

Numerical Integration.

Trapezoidal Approximations.

Simpson's Rule: Approximations Using Parabolas

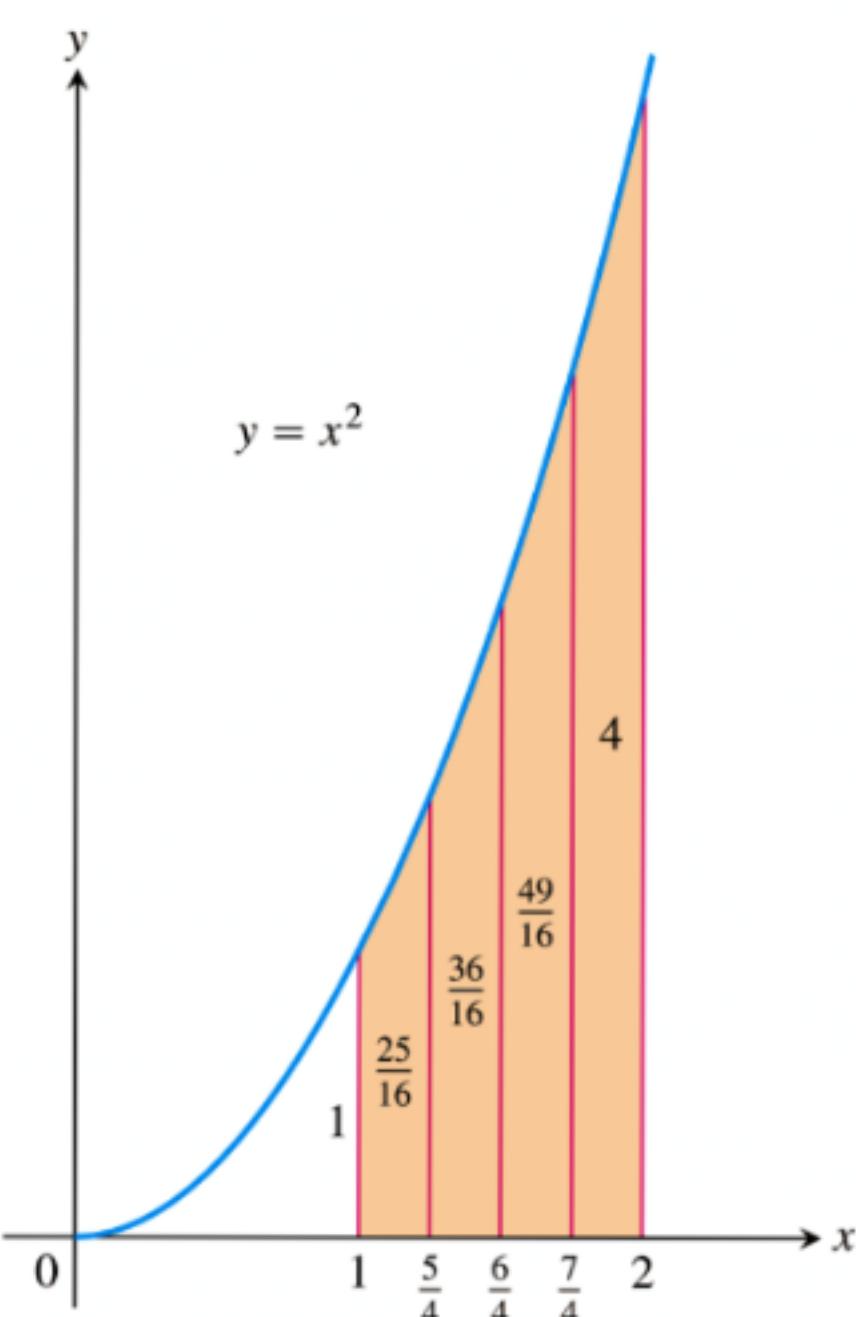
The Trapezoidal Rule

To approximate $\int_a^b f(x) dx$, use

$$T = \frac{\Delta x}{2} \left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n \right).$$

The y 's are the values of f at the partition points

$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n - 1)\Delta x, x_n = b$,
where $\Delta x = (b - a)/n$.



EXAMPLE 1 Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2 dx$. Compare the estimate with the exact value.

Solution Partition $[1, 2]$ into four subintervals of equal length (Figure 8.8). Then evaluate $y = x^2$ at each partition point (Table 8.2).

Using these y values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2} \left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4 \right) \\ &= \frac{1}{8} \left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4 \right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

FIGURE 8.8 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

Since the parabola is concave up, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. The exact value of the integral is

$$\int_1^2 x^2 dx = \left[\frac{x^3}{3} \right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

TABLE 8.2

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

Simpson's Rule

To approximate $\int_a^b f(x) dx$, use

$$S = \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y 's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n - 1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

EXAMPLE 2 Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4 dx$.

Solution Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.3). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned} S &= \frac{\Delta x}{3} \left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4 \right) \\ &= \frac{1}{6} \left(0 + 4 \left(\frac{5}{16} \right) + 2(5) + 4 \left(\frac{405}{16} \right) + 80 \right) \\ &= 32 \frac{1}{12}. \end{aligned}$$

This estimate differs from the exact value (32) by only $1/12$, a percentage error of less than three-tenths of one percent, and this was with just four subintervals.

TABLE 8.3

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

I. Using the Trapezoidal Rule

- a. Estimate the integral with $n = 4$

II. Using Simpson's Rule

- a. Estimate the integral with $n = 4$

1. $\int_1^2 x \, dx$

3. $\int_{-1}^1 (x^2 + 1) \, dx$

5. $\int_0^2 (t^3 + t) \, dt$

7. $\int_1^2 \frac{1}{s^2} \, ds$

9. $\int_0^\pi \sin t \, dt$

2. $\int_1^3 (2x - 1) \, dx$

4. $\int_{-2}^0 (x^2 - 1) \, dx$

6. $\int_{-1}^1 (t^3 + 1) \, dt$

8. $\int_2^4 \frac{1}{(s - 1)^2} \, ds$

10. $\int_0^1 \sin \pi t \, dt$

