theta = 2\cos\theta$

$y = r\sin\theta = 2\sin\theta$

$z = z$

$\therefore dS_3 = 2\,d\theta\,dz$

Formula,

$2\cos^2\theta = 1 + \cos 2\theta$

$4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$$\therefore \iint_S \underline{F}.\,\underline{n}\,dS = \iint_{S1} \underline{F}.\,\underline{n}\,dS_1 + \iint_{S2} \underline{F}.\,\underline{n}\,dS_2 + \iint_{S3} \underline{F}.\,\underline{n}\,dS_2$$
$$= 0 + 36\pi + 48\pi = 84\pi = \text{R.H.S (verified)}$$

Stoke's theorem

Stoke's theorem: If the components of a vector field $\underline{F}(x,y,z) = f(x,y,z)\underline{i} + g(x,y,z)\underline{j} + h(x,y,z)\underline{k}$ are continuous and have continuous first partial derivatives on some open set S and C be smooth closed curve then,

$$\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\underline{\nabla}\times\underline{F})\cdot\underline{n}\,dS$$

relationship between line and surface integrals a generalization of Green's theorem to three dimensions is called Stoke's theorem.

Exr. Verify stoke's theorem for the function $\underline{F} = x^2\underline{i} + xy\underline{j}$ taken round the square in the plane $z=0$ whose sides are along the lines $x=0,\ y=0$

Solution:

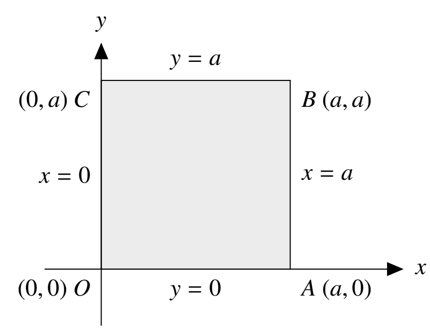
$$\text{L.H.S} = \oint_C \underline{F}\cdot d\underline{r} = \int_{OABCO} x^2\,dx + xy\,dy$$

$$= \int_{\substack{OA\\y=0\\dy=0}} x^2\,dx + \int_{\substack{AB\\x=a\\dx=0}} ay\,dy + \int_{\substack{BC\\y=a\\dy=0}} x^2\,dx + \int_{\substack{CO\\x=0\\dx=0}} 0$$

$$= \int_0^a x^2\,dx + \int_0^a ay\,dy + \int_a^0 x^2\,dx + \int 0$$

$$= \left[\frac{x^3}{3}\right]_0^a + \left[\frac{ay^2}{2}\right]_0^a + \left[\frac{x^3}{3}\right]_a^0 + 0$$

$$= \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} \qquad = \frac{a^3}{2}$$



Here,

$\underline{F} = x^2\underline{i} + xy\underline{j}$ and $\underline{r} = x\underline{i} + y\underline{j}$

$\therefore\ d\underline{r} = dx\underline{i} + dy\underline{j}$

$\therefore\ \underline{F}.\ d\underline{r} = x^2\,dx + xy\,dy$

$$\text{R.H.S} = \iint_R (\underline{\nabla}\times\underline{F})\cdot\underline{n}\,dS = \iint_S y\underline{k}\cdot\underline{k}\,dS = \int_{y=0}^a\int_{x=0}^a y\,dx\,dy$$

$$= \int_{y=0}^a [x]_0^a\,y\,dy = \int_{y=0}^a ay\,dy = a\left[\frac{y^2}{2}\right]_0^a$$

$$= a\cdot\frac{a^2}{2} = \frac{a^3}{2}$$

= L.H.S (verified)

$$\underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$= \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(y-0)$

$= y\underline{k}$

20(a) Using Stoke's theorem or otherwise evaluate $\oint_C \underline{F} \cdot d\underline{r}$ where $\underline{F} = (x^2 + y^2)\underline{i} - 2xy\underline{j}$ taken round the rectangle bounded by the lines $x = \pm a,\ y = 0,\ y = b$.

Solution:

$$\iint_R (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dS = \iint_S -4y\,\underline{k} \cdot \underline{k}\, dS$$

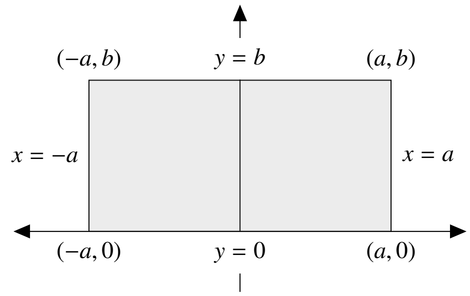
$$= \int_{y=0}^{b} \int_{x=-a}^{a} -4y\, dx\, dy$$

$$= -4 \int_{y=0}^{b} y\, dy \int_{x=-a}^{a} dx$$

$$= -4 \cdot \left[\frac{y^2}{2}\right]_0^b \cdot [x]_{-a}^{a}$$

$$= -4 \cdot \frac{b^2}{2} \cdot (a + a)$$

$$= -4 \cdot \frac{b^2}{2} \cdot 2a \quad = -4ab^2$$



$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \underline{i}(0 - 0) - \underline{j}(0 - 0) + \underline{k}(-2y - 2y)$$

$$= -4y\,\underline{k}$$

Coordinate System

		Cartesian – Polar	Rectangular – Cylindrical	Rectangular – Spherical
Cartesian	: (x, y)	$x = r\cos\theta$	$x = \rho\cos\phi$	$x = r\sin\theta\cos\phi$
Polar	: (r, θ)	$y = r\sin\theta$	$y = \rho\sin\phi$	$y = r\sin\theta\sin\phi$
Rectangular	: (x, y, z)		$z = z$	$z = r\cos\theta$
Cylindrical	: (ρ, ϕ, z)			
Spherical	: (r, θ, ϕ)	$\lvert J\rvert = r\,d\theta\,dz$	$\lvert J\rvert = \rho\;d\rho\,d\phi\,dz$	$\lvert J\rvert = r^2\sin\theta\,dr\,d\theta\,d\phi$

Exr. Use triple integration in cylindrical coordinates to find the volume of the solid G that is bounded above by the hemisphere $Z = \sqrt{25 - x^2 - y^2}$, below by xy-plane and laterally by the cylinder $x^2 + y^2 = 9$

Solution:

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

$x^2 + y^2 = 9$

$\Rightarrow \rho^2(\cos^2\phi + \sin^2\phi) = 9$

$\Rightarrow \rho = \pm 3$

Here, the upper surface $z = \sqrt{25 - x^2 - y^2}$
$= \sqrt{25 - \rho^2}$

the lower surface $z = 0$

$$dV = dx\,dy\,dz = |J|\;d\rho\,d\phi\,dz = \rho\;d\rho\,d\phi\,dz$$

$$|J| = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\left(\cos^2\phi + \sin^2\phi\right) = \rho$$

$$\text{Volume } v = \iiint_G dV = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{3}\int_{z=0}^{\sqrt{25-\rho^2}} \rho\;dz\,d\rho\,d\phi$$

$$= \int_0^{2\pi}\int_0^3 [z]_0^{\sqrt{25-\rho^2}}\;\rho\,d\rho\,d\phi$$

$$= \int_0^{2\pi}\int_0^3 \sqrt{25-\rho^2}\,\rho\,d\rho\,d\phi$$

$$= \int_0^{2\pi}\int_{25}^{16} \sqrt{y}\,\frac{dy}{-2}\,d\phi \quad = \int_{\phi=0}^{2\pi} d\phi \int_{y=25}^{16} \sqrt{y}\,\frac{dy}{-2}$$

$$= \int_{\phi=0}^{2\pi} d\phi \cdot -\frac{1}{2}\int_{y=25}^{16} y^{\frac{1}{2}}\,dy \quad = [\phi]_0^{2\pi} \cdot -\frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_{25}^{16}$$

$$= [\phi]_0^{2\pi} \cdot -\frac{1}{2}\cdot\frac{2}{3}\left[y^{\frac{3}{2}}\right]_{25}^{16} \quad = 2\pi \cdot -\frac{1}{3}\left(16^{\frac{3}{2}} - 25^{\frac{3}{2}}\right)$$

$$= 2\pi \cdot -\frac{1}{3}\cdot -61 \quad = \frac{122\pi}{3}$$

Here,

$$dV = dx\,dy\,dz = |J|\;d\rho\,d\phi\,dz = \rho\;d\rho\,d\phi\,dz$$

Let,

$25 - \rho^2 = y$

$\Rightarrow 0 - 2\rho\,d\rho = dy$

$\therefore \rho\,d\rho = \dfrac{dy}{-2}$

$\rho \rightarrow 0$ to 3

$y \rightarrow 25$ to 16

Exr. Use cylindrical coordinates to evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} x^2\,dz\,dy\,dx$

Solution:

Here, $z=\sqrt{4-x^2-y^2}$	$x^2+y^2=4$	$x=\rho\cos\phi$	$dV = dx\,dy\,dz$
$\Rightarrow x^2+y^2+z^2=4$	$\Rightarrow \rho^2=4$	$y=\rho\sin\phi$	$= \lvert J\rvert\ d\rho\,d\phi\,dz$
Sphere	$\Rightarrow \rho=\pm 2$	$z=z$	$=\rho\ d\rho\,d\phi\,dz$

Volume $V = \iiint\limits_G x^2\,dV$

$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{2}\int_{z=0}^{\sqrt{4-\rho^2}} (\rho\cos\phi)^2\,\rho\,dz\,d\rho\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\cos^2\phi\,d\phi \quad \int_{\rho=0}^{2}[z]_0^{\sqrt{4-\rho^2}}\rho^3\,d\rho$$

$$= \int_{\phi=0}^{2\pi}\frac{1}{2}2\cos^2\phi\,d\phi \quad \int_{\rho=0}^{2}\sqrt{4-\rho^2}.\,\rho\,d\rho.\,\rho^2$$

$$= \frac{1}{2}\int_{\phi=0}^{2\pi}2\cos^2\phi\,d\phi \quad \int_{y=2}^{0} y.\,(-y\,dy).\,(4-y^2)$$

$$= \frac{1}{2}\int_{\phi=0}^{2\pi}(1+\cos 2\phi)\,d\phi \int_{y=2}^{0} -\left(4y^2-y^4\right)\,dy$$

$$= \frac{1}{2}\int_{\phi=0}^{2\pi}(1+\cos 2\phi)\,d\phi \int_{y=0}^{2} \left(4y^2-y^4\right)\,dy$$

$$= \frac{1}{2}\left[\phi+\frac{\sin 2\phi}{2}\right]_0^{2\pi}\cdot\left[\frac{4y^3}{3}-\frac{y^5}{5}\right]_0^2$$

$$= \frac{1}{2}\cdot 2\pi\cdot\left[\frac{4(2)^3}{3}-\frac{2^5}{5}\right]$$

$$= \frac{64\pi}{15}$$

Let,

$4-\rho^2 = y^2$

$\Rightarrow \rho^2 = 4-y^2$

$\Rightarrow 2\rho\,d\rho = -2y\,dy$

$\therefore \rho\,d\rho = -y\,dy$

When,

$p \rightarrow$ 0 to 2

$y \rightarrow$ 2 to 0

Exr. Use spherical coordinates to find the volume of the solid G bounded above by the sphene $x^2+y^2+z^2 = 4^2$ and below by the cone $z = \sqrt{x^2+y^2}$

Solution:

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\begin{aligned} & x^2+y^2+z^2 \\ &= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ &= r^2\sin^2\theta\cdot 1 + r^2\cos^2\theta \\ &= r^2 \end{aligned}$$

$$\begin{aligned} & x^2+y^2+z^2 = 4^2 \\ &\Rightarrow r^2 = 4^2 \\ &\therefore r = 4 \end{aligned}$$

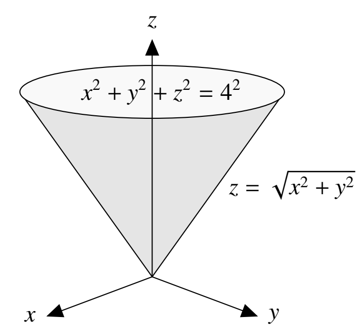
Since, $z = \sqrt{x^2+y^2}$

$$\begin{aligned} &\Rightarrow r\cos\theta = \sqrt{r^2\sin^2\theta} \\ &\Rightarrow r\cos\theta = r\sin\theta \\ &\Rightarrow \tan\theta = 1 = \tan\frac{\pi}{4} \\ &\therefore \theta = \frac{\pi}{4} \end{aligned}$$

$$V = \iiint_G dV = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{4} r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\frac{\pi}{4}} \sin\theta\, d\theta \int_{r=0}^{4} r^2\, dr \quad = [\phi]_0^{2\pi}\cdot[-\cos\theta]_0^{\frac{\pi}{4}}\cdot\left[\frac{r^3}{3}\right]_0^4$$

$$= 2\pi\cdot\left(-\frac{1}{\sqrt{2}}+1\right)\cdot\frac{4^3}{3} \quad = \frac{128\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)\cdot \quad = \frac{128\pi}{3}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$



Exr. Use spherical coordinates to evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dV$

Solution:

$$\begin{aligned} & x^2+y^2 = 4 \\ &\therefore r = 2 \end{aligned}$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\begin{aligned} & x^2+y^2+z^2 \\ &= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ &= r^2\sin^2\theta\cdot 1 + r^2\cos^2\theta \\ &= r^2 \end{aligned}$$

Since, $z = 0$

$$\begin{aligned} &\Rightarrow r\cos\theta = 0 \\ &\Rightarrow \cos\theta = \cos\frac{\pi}{2} \\ &\therefore \theta = \frac{\pi}{2} \end{aligned}$$

Now,

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2} (r\cos\theta)^2\sqrt{r^2}\, r^2\sin\theta\ \, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi} d\phi \quad \int_{r=0}^{2} r^5\, dr \quad \int_{\theta=0}^{\frac{\pi}{2}} \cos^2\theta\sin\theta\, d\theta$$

$$= \int_{\phi=0}^{2\pi} d\phi \quad \int_{r=0}^{2} r^5\, dr \quad \int_{1}^{0} z^2(-dz)$$

$$= [\Phi]_0^{2\pi} \quad \cdot\left[\frac{r^6}{6}\right]_0^2 \quad \cdot\left[-\frac{z^3}{3}\right]_1^0$$

$$= 2\pi\cdot\frac{64}{6}\cdot\frac{1}{3} \quad = \frac{64\pi}{9}$$

Let

$\cos\theta = z$

$\Rightarrow -\sin\theta\, d\theta = dz$

$\therefore \sin\theta\, d\theta = -dz$

when $\theta \to 0$ to $\frac{\pi}{2}$

then $z \to 1$ to 0

Exr. Use spherical coordinates to evaluate $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

Solution:

$x^2+y^2=9$

$\therefore r=3$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$x^2+y^2+z^2$

$= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$

$= r^2\sin^2\theta\cdot 1 + r^2\cos^2\theta$

$= r^2$

Since, $z=0$

$\Rightarrow r\cos\theta = 0$

$\Rightarrow \cos\theta = \cos\dfrac{\pi}{2}$

$\therefore \theta = \dfrac{\pi}{2}$

For whole sphere, $\theta=\pi$

Now,

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{3} \sqrt{r^2}\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_{r=0}^{3} r^3\,dr \quad = [\phi]_0^{2\pi} \quad [-\cos\theta]_0^{\pi} \quad \left[\frac{r^4}{4}\right]_0^3$$

$$= 2\pi\cdot(-\cos\pi+\cos 0)\cdot\frac{3^4}{4} \quad = 2\pi\cdot(1+1)\cdot\frac{81}{4} \quad = 81\pi$$

Extremum Principle

Let $f(x, y)$ and $g(x, y)$ be two function of two variables x and y with continuous partial derivative on some open set containing the constraint curve $g(x, y) = 0$ and assuming that $\underline{\nabla} g(x, y) \neq 0$ at any point on the curve. If $f(x, y)$ has a relative extremum at a point (x_o, y_o) on the constraint curve $g(x, y)$ where the gradient vectors $\underline{\nabla} f\,(x_o, y_o)$ and $\underline{\nabla} g\,(x_o, y_o)$ this are parallel. That is, $\underline{\nabla} f = \lambda\,\underline{\nabla} g$, for some sealer λ. This λ is called the **Lagrange** multiplies.

Green's Theorem:

Let C be a simple closed curve in the xy plane such that a line parallel to either axis cuts C in at most two points. Let $M(x, y)$, $N(x, y)$, $\dfrac{\partial N}{\partial x}$, $\dfrac{\partial M}{\partial y}$ be continuous function of x and y inside and on C and R be the region inside C then,

$$\oint_c M(x, y)\,dx + N(x, y)\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Gauss Theorem

The surface integral of the normal component of a continuous diffentiable vector $\underline{\mathrm{F}}$ taken over a closed surface S is equal to the integral of the divergence of $\underline{\mathrm{F}}$ taken over the volume V enclosed by the surface.

Mathematically $$\iint_S (\underline{\mathrm{F}} \cdot \underline{\mathrm{n}})\; dS = \iiint_V (\underline{\nabla} \cdot \underline{\mathrm{F}})\; dV$$

where n is the positive normal to s.

Stoke's theorem

If the components of a vector field $\underline{\mathrm{F}}(x, y, z) = f(x, y, z)\,\underline{\mathrm{i}} + g(x, y, z)\,\underline{\mathrm{j}} + h(x, y, z)\,\underline{\mathrm{k}}$ are continuous and have continuous first partial derivatives on some open set S and C be smooth closed curve then,

$$\oint_C \underline{\mathrm{F}} \cdot d\underline{\mathrm{r}} = \iint_S (\underline{\nabla} \times \underline{\mathrm{F}}) \cdot \underline{\mathrm{n}}\, dS$$

relationship between line and surface integrals a generalization of Green's theorem to three dimensions is called Stoke's theorem.

Coordinate System

		Cartesian – Polar	Rectangular – Cylindrical	Rectangular – Spherical
Cartesian	: (x, y)			
Polar	: (r, θ)	$x = r\cos\theta$	$x = \rho\cos\phi$	$x = r\sin\theta\cos\phi$
Rectangular	: (x, y, z)	$y = r\sin\theta$	$y = \rho\sin\phi$	$y = r\sin\theta\sin\phi$
Cylindrical	: (ρ, ϕ, z)		$z = z$	$z = r\cos\theta$
Spherical	: (r, θ, ϕ)	$\lvert J\rvert = r\,dr\,d\theta$	$\lvert J\rvert = \rho\;\; d\rho\, d\phi\, dz$	$\lvert J\rvert = r^2\sin\theta\, dr\, d\theta\, d\phi$