Maximum & Minimum

Maxima and minima of functions of two or more variables. Also find the absolute maximum and minimum in $[-6, 4]$

Solution:

$f(x) = x^3 + 3x^2 - 9x - 7$

$f'(x) = 3x^2 + 6x - 9 - 0$

$f''(x) = 6x + 6$

Taking $f'(x) = 0$ to get the critical values of $f(x)$

$\Rightarrow 3x^2 + 6x - 9 - 0 = 0$

$\Rightarrow 3(x^2 + 2x - 3) = 0$

$\Rightarrow (x + 3)(x - 1) = 0$

$\therefore x = -3, 1$

At $x = -3$, $f''(-3) = -18 + 6 = -12$ $\quad 12 < 0$ $\quad$ so $f(x)$ is maximum at $x = -3$

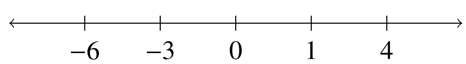
At $x = 1$, $f''(1) = 6 + 6 = 12$ $\quad 12 > 0$ $\quad$ so $f(x)$ is minimum at $x = 1$

Maximum value at $x = -3$ is $f(-3)$ $= (-3)^3 + 3(-3)^2 - 4(-3) - 7$

$= -27 + 27 + 27 - 7 = 20$

Minimum value at $x = 1$ is $f(1)$ $= 1^3 + 3(1)^2 - 9(1) - 7 = -12$

Since, $-3, 1$ is in between $-6, 4$. Therefore there are 4 points $-6, -3, 1, 4$.



Now,

$f(-6) = -216 + 108 + 54 - 7 = -61$

$f(4) = 64 + 48 - 36 - 7 = 69$

$\therefore$ Absolute maximum is 69 (highest) and the absolute minimum is -61 (lowest)

Find the relative extrema of the function $f(x, y) = 3x^2 - 2xy + y^2 - 8y$

Solution:

$f_x(x, y) = 6x - 2y + 0 - 0$

$f_y(x, y) = 0 - 2x + 2y - 8$

Taking $f_x(x, y) = 0$ and $f_y(x, y) = 0$ to get the values of x and y

$6x - 2y = 0 \cdots (1)$ and $-2x + 2y = 8 \cdots (2)$

Solving (1) and (2)

$$\begin{aligned} 6x - 2y &= 0 \\ -2x + 2y &= 8 \\ \hline 4x &= 8 \\ \therefore x &= 2 \end{aligned}$$

Now, $6(2) - 2y \Rightarrow 12 = 2y \therefore y = 6$ $\quad \therefore$ Critical point $(2, 6)$

	$f_{xx}(x, y) = 6$	$f_{xy}(x, y) = -2$	$f_{yy}(x, y) = 2$
At $(x, y) = (2, 6)$	$f_{xx}(x, y) = 6$	$f_{xy}(x, y) = -2$	$f_{yy}(x, y) = 2$
	A	B	C

Let $D = AC - B^2 = 12 - 4 = 8 > 0$

(a) if $D > 0,\ A > 0$ then $f(x, y)$ has a relative minimum.
(b) if $D > 0,\ A < 0$, the $f(x, y)$ has a relative maximum.
(c) if $D < 0$, then $f(x, y)$ has a sattle point.
(d) if $D = 0$, then $f(x, y)$ has no conclusion.

But $A = 6 > 0$ So, $f(x, y)$ has a relative minimum at $(2, 6)$.

Find the relative extrema of the function $f(x, y) = 4xy - x^4 - y^4$

Solution:

$f_x(x, y) = 4y - 4x^3 - 0$

$f_y(x, y) = 4x - 0 - 4y^3$

Taking $f_x(x, y) = 0$ and $f_y(x, y) = 0$ to get the values of x and y

$6x - 2y = 0 \cdots (1)$ and $-2x + 2y = 8 \cdots (2)$

Taking $f_x(x, y) = 0$

$4y - 4x^3 = 0$

$y = x^3$

Putting $x = -1, 0, 1$ in $y = x^3$

$y = (-1)^3, \ 0^3, \ 1^3$

$\therefore y = -1, 0, 1$

Taking $f_y(x, y) = 0$

$4x - 4y^3 = 0$

$x = y^3 \Rightarrow x = (x^3)^3 \Rightarrow x^9 - x = 0 \Rightarrow x(x^8 - 1) = 0$

$x = 0, \quad x^8 - 1 = 0$

Now,

$x^8 - 1 = 0$

$\Rightarrow \left(x^4 + 1\right)\left(x^4 - 1\right) = 0$

$\Rightarrow \left(x^4 + 1\right)\left(x^2 + 1\right)\left(x^2 - 1\right) = 0$

$\Rightarrow \left(x^4 + 1\right)\left(x^2 + 1\right)(x + 1)(x - 1) = 0$

$\Rightarrow x = -1, 1$

$\therefore x = 0, -1, 1$

$\therefore$ Critical points are $(-1, -1), \ (0, 0), \ (1, 1)$

Now, $f_{xx} = -12x^2 \qquad f_{xy} = y \qquad f_{yy} = -12y^2$

At $(-1, -1)$

$A = f_{xx} = -12$

$B = f_{xy} = 4$

$C = f_{yy} = -12$

Now, $D = AC - B^2 = 144 - 16$

$D = 128 > 0$

$A = -12 < 0$

$f(x, y)$ has relative max at $(-1, -1)$

At $(0, 0)$

$A = f_{xx} = 0$

$B = f_{xy} = 4$

$C = f_{yy} = 0$

Now, $D = AC - B^2 = 0 - 16$

$D = -16$

$f(0, 0)$ is a sattle point.

At $(1, 1)$

$A = f_{xx} = -12$

$B = f_{xy} = 4$

$C = f_{yy} = -12$

Now, $D = AC - B^2 = 144 - 16$

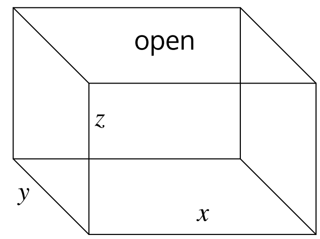
$D = 128 > 0$

$A = -12 < 0$

$f(x, y)$ has relative max at $(1, 1)$

Determine the dimensions of a rectangular box, open at the top, having a volume of 32ft^3, and requiring the least amount of materials for its construction.

Solution:



Let,
x = length of the box in feet.
y = width of the box in feet.
z = height of the box in feet.

Therefore, Volume, $V = xyz = 32 \quad \therefore z = \dfrac{32}{xy}$

Surface Area, $S = xy + 2yz + 2zx \quad \Rightarrow S = xy + 2y\dfrac{32}{xy} + 2\dfrac{32}{xy}x$

$$\therefore S_x = y - \frac{64}{x^2} + 0$$
$$\therefore S_y = x + 0 - \frac{64}{y^2}$$

$$\therefore S_{xx} = 0 - \frac{128}{x^3}$$
$$\therefore S_{xy} = 1 - 0$$
$$\therefore S_{yy} = 0 + \frac{128}{y^3}$$

Taking $S_x = 0 \;\therefore y = \dfrac{64}{x^2}$ and $S_y = 0 \;\therefore x = \dfrac{64}{y^2}$

$$x = \frac{64}{\left(\frac{64}{x^2}\right)^2}$$
$$\Rightarrow x = \frac{64}{64^2} \times x^4$$
$$\Rightarrow x^4 = 64x$$
$$\Rightarrow x\left(x^3 - 64\right) = 0$$
$$\Rightarrow x = 0, 4$$

Since product xy can't be 0.

Hence, $x = 0$ not possible

$\therefore x = 4$

Putting the value in y,

$\therefore y = \dfrac{64}{4^2} = 4$

$\therefore (4, 4)$ is the critical point

Volume $z = \dfrac{32}{xy} = \dfrac{32}{4.4} = 2$

Now, at $(4, 4)$

$A = S_{xx} = \dfrac{128}{4^3} = 2$

$B = S_{xy} = 1$

$C = S_{yy} = 2 \quad \dfrac{128}{4^3} = 2$

$D = AC - B^2 = 4 - 1 = 3 > 0$ and $A = 2 > 0$

So, S is minimum (least amount) at $x = 4\text{ft}$, $y = 4\text{ft}$, and $z = 2\text{ft}$

$\therefore$ Surface are, $S = xy + 2yz + 2zx = 4.4 + 2.4.2 + 2.2.4 = 48$

Extremum Principle

Statement: Let $f(x,y)$ and $g(x,y)$ be two function of two variables x and y with continuous partial derivative on some open set containing the constraint curve $g(x,y)=0$ and assuming that $\underline{\nabla} g(x,y) \neq 0$ at any point on the curve. If $f(x,y)$ has a relative extremum at a point (x_o, y_o) on the constraint curve $g(x,y)$ where the gradient vectors $\underline{\nabla} f\,(x_o, y_o)$ and $\underline{\nabla} g\,(x_o, y_o)$ this are parallel. That is, $\underline{\nabla} f = \lambda \underline{\nabla} g$, for some sealer λ. This λ is called the **Lagrange** multiplies.

7(b) At what points on the circle $x^2+y^2=1$ does the product xy have extremum?

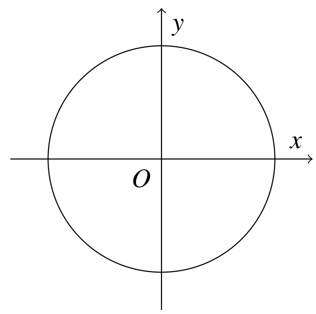
Solution:

Here, $f(x,y) = xy \cdots\cdots\cdots (i)$

$g(x,y) = x^2+y^2 = 1 \cdots\cdots\cdots (ii)$

$\therefore \underline{\nabla} f = y\underline{i} + x\underline{j}$

$\therefore \underline{\nabla} g = 2x\underline{i} + 2y\underline{j}$



Set, $\underline{\nabla} g = 0 \Rightarrow 2x\underline{i} + 2y\underline{j} = 0 = 0\underline{i} + 0\underline{j}$

From equation (i) & (ii), $2x = 0 \;\therefore x = 0$ $\qquad 2y = 0 \;\therefore y = 0$

So, $\underline{\nabla} g \neq 0$ at any points on the circle $x^2+y^2=1$

Then $f(x,y)$ has relative extremum only when $\underline{\nabla} f = \lambda \underline{\nabla} g$

$\Rightarrow y\underline{i} + x\underline{j} = \lambda(2x\underline{i} + 2y\underline{j})$

Equating both side, $y = 2x\lambda \;\therefore \lambda = \dfrac{y}{2x}$ and $x = 2y\lambda \;\therefore \lambda = \dfrac{x}{2y}$

Now,

$\therefore \dfrac{y}{2x} = \dfrac{x}{2y}$

$\Rightarrow x^2 = y^2$

$\Rightarrow x^2 - y^2 = 0$

$\Rightarrow (x+y)(x-y) = 0$

$\therefore x = \pm y$

Then,

$x^2+y^2=1$

$\Rightarrow y^2+y^2=1$

$\Rightarrow 2y^2 = 1$

$\Rightarrow y^2 = \dfrac{1}{2}$

$\therefore y = \pm\dfrac{1}{\sqrt{2}}$

$\therefore x = \pm\dfrac{1}{\sqrt{2}}$

$\therefore$ Points are $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right), \left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right), \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right), \left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$

$\therefore f(x,y) = \dfrac{1}{2}$ at $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$\therefore f(x,y) = \dfrac{1}{2}$ at $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$

Maximum

$\therefore f(x,y) = -\dfrac{1}{2}$ at $\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$

$\therefore f(x,y) = -\dfrac{1}{2}$ at $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

Minimum

6(b) Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are nearest to and farthest from $(1, 2, 2)$.

Solution:

Let, $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2 \cdots\cdots\cdots (i)$

$g(x, y, z) = x^2 + y^2 + z^2 - 36 = 0 \cdots\cdots\cdots (ii)$

$\therefore \underline{\nabla} f = 2(x-1)\underline{i} + 2(y-2)\underline{j} + 2(z-2)\underline{k}$

$\therefore \underline{\nabla} g = 2x\underline{i} + 2y\underline{j} + 2z\underline{k}$

Set, $\underline{\nabla} g = \underline{0} \Rightarrow 2x\underline{i} + 2y\underline{j} + 2z\underline{k} = \underline{0} = 0\underline{i} + 0\underline{j} + 0\underline{k}$

Equating both side, $2x = 0 \quad 2y = 0 \quad 2z = 0$

$\therefore x = 0 \quad \therefore y = 0 \quad \therefore z = 0$

So, $\underline{\nabla} g \neq 0$, at any points on the sphere.

Then $f(x, y, z)$ has relative extremum only when $\underline{\nabla} f = \lambda \underline{\nabla} g$

$\Rightarrow 2(x-1)\underline{i} + 2(y-2)\underline{j} + 2(z-2)\underline{k} = \lambda(2x\underline{i} + 2y\underline{j} + 2z\underline{k})$

Equating both side, $2(x-1) = 2x$ | $2(y-2) = 2y$ | $2(z-2) = 2z$

$\therefore \lambda = \dfrac{x-1}{x}$ | $\therefore \lambda = \dfrac{y-2}{y}$ | $\therefore \lambda = \dfrac{z-2}{z}$

$$\therefore \frac{x-1}{x} = \frac{y-2}{y} = \frac{z-2}{z}$$

$\dfrac{x-1}{x} = \dfrac{y-2}{y}$

$\Rightarrow xy - y = xy - 2x$

$\Rightarrow y = 2x$

$\dfrac{x-1}{x} = \dfrac{z-2}{z}$

$\Rightarrow xz - z = xz - 2x$

$\Rightarrow z = 2x$

$x^2 + y^2 + z^2 = 36$

$\Rightarrow x^2 + 4x^2 + 4x^2 = 36$

$\Rightarrow 4x^2 = 36$

$\Rightarrow x^2 = 4$

$\therefore x = \pm 2, \quad y = \pm 4, \quad z = \pm 4$

$\therefore$ Points are $(2, 4, 4)$ and $(-2, -4, -4)$

Putting them in equation (i),

$\therefore f(2, 4, 4) = (2-1)^2 + (4-2)^2 + (4-2)^2 = 1 + 4 + 4 = 9$ (nearest) and

$\therefore f(-2, -4, -4) = (-2-1)^2 + (-4-2)^2 + (-4-2)^2 = 9 + 36 + 36 = 81$ (farthest)

9(b) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Solution:

Let, $f(x, y, z) = x^2 + y^2 + z^2 = 9 \cdots\cdots\cdots (i)$

$g(x, y, z) = x^2 + y^2 - z - 3 = 0 \cdots\cdots\cdots (ii)$

$\therefore \underline{u} = \underline{\nabla} f = 2x\underline{i} + 2y\underline{j} + 2z\underline{k}$

$\therefore \underline{v} = \underline{\nabla} g = 2x\underline{i} + 2y\underline{j} - 1\underline{k}$

At point $(2, -1, 2)$

$\therefore \underline{u} = \underline{\nabla} f = 2(2)\underline{i} + 2(-1)\underline{j} + 2(2)\underline{k} = 4\underline{i} - 2\underline{j} + 4\underline{k}$

$\therefore \underline{v} = \underline{\nabla} g = 2(2)\underline{i} + 2(-1)\underline{j} - 1\underline{k} = 4\underline{i} - 2\underline{j} - 1\underline{k}$

Let θ be the angle between $\underline{u}$ and $\underline{v}$, then,

Now,

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\underline{u} \cdot \underline{v}}{\| \underline{u} \| \| \underline{v} \|}\right)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)$$

$$\therefore \theta = 54.414^\circ$$

Here,

$\| \underline{u} \| = \sqrt{16 + 4 + 16} = 6$

$\| \underline{v} \| = \sqrt{16 + 4 + 1} = \sqrt{21}$

$\underline{u} \cdot \underline{v} = 16 + 4 - 4 = 16$