Line, Double, Triple Integral

Line Integral

15(b) Calculate the work done when a force $\underline{F} = 3xy\,\underline{i} - y^2\,\underline{j}$ moves a particle in the xy plane from $(0,0)$ to $(1,2)$ along the parabola $y = 2x^2$.

Solution:

Work done $= \oint F.\ d\underline{r}$

$$\therefore \oint F.\ d\underline{r} = \oint 3xy\,dx - y^2\,dy$$
$$= \int_0^1 3x \cdot 2x^2\,dx - \left(2x^2\right)^2 4x\,dx$$
$$= \int_0^1 \left(6x^3 - 16x^5\right)\,dx$$
$$= \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$$
$$= \frac{6}{4} - \frac{16}{6}$$
$$= -\frac{7}{6}$$

but, $y = 2x^2$

$\therefore dy = 4x\,dx$

Given that,

$\underline{F} = 3xy\,\underline{i} - y^2\,\underline{j}$

$\underline{r} = x\,\underline{i} + y\,\underline{j}$

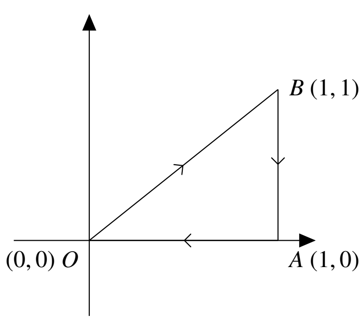
$\therefore d\underline{r} = dx\,\underline{i} + dy\,\underline{j}$

$\therefore F.\,d\underline{r} = 3xy\,dx - y^2\,dy$

10(b) Evaluate $\oint (x^2y^2\,dx - xy^3\,dy)$ where C is the triangle with vertices $(0,0),\ (1,0),\ (1,1)$.

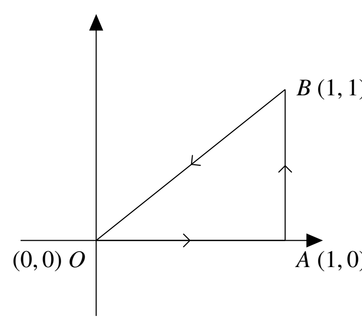
Concept:

Clockwise direction



$$\oint_{OBAO} = \int_{OB} + \int_{BA} + \int_{AO}$$

Anti-Clockwise direction



$$\oint_{OABO} = \int_{OA} + \int_{AB} + \int_{BO}$$

$$\begin{array}{lll} \text{OA, AO} & y = 0 & \therefore dy = 0 \\ \text{AB, BA} & x = 1 & \therefore dx = 0 \\ \text{BO, OB} & x = y & \therefore dx = dy \end{array}$$

Solution:

$$\oint_{OABO} x^2y^2\,dx - xy^3\,dy$$

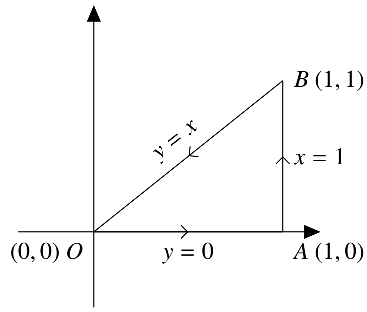
$$= \int_{\substack{OA\\y=0\\dy=0}} x^2y^2\,dx - xy^3\,dy + \int_{\substack{AB\\x=1\\dx=0}} x^2y^2\,dx - xy^3\,dy + \int_{\substack{BO\\x=y\\dx=dy}} x^2y^2\,dx - xy^3\,dy$$

$$= \int 0 + \int_0^1 0 - 1\cdot y^3\,dy + \int_1^0 x^4\,dx - x^4\,dx$$

$$= 0 - \left[\frac{y^4}{4}\right]_0^1 + 0$$

$$= -\left(\frac{1}{4} - 0\right)$$

$$= -\frac{1}{4}$$



Evaluate $\oint \left(xy - x^2\right)\,dx + x^2y\,dy$ where C is the triangle bounded by the line $y = 0,\ x = 1,\ y = x$.

Solution:

$$\oint_C \left(xy - x^2\right)\,dx + x^2y\,dy$$

$$= \int_{OA} C \quad + \int_{AB} C \quad + \int_{BO} C$$

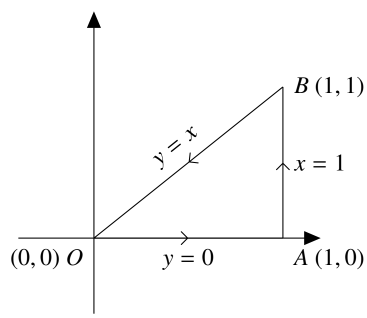
$$= \int_{\substack{OA\\y=0\\dy=0}} (0 - x^2)\,dx + 0 + \int_{\substack{AB\\x=1\\dx=0}} 0 + 1^2y\,dy + \int_{\substack{BO\\x=y\\dx=dy}} (x^2 - x^2)\,dx + x^3\,dx$$

$$= \int_0^1 -x^2\,dx + \int_0^1 y\,dy + \int_1^0 x^3\,dx$$

$$= \left[\frac{-x^3}{3}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 + \left[\frac{x^4}{4}\right]_1^0$$

$$= \left(-\frac{1}{3} - 0\right) + \left(\frac{1}{2} - 0\right) + \left(0 - \frac{1}{4}\right)$$

$$= -\frac{1}{3} + \frac{1}{2} - \frac{1}{4} \quad = -\frac{1}{12}$$



10(a) If the vector field is given by $\underline{F} = (2x - y + z)\underline{i} + (x + y - z^2)\underline{j} + (3x - 2y + 4z)\underline{k}$, evaluate the line integral over a circular path given by $x^2 + y^2 = 16,\ z = 0$.

Solution:

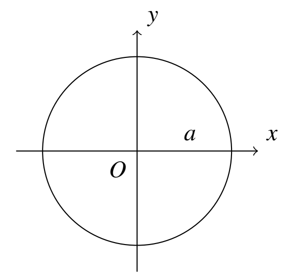
Since $z = 0$, $\therefore\ \underline{F} = (2x - y)\underline{i} + (x + y)\underline{j} + (3x - 2y)\underline{k}$

Here,

$\underline{r} = x\underline{i} + y\underline{j} + 0\underline{k}$

$\therefore\ d\underline{r} = dx\underline{i} + dy\underline{j} + 0\underline{k}$

$\therefore\ \text{F}.\,\text{d}\underline{r} = (2x - y)\,dx + (x + y)\,dy$



Now,

Line integral $= \oint \text{F}.\ \text{d}\underline{r}$

Parametric equation of a circle,

$x = a\cos\theta$ and $y = a\sin\theta$

$\therefore\ dx = -a\sin\theta$ and $\therefore\ dy = a\cos\theta$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} (2a\cos\theta - a\sin\theta)(-a\sin\theta\, d\theta) + (a\cos\theta + a\sin\theta)(a\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} \left[-2a^2\sin\theta\cos\theta + a^2\sin^2\theta + a^2\cos^2\theta + a^2\sin\theta\cos\theta\right] d\theta$$

$$= \int_0^{2\pi} \left[a^2\left(\sin^2\theta + \cos^2\theta\right) - 2a^2\sin\theta\cos\theta + a^2\sin\theta\cos\theta\right] d\theta$$

$$= \int_0^{2\pi} a^2 - a^2\sin\theta\cos\theta\, d\theta \quad = \int_0^{2\pi} a^2\,(1 - \sin\theta\cos\theta)\ d\theta$$

$$= a^2\int_0^{2\pi}\left(1 - \frac{\sin 2\theta}{2}\right) d\theta \quad = a^2\int_0^{2\pi}\left(1 - \frac{1}{2}\sin 2\theta\right) d\theta$$

$$= a^2\left[\theta + \frac{1}{2}\cos 2\theta \cdot \frac{1}{2}\right]_0^{2\pi} \quad = a^2\left[\theta + \frac{1}{4}\cos 2\theta\right]_0^{2\pi}$$

$$= a^2\left(\left(2\pi + \frac{1}{4}\ \cos 4\pi\right) - \left(0 + \frac{1}{4}\ \cos 0\right)\right)$$

$$= a^2\left(2\pi + \frac{1}{4}\cdot 1 - 0 - \frac{1}{4}\cdot 1\right)$$

$$= 2\pi a^2$$

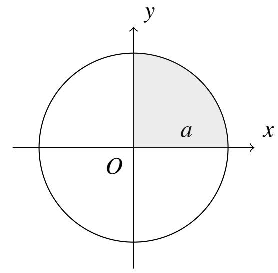
17 Evaluate the integral $\iint_R xy\,dx\,dy$ where R is the 1st quadrant of the circle $x^2 + y^2 = a^2$

Solution:

$$\iint_R xy\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a x \cdot \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx \quad = \int_0^a x \cdot \frac{1}{2}\left(a^2 - x^2\right) dx \quad = \frac{1}{2}\int_0^a \left(a^2 x - x^3\right) dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a \quad = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] \quad = \frac{1}{2}\left[\frac{2a^4 - a^4}{4}\right] = \frac{a^4}{8}$$



Given that,

$$x^2 + y^2 = a^2$$

$$\therefore y^2 = a^2 - x^2$$

$$\therefore y = \pm\sqrt{a^2 - x^2}$$

and,

$$x^2 = a^2$$

$$\therefore x = \pm a$$

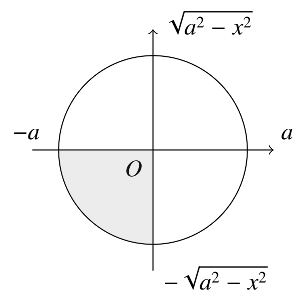
17 Evaluate the integral $\iint_R xy\,dx\,dy$ where R is the 3rd quadrant of the circle $x^2 + y^2 = a^2$

Solution:

$$\iint_R xy\,dx\,dy = \int_{x=-a}^{0} \int_{y=-\sqrt{a^2-x^2}}^{0} xy\,dy\,dx$$

$$= \int_{-a}^0 x \cdot \left[\frac{y^2}{2}\right]_{\sqrt{a^2-x^2}}^{0} dx \quad = \int_{-a}^0 x \cdot -\frac{1}{2}\left(a^2 - x^2\right) dx \quad = -\frac{1}{2}\int_{-a}^0 \left(a^2 x - x^3\right) dx$$

$$= -\frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_{-a}^0 \quad = -\frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] \quad = -\frac{1}{2}\left[\frac{2a^4 - a^4}{4}\right] = -\frac{a^4}{8}$$



Given that,

$$x^2 + y^2 = a^2$$

$$\therefore y^2 = a^2 - x^2$$

$$\therefore y = \pm\sqrt{a^2 - x^2}$$

and,

$$x^2 = a^2$$

$$\therefore x = \pm a$$

Calculate the volume of the solid bounded by the surface $x = 0,\ y = 0,\ x + y + z = 1$ and $z = 0$

Solution:

Volume $= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{0}^{1-x-y} dz\, dy\, dx$

$= \int_{x=0}^{1} \int_{y=0}^{1-x} [z]_0^{1-x-y}\, dy\, dx$

$= \int_{x=0}^{1} \int_{y=0}^{1-x} (1 - x - y)\, dy\, dx$

$= \int_{x=0}^{1} \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$

$= \int_0^1 \left[1 - x - x(1-x) - \frac{1}{2}(1-x)^2 \right] dx$

$= \int_0^1 \left[1 - x - x + x^2 - \frac{1}{2}\left(1^2 - 2.1.x + x^2\right) \right] dx$

$= \int_0^1 \left[1 - 2x + x^2 - \frac{1}{2} + x - \frac{1}{2}x^2 \right] dx$

$= \int_0^1 \left[\frac{1}{2} - x + \frac{1}{2}x^2 \right] dx \quad = \left[\frac{1}{2}x - \frac{x^2}{2} + \frac{1}{2}\frac{x^3}{3} \right]_0^1$

$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \quad = \frac{1}{6}$

Give that

$x + y + z = 1$

$\Rightarrow z = 1 - x - y$

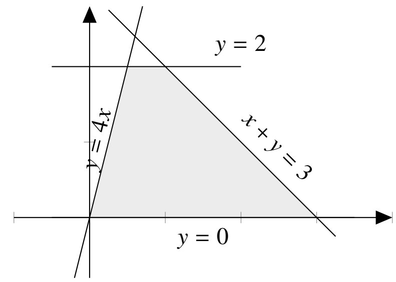
and $x + y = 1$

$\Rightarrow y = 1 - x$

and also

$x = 1$

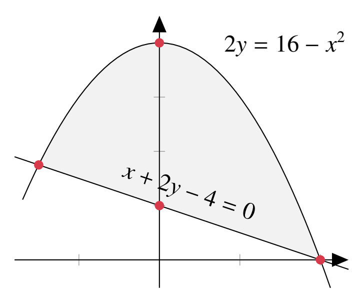
Evaluate $\iint_R \left(x^2+y^2\right)\,dx\,dy$ throughout the area enclosed by the curves $y=4x,\ x+y=3,\ y=0$ and $y=2$.



$$\text{Area} = \iint_R \left(x^2+y^2\right)\,dx\,dy$$
$$= \int_{y=0}^{2}\int_{x=\frac{y}{4}}^{3-y} \left(x^2+y^2\right)\,dx\,dy$$
$$= \int_{y=0}^{2}\left[\frac{x^3}{3}+xy^2\right]_{x=\frac{y}{4}}^{3-y} dy$$
$$= \int_{y=0}^{2}\left[\left(\frac{(3-y)^3}{3}+(3-y)\,y^2\right)-\left(\frac{\left(\frac{y}{4}\right)^3}{3}+\left(\frac{y}{4}\right)y^2\right)\right] dy$$
$$= \int_{y=0}^{2}\left[\frac{1}{3}(3-y)^3+(3y^2-y^3)\ -\frac{y^3}{4^3.\,3}-\frac{y^3}{4}\right] dy$$
$$= \left[\frac{1}{3}\cdot-\frac{(3-y)^4}{4}+\frac{3y^3}{3}-\frac{y^4}{4}\ -\frac{1}{4^3.\,3}\frac{y^4}{4}-\frac{1}{4}\frac{y^4}{4}\right]_{y=0}^{2}$$
$$= \left[\frac{1}{3}\cdot-\frac{(3-2)^4}{4}+\frac{3.\,2^3}{3}-\frac{2^4}{4}\ -\frac{1}{4^3.\,3}\frac{2^4}{4}-\frac{1}{4}\frac{2^4}{4}\right]$$
$$= -\frac{1}{12}+8-4-\frac{1}{48}-1+\frac{81}{12}$$
$$= \frac{463}{48}\quad \text{sq. unit area}$$

Use double integral to find the area bounded by $x+2y-4=0$ and $2y=16-x^2$

Solution:



Given that

$x+2y-4=0 \quad \Rightarrow\ 2y=4-x \quad \Rightarrow\ y=\dfrac{4-x}{2}\ \cdots\ \cdots\ \cdots\ (i)$

Put $x=0,\ \therefore y=2$ and Put $y=0\ \therefore x=4$

$2y=16-x^2 \quad \Rightarrow\ y=8-\dfrac{1}{2}x^2\ \cdots\ \cdots\ \cdots\ (ii)$

Put $x=0,\ \therefore y=8$

Solving equation (i) and (ii),

$4-x=16-x^2\ \Rightarrow\ x^2-x-12=0\ \Rightarrow\ (x-4)(x+3)=0\ \therefore\ x=-3,4$

$$\therefore \text{Area} = \iint_R dx\,dy = \int_{-3}^{4}\int_{y=2-\frac{x}{2}}^{8-\frac{x^2}{2}} dy\,dx$$

$$= \int_{-3}^{4} [y]_{y=2-\frac{x}{2}}^{8-\frac{x^2}{2}}\, dx$$

$$= \int_{-3}^{4}\left(8 - \frac{x^2}{2} - 2 + \frac{x}{2}\right) dx$$

$$= \left[8x - \frac{x^3}{2.3} - 2x + \frac{x^2}{2.2}\right]_{-3}^{4} = \left[6x - \frac{x^3}{6} + \frac{x^2}{4}\right]_{-3}^{4}$$

$$= \left(6.4 - \frac{4^3}{6} + \frac{4^2}{4}\right) - \left(6.(-3) - \frac{(-3)^3}{6} + \frac{(-3)^2}{4}\right)$$

$$= 24 - \frac{64}{6} + 4 + 18 - \frac{27}{6} + \frac{9}{4}$$

$$= \frac{343}{12} \quad \text{sq. unit area}$$

12 Evaluate the followings:

$$(i)\ \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{1} \rho^3 \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta \qquad (ii)\ \int_1^{3}\int_x^{x^2}\int_0^{lnz} xe^y\, dy\, dz\, dx$$

Solution (i):

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^1 \sin\phi\cos\phi\, d\phi\, d\theta \quad = \int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}} \frac{1}{4}\cdot\frac{\sin 2\phi}{2}\, d\phi\, d\theta \quad = \int_{\theta=0}^{\frac{\pi}{2}} \frac{1}{8}\left[\frac{-\cos 2\phi}{2}\right]_0^{\frac{\pi}{2}} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{1}{8}\cdot\frac{-1}{2}(\cos\pi - \cos 0)\, d\theta \quad = -\frac{1}{16}\int_{\theta=0}^{\frac{\pi}{2}}(-1-1)\, d\theta \quad = -\frac{1}{16}\int_0^{\frac{\pi}{2}} -2\, d\theta \quad = -\frac{-2}{16}\int_0^{\frac{\pi}{2}} 1\, d\theta \quad = \frac{1}{8}[\theta]_0^{\frac{\pi}{2}} \quad = \frac{\pi}{16}$$

Solution (ii):

$$\int_1^{3}\int_x^{x^2}\int_0^{lnz} xe^y\, dy\, dz\, dx \quad = \int_1^{3}\int_x^{x^2} x\,[e^y]_0^{\ln z}\, dz\, dx \quad = \int_1^{3}\int_x^{x^2} x\left(e^{\ln z} - e^0\right) dz\, dx \quad = \int_1^{3}\int_x^{x^2} x(z-1)\, dz\, dx$$

$$= \int_1^{3} x\left[\frac{z^2}{2} - z\right]_x^{x^2} dx \quad = \int_1^{3} x\left(\frac{x^4}{2} - x^2 - \frac{x^2}{2} + x\right) dx \quad = \int_1^{3}\left(\frac{1}{2}x^5 - \frac{3}{2}x^3 + x^2\right) dx \quad = \left[\frac{1}{2}\cdot\frac{x^6}{6} - \frac{3}{2}\cdot\frac{x^4}{4} + \frac{x^3}{3}\right]_1^3$$

$$= \left[\frac{x^6}{12} - \frac{3x^4}{8} + \frac{x^3}{3}\right]_1^3 \quad = \left(\frac{3^6}{12} - \frac{3.3^4}{8} + \frac{3^3}{3}\right) - \left(\frac{1}{12} - \frac{3}{8} + \frac{1}{3}\right) \quad = \frac{729}{12} - \frac{243}{8} + \frac{27}{3} - \frac{1}{12} + \frac{3}{8} - \frac{1}{3} \quad = \frac{118}{3}$$

13 Let $\phi = y^2 z$ and V denotes the region bounded by the plane $x + 4y + 2z = 4,\ x = 0,\ y = 0,\ z = 0$. Evaluate $\iiint\limits_V \phi \, dV$.

Solution:

$$\int_{y=0}^{1} \int_{z=0}^{2-2y} \int_{x=0}^{4-4y-2z} y^2 z \; dx\, dz\, dy$$

$$= \int_{y=0}^{1} \int_{z=0}^{2-2y} y^2 z \; [x]_0^{4-4y-2z} \; dz\, dy$$

$$= \int_{y=0}^{1} \int_{z=0}^{2-2y} y^2 z\,(4 - 4y - 2z)\, dz\, dy$$

$$= \int_{y=0}^{1} \int_{z=0}^{2-2y} \left(4y^2 z - 4y^3 z - 2y^2 z^2\right) dz\, dy$$

$$= \int^{1} \left[4y^2 \frac{z^2}{2} - 4y^3 \frac{z^2}{2} - 2y^2 \frac{z^3}{3}\right]_0^{2-2y} dy$$

$$= \int_{y=0}^{1} \left[2y^2(2-2y)^2 - 2y^3(2-2y)^2 - \frac{2}{3}y^2(2-2y)^3\right] dy$$

$$= \int_{y=0}^{1} \left[2y^2\left(4 - 8y + 4y^2\right) - 2y^3\left(4 - 8y + 4y^2\right) - \frac{2}{3}y^2\left(8 - 24y + 24y^2 - 8y^3\right)\right] dy$$

$$= \int_{y=0}^{1} \left(8y^2 - 16y^3 + 8y^4 - 8y^3 + 16y^4 - 8y^5 - \frac{16}{3}y^2 + 16y^3 - 16y^4 + \frac{16}{3}y^5\right)$$

$$= \int_{y=0}^{1} \left(\frac{8}{3}y^2 - 8y^3 + 8y^4 - \frac{8}{3}y^5\right) dy$$

$$= \left[\frac{8}{3}\frac{y^3}{3} - 8\frac{y^4}{4} + 8\frac{y^5}{5} - \frac{8}{3}\frac{y^6}{6}\right]_0^1$$

$$= \frac{8}{9} - 2 + \frac{8}{5} - \frac{4}{5}$$

$$= \frac{2}{45}$$

Given that,

$x + 4y + 2z = 4$

$\Rightarrow\ x = 4 - 4y - 2z$

Now,

$4y - 2z = 4$

$\Rightarrow\ z = 2 - 2y$

Also,

$2 - 2y = 0$

$\Rightarrow\ y = 1$