Class work 3

Task 1. Find a closed form for the generating function for each of these sequences.

(a) $1, 3, 9, 27, 81, 243, 729, \ldots$

(b) $1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots$

(c) $0, 1, -2, 4, -8, 16, -32, 64, \ldots$

(d) $1, 1, 0, 1, 1, 0, 1, 1, 0, 1, \ldots$

(e) $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$

(f) $a_n = n - 1$ for $n = 0, 1, 2, \ldots$

(g) $a_n = C^2_{n+2}$ for $n = 0, 1, 2, \ldots$

(h) $a_n = 3^n$ for $n = 0, 1, 2, \ldots$

(i) $a_n = \frac{1}{n+1}$ for $n = 0, 1, 2, \ldots$

(j) $a_n = n(n + 1)$ for $n = 0, 1, 2, \ldots$

(k) $a_n = 4n^2 + 3n + 2$

(l) $a_n = 4n^2 + 3n + 2$

(m) $a_n = (2n + 1)3^n$

(n) $a_n = (-1)^n + 2n$

(o) $a_n = n^2 2^n$

(p) $a_n = (n + 2)5^n$

(q) $a_n = (-2)^n + n^2$

(r) $a_0 = 1, \quad a_1 = 4, \quad a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$

(s) $a_0 = 2, \quad a_1 = 3, \quad a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$

(t) $a_0 = 0, \quad a_1 = 1, \quad a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$

(u) $a_n = (-1)^n n$

(v) $a_n = 3^n + 2^n$

(w) $a_n = (n^2 + n)4^n$

(x) $a_0 = 5, \quad a_1 = 7, \quad a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$

(y) $a_n = (-1)^n + n2^n$

(z) $a_0 = 3, \quad a_1 = 1, \quad a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$

Task 2.

(a) Find the coefficient of x^{12} in the power series of
$$(x^2 + x^3 + x^4)(x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + \cdots).$$

(b) Find the coefficient of x^{15} in the power series of
$$(x^3 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + \cdots).$$

(c) Find the coefficient of x^{20} in the power series of
$$(x^5 + x^6 + x^7)(x^4 + x^5 + x^6 + x^7 + x^8)(1 + x + x^2 + x^3 + \cdots).$$

(d) Find the coefficient of x^{18} in the power series of
$$(x^2 + x^4 + x^6)(x^3 + x^5 + x^7)(1 + x + x^2 + x^3 + \cdots).$$

(e) Find the coefficient of x^{14} in the power series of
$$(x + x^2 + x^3)(x^4 + x^5 + x^6 + x^7)(1 + x + x^2 + x^3 + \cdots).$$

(f) Find the coefficient of x^{16} in the power series of
$$(x^4 + x^5)(x^2 + x^3 + x^4 + x^5 + x^6)(1 + x + x^2 + x^3 + \cdots).$$

(g) Find the coefficient of x^{22} in the power series of
$$(x^6 + x^7 + x^8)(x^3 + x^4 + x^5 + x^6)(1 + x + x^2 + x^3 + \cdots).$$

(h) Find the coefficient of x^{25} in the power series of
$$(x^5 + x^6 + x^7 + x^8)(x^4 + x^5 + x^6)(1 + x + x^2 + x^3 + \cdots).$$

(i) Find the coefficient of x^{19} in the power series of

$$(x^3 + x^4 + x^5)(x^5 + x^6 + x^7 + x^8 + x^9)(1 + x + x^2 + x^3 + \cdots).$$

(j) Find the coefficient of x^{17} in the power series of

$$(x^2 + x^3 + x^4 + x^5)(x^6 + x^7)(1 + x + x^2 + x^3 + \cdots).$$

(k) Find the coefficient of x^{21} in the power series of

$$(x^4 + x^5 + x^6)(x^7 + x^8 + x^9)(1 + x + x^2 + x^3 + \cdots).$$

(l) Find the coefficient of x^{13} in the power series of

$$(x^2 + x^3)(x^3 + x^4 + x^5 + x^6 + x^7)(1 + x + x^2 + x^3 + \cdots).$$

(m) Find the coefficient of x^{24} in the power series of

$$(x^6 + x^7 + x^8 + x^9)(x^5 + x^6 + x^7)(1 + x + x^2 + x^3 + \cdots).$$

(n) Find the coefficient of x^{10} in

$$\frac{1}{(1-3x)^5}.$$

(o) Find the coefficient of x^{12} in

$$\frac{1}{(1+2x)^6}.$$

(p) Find the coefficient of x^9 in

$$\frac{1}{(1-4x)^3}.$$

(q) Find the coefficient of x^{15} in

$$\frac{1}{(1-5x)^7}.$$

(r) Find the coefficient of x^{11} in

$$\frac{1}{(1+3x)^4}.$$

(s) Find the coefficient of x^{14} in

$$\frac{1}{(1-2x)^6}.$$

(t) Find the coefficient of x^{13} in
$$\frac{1}{(1-6x)^4}.$$

(u) Find the coefficient of x^7 in
$$\frac{1}{(1+4x)^5}.$$

(v) Find the coefficient of x^{16} in
$$\frac{1}{(1-3x)^8}.$$

(w) Find the coefficient of x^{18} in
$$\frac{1}{(1-2x)^9}.$$

(x) Find the coefficient of x^{10} in
$$\frac{1}{(1+5x)^3}.$$

(y) Find the coefficient of x^{20} in
$$\frac{1}{(1-7x)^5}.$$

(z) Find the coefficient of x^9 in
$$\frac{1}{(1-2x)^8}.$$

Task 3.

(a) Use generating functions to determine the number of different ways 10 identical balloons can be given to four children if each child receives at least two balloons.

(b) Use generating functions to determine the number of different ways 12 identical action figures can be given to five children so that each child receives at most three action figures.

(c) Use generating functions to determine the number of different ways 15 identical stuffed animals can be given to six children so that each child receives at least one but no more than three stuffed animals.

(d) Use generating functions to find the number of ways to choose a dozen bagels from three varieties—egg, salty, and plain—if at least two bagels of each kind but no more than three salty bagels are chosen.

(e) In how many ways can 25 identical donuts be distributed to four police officers so that each officer gets at least three but no more than seven donuts?

(f) Use generating functions to find the number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls, and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are drawn does not matter.

(g) What is the generating function for the sequence $\{c_k\}$,where c_k is the number of ways to make change for k dollars using \$1 bills, \$2 bills, \$5 bills, and \$10 bills?

(h) What is the generating function for the sequence $\{c_k\}$, where c_k represents the number of ways to make change for k pesos using bills worth 10 pesos, 20 pesos, 50 pesos, and 100 pesos?

(i) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of

$$x_1 + x_2 + x_3 = k$$

when x_1, x_2, and x_3 are integers with $x_1 \geq 2$, $0 \leq x_2 \leq 3$, and $2 \leq x_3 \leq 5$?

(j) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = k$$

when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3$, $1 \leq x_2 \leq 5$, $0 \leq x_3 \leq 4$, and $x_4 \geq 1$?

(k) Use generating functions to determine the number of ways 18 identical candies can be distributed to five children if each child receives at least one candy and at most five candies.

(l) Use generating functions to determine the number of ways 20 identical pencils can be distributed to four students so that each student receives at least two pencils.

(m) Use generating functions to determine the number of ways 16 identical books can be distributed to six students so that no student receives more than four books.

(n) Use generating functions to determine the number of ways 14 identical cookies can be given to four children if exactly two children receive at least four cookies each.

(o) Use generating functions to determine the number of ways 30 identical tickets can be distributed among five people if each person receives between three and eight tickets inclusive.

(p) Use generating functions to find the number of ways to select 20 balls from a jar containing red, blue, and green balls if at least 5 red balls and at most 8 green balls are selected.

(q) Use generating functions to find the number of ways to choose 15 pieces of fruit from apples, oranges, bananas, and pears if at least two apples and no more than three bananas are chosen.

(r) Use generating functions to determine the number of nonnegative integer solutions of
$$x_1 + x_2 + x_3 + x_4 = k$$
where $0 \leq x_1 \leq 4$, $x_2 \geq 2$, $1 \leq x_3 \leq 5$, and $x_4 \geq 0$.

(s) What is the generating function for the sequence $\{a_k\}$, where a_k is the number of ways to pay k dollars using \$1, \$3, \$4, and \$7 bills?

(t) What is the generating function for the sequence $\{a_k\}$, where a_k is the number of ways to make change for k euros using 2, 5, and 10 euro coins?

(u) Use generating functions to determine the number of integer solutions of
$$x_1 + x_2 + x_3 = 25$$
if $x_1 \geq 4$, $0 \leq x_2 \leq 6$, and $x_3 \geq 2$.

(v) Use generating functions to determine the number of ways 12 identical toys can be distributed among three children if each child receives an even number of toys.

(w) Use generating functions to determine the number of ways 21 identical balls can be distributed among four boxes if each box contains at least two balls and at most nine balls.

(x) What is the generating function for the number of partitions of n into parts of size at most 4?

(y) What is the generating function for the number of ways to write n as a sum of even positive integers?

(z) Use generating functions to determine the number of ways to select 18 candies from three types if at least 3 of each type and no more than 9 of any type are selected.