Computability and Complexity

Problem Set 4

Rice's theorem and applications

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1. Let $A = \{i \mid \exists x, y : \varphi_i(x) = \varphi_i(y) \neq \perp, x \neq y\}$ be the set of programs that halts with returning the same value for at least two different inputs. Using Rice's theorem, show that A is not recursive.
2. Let $A \subseteq \mathbb{N}$, $A \neq \mathbb{N}$ and $A \neq \emptyset$. If it exists $i \in A$ and $j \in \mathbb{N} \setminus A$ such that $\varphi_i = \varphi_j$, can we say that A is recursive ? Prove your answer.
3. Let $A \subseteq \mathbb{N}$ not recursive ($A \neq \mathbb{N}$ and $A \neq \emptyset$). Can we say that $\forall i \in A$ and $\forall j \in \mathbb{N} \setminus A$, $\varphi_i \neq \varphi_j$? Prove your answer.
4. Let $A = \{i \mid P_i(x)$ halts for every input $x\}$.
   (a) Is A recursive ? Why ?
   (b) Is A recursively enumerable ? Why ? (Hint : use the reduction method.)

Mini project
Write in Python two different programs. Both take one argument which is a text file containing the source code of a valid deterministic Python program that has no argument. The first one has to check if the source code of the input file has exactly 2 lines of code. The second one has to check if the program described by the source code of the input file is will always return 2. **Read all subquestions before starting !**

- What do you think the easier program to write will be ?
- Look up for the **compile()** Python function [1]. Do you think you can write the second program without it ? Explain.
- Test your programs on some examples.
- What does Rice's theorem tells you about those two programs ? What is the difference between the two in terms of computability ?
- How does that reflect in the code of your programs ?

1. Be extremely careful with it ! This is a very dangerous function !

1) • $A \neq 0 \rightarrow P(x) \equiv$ return 1

• $A \neq \mathbb{N} \rightarrow P(x) \equiv$ return x

• $\forall i \in A, j \in \bar{A} \mid \varphi_i \neq \varphi_j$:

explicitons ce qu'appartenir à $\bar{A}$ signifie

$\Rightarrow \forall j \in \bar{A}$:

$\forall x, y : x \neq y \Rightarrow \varphi_j(x) \neq \varphi_j(y) \vee \varphi_j(x) = \perp \vee \varphi_j(y) = \perp$

2) Prenons ensemble non récursif (HALT, K, ...)

$\Rightarrow$ l'ensemble $K = \{ n \mid P_n(n)$ se termine$\}$

$P_{101} \equiv$ [return 1 $\quad P_i(x) \equiv$ [$x = 1$; return x

$A \neq 0$, $A \neq \mathbb{N}$ et $\exists i \in A, j \in \bar{A} : \varphi_i = \varphi_j$

$\Rightarrow$ Prouvons que cette ensemble est non récursif.

Par l'absurde, si A récursif, $P_n(n)$ décide A

$\Rightarrow$ on peut décider HALT.

$P_{HALT}(n,h) \equiv$ Voir slides

3. Faux, contraposée de la quest° précédente

$\Rightarrow$ contre exemple : celui de la quest° 2