MTH 103: Linear Alzebra

Matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}_{2\times 2}$$

Addition and Subtraction:

Two matrices can be added or subtracted if,

• The order of the matrices is equal. or • Their corresponding entries are equal.

Multiplication:

Two matrices can be multiplied if the number of the column in the first matrix is equal to the row in the second matrix.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}_{2\times 3} \qquad B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix}_{3\times 2} \qquad \therefore AB = \begin{pmatrix} 1+4+9 & 1+0-3 \\ 2+6+12 & 2+0-4 \end{pmatrix} = \begin{pmatrix} 19 & -2 \\ 20 & -2 \end{pmatrix}_{2\times 2}$$

Different types of matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zero matrix Diagonal matrix Identity matrix

Determinant

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Determinant of A, $|A| = (a_{11}a_{22} - a_{12}a_{21}) \quad \in R$

Minors and Co-factor:

$$\begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

$$\therefore M_{32} = \begin{pmatrix} 3 & 4 \\ 2 & 6 \end{pmatrix} = 26 \qquad \therefore C_{32} = (-1)^{3+2} \cdot \begin{pmatrix} 3 & 4 \\ 2 & 6 \end{pmatrix} = -26$$

Determinant Properties:

$$|A| = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

$|A| = 0$; if one row/column is zero.

$|A| = 0$; if two rows/columns are identical.

$|AB| = |A| \cdot |B|$

$|A + B| \neq |A| + |B|$

$|A^{-1}| = \dfrac{1}{|A|}$

$|KA| = (K)^n \, |A|$

$|KA^{-1}| = K^n \, \dfrac{1}{|A|}$

$|(KA)^{-1}| = \dfrac{1}{K^n \cdot |A|}$

Inverse matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

A^{-1} exists if $|A| \neq 0$

$\boxed{\text{Verification:}}$

If $A^{-1} = B$ then $AB = BA = I_n$

or, $A \cdot A^{-1} = I_n$

Inverse matrix of (A) using Adjoint/Co-factor matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$$

$|A| = 2(-3-2) + 2(-7+6) + 0 = -12$

since $|A| \neq 0$, then A^{-1} exists.

Co-factors of the matrix,

$C_{11} = -5 \qquad C_{21} = 6 \qquad C_{31} = 2$

$C_{12} = -1 \qquad C_{22} = -6 \qquad C_{32} = -2$

$C_{13} = 11 \qquad C_{23} = -18 \qquad C_{33} = -2$

$$\therefore \ \text{Adj}(A) = \left[C_{ij}\right]^T = \begin{pmatrix} -5 & 6 & 2 \\ -1 & -6 & -2 \\ 11 & -18 & -2 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{AdJ}(A) = \frac{1}{-12}\begin{pmatrix} -5 & 6 & 2 \\ -1 & -6 & -2 \\ 11 & -18 & -2 \end{pmatrix}$$

Alternative way for determinant:

Determinant of (A) by co-factor expansion along first row:

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2(-5) + 2(-1) + 0(11) = -12$

Determinant of (A) by co-factor expansion along second column:

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = 2(-1) + 1(-6) + 2(-2) = -12$

Determinant of (A) by co-factor expansion along second row:

$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 2(6) + 1(-6) + 1(-18) = -12$

Solving system of linear equation : Crammer's rule

If $AX = B$ is a system of m linear equations in n unknowns such that determinant $\boxed{D \neq 0}$, the system has a unique solution. This solution is,

$$x = \frac{Dx}{D}, \qquad y = \frac{Dy}{D}, \qquad z = \frac{Dz}{D}$$

Problem: Solve the system of linear equations:

$$\left.\begin{aligned} 2x + 3y &= 60 \\ -6x + 7y &= 40 \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

Write (1) in matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 2 & 3 \\ -6 & 7 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 60 \\ 40 \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 2 & 3 \\ -6 & 7 \end{pmatrix} = 14 + 18 = 32$$

$$\therefore Dx = \begin{pmatrix} 60 & 3 \\ 40 & 7 \end{pmatrix} = 420 - 120 = 300 \qquad \therefore x = \frac{Dx}{D} = \frac{300}{32} = \frac{75}{8}$$

$$\therefore Dy = \begin{pmatrix} 2 & 60 \\ -6 & 40 \end{pmatrix} = 80 + 360 = 440 \qquad \therefore y = \frac{Dy}{D} = \frac{440}{32} = \frac{55}{4}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 75/8 \\ 55/4 \end{pmatrix}$$

$\boxed{\text{Problem:}}$ Solve the system of linear equations,

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ 2x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 - x_3 &= -6 \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

Write (1) in matrix form, $A\mathbf{x} = \mathbf{b}$

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ -6 \end{pmatrix}$$

$$\therefore D = |A| = 1(-1-1) + 2(-2+1) + 3(-2-1) = -13$$

Since $D \neq 0$, the system has a unique solution.

$$\therefore Dx_1 = \begin{vmatrix} 7 & -2 & 3 \\ 1 & 1 & -1 \\ -6 & -1 & -1 \end{vmatrix} = -13 \qquad \therefore x_1 = \frac{Dx_1}{D} = \frac{-13}{-13} = 1$$

$$\therefore Dx_2 = \begin{vmatrix} 1 & 7 & 3 \\ 2 & 1 & -1 \\ 1 & -6 & -1 \end{vmatrix} = -39 \qquad \therefore x_2 = \frac{Dx_2}{D} = \frac{-39}{-13} = 3$$

$$\therefore Dx_3 = \begin{vmatrix} 1 & -2 & 7 \\ 2 & 1 & 1 \\ 1 & -1 & -6 \end{vmatrix} = -52 \qquad \therefore x_3 = \frac{Dx_3}{D} = \frac{-52}{-13} = 4$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Elementary Matrices for Fing A^{-1}

Elementary row operation (e.r.o):

(1) Multiply a row by a non-zero constant C

(2) Interchange two rows

(3) Add C times one row to another.

Row equivalent matrix:

Matrices A and B are said to be row equivalent if each of them can be obtained from the other by a sequence of elementary row operations (e.r.o.)

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_3' = -2R_3} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 0 & -2 & -10 \end{pmatrix} = B$$

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 0 & -2 & -10 \end{pmatrix} \xrightarrow{R_3' = -\frac{1}{2}R_3} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 2 & 3 & 4 \end{pmatrix} = A$$

Elementary matrix:

A matrix E is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation. $I \xrightarrow{\text{e.r.o}} E$.

Problem Find an elementary matrix E that satisfies $EB = D$

$$B = \begin{pmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2' = -3R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

$$\therefore EB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 9 & 1 \end{pmatrix}_{\text{(proved)}}$$

Row Echelon form (REF) / Gaussian Elimination

A matrix is in echelon form if -

(1) Non-zero rows appear above the zero rows.

(2) In any non-zero row, the first non-zero element (called the leading element) appears to the left of the leading element in any lower row.

Problem: Determine whether the matrix is in Row Echelon Form or not.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \times \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 9 \end{pmatrix} \checkmark \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 5 \end{pmatrix} \times \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{pmatrix} \times \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \checkmark$$

Reduced Row Echelon Form (RREF) / Gauss-Jordan Elimination

If a column contains a leading element, all the other elements in that column are zero.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \times \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix} \checkmark \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \quad \begin{pmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \checkmark$$

Augmented matrix

An augmented matrix is a matrix obtained by appending the columns of two given matrices, usually to perform the same elementary row operation (e.r.o) on each of the given matrices.

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 5 & 2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \quad \therefore (A \mid B) = \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 5 & 2 & 2 & 1 \end{array}\right)$$

Problem Find the inverse of the given matrix.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

reduce it to reduce row echelon form by e.r.o

$$\therefore (A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3' = R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \xrightarrow{R_3' = -R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow[R_1' = R_1 - 3R_3]{R_2' = R_2 + 3R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow{R_1' = R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

System of linear equation

$ax + by + c$ — linear equation

$x^2 + y^2 = 1$ — polynomial equation

$\boxed{\text{variables} \to \text{unknowns}}$

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_n \end{array}$$

It can be expressed in matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = [x_i] = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = [b_i] = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$a_{ij} \in R; \qquad b_i \in R$$

m equations & n unknowns

One \ Unique solution	No solution	Infinitely many solutions
$x - y = 1$ $2x + y = 6$	$x + y = 4$ $3x + 3y = 6$	$4x - 2y = 1$ $16x - 8y = 4$
$2x - 2y = 2$ $2x + y = 6$ $-3y = -4$	$3x + 3y = 12$ $3x + 3y = 6$ $0 = 6$	$4x - 2y = 1$ $4x - 2y = 1$ $0 = 0$
$\therefore x = 7/3 \quad y = 4/3$	$\therefore 0 = b$ and $b \neq 0$	$\therefore 0 = b$ and $b = 0$
equations (m) = unknowns (n)	equations (m) > unknowns (n)	equations (m) < unknowns (n)
One \ Unique solution	No solution	Infinitely many solutions

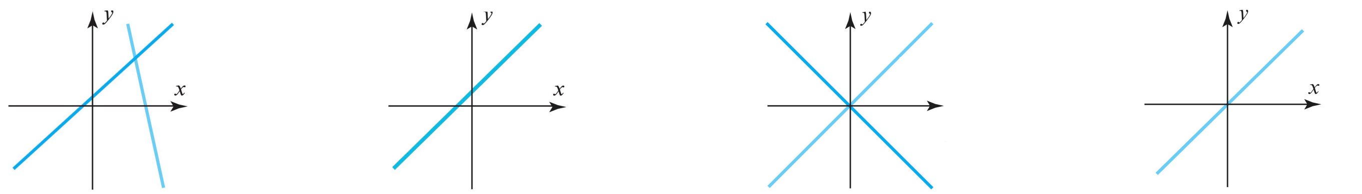
$\boxed{\text{Free variables: } n - m}$

Types of solution for the system of linear equation

Consistent system

Non-homogeneous $(C \neq 0)$ — Unique solution; Infinitely many solutions

Homogeneous $(C = 0)$ — Trivial solution; Non-trivial solution



System of linear equations: Exercises

Problem: Solve the system of linear equations

$$
\begin{aligned}
x + y + 2z &= 9\\
2x + 4y - 3z &= 1\\
3x + 6y - 5z &= 0
\end{aligned}
$$

Solution: Let,

$$
\left.
\begin{aligned}
x + y + 2z &= 9\\
2x + 4y - 3z &= 1\\
3x + 6y - 5z &= 0
\end{aligned}
\right\} \quad \cdots\ \cdots\ \cdots\ (1)
$$

In matrix form $A\mathbf{x} = \mathbf{b}$,

$$
A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 9 \\ 1 \\ 0 \end{pmatrix}
$$

Reduce augmented matrix $(A \mid \mathbf{b})$ to row echelon form by e.r.o

$$(A \mid \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right) \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right) \xrightarrow{R_3' = 2R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1 & -3 \end{array}\right)$$

This is in row echelon form.

Corresponding linear equations,

$$x + y + 2z = 9$$

$$2y - 7z = -17$$

$$-z = -3$$

There is no equations in the form $0 = b$ with $b \neq 0$

So the system is consistent and the system has a solution.

Since there are 3 unknowns is 3 equations, so the system has a unique solution.

By backward substition,

$$z = 3$$

$$y = 2$$

$$x = 1$$

$$\therefore \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$ is the solution.

Problem: Solve the system of linear equations

$$
\begin{array}{rcrcrcrcrcrcr}
x_1 & - & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}
$$

Solution: Let,

$$
\left.
\begin{array}{r}
x_1 - 3x_2 - 2x_3 + 2x_5 = 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\
5x_3 + 10x_4 + 15x_6 = 5 \\
2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6
\end{array}
\right\} \quad \cdots\cdots\cdots (1)
$$

In matrix form $A\mathbf{x} = \mathbf{b}$,

$$
A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ 5 \\ 6 \end{pmatrix}
$$

Reduce augmented matrix $(A \mid \mathbf{b})$ to reduced row echelon form by e.r.o

$$(A \mid \mathbf{b}) = \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right) \xrightarrow[R_4' = R_4 - 2R_1]{R_2' = R_2 - 2R_1} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

$$\xrightarrow[R4' = R4 + 4R2]{R_3' = \frac{R3}{5}} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right) \xrightarrow{R_3' = R3 + R2} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

$$\xrightarrow[R3 \leftrightarrow R4]{R3' = -R3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R1' = R1 + R2]{R2' = 2R2 - R1} \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This is in reduced row echelon form.

Corresponding linear equations,

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + 2x_5 &= 0 \\ 2x_3 + 4x_4 &= 0 \\ 6x_6 &= 2 \end{aligned}$$

There is no equations in the form $0 = b$ with $b \neq 0$

So the system is consistent and the system has a solution.

Since there are 6 unknowns is 3 equations, so the system has infinitely many solutions and have $(6 - 3) = 3$ free variables, namely x_2, x_4, x_5.

Let, $\quad x_2 = r, \quad x_4 = s, \quad x_5 = t$

By backward substition, $\quad x_6 = \frac{1}{3} \qquad x_3 = -2s \qquad x_1 = -3r - 4s - 2t$

The solution is,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix} + r \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Problem: determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

$$\begin{aligned} x + 2y - \quad 3z &= 4 \\ 3x - \quad y + \quad 5z &= 2 \\ 4x + \quad y + (a^2 - 14)z &= a + 2 \end{aligned}$$

In matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & a^2 - 14 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ a + 2 \end{pmatrix}$$

Reduce augmented matrix $(A \mid \mathbf{b})$ to row echelon form by e.r.o

$$(A \mid \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2 - 14 & a+2 \end{array}\right) \xrightarrow[R_3' = R_3 - 4R_1]{R_2' = R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2 - 2 & a - 14 \end{array}\right)$$

$$\xrightarrow{R_3' = R3 - R2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2 - 16 & a - 4 \end{array}\right)$$

This is in row echelon form.

Corresponding linear equations,

$$x + 2y - 3z = 4$$

$$-7y + 14z = -10$$

$$(a-4)(a+4)z = a - 4$$

Conclusion:

$a = -4$: There is an equation of the form $0 = b$ with $b \neq 0$

So, the system is inconsistent and has no solution.

$a \neq \pm 4$: There are 3 unknowns in 3 equations.

So, the system has exactly one solution.

$a = 4$: There are 3 unknowns in 2 equations.

So, the system has infinitely many solutions.

Solving system of linear equation : Inverting the coefficient matrix / using A^{-1}

Reference: *Exercise Set-1.6 (1-8)* | *Chapter-1* | *Page-66*

Problem: Solve the system of linear equations using A^{-1}

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

In matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}$$

Finding A^{-1} by reducing it to reduce row echelon form,

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3' = R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \xrightarrow{R_3' = -R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow[R_1' = R_1 - 3R_3]{R_2' = R_2 + 3R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow{R_1' = R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

The solution is,

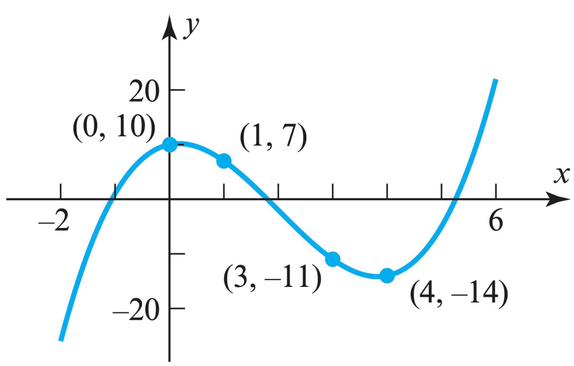
$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Finding coefficient

Reference: *Exercise Set-1.2 (37, 38)* | *Chapter-1* | *Page-22*

Problem: Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.



Solution:

$$y = ax^3 + bx^2 + cx + d \quad \cdots\cdots\cdots (1)$$

If the poly (1) passes through the points $(0, 10)$, $(1, 7)$, $(3, -11)$, $(4, -14)$ then,

$$d = 10$$

$$a + b + c + d = 7$$

$$27a + 9b + 3c + d = -11$$

$$64a + 16b + 4c + d = -14$$

$$\left.\begin{aligned} a+b+c &= -3 \\ 27a+9b+3c &= -21 \\ 64a+16b+4c &= -24 \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

In matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 27 & 9 & 3 \\ 64 & 16 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -3 \\ -21 \\ -24 \end{pmatrix}$$

$$(A \mid \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 27 & 9 & 3 & -21 \\ 64 & 16 & 4 & -24 \end{array}\right) \xrightarrow[R3' = \frac{R3}{4}]{R2' = \frac{R2}{3}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 9 & 3 & 1 & -7 \\ 16 & 4 & 1 & -6 \end{array}\right)$$

$$\xrightarrow[R3' = R3 - 16R1]{R2' = R2 - 9R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -6 & -8 & 20 \\ 0 & -12 & -15 & 42 \end{array}\right) \xrightarrow{R3' = R3 - 2R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -6 & -8 & 20 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

The corresponding linear system,

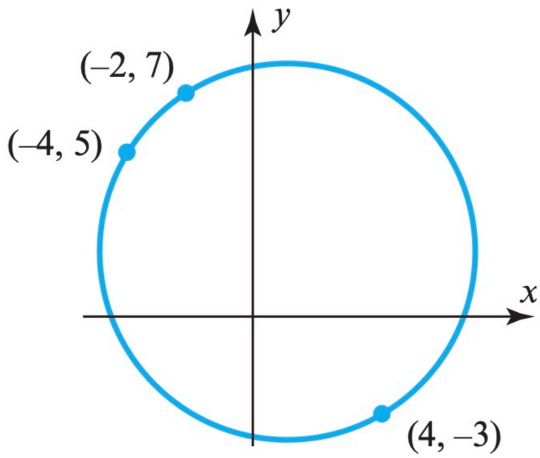
$$\begin{aligned} a+b+c &= -3 \\ -6b-8c &= 20 \\ c &= 2 \end{aligned}$$

By backward substitution,

$$\begin{aligned} c &= 2 \\ b &= -6 \\ a &= 1 \end{aligned}$$

$\therefore$ The required poly is $\quad y = x^3 - 6x^2 + 2x + 10$

Problem: Find the coefficients a, b, c, and d so that the circle shown in the accompanying figure is given by the equation $ax^2 + ay^2 + bx + cy + d = 0$.



Solution:

$$ax^2 + ay^2 + bx + cy + d = 0 \qquad \cdots\ \cdots\ \cdots\ (1)$$

If the poly (1) passes through the points $(-4, 5)$, $(-2, 7)$, $(4, 3)$ then,

$$\begin{aligned} 16a + 25a - 4b + 5c + d &= 0 \\ 4a + 49a - 2b + 7c + d &= 0 \\ 16a + 9a + 4b - 3c + d &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} 41a - 4b + 5c + d &= 0 \\ 53a - 2b + 7c + d &= 0 \\ 25a + 4b - 3c + d &= 0 \end{aligned}$$

$$\left.\begin{aligned} d + 5c - 4b + 41a &= 0 \\ d + 7c - 2b + 53a &= 0 \\ d - 3c + 4b + 25a &= 0 \end{aligned}\right\} \qquad \cdots\ \cdots\ \cdots\ (1)$$

In matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 1 & 5 & -4 & 41 \\ 1 & 7 & -2 & 53 \\ 1 & -3 & 4 & 25 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A \mid \mathbf{b}) = \left(\begin{array}{cccc|c} 1 & 5 & -4 & 41 & 0 \\ 1 & 7 & -2 & 53 & 0 \\ 1 & -3 & 4 & 25 & 0 \end{array}\right) \xrightarrow[R3' = R3 - R1]{R2' = R2 - R1} \left(\begin{array}{cccc|c} 1 & 5 & -4 & 41 & 0 \\ 0 & 2 & 2 & 12 & 0 \\ 0 & -8 & 8 & -16 & 0 \end{array}\right)$$

$$\xrightarrow{R3' = R3 + 4R2} \left(\begin{array}{cccc|c} 1 & 5 & -4 & 41 & 0 \\ 0 & 2 & 2 & 12 & 0 \\ 0 & 0 & 16 & 32 & 0 \end{array}\right) \xrightarrow[R2' = \frac{R2}{2}]{R3' = \frac{R3}{16}} \left(\begin{array}{cccc|c} 1 & 5 & -4 & 41 & 0 \\ 0 & 1 & 1 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

The corresponding linear system,

$$d + 5c - 4b + 41a = 0$$

$$c + b + 6a = 0$$

$$b + 2a = 0$$

Let $a = r$, therefore $b = -2r$; $c = -4r$; $d = -29r$

Substituting this values in equation (1),

$$= r(x^2 + y^2 - 2x - 4y - 29)$$

$$= x^2 + y^2 - 2x - 4y - 29 \qquad \text{[where } r = 1]$$

$\therefore$ The required poly is $x^2 + y^2 - 2x - 4y - 29$