Continuity

Definition Of Continuity

A function $f(x, y)$ is said to be continuous at (x_o, y_o) provided the following conditions are satisfied:

1. $f(x_o, y_o)$ is defined.
2. $\underset{(x,y)\to(x_o,y_o)}{\mathrm{Lt}} f(x, y)$ exists.
3. $\underset{(x,y)\to(x_o,y_o)}{\mathrm{Lt}} f(x, y) = f(x_o, y_o)$.

Mathematical Problems

Exr1. Test the continuity for $f(x, y)$ at $(x, y) = (0, 0)$ where,

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & when\ (x, y) \neq (0, 0) \\ 0, & when\ (x, y) = (0, 0) \end{cases}$$

Solution:

At, $(x, y) = (0, 0) \quad f(x, y) = 0$ defined.

At, $(x, y) \neq (0, 0) \quad f(x, y) = \dfrac{2xy}{x^2 + y^2}$

Now,

$$\begin{aligned} \underset{(x,y)\to(0,0)}{\mathrm{Lt}} f(x, y) &= \underset{(x,y)\to(0,0)}{\mathrm{Lt}} \frac{2xy}{x^2 + y^2} \\ &= \underset{(x,y)\to(0,0)}{\mathrm{Lt}} \frac{2xmx}{x^2 + m^2x^2} \qquad [\text{along } y = mx] \\ &= \underset{(x,y)\to(0,0)}{\mathrm{Lt}} \frac{x^2\, 2m}{x^2(1 + m^2)} = \underset{(x,y)\to(0,0)}{\mathrm{Lt}} \frac{2m}{(1 + m^2)} \end{aligned}$$

Since, this limit changes with the value of m. So, there is no single value of m.

Hence, $\underset{(x,y)\to(0,0)}{\mathrm{Lt}} f(x, y)$ does not exist.

$\therefore f(x, y)$ is not continuous at $(0, 0)$.

Exr2. Test the continuity for $f(x, y)$ at $(x, y) = (0, 0)$ where,

$$f(x,y) = \begin{cases} xy \ln(x^2 + y^2), & when\ (x, y) \neq (0, 0) \\ 0, & when\ (x, y) = (0, 0) \end{cases}$$

Solution:

At, $(x, y) = (0, 0) \quad f(x, y) = 0$ defined.

At, $(x, y) \neq (0, 0) \quad f(x, y) = xy \ln(x^2 + y^2)$

Now,

$$\underset{(x,y)\to(0,0)}{\text{Lt}} f(x,y) = \underset{(x,y)\to(0,0)}{\text{Lt}} xy\ln(x^2+y^2)$$

$$= \underset{r\to 0^+}{\text{Lt}}\ r\cos\theta \cdot r\sin\theta \cdot \ln r^2$$

$$= \underset{r\to 0^+}{\text{Lt}}\ r^2 \cdot \frac{\sin 2\theta}{2} \cdot \ln r^2 \quad \cdots \cdots \cdots \ (1)$$

using polar coordinates,

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

when $(x,y) = (0,0)$, then $r \to 0^+$.

Since: $\sin 2\theta \le 1$,

$$\left|xy\ln(x^2+y^2)\right| = \left|\frac{r^2\sin 2\theta\ \ln r^2}{2}\right| = \left|\frac{r^2\ln r^2}{2}\right|$$

from (1),

$$\underset{r\to 0}{\text{Lt}} \frac{r^2\ln r^2}{2} = \underset{r\to 0}{\text{Lt}} \frac{\ln r^2}{\frac{2}{r^2}} = \underset{r\to 0}{\text{Lt}} \frac{\frac{1}{r^2}2r}{\frac{-4}{r^3}} = \underset{r\to 0}{\text{Lt}} \frac{-r^2}{2} = 0$$

Since, $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x,y)$ exist and also, $\underset{(x,y)\to(0,0)}{\text{Lt}} f(x,y) = f(0,0)$

$\therefore f(x,y)$ is continuous at $(0,0)$.

Exr3. Show that the function $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ has no limit as (x,y) approaches $(0,0)$.

Solution:

Along the curve $y = mx^2$, $x \neq 0$, the function,

$$f(x,y)|_{y=mx^2} = \frac{2x^2(mx^2)}{x^4+(mx^2)^2} = \frac{2mx^4}{x^4(1+m^2)} = \frac{2m}{1+m^2}$$

$$\underset{(x,y)\to(0,0)}{\text{Lt}} f(x,y) = \underset{(x,y)\to(0,0)}{\text{Lt}} f(x,y)|_{y=mx^2} = \frac{2m}{1+m^2} = constant$$

This limit varies with the path of approaches.

If (x,y) approaches $(0,0)$ along the parabola $y = mx^2$ for $(m = 1)$ and the limit is 1 and

if (x,y) approaches $(0,0)$ along the x-axis $y = 0$ for $(m = 0)$ and the limit is 0.

By two path test f has no limit as (x,y) approaches $(0,0)$.

Exr4. Test the continuity for $f(x,y)$ at $(x,y)=(0,0)$ where,

$$f(x,y)=\begin{cases}\dfrac{\sin(x^2+y^2)}{x^2+y^2}, & when\ (x,y)\neq(0,0)\\ 1, & when\ (x,y)=(0,0)\end{cases}$$

Solution:

At, $(x,y)=(0,0)$ $\quad f(x,y)=1$ $\quad$ defined.

At, $(x,y)\neq(0,0)$ $\quad f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$

Now, $\underset{(x,y)\to(0,0)}{\mathrm{Lt}} f(x,y) = \underset{(x,y)\to(0,0)}{\mathrm{Lt}} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = 1$

Also, $\underset{(x,y)\to(0,0)}{\mathrm{Lt}} f(x,y)=f(0,0)$

$\therefore f(x,y)$ is continuous at $(0,0)$.

Partial Derivatives

If $y=f(x)$, then the derivative of $y=f(x)$ with respect to x,

$\dfrac{dy}{dx}$ or $f'(x)=\underset{h\to0}{\mathrm{Lt}}\dfrac{f(x+h)-f(x)}{h}$ if exists.

If $u=f(x,y)$ then the partial derivatives, of $u=f(x,y)$ with respect to x, keeping y as constant, denoted by $\dfrac{\partial y}{\partial x}$ or $f_x(x,y)=\underset{h\to0}{\mathrm{Lt}}\dfrac{f(x+h,y)-f(x,y)}{h}$ if exists.

Exr1. Given $f(x,y)=x^3+3xy^2+6y^3$. Now, find the following,

$(i)\ f_x(x,y),\quad (ii)\ f_y(x,y)\quad (iii)\ f_{xy}(x,y),\quad (iv)\ f_{yx}(x,y)\quad (v)\ f_{xx}(x,y),\quad (vi)\ f_{yy}(x,y).$

Solution:

$$(i)\ f_x(x,y) = \frac{\partial}{\partial x}[f(x,y)] = \frac{\partial}{\partial x}\left[x^3+3xy^2+6y^3\right] = 3x^2+3y^2+0 = 3x^2+3y^2$$

$$(ii)\ f_y(x,y) = \frac{\partial}{\partial y}[f(x,y)] = \frac{\partial}{\partial y}\left[x^3+3xy^2+6y^3\right] = 0+6xy+18y^2 = 6xy+18y^2$$

$$(iii)\ f_{xy}(x,y) = \frac{\partial}{\partial y}[f_x(x,y)] = \frac{\partial}{\partial y}\left[3x^2+3y^2\right] = 0+6y = 6y$$

$$(iv)\ f_{yx}(x,y) = \frac{\partial}{\partial x}\left[f_y(x,y)\right] = \frac{\partial}{\partial x}\left[6xy+18y^2\right] = 6y+0 = 6y$$

$$(v)\quad f_{xx}(x,y) = \frac{\partial}{\partial x}[f_x(x,y)] = \frac{\partial}{\partial x}\left[3x^2+3y^2\right] = 6x+0 = 6x$$

$$(vi)\quad f_{yy}(x,y) = \frac{\partial}{\partial y}\left[f_y(x,y)\right] = \frac{\partial}{\partial y}\left[6xy+18y^2\right] = 6x+36y$$

$$(vii)\quad f_{xx}(1,2) = 6x = 6(1) = 6$$

$$(viii)\quad f_{yy}(1,2) = 6(1)+36(2) = 6+72 = 78$$

Exr2. If $Z = f(x,y) = \ln(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right)$. Prove that $Z_{xx} + Z_{yy} = 0$

Solution:

$$Z_x = \frac{\partial}{\partial x}[Z] = \frac{\partial}{\partial x}\left[\ln(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{2x}{(x^2+y^2)} + 2.\left(\frac{1}{1+\frac{y^2}{x^2}}\right).\left(-\frac{y}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{2y}{x^2+y^2} = \frac{2x-2y}{x^2+y^2}$$

$$Z_y = \frac{\partial}{\partial y}[Z] = \frac{\partial}{\partial y}\left[\ln(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{2y}{(x^2+y^2)} + 2.\left(\frac{1}{1+\frac{y^2}{x^2}}\right).\left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{2x}{x^2+y^2} = \frac{2x+2y}{x^2+y^2}$$

$$Z_{xx} = \frac{\partial}{\partial x}[Z_x] = \frac{\partial}{\partial x}\left[\frac{2x-2y}{x^2+y^2}\right] = \frac{(x^2+y^2)(2)-(2x-2y)(2x)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4x^2+4xy}{(x^2+y^2)^2} = \frac{2y^2-2x^2+4xy}{(x^2+y^2)^2}$$

$$Z_{yy} = \frac{\partial}{\partial y}[Z_y] = \frac{\partial}{\partial y}\left[\frac{2x+2y}{x^2+y^2}\right] = \frac{(x^2+y^2)(2)-(2x+2y)(2y)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4y^2-4xy}{(x^2+y^2)^2} = \frac{2x^2-2y^2-4xy}{(x^2+y^2)^2}$$

$$L.H.S = Z_{xx}+Z_{yy} = \frac{2y^2-2x^2+4xy}{(x^2+y^2)^2} + \frac{2x^2-2y^2-4xy}{(x^2+y^2)^2} = \frac{2y^2-2x^2+4xy+2x^2-2y^2-4xy}{(x^2+y^2)^2} = 0 = R.H.S$$

(Proved)

Exr3. If the resistors of R_1, R_2, and R_3 ohms are connected in paraller to make an R ohms resistor, the value of R can be found from the equation: $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.
Find The value of $\frac{\partial R}{\partial R_2}$ where, $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

Solution:

We have to find $\frac{\partial R}{\partial R_2} = \frac{\partial}{\partial R_2}(R)$

Now,

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\Rightarrow -\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2^2} + 0$$

$$\Rightarrow \frac{\partial R}{\partial R_2} = \left(\frac{R}{R_2}\right)^2 = \left(\frac{15}{45}\right)^2 = \frac{1}{9}$$

Since,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$= \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3+2+1}{90} = \frac{1}{15}$$

$$\therefore R = 15$$

Curl, Gradient, Divergence & Laplacian

Let ϕ be a scalar function and V be a vector, then,

Gradient: The gradient of ϕ, denoted by $\text{grad}\,\phi$ or $\underline{\nabla}\phi$ and defined as

$$\text{grad}\,\phi = \underline{\nabla}\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

Divergence: The divergence of V, denoted by $\text{div}\,V$ and defined as

$$\text{div}\,V = \underline{\nabla}\cdot\underline{\text{V}} = \left(\underline{\text{i}}\,\frac{\partial}{\partial x} + \underline{\text{j}}\,\frac{\partial}{\partial y} + \underline{\text{k}}\,\frac{\partial}{\partial z}\right)\cdot\left(v_1\,\underline{\text{i}} + v_2\,\underline{\text{j}} + v_3\,\underline{\text{k}}\right) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad \text{(scaler)}$$

Carl: The curl of V, denoted by $\text{curl}\,V$ and defined as

$$\text{curl}\,V = \underline{\nabla}\times\underline{\text{v}} = \left(\underline{\text{i}}\,\frac{\partial}{\partial x} + \underline{\text{j}}\,\frac{\partial}{\partial y} + \underline{\text{k}}\,\frac{\partial}{\partial z}\right)\times\left(v_1\,\underline{\text{i}} + v_2\,\underline{\text{j}} + v_3\,\underline{\text{k}}\right) = \begin{vmatrix} \underline{\text{i}} & \underline{\text{j}} & \underline{\text{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Laplacian:

$$\underline{\nabla}\cdot(\underline{\nabla}\phi) = \underline{\nabla}\cdot\left(\underline{\text{i}}\frac{\partial\phi}{\partial x} + \underline{\text{j}}\frac{\partial\phi}{\partial y} + \underline{\text{k}}\frac{\partial\phi}{\partial z}\right) = \left(\underline{\text{i}}\frac{\partial}{\partial x} + \underline{\text{j}}\frac{\partial}{\partial y} + \underline{\text{k}}\frac{\partial}{\partial z}\right)\cdot\left(\underline{\text{i}}\frac{\partial\phi}{\partial x} + \underline{\text{j}}\frac{\partial\phi}{\partial y} + \underline{\text{k}}\frac{\partial\phi}{\partial z}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}$ is called the laplacian operator.

Exr-1: If $\phi(x,y,z) = 3x^2y - y^3z^2$ find the grad ϕ at the point $(1,-2,-1)$

Solution:

$$\begin{aligned}
\text{grad}\,\phi &= \underline{i}\,\frac{\partial\phi}{\partial x} + \underline{j}\,\frac{\partial\phi}{\partial y} + \underline{k}\,\frac{\partial\phi}{\partial z}\\
&= \underline{i}\,\frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right) + \underline{j}\,\frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right) + \underline{k}\,\frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)\\
&= \underline{i}\,(6xy - 0) + \underline{j}\,\left(3x^2 - 3y^2z^2\right) + \underline{k}\,\left(0 - 2y^3z\right)
\end{aligned}$$

At the point $(1,-2,-1)$,

$$\text{grad}\,\phi = \underline{i}\,\{6(1)(-2) - 0\} + \underline{j}\,\left\{3(1)^2 - 3(-2)^2(-1)^2\right\} + \underline{k}\,\left\{0 - 2(-2)^3(-1)\right\} = -12\underline{i} - 9\underline{j} - 16\underline{k}$$

Now, $\text{curl}\,(\text{grad}\,\phi) = \underline{\nabla} \times (\text{grad}\,\phi)$

$$\begin{aligned}
&= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy & 3x^2 - 3y^2z^2 & -2y^3z \end{vmatrix}\\
&= \underline{i}\,\left\{-6y^2z - 0 + 6y^2z\right\} - \underline{j}\,\{0 - 0\} + \underline{k}\,\{6x - 0 - 6x\}\\
&= 0\,\underline{i} + 0\,\underline{j} + 0\,\underline{k} = \underline{0}
\end{aligned}$$

Exr-2: Prove that (i) the curl of the gradient of scaler function ϕ is zero

(ii) the divergence of the carl of a vector $\underline{u}$ is zero(sealer).

Solution:

(i) Let ϕ be a scalar function then $\text{grad}\,\phi = \underline{\nabla}\phi = \underline{i}\dfrac{\partial\phi}{\partial x} + \underline{j}\dfrac{\partial\phi}{\partial y} + \underline{k}\dfrac{\partial\phi}{\partial z}$

Now, $\text{curl}\,(\text{grad}\,\phi) = \underline{\nabla} \times (\underline{\nabla}\phi)$

$$\begin{aligned}
&= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix}\\
&= \underline{i}\left\{\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right\} - \underline{j}\left\{\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right\} + \underline{k}\left\{\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right\}\\
&= \underline{0}
\end{aligned}$$

(ii) Here, $\underline{u} = u_1\,\underline{i} + u_2\,\underline{j} + u_3\,\underline{k}$

$$\text{curl}\,\underline{u} = \underline{\nabla} \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{v} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= \underline{i}\left\{\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right\} - \underline{j}\left\{\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right\} + \underline{k}\left\{\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right\}$$

Now,

$$\text{div}\left(\text{curl}\,\underline{u}\right) = \underline{\nabla}\cdot(\text{curl}\,\underline{u})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(\underline{i}\left\{\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right\} - \underline{j}\left\{\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right\} + \underline{k}\left\{\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right\}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right]$$

$$= \frac{\partial^2 u_3}{\partial x\partial y} - \frac{\partial^2 u_2}{\partial x\partial z} - \frac{\partial^2 u_3}{\partial y\partial x} + \frac{\partial^2 u_1}{\partial y\partial z} + \frac{\partial^2 u_2}{\partial z\partial x} - \frac{\partial^2 u_1}{\partial z\partial y}$$

$$= 0$$

Exr-3: If $\underline{u} = 3x^2y\,\underline{i} + 5xy^2z\,\underline{j} + xyz^3\,\underline{k}$ find the divergence of $\underline{u}$ at $(1,2,3)$ and gradient of that divergence.

Solution:

$$\text{div}\,\underline{u} = \underline{\nabla}\cdot\underline{u} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(3x^2y\,\underline{i} + 5xy^2z\,\underline{j} + xyz^3\,\underline{k}\right)$$

$$= \frac{\partial}{\partial x}\left(3x^2y\right) + \frac{\partial}{\partial y}\left(5xy^2z\right) + \frac{\partial}{\partial z}\left(xyz^3\right)$$

$= 6xy + 10xyz + 3xyz^2 \rightarrow$ scaler quantity

At the point $(1,2,3)$, $\text{div}\,\underline{u} = (6.1.2) + (10.1.2.3) + (3.1.2.9) = 12 + 60 + 59 = 126$

Now,

$$\text{grad}(\text{div}\,\underline{u}) = \underline{i}\,\frac{\partial}{\partial x}\left(6xy + 10xyz + 3xyz^2\right) + \underline{j}\,\frac{\partial}{\partial y}\left(6xy + 10xyz + 3xyz^2\right) + \underline{k}\,\frac{\partial}{\partial z}\left(6xy + 10xyz + 3xyz^2\right)$$

$$= \left(6y + 10yz + 3yz^2\right)\underline{i} + \left(6x + 10xz + 3xz^2\right)\underline{j} + (0 + 10xy + 6xyz)\,\underline{k}$$

Solenoidal vector: If $\underline{\nabla} \cdot \underline{A} = 0$ Then $\underline{A}$ is called solenoidal.

Irrotational vector: If $\underline{\nabla} \times \underline{A} = \underline{0}$ Then $\underline{A}$ is called, irrotational.

Exr-4: Find the directional derivation of $\phi(x, y, z) = 4xz^3 - 3x^2y^2z$ at point $(2, -1, 2)$ in the direction $2\underline{i} - 3\underline{j} + 6\underline{k}$

Here,

$$\underline{\nabla}\phi = \underline{i}\,\frac{\partial}{\partial x}\left(4xz^3 - 3x^2y^2z\right) + \underline{j}\,\frac{\partial}{\partial y}\left(4xz^3 - 3x^2y^2z\right) + \underline{k}\,\frac{\partial}{\partial z}\left(4xz^3 - 3x^2y^2z\right)$$

$$= \left(4z^3 - 6xy^2z\right)\underline{i} + \left(0 - 6x^2yz\right)\underline{j} + \left(12xz^2 - 3x^2y^2\right)\underline{k}$$

At $(2, -1, 2)$, $\underline{\nabla}\phi = (32 - 24)\underline{i} - (0 - 6.4.(-1).2)\,\underline{j} + (96 - 12)\underline{k} = 8\underline{i} + 48\underline{j} + 84\underline{k}$

$$\therefore \underline{a} = \frac{2i + 3\underline{j} + 6\underline{k}}{\sqrt{4 + 9 + 36}} = = \frac{2}{7}\underline{i} - \frac{3}{7}\underline{j} + \frac{6}{7}\underline{k}$$

Now,

Directional Derivation $D.D = \underline{\nabla}\phi \cdot \underline{a}$

$$= \left(8\underline{i} + 48\underline{j} + 84\underline{k}\right) \cdot \left(\frac{2}{7}\underline{i} - \frac{3}{7}\underline{j} + \frac{6}{7}\underline{k}\right)$$

$$= 8 \cdot \frac{2}{7} + 48 \cdot \frac{-3}{7} + 84 \cdot \frac{6}{7} = \frac{16 - 144 + 504}{7} = \frac{376}{7}$$