Computability and Complexity
Problem Set 5
Fixed point theorem and applications

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1. The **S** property is defined as follows :

$$\forall k \; \exists S \text{ total \& computable } : \; \varphi_k(x, y) = \varphi_{S(x)}(y)$$

   Prove that the **S** property is a particular case of S-m-n (i.e. prove that S-m-n implies **S** for $m = n = 1$).

2. Using the fixed point theorem, show that there exists a program P_n such that P_n terminates only for input n. (Hint : use the function $g(n, x) = 1$ if $x = n$, $\perp$ otherwise together with the **S** property)

3. Using the fixed point theorem, show that there exists a program P_n that always outputs n (i.e. that prints its source code).

4. Prove Rice's theorem using the fixed point theorem.
   (Hint : define the function $f(x) = i$ if $x \in A$, j if $x \in \overline{A}$, with $i \in \overline{A}$ and $j \in A$)

5. Prove that $K = \{n \in \mathbb{N} \mid \varphi_n(n) \neq \perp\}$ is not recursive using the fixed point theorem.

Challenge

Show that, for any computable total function f, there exist an infinity of k's such that $\varphi_k = \varphi_{f(k)}$.

(Hint : Show that if it was not the case, we could find a computable total function that would not satisfy the fixed point theorem)

1. $\underline{S-m-n}$ $\to$ $\exists f$ totale calculable

$\forall k$; $\varphi_k(x,y) = \varphi_{f(k,x)}(y)$

$\underline{S}$ $\to$ $\forall k$ $\exists f$ totale calculable : $\varphi_k(x,y) = \varphi_{f(k,x)}(y)$

$f(k,x) = \varphi_S(k,x) = \varphi_{k'}(x)$

2. Soit $g(n,x) = 1$ $\begin{cases} \text{si } n = x \\ \perp \text{ sinon} \end{cases}$

- Avec S, $\exists$ S totale calculable : $\varphi_{g(n,x)} = \varphi_{S(n)}(x)$

- $\underline{\text{Point fixe}}$: $\exists$ pgm n ; $\varphi_n = \varphi_{S(n)}$

$\to$ $\varphi_n(x) = \varphi_{S(n)}(x) = \varphi_{g(n,x)}$

$\to$ g calculable $\to$ p existe

3. $f(n,x) = $ calculable

- avec S, $\exists$ S totale calculable : $\varphi_{f(n,x)} = \varphi_{S(n)}(x)$

- $\underline{\text{pt fixe}}$: $\exists$ pgm n : $\varphi_n = \varphi_{S(n)}$

$\to$ $\exists n$: $\varphi_{f(n,x)} = \varphi_{S(n)}(x) = \varphi_n(x)$

$\to f$ calculable $\to \varphi$ existe

4. Soit $i \in A$, $j \in \bar{A}$

$$\to f(x) = \begin{cases} j & \text{si } x \in A \\ i & x \in \bar{A} \end{cases}$$

Si A récursif, f calculable

Point fixe : $\exists\, h$; $\varphi_h = \varphi_{f(h)}$

si $h \in A$; $\varphi_h = \varphi_j$

si $h \in \bar{A}$; $\varphi_h = \varphi_i$