Class work 1

Task 1. Prove the following statements by induction

(a) $\forall n \geq 1: \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(b) $\forall n \geq 1: \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) $\forall n \geq 1: \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

(d) $\forall n \geq 1: \sum_{k=1}^{n} k^4 = \frac{(3n^2+3n-1)(2n+1)(n+1)n}{30}$.

(e) $\forall n \geq 1: \sum_{k=1}^{n} k^5 = \frac{(2n^2+2n-1)(n+1)^2n^2}{12}$.

(f) $\forall n \geq 1: \sum_{k=1}^{n}(2k-1) = n^2$.

(g) $\forall n \geq 1: \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

(h) $\forall n \geq 1: \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$.

(i) $\forall n \geq 4: n! > 2^n$.

(j) $\forall n \geq 5: 2^n > n^2$.

(k) $\forall n \geq 1: 3 \mid (n^3 - n)$.

(l) $\forall n \geq 1: 7 \mid (8^n - 1)$.

(m) $\forall n \geq 1: 5 \mid (n^5 - n)$.

(n) $\forall n \geq 1: 3^{n+1} \mid (2^{3^n} + 1)$.

(o) $\forall n \geq 1: F_{n+2} = \sum_{k=0}^{n} F_k$,
where (F_n) — Fibbonachi numbers defined by: $F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n$

(p) $\forall n \geq 1: F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

(q) $\forall n \geq 1: F_n \mid F_{2n}$.

(r) $\forall n \geq 1: F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$.

(s) $\forall n \geq 1: \sum_{i=1}^{2n-1} F_i F_{i+1} = F_{2n}^2$.

(t) $\forall n \geq 1: \sum_{i=1}^{2n} F_i F_{i+1} = F_{2n+1}^2 - 1$.

(u) $\forall n \geq 1: 2 \mid F_{3n}$.

(v) $\forall n \geq 1: \; 3 \mid F_{4n}$.

(w) $\forall n \geq 1: \; 4 \mid F_{6n}$.

(x) $\forall n \geq 1: \; 5 \mid F_{5n}$.

(y) $\forall n \geq 1: \; 16 \mid F_{12n}$.

Task 2. Count the followings:

(a) At a university there are 24 physics majors and 180 engineering majors. In how many ways can two representatives be chosen so that one is a physics major and the other is an engineering major? In how many ways can one representative be chosen who is either a physics major or an engineering major?

(b) A hotel has 18 floors, and each floor contains 42 rooms. How many rooms are there in the hotel?

(c) A university library has 6 levels, and each level contains 125 study desks. How many study desks are in the library?

(d) A conference hall has 28 rows, and each row contains 40 seats. How many seats are in the conference hall?

(e) A particular model of sports shoes comes in 8 colors, has a men's version and a women's version, and is available in 5 sizes for each version. How many different types of these sports shoes are made?

(f) There are 8 different airlines that fly from London to Paris and 5 different airlines that fly from Paris to Rome. How many different pairs of airlines can be chosen to book a trip from London to Rome via Paris, when you select one airline for the flight to Paris and another airline for the continuation flight to Rome?

(g) How many different four-letter codes can be formed from the English alphabet if no letter is repeated?

(h) How many different five-letter passwords can be formed from the English alphabet if no letter appears more than once?

(i) How many bit strings of length 12 both begin with a 1 and end with a 0?

(j) How many bit strings with length not exceeding n, where n is a positive integer, consist entirely of 1s, not counting the empty string?

(k) How many positive integers between 10 and 60 are divisible by both 5 and 6? Which integers are these?

(l) How many positive integers less than 1000
- are divisible by 7 but not by 11?
- are divisible by either 7 or 11?
- have distinct digits and are even?

(m) How many positive integers less than 1000
- are divisible by 8 but not by 12?
- are divisible by either 8 or 12?
- have distinct digits and are divisible by 5?

(n) How many strings of three decimal digits
- do not contain the same digit three times?
- begin with an odd digit?
- have exactly two digits that are 4s?

(o) How many strings of four decimal digits
- have no repeated digits?
- have the first digit odd and the last digit even?
- contain exactly two 3s?

(p) How many strings of four decimal digits
- do not contain the same digit twice?
- end with an even digit?
- have exactly three digits that are 9s?

(q) How many different functions are there from a set with 10 elements to sets with 5 elements?

(r) How many one-to-one functions are there from a set with five elements to sets with seven elements?

(s) How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$
- that are one-to-one?
- that assign 0 to both 1 and n?
- that assign 1 to exactly one of the positive integers less than n?

(t) At a large university, 434 freshmen, 883 sophomores, and 43 juniors are enrolled in an introductory algorithms course. How many sections of this course need to be scheduled to accommodate all these students if each section contains 34 students?

(u) How many positive integers not exceeding 1000 are divisible either by 4 or by 6 or by 7?

(v) Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase

English letter, an uppercase English letter, a digit, or one of the six special characters $*$, $>$, $<$, !, $+$, and $=$.

- How many different passwords are available for this computer system?
- How many of these passwords contain at least one occurrence of at least one of the six special characters?
- Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that it takes one nanosecond for a hacker to check each possible password.

(w) The name of a variable in the JAVA programming language is a string of between 1 and 65,535 characters, inclusive, where each character can be an uppercase or a lowercase letter, a dollar sign, an underscore, or a digit, except that the first character must not be a digit. Determine the number of different variable names in JAVA.

(x) A key in the Vigenere cryptosystem is a string of English letters, where the case of the letters does not matter. How many different keys for this cryptosystem are there with three, four, five, or six letters?

(y) Determine the number of matches played in a singleelimination (knock-out) tournament with n players, where for each game between two players the winner goes on, but the loser is eliminated.

(z) Determine the minimum and the maximum number of matches that can be played in a double-elimination tournament with n players, where after each game between two players, the winner goes on and the loser goes on if and only if this is not a second loss.