Computer Systems Principles

Data Representation - Bits and Bytes

UMASSCS

SCHOOL OF COMPUTER SCIENCE

Today’s Class

1. Learn data representation in binary and hexadecimal
   – represent negative numbers.
2. Perform operations
   – addition, subtraction, multiplication

- Announcements: Moodle Quiz 2 & HW 1 are out.

Group Task

- Form partners
- Using only the three symbols @#& represent:
  - integers 0 – 10

How do we think about numbers?

- Representing and reasoning about numbers
  - Computers store variables (data).
  - Data is typically composed of numbers and characters.
  - Want a representation that is sensible.

Decimal Number Systems

- Digits 0-9

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- 171_{10}=1*100+7*10+1= 171

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Decimal Number Systems

- Digits 0-9
- 171_{10}=1*100+7*10+1= 171
- Every time we move to the left we're thinking in bundles of 10 of the space to the right
- Power of the base 10 system

Given four positions: [X X X X]$_{10}$ what is the largest number you can represent?

- 9999_{10}

Binary Digits: (BITs)

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7 6 5 4 3 2 1 0

Most significant bit → <u>1</u>000111<u>1</u> ← Least significant bit

- Sequence of eight bits: byte

Binary Digits: (BITS)

7 6 5 4 3 2 1 0

Most significant bit → 10001111 ← Least significant bit

$b_7b_6b_5b_4b_3b_2b_1b_0$

b_0 1s place

b_1 2s place

b_2 4s place

b_3 8s place

b_4 16s place

b_5 32s place

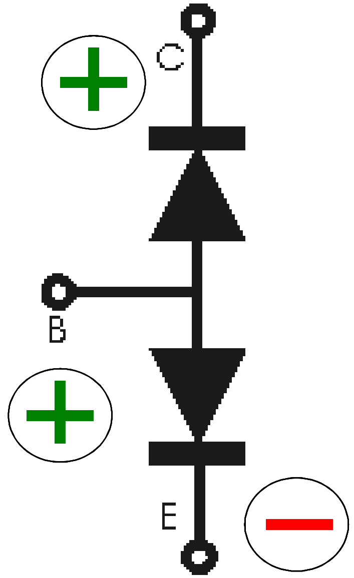
b_6 64s place

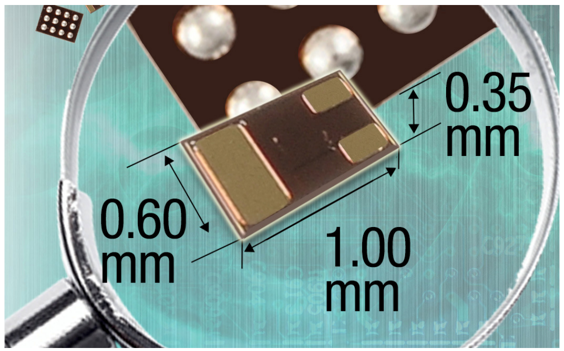
b_7 128s place

- Sequence of eight bits: byte

Binary Number Systems

- Transistors: On or Off
- Optical: Light or No light
- Magnetic: Positive or Negative





Conversion: Binary to Decimal

Method:

Multiply each of the binary digits by the appropriate power of 2：

Example:

$11111111_2 = 128+64+32+16+8+4+2+1 = 255_{10}$

Conversion: Decimal to Binary

Method :

We have a decimal number (x):

1. Highest power of two less than or equal to the decimal number (y)
2. Subtract the power of two (y) from the decimal number (x) as $x = x-y$.
3. If $x = 0$, we have our result! Else: Repeat

Example:

$19_{10}=16+2+1=00010011_2$

Hexadecimal Representation

- Hexadecimal numbers use 16 digits: {0-9, A-F}
  - does not distinguish between upper and lower case!

DEC	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
HEX	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Hexadecimal Representation

- Hexadecimal numbers use 16 digits: {0-9, A-F}
  - does not distinguish between upper and lower case!
- AB_{16}=A*16+B*1

DEC	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
HEX	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Hexadecimal Representation

- Hexadecimal numbers use 16 digits: {0-9, A-F}
  - does not distinguish between upper and lower case!
- AB_{16}=A*16+B*1
  =10*16+11*1
  =171

DEC	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
HEX	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Conversion from Binary to Hexadecimal

- Splitting into groups of 4 bits each.
- If not a multiple of 4:
  – pad the number with leading zer
- Example:

Hexadecimal to Binary is similar: write each hexadecimal digit into 4 binary digits

$$3CADB3_{16} = 0011\ 1100\ 1010\ 1101\ 1011\ 0011_2$$

Bin	0011	1100	1010	1101	1011	0011
Hex	3	C	A	D	B	3

iClicker Activity

Convert the decimal number 231 to binary and hexadecimal equivalents.

a) 1110 1111, F7

b) 1110 0110, E6

c) 1110 0111, E7

d) 0110 0111, 57

DEC	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
HEX	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Binary Addition

Rules

- 1+0 = 1
- 1+1 = 10
- 1+1+1 = 11

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	1	1	1	0		Carry in
		1	0	1	1	Augend (A)
+		1	1	1	0	Addend (B)
	1	1	0	0	1	Sum

Binary Addition

Rules

- 1+0 = 1
- 1+1 = 10
- 1+1+1 = 11

		1	0	1	1	= 1 + 2 + 8 = 11 (dec)
+		1	1	1	0	= 2 + 4 + 8 = 14 (dec)
	1	1	0	0	1	= 1 + 8 + 16 = 25 (dec)

Binary Subtraction

Rules

- 1-0 = 1
- 1-1 = 0
- 0-1 = 1 (borrow 1)
- 1-1 = 0

	1	0	1	1	= 1 + 2 + 8 = 11 (dec)
-	0	1	1	0	= 2 + 4 = 6 (dec)
	0	1	0	1	= 1 + 4 = 5 (dec)

Binary Multiplication

Example:

$$\begin{array}{cccc} & & 1 & 1 \\ & \times & 1 & 3 \\ \hline & & 3 & 3 \\ + & 1 & 1 & - \\ \hline & 1 & 4 & 3 \end{array}$$

- 0X0 = 0
- 1X0 = 0
- 1X1 = 1

Binary Multiplication

Example:

$$
\begin{array}{cccc}
 & & 1 & 1 \\
 & \times & 1 & 3 \\
\hline
 & & 3 & 3 \\
+ & 1 & 1 & - \\
\hline
 & 1 & 4 & 3
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & & 1 & 0 & 1 & 1 \\
 & & & & \times & 1 & 1 & 0 & 1 \\
\hline
 & & & & & 1 & 0 & 1 & 1 \\
 & & & & {}^{1}0 & 0 & 0 & 0 & - \\
 & & & {}^{1}1 & 0 & 1 & 1 & - & - \\
+ & & {}^{1}1 & 0 & 1 & 1 & - & - & - \\
\hline
 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}
$$

- $0 \times 0 = 0$
- $1 \times 0 = 0$
- $1 \times 1 = 1$

Binary Multiplication

Example:

		1	1
	X	1	3
		3	3
+	1	1	-
	1	4	3

					1	0	1	1	= 8 + 2 + 1 = 11
				X	1	1	0	1	= 8 + 4 + 1 = 13
					1	0	1	1	
				$^{1}0$	0	0	0	-	
			$^{1}1$	0	1	1	-	-	
+		$^{1}1$	0	1	1	-	-	-	
	1	0	0	0	1	1	1	1	= 128 + 8 + 4 + 2 + 1 = 143

- 0X0 = 0
- 1X0 = 0
- 1X1 = 1

Binary Multiplication

Exam

Since we always multiply by either 0 or 1, the partial products are always either 0000 or the multiplicand (in this example: 1011).

					1	0	1	1	= 8 + 2 + 1 = 11
				X	1	1	0	1	= 8 + 4 + 1 = 13
					1	0	1	1	
				¹0	0	0	0	-	
			¹1	0	1	1	-	-	
+		¹1	0	1	1	-	-	-	
	1	0	0	0	1	1	1	1	= 128 + 8 + 4 + 1 = 141

+ 1 1

1 4 3

- 0X0 = 0
- 1X0 = 0
- 1X1 = 1

iClicker Activity

Multiply the two numbers: 1101 x 1001

a) 110 1001
b) 100 1111
c) 110 1101
d) 111 0101

iClicker Activity

- Largest binary integer that can be stored in 3 bits?

a) 001

b) 100

c) 111

d) None of these

What about the smallest?

Unsigned Binary representation

- Unsigned binary:

7 6 5 4 3 2 1 0

Most significant bit → 10001111 ← Least significant bit

Representation:

$1 \times 2^7 + 0 \times 2^6 \ldots.. + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

$128 + 0 \ldots + 8 + 4 + 2 + 1$

$10001111 = 143$

- We add/subtract/multiply using the normal rules that we use for decimal addition/subtraction/multiplication

Signed binary representation

- Sign bit
  - Left-most bit: $b_7 b_6 b_5 b_4 b_3 b_2 b_1 b_0$
  - Also called the most significant bit

Negative Binary: Sign-magnitude

To get the negative number, set the sign bit to 1 and leave all the other bits unchanged.

81 =	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1
-81=	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1

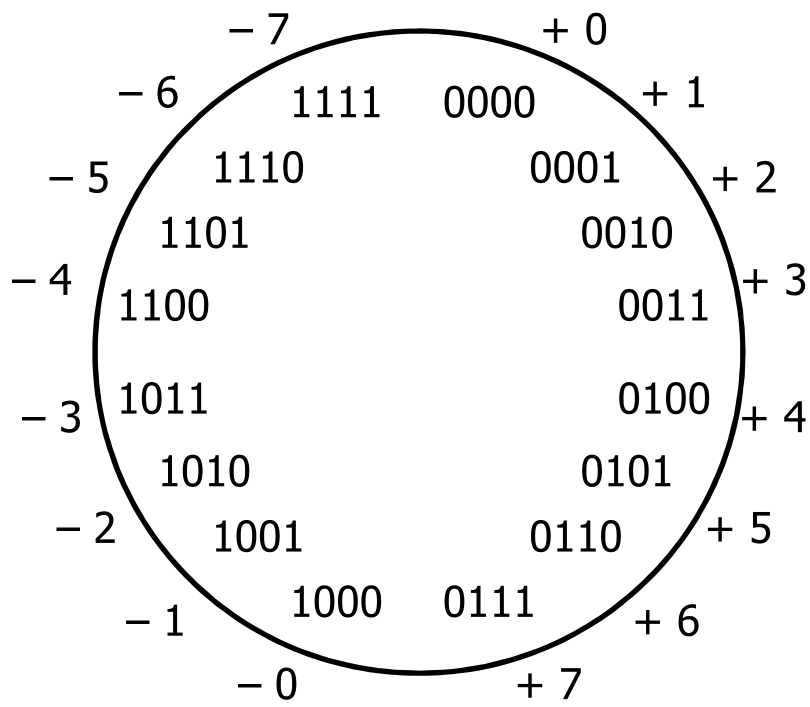
Called sign-magnitude representation: sign bit + value

0 =	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0 =	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Two zeroes! Not fun ….

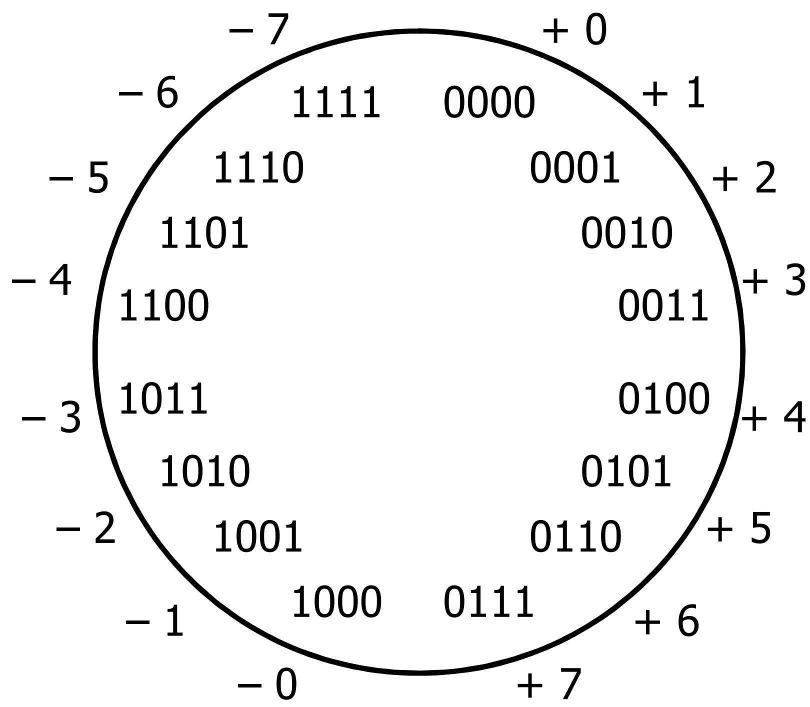
More difficulties …

- It is not easy to add, subtract, or compare numbers in sign-magnitude format
    - ❖5 -2 = 5 + (-2) = 0101 + 1010 = 1111 = -7



More difficulties …

- It is not easy to add, subtract, or compare numbers in sign-magnitude format
  - ❖ 5 -2 = 5 + (-2) = 0101 + 1010 = 1111 = -7
- Values do not fall in a natural order as compared with unsigned numbers



An alternative: Ones Complement

To get the negative of a number, invert or complement all the bits.

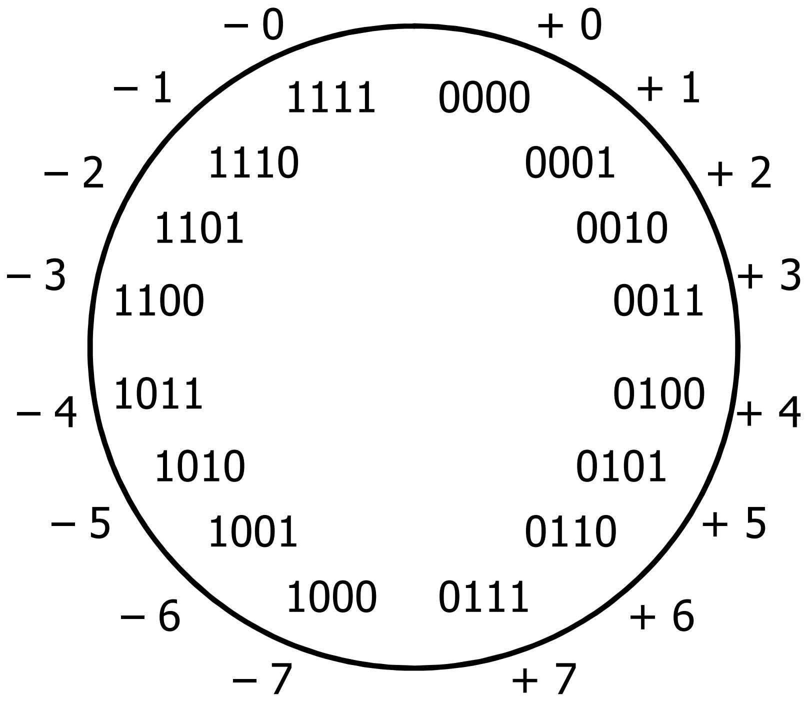
Complement = (~)

- ~1 = 0 and ~0 = 1

81 =	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1
-81=	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	0

An alternative: Ones Complement

- This solves the “unnatural order” problem
- But we still have two zeroes!



Two's Complement

To get the negative of a number, invert all the bits and then add 1.
If the addition causes a carry bit past the most significant bit, discard the high carry.

81 =	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1

~81 =	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	0

+1

-81 =	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1

Two's Complement

Do we have a unique representation of the value zero? Yes!

0 =	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
~0 =	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
																+1
-0 =	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Two’s Complement: Advantages

- ✓ Unique representation of zero
- ✓ Exactly same method for addition, multiplication, etc. as unsigned integers except throw away the high carry (high borrow for subtraction).

Two's Complement: Addition, Subtraction

45 – 14 = 45 + (-14) = 31

45 =	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	1

-14 =	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	0

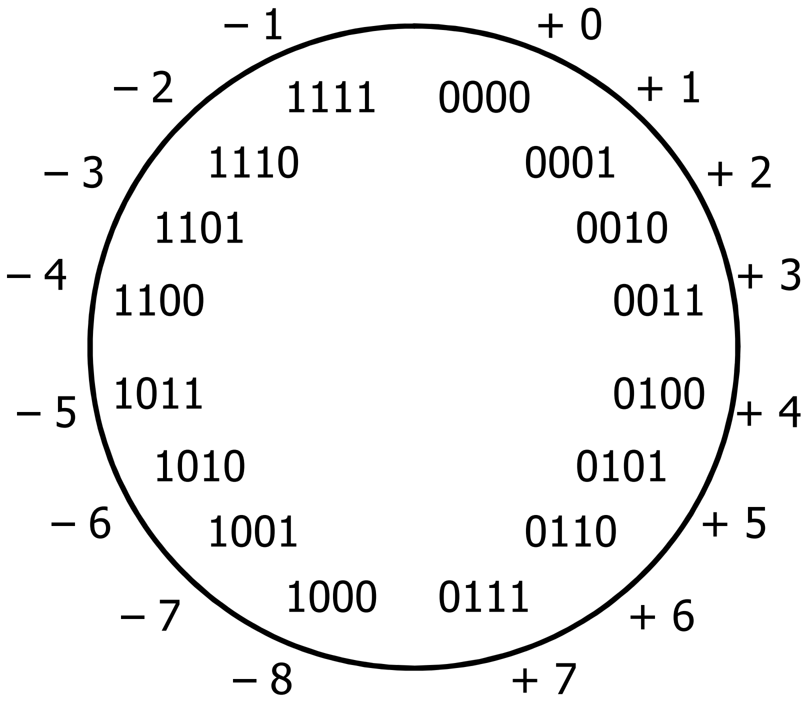
---

31 =	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1

ignore high carry 1!

Two's Complement: Advantages

- This solves the "unnatural order" problem



Conversion: Binary to Decimal

Method:

Multiply each of the binary digits by the appropriate power of 2：

Example:

11111111_2=128+64+32+16+8+4+2+1=255_{10}

Two’s complement: Conversion from Binary to Decimal

Method:

Multiply each of the binary digits by the appropriate power of 2：

b-bit word x in Two’s complement

1) For bit $0 \leq i \leq b-2$, multiply 2^i
2) For bit b-1, multiply -2^{b-1}

Example:

unsigned $11111111_2=128+64+32+16+8+4+2+1=255_{10}$

signed $11111111_2=-128+64+32+16+8+4+2+1=-1_{10}$

Signed	Bits	UnSigned
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
-8	1000	8
-7	1001	9
-6	1010	10
-5	1011	11
-4	1100	12
-3	1101	13
-2	1110	14
-1	1111	15

Integer Value Range

- Representing negative and positive numbers in b bits:
  - Unsigned: 0 to 2^b-1 = 00…00 to 11…11
  - Signed: $-2^{(b-1)}$ to $2^{(b-1)}$ -1 = 100..00 to 011…11

Integer Value Range

- Representing negative and positive numbers in b bits:
  - Unsigned: 0 to 2^b-1 = 00…00 to 11…11
  - Signed: $-2^{(b-1)}$ to $2^{(b-1)}$ -1 = 100..00 to 011…11
- Example: range for 8 bits is:
  - Unsigned: 0 to 2^8 - 1 = 0 …. 255
  - Signed: $-2^{(7)}$ to $2^{(7)}$ -1 = -128 …. 127

i-clicker question

- What is the range of a signed 3-bit number?
  A. -2^1 to +2^1-1
  B. -2^2 to +2^2-1
  C. -2^3 to +2^3-1
  D. -2^4 to +2^4-1

Two's Complement Overflow & Underflow

- Overflow is caused by a value near the upper limit of the range, while an underflow is caused by values near the lower limit of the range.

Overflow: Example

Consider the 8-bit two’s complement addition:

				1	1	1	1	1	1	
1	2	7	0	1	1	1	1	1	1	1
	+	1							+	1
1	2	8	1	0	0	0	0	0	0	0

- The result should be +128, but the leftmost bit is 1, therefore the result is -128!
- This is an overflow: an arithmetic operation that should be positive gives a negative result.

Underflow: Example

Consider the 8-bit two's complement addition:

-128		1	0	0	0	0	0	0	0
-1	-								1
-129		0	1	1	1	1	1	1	1

- The result should be -129, but the leftmost bit is 0, therefore the result is +127!
- This is an underflow: as an arithmetic operation that should be negative gives a positive result.

iClicker Activity

- What is the result of the following 8-bit two's complement addition 1000 0000 - 1?

a) 0111 1111

b) 1111 1111

c) 0000 0000

d) 0000 0001

Hint: Try converting it back to decimal and compare!

Next Class

- Lets represent Binary operations in C!
- Overflow conditions
  - How do we represent overflow?
  - What’s the impact of an overflow?
  - How do we mitigate it?
- Readings posted on website