Introduction to Set Theory: Operations, Relations, Functions.

Birzhan Kalmurzayev

Ac. year 2025-2026

Motivation

Set theory provides a unified language for modern mathematics
and computer science.

Motivation

Set theory provides a unified language for modern mathematics
and computer science.

- Data structures and databases
- Logic and formal languages
- Algorithms and computability theory
- Foundations of mathematics

Basic Notions

Definition

A *set* is a collection of distinct objects, called *elements*.

Basic Notions

Definition

A *set* is a collection of distinct objects, called *elements*.

- Sets are usually denoted by capital letters: A, B, X
- Elements are denoted by lowercase letters: a, b, x
- Membership: $a \in A$, $a \notin A$

- **Listing elements:**
$$A = \{1, 2, 3, 4\}$$

Ways to Describe Sets

- **Listing elements:**
$$A = \{1, 2, 3, 4\}$$

- **Set-builder notation:**
$$B = \{x \in \mathbb{N} \mid x \text{ is even}\}$$

Let A and B be sets.

Basic Set Operations

Let A and B be sets.

- Union:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Basic Set Operations

Let A and B be sets.

- Union:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- Intersection:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Basic Set Operations

Let A and B be sets.

- Union:
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- Intersection:
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- Difference:
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Subsets

Definition

A set A is called a *subset* of a set B if every element of A is also an element of B.

$$A \subseteq B \iff \forall x\,(x \in A \Rightarrow x \in B)$$

Subsets

Definition

A set A is called a *subset* of a set B if every element of A is also an element of B.

$$A \subseteq B \iff \forall x \, (x \in A \Rightarrow x \in B)$$

- Proper subset: $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$

Subsets

Definition

A set A is called a *subset* of a set B if every element of A is also an element of B.

$$A \subseteq B \iff \forall x \, (x \in A \Rightarrow x \in B)$$

- Proper subset: $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$
- The empty set is a subset of every set: $\varnothing \subseteq A$

Subsets

Definition

A set A is called a *subset* of a set B if every element of A is also an element of B.

$$A \subseteq B \iff \forall x \, (x \in A \Rightarrow x \in B)$$

- Proper subset: $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$
- The empty set is a subset of every set: $\varnothing \subseteq A$
- We say that two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$.

Subsets

Definition

A set A is called a *subset* of a set B if every element of A is also an element of B.

$$A \subseteq B \iff \forall x \, (x \in A \Rightarrow x \in B)$$

- Proper subset: $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$
- The empty set is a subset of every set: $\varnothing \subseteq A$
- We say that two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$.

Example

If $B = \{1, 2, 3\}$, then $\{1, 2\} \subseteq B$, $\quad \{1, 4\} \nsubseteq B$.

Equality of Sets

Principle of Set Equality

Two sets A and B are equal if and only if they have the same elements:

$$A = B \iff (\forall x \, (x \in A \iff x \in B)).$$

Equality of Sets

Principle of Set Equality

Two sets A and B are equal if and only if they have the same elements:

$$A = B \iff (\forall x \, (x \in A \iff x \in B)).$$

Proof Strategy

To prove that $A = B$, it is sufficient to show:

- $A \subseteq B$
- $B \subseteq A$

Definition

A *universal set U* is a set that contains all objects under consideration in a given context.

Definition

A *universal set* U is a set that contains all objects under consideration in a given context.

- All sets discussed are assumed to be subsets of U:

$$A \subseteq U$$

Universal Set

Definition

A *universal set U* is a set that contains all objects under consideration in a given context.

- All sets discussed are assumed to be subsets of U:

$$A \subseteq U$$

- The choice of U depends on the problem and is not absolute

Universal Set

Definition

A *universal set U* is a set that contains all objects under consideration in a given context.

- All sets discussed are assumed to be subsets of U:

$$A \subseteq U$$

- The choice of U depends on the problem and is not absolute
- Operations such as complement are defined with respect to U

Universal Set

Definition

A *universal set* U is a set that contains all objects under consideration in a given context.

- All sets discussed are assumed to be subsets of U:

$$A \subseteq U$$

- The choice of U depends on the problem and is not absolute
- Operations such as complement are defined with respect to U

For example:

- If we study subsets of natural numbers, we may take $U = \mathbb{N}$.
- If we study sets of real numbers, we take $U = \mathbb{R}$. (For example in Calculus)

Complement of a Set

Definition

Let U be a universal set. The complement of A is

$$A^c = U \setminus A.$$

Complement of a Set

Definition

Let U be a universal set. The complement of A is

$$A^c = U \setminus A.$$

- $A \cup A^c = U$
- $A \cap A^c = \varnothing$

Informal Explanation

A *predicate* is a statement that depends on one or more variables and becomes either true or false once the variables are specified.

What Is a Predicate?

Informal Explanation

A *predicate* is a statement that depends on one or more variables and becomes either true or false once the variables are specified.

- A predicate is not a statement by itself
- It turns into a statement after substituting concrete values
- Predicates are used to describe properties and relations

What Is a Predicate?

Informal Explanation

A *predicate* is a statement that depends on one or more variables and becomes either true or false once the variables are specified.

- A predicate is not a statement by itself
- It turns into a statement after substituting concrete values
- Predicates are used to describe properties and relations

Example

Let $P(x)$ be the predicate:

$$P(x): \quad \text{"}x \text{ is an even number"}.$$

$P(4)$ is true; $\qquad\qquad P(5)$ is false

Main operation on predicates

Let predicates $P(x)$ and $Q(x)$ is given, then

- $(\neg P)(x)$ is mean "not $P(x)$",
  and $(\neg P)(x)$ is true iff $P(x)$ is false;

- $(P \wedge Q)(x)$ is mean "$P(x)$ and $Q(x)$",
  and $(P \wedge Q)(x)$ is true iff both $P(x)$ and $Q(x)$ are true;

- $(P \vee Q)(x)$ is mean "$P(x)$ or $Q(x)$",
  and $(P \vee Q)(x)$ is true iff at least on of $P(x)$ or $Q(x)$ is true

- $(P \Rightarrow Q)(x)$ is mean "if $P(x)$, then $Q(x)$",
  and $(P \Rightarrow Q)(x)$ is false iff $P(x)$ is true and $Q(x)$ is false.

Predicate Equivalences: Basic Logical Equivalences

For any predicates $P(x)$ and $Q(x)$ the following equivalences are hold:

$$P \wedge Q \iff Q \wedge P,$$
$$P \vee Q \iff Q \vee P$$
$$(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \iff P \vee (Q \vee R)$$
$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$$
$$\neg(P \wedge Q) \iff \neg Q \vee \neg P,$$
$$\neg(P \vee Q) \iff \neg Q \wedge \neg P,$$
$$(P \Rightarrow Q) \iff \neg Q \vee P,$$

Trivial statements

By **1** lets denote always true statement, and by **0** lets denote always false statement. Then the following equivalences are hold

- **1** $\iff x \in U$;
- **0** $\iff x \in \emptyset$;
- $P \vee \neg P \iff$ **1**;
- $P \wedge \neg P \iff$ **0**;
- $\neg$**1** $\iff$ **0**;
- $\neg$**0** $\iff$ **1**;
- **0** $\wedge P \iff$ **0**;
- **0** $\vee P \iff P$;
- **1** $\wedge P \iff P$;
- **1** $\vee P \iff$ **1**

Definition

Let $P(x)$ be a predicate. Then we say the statement

$$\exists x \, P(x)$$

is true iff there is an element $x \in U$ with true $P(x)$.
The statement

$$\forall x \, P(x)$$

is true iff for all elements $x \in U$ the statement $P(x)$ is true.

Quantifiers Equivalences

$$\neg(\forall x \, P(x)) \iff \exists x \, \neg P(x)$$

$$\neg(\exists x \, P(x)) \iff \forall x \, \neg P(x)$$

logic $\leftrightarrow$ set operations

Assume A and B are sets and x is some element. Then the following equivalences are hold

- $x \notin A \iff \neg(x \in A)$;
- $x \in A \cup B \iff x \in A \lor x \in B$;
- $x \in A \cap B \iff x \in A \land x \in B$;

For any sets A, B, and C:

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

For any sets A, B, and C:

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Proof (element-wise)

Let x be an arbitrary element.

Example

For any sets A, B, and C:

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Proof (element-wise)

Let x be an arbitrary element.

$$
\begin{aligned}
x \in A \setminus (B \cap C) &\iff x \in A \land x \notin (B \cap C) \\
&\iff x \in A \land \neg(x \in B \land x \in C) \\
&\iff x \in A \land (x \notin B \lor x \notin C) \\
&\iff (x \in A \land x \notin B) \lor (x \in A \land x \notin C) \\
&\iff x \in A \setminus B \lor x \in A \setminus C \\
&\iff x \in (A \setminus B) \cup (A \setminus C).
\end{aligned}
$$

Ordered Pairs and Cartesian Product

Definition

An *ordered pair* (a, b) is a pair of elements in which the order matters:

$$(a, b) = (c, d) \iff a = c \text{ and } b = d.$$

Ordered Pairs and Cartesian Product

Definition

An *ordered pair* (a, b) is a pair of elements in which the order matters:

$$(a, b) = (c, d) \iff a = c \text{ and } b = d.$$

For sets A and B, the *Cartesian product* is the set

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}.$$

Ordered Pairs and Cartesian Product

Definition

An *ordered pair* (a, b) is a pair of elements in which the order matters:

$$(a, b) = (c, d) \iff a = c \text{ and } b = d.$$

For sets A and B, the *Cartesian product* is the set

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}.$$

Example. If $A = \{1, 2, 3\}$ and $B = \{x, y\}$, then

$$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

Relations

Definition

A *relation R* between sets *A* and *B* is a subset of the Cartesian product:

$$R \subseteq A \times B.$$

Relations

Definition

A *relation R* between sets A and B is a subset of the Cartesian product:

$$R \subseteq A \times B.$$

Example

Let $A = \{1, 2, 3\}$ and

$$R = \{(1,1), (2,4), (3,9)\} \subseteq A \times \mathbb{N}.$$

Notions in Relation:

Let $R \subseteq A \times B$, $Q \subseteq B \times C$ be a relations.

- $\mathrm{Dom}(R) = \{a \in A \mid \exists b \in B \ (a, b) \in R\}$,
- $\mathrm{Ran}(R) = \{b \in B \mid \exists a \in A \ (a, b) \in R\}$,
- $R^{-1} = \{(b, a) \mid (a, b) \in R\} \subseteq B \times A$,
- $R \circ Q = \{(a, c) \mid \exists b \in B \ (a, b) \in R \land (b, c) \in Q\} \subseteq A \times C$.

Examples

Let
$$A = \{0, 1, 2, 3\}, \quad B = \{a, b, c\}, \quad C = \{x, y, z\},$$

and define relations

$$R = \{(1, a), (2, a), (3, b)\} \subseteq A \times B, \qquad S = \{(a, x), (b, y)\} \subseteq B \times C.$$

Examples

Let
$$A = \{0, 1, 2, 3\}, \quad B = \{a, b, c\}, \quad C = \{x, y, z\},$$

and define relations

$$R = \{(1, a), (2, a), (3, b)\} \subseteq A \times B, \qquad S = \{(a, x), (b, y)\} \subseteq B \times C.$$

- $\mathrm{Dom}(R) = \{1, 2, 3\}$,
- $\mathrm{Ran}(R) = \{a, b\}$,
- $R^{-1} = \{(a, 1), (a, 2), (b, 3)\} \subseteq B \times A$,
- $R \circ S = \{(1, x), (2, x), (3, y)\} \subseteq A \times C$.

A relation $R \subseteq A \times A$ may have the following properties:

Relation types

A relation $R \subseteq A \times A$ may have the following properties:

- Reflexive: $\forall a \in A \ [(a, a) \in R]$;
- Irreflexive: $\forall a \in A \ [(a, a) \notin R]$;
- Symmetric: $\forall a, b \in A \ [(a, b) \in R \Rightarrow (b, a) \in R]$;
- Anti-symmetric: $\forall a, b \in A \ [(a, b) \in R \land (b, a) \in R \Rightarrow a = b]$;
- Transitive: $\forall a, b, c \in A \ [(a, b) \in R \land (b, c) \in R \Rightarrow (a, c) \in R]$

Definition

A relation R is an *equivalence relation* if it is:

- Reflexive
- Symmetric
- Transitive

Definition

A relation R is an *equivalence relation* if it is:

- Reflexive
- Symmetric
- Transitive

A relation R is an *partial order* if it is:

- Reflexive
- Anti-symmetric
- Transitive

Definition

A relation R is an *equivalence relation* if it is:

- Reflexive
- Symmetric
- Transitive

A relation R is an *partial order* if it is:

- Reflexive
- Anti-symmetric
- Transitive

A relation R is an *linear order* if it is:

- Partial order, and
- $\forall x, y \, [(x, y) \in R \lor (y, x) \in R]$.

Partitions of a Set

Definition

A *partition* of a set A is a collection of nonempty sets $A_1, A_2, \ldots$ such that for all i, i

- $A_i \subseteq A$;
- $A_i \cap A_j = \varnothing$ for $i \neq j$
- $\displaystyle\bigcup_i A_i = A$

Partitions of a Set

Definition

A *partition* of a set A is a collection of nonempty sets $A_1, A_2, \ldots$ such that for all i, i

- $A_i \subseteq A$;
- $A_i \cap A_j = \varnothing$ for $i \neq j$
- $\bigcup_i A_i = A$

Intuition

A partition divides a set into disjoint *blocks* so that every element belongs to exactly one block.

Partition: Example

Example

Let

$$A = \{1, 2, 3, 4\}.$$

The collection

$$\mathcal{P} = \big\{\{1, 3\}, \{2\}, \{4\}\big\}$$

is a partition of A.

Partition: Example

Example

Let

$$A = \{1, 2, 3, 4\}.$$

The collection

$$\mathcal{P} = \big\{\{1, 3\}, \{2\}, \{4\}\big\}$$

is a partition of A.

- Every element of A belongs to exactly one block
- Blocks do not overlap
- No element is missing

Equivalence Relations and Partitions

Theorem

Let A be a set.

- Every equivalence relation on A determines a partition of A into equivalence classes.
- Conversely, every partition of A determines an equivalence relation on A.

Thus, there is a one-to-one correspondence:

$$\text{equivalence relations on } A \iff \text{partitions of } A.$$

Example: Equivalence Relation and Partition

Example

Consider the set A of all Kazakhs.
Define a relation $\sim$ on A by:

$x \sim y$ if and only if x and y belong to the same clan (ru).

Example: Equivalence Relation and Partition

Example

Consider the set A of all Kazakhs.
Define a relation $\sim$ on A by:

$x \sim y$ if and only if x and y belong to the same clan (ru).

Induced Partition

This equivalence relation partitions the set A into disjoint subsets, each subset consisting of all members of one clan.

Functions

Definition

A *function* $f : A \to B$ is a relation (i.e. $f \subseteq A \times B$) such that

For any $a \in A$ there is at most one $b \in B$ with $(a, b) \in f$

Functions

Definition

A *function* $f : A \to B$ is a relation (i.e. $f \subseteq A \times B$) such that

For any $a \in A$ there is at most one $b \in B$ with $(a, b) \in f$

Let $f : A \to B$.

- Domain is a domain of the relation f;
- Range is a range of the relation f.

Functions

Definition

A *function* $f : A \to B$ is a relation (i.e. $f \subseteq A \times B$) such that

   For any $a \in A$ there is at most one $b \in B$ with $(a, b) \in f$

Let $f : A \to B$.

- Domain is a domain of the relation f;
- Range is a range of the relation f.

Usually, instead of writing $(x, y) \in f$, we write $f(x) = y$.

Types of Functions

Assume $f : A \rightarrow B$ be a function, then we call that the function f is

- Injective (one-to-one), if $\mathrm{Dom}(f) = A$ and

$$\forall x_1, x_2 \in A\,[f(x_1) = f(x_2) \Rightarrow x_1 = x_2];$$

- Surjective (onto), if

$$\forall y \in B\,\exists x \in A\,[f(x) = y];$$

- Bijective, if it is Injective and Surjective

◀ □ ▶ ◀ 🗗 ▶ ◀ 🗉 ▶ ◀ 🗉 ▶   🗉   ᕞᘁᗕ

Types of Functions

Assume $f : A \to B$ be a function, then we call that the function f is

- Injective (one-to-one), if $\mathrm{Dom}(f) = A$ and

$$\forall x_1, x_2 \in A \, [f(x_1) = f(x_2) \Rightarrow x_1 = x_2];$$

- Surjective (onto), if

$$\forall y \in B \, \exists x \in A \, [f(x) = y];$$

- Bijective, if it is Injective and Surjective

Remark

A bijection establishes a one-to-one correspondence between sets.

Thank you for your attention!