Linear combination

Linear combination: vector

Linear Combination of vectors: Let V be a vector space over the field F and let $\mathrm{v}_1, \mathrm{v}_2, \ldots\ldots\ldots, \mathrm{v}_\mathrm{n} \in \mathrm{V}$. Then any vector $\mathrm{v} \in \mathrm{V}$ is called a linear combination of $\mathrm{v}_1, \mathrm{v}_2, \ldots\ldots\ldots, \mathrm{v}_\mathrm{n}$ if and only if there exists scalar $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n$ in F such that, $\mathrm{v} = \alpha_1 \mathrm{v}_1 + \alpha_2 \mathrm{v}_2 + \cdots\ldots\ldots + \alpha_n \mathrm{v}_\mathrm{n}$

Example 1: Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\mathbb{R}^3$. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution:

$$
\begin{aligned}
\text{Let, } \quad w &= \alpha_1 u + \alpha_2 v \qquad \alpha_i \in \mathbb{R} \\
\Rightarrow \quad (9, 2, 7) &= \alpha_1 (1, 2, -1) + \alpha_2 (6, 4, 2) \\
\Rightarrow \quad (9, 2, 7) &= (\alpha_1, 2\alpha_1, -\alpha_1) + (6\alpha_2, 4\alpha_2, 2\alpha_2) \\
&= (\alpha_1 + 6\alpha_2, \quad 2\alpha_1 + 4\alpha_2, \quad -\alpha_1 + 2\alpha_2)
\end{aligned}
$$

Equating corresponding components:

$$
\left.
\begin{aligned}
\alpha_1 + 6\alpha_2 &= 9 \\
2\alpha_1 + 4\alpha_2 &= 2 \\
-\alpha_1 + 2\alpha_2 &= 7
\end{aligned}
\right\} \quad \cdots\cdots\cdots (1)
$$

Writing (1) in matrix form $A\mathbf{x} = \mathbf{b}$,

$$
A = \begin{pmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix}
$$

Reduce augmented matrix $(A \mid \mathbf{b})$ to echelon form by e.r.o

$$
(A \mid \mathbf{b}) = \left(\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)
$$

Corresponding system,

$$\begin{aligned} \alpha_1 + 6\alpha_2 &= 9 \\ \alpha_2 &= 2 \end{aligned}$$

By backward substitution,

$$\alpha_1 = -3 \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\therefore w = -3u + 2v$$

Therefore, w is a linear combination of u and v

Example 2: Is the vector $v = (2, -5, 3)$ in $\mathbb{R}^3$ is a linear combination of the vectors $v_1 = (1, -3, 2)$, $v_2 = (2, -4, -1)$ and $v_3 = (1, -5, 7)$

Solution:

$$\begin{aligned} \text{Let,} \quad v &= \alpha_1 v_1 + \alpha_2 v_2 + \alpha_2 v_3 \qquad \alpha_i \in \mathbb{R} \\ \Rightarrow \quad (2, -5, 3) &= \alpha_1(1, -3, 2) + \alpha_2(2, -4, -1) + \alpha_2(1, -5, 7) \\ \Rightarrow \quad (2, -5, 3) &= (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -4\alpha_2, -\alpha_2) + (\alpha_3, -5\alpha_3, 7\alpha_3) \\ &= (\alpha_1 + 2\alpha_2 + \alpha_3, \quad -3\alpha_1 - 4\alpha_2 - 5\alpha_3, \quad 2\alpha_1 - \alpha_2 + 7\alpha_3) \end{aligned}$$

Equating corresponding components:

$$\left.\begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 2 \\ -3\alpha_1 - 4\alpha_2 - 5\alpha_3 &= -5 \\ 2\alpha_1 - \alpha_2 + 7\alpha_3 &= 3 \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

Writing (1) in matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

Reduce augmented matrix $(A \mid \mathbf{b})$ to echelon form by e.r.o

$$(A \mid \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

Corresponding system,

$$\begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 2 \\ 2\alpha_2 - 2\alpha_3 &= 1 \\ 0 &= 3 \end{aligned}$$

There is an equation in the form $0 = b$ and $b \neq 0$

Hence the above system is inconsistent and it has no solution.

Therefore, the vector v is not a linear combination of v_1, v_2 and v_3 .

Linear combination: Matrix

Example: Write the matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ as a linear combination of the matrices $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$

Solution: Let, $S = \{A_1, A_2, A_3\}$ and $M = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

Set, $M = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$

$$\Rightarrow \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_1 \\ \alpha_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_2 & \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 & 2\alpha_3 \\ 0 & -\alpha_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 & \alpha_1 + 2\alpha_3 \\ \alpha_1 + \alpha_2 & \alpha_2 - \alpha_3 \end{pmatrix}$$

Equating corresponding components:

$$\left.\begin{aligned} \alpha_1 &= 3 \\ \alpha_1 + 2\alpha_3 &= 1 \\ \alpha_1 + \alpha_2 &= 1 \\ \alpha_2 - \alpha_3 &= -1 \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

Writing (1) in matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

Reduce augmented matrix $(A \mid \mathbf{b})$ to echelon form by e.r.o

$$(A \mid \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right) \xrightarrow[R_4 \leftrightarrow R_2]{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 \end{array}\right) \xrightarrow[R_4' = R_4 - R_1]{R_3' = R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 2 \\ 0 & -1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[R_4' = R_4 + R_2]{R_3' = R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R_4' = R_4 + R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Corresponding system,

$$\begin{aligned} \alpha_1 + \alpha_2 \qquad &= 1 \\ \alpha_2 - \alpha_3 &= -1 \\ -\alpha_3 &= 1 \end{aligned}$$

By backward substitution,

$$\begin{aligned} \alpha_3 &= -1 \\ \alpha_2 &= -2 \\ \alpha_1 &= 3 \end{aligned} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

$$\therefore A = 3A_1 - 2A_2 - A_3$$

Therefore, the matrix A is a linear combination of $A1$, A_2 and A_3

Spanning: Polynomial

Example: Determine whether the polynomials:

$$\left.\begin{aligned} P_1 &= 1 - x + 2x^2 \\ P_2 &= 3 + x \\ P_3 &= 5 - x + 4x^2 \\ P_4 &= -2 - 2x + 2x^2 \end{aligned}\right\} \text{spans } \mathbb{P}_2$$

Solution: Let $P \in \mathbb{P}_2$ be anbitary, then:

$$P = a + bx + cx^2.$$

$$\text{Set, } P = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4$$

$$\begin{aligned} a + bx + cx^2 &= \alpha_1(1 - x + 2x^2) + \alpha_2(3 + x) + \alpha_3(5 - x + 4x^2) + \alpha_4(-2 - 2x + 2x^2) \\ &= (\alpha_1 + 3\alpha_2 + 5\alpha_3 - 2\alpha_4) + (-\alpha_1 + \alpha_2 - \alpha_3 - 2\alpha_4)x + (2\alpha_1 + 0 + 4\alpha_3 + 2\alpha_4)x^2 \end{aligned}$$

Equating corresponding components:

$$\begin{aligned} \alpha_1 + 3\alpha_2 + 5\alpha_3 - 2\alpha_4 &= a \\ -\alpha_1 + \alpha_2 - \alpha_3 - 2\alpha_4 &= b \\ 2\alpha_1 + 0 + 4\alpha_3 + 2\alpha_4 &= c \end{aligned}$$

Writing (1) in matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 1 & 3 & 5 & -2 \\ -1 & 1 & -1 & -2 \\ 2 & 0 & 4 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Reduce augmented matrix $(A \mid \mathbf{b})$ to echelon form by e.r.o

$$(A \mid \mathbf{b}) = \left(\begin{array}{cccc|c} 1 & 3 & 5 & -2 & a \\ -1 & 1 & -1 & -2 & b \\ 2 & 0 & 4 & 2 & c \end{array}\right) \xrightarrow{R_2' = R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 3 & 5 & -2 & a \\ 0 & 4 & 4 & -4 & a+b \\ 0 & -6 & -6 & 6 & c-2a \end{array}\right)$$

$$\xrightarrow{R_3' = 4R_3 + 6R_2} \left(\begin{array}{cccc|c} 1 & 3 & 5 & -2 & a \\ 0 & 4 & 4 & -4 & a+b \\ 0 & 0 & 0 & 0 & 6c-8a+4 \end{array}\right)$$

Corresponding system,

$$\begin{aligned} \alpha_1 + 3\alpha_2 + 5\alpha_3 - 2\alpha_4 &= a \\ 4\alpha_2 + 4\alpha_3 - 4\alpha_4 &= a+b \\ 0 &= 6c - 8a + 4b \end{aligned}$$

There is an equation in the form $0 = b$ and $b \neq 0$

Hence the above system is inconsistent and it has no solution.

Therefore, the polynomials doesn't not spans $\mathbb{P}_2$

Linear Independence: vector

Example: Determine wheather the vactors $v_1 = (1, -2, 3)$, $v_2 = (5, 6, -1)$, $v_3 = (3, 2, 1)$ are Linearly Dependent or Independent in $\mathbb{R}^3$

Solution: Let, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \quad \alpha_i \in \mathbb{R}$

$$\begin{aligned} \alpha_1(1,-2,3) + \alpha_2(5,6,-1) + \alpha_3(3,2,1) &= (0,0,0) \\ (\alpha_1, -2\alpha_1, 3\alpha_1) + (5\alpha_2, 6\alpha_2, -\alpha_2) + (3\alpha_3, 2\alpha_3, \alpha_3) &= (0,0,0) \\ (\alpha_1 + 5\alpha_2 + 3\alpha_3, \quad -2\alpha_1 + 6\alpha_2 + 2\alpha_3, \quad 3\alpha_1 - \alpha_2 + \alpha_3) &= (0,0,0) \end{aligned}$$

Equating corresponding components:

$$\left.\begin{aligned} \alpha_1 + 5\alpha_2 + 3\alpha_3 &= 0 \\ -2\alpha_1 + 6\alpha_2 + 2\alpha_3 &= 0 \\ 3\alpha_1 - \alpha_2 + \alpha_3 &= 0 \end{aligned}\right\} \quad \cdots\cdots\cdots (1)$$

Writing (1) in matrix form $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Reduce augmented matrix $(A \mid \mathbf{b})$ to echelon form by e.r.o

$$(A \mid \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right) \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{array}\right) \xrightarrow[R_3' = R_3 + R_2]{R_2' = \frac{R_2}{8}} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Corresponding system,

$$\begin{aligned} \alpha_1 + 5\alpha_2 + 3\alpha_3 &= 0 \\ 2\alpha_2 + \alpha_3 &= 0 \end{aligned}$$

The system has a non-trivial solution. Therefore the vectors v_1, v_2, v_3 are Linearly Dependent. And, at least one of them is a linear combination of the others.

Let, $\alpha_3 = -2$ Now, $\alpha_2 = 1, \quad \alpha_1 = 1$

Therefore, $v_1 + v_2 - 2v_3 = 0$

$$v_1 = 2v_3 - v_2$$

Therefore, v_1 is a linear combination of v_2 and v_3