SEMINAR LESSON 2

Relations and functions

1) Prove that there exist A, B and C such that:
   (a) $A \times B \neq B \times A$;
   (b) $A \times (B \times C) \neq (A \times B) \times C$.
2) Find a geometric interpretation of the following set: $[a; b] \times [c; d]$, where $[a; b]$ and $[c; d]$ are segments of the real line $\mathbb{R}$;
3) Prove that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.
4) Prove that:
   (a) $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
   (b) $(A \backslash B) \times C = (A \times C) \backslash (B \times C)$.
5) Let $A, B \neq \emptyset$ and $(A \times B) \cup (B \times A) = C \times D$. Prove that $A = B = C = D$.
6) Find $dom(R), range(R), R^{-1}, R \cdot R, R \cdot R^{-1}, R^{-1} \cdot R$ for the following relations:
   (a) $R = \{(x, y) | x, y \in \mathbb{N}\ and\ x\ divides\ y\}$;
   (b) $R = \{(x, y) | x, y \in \mathbb{R}\ and\ x + y \leq 0\}$.
7) Prove that
   (a) $dom(R_1 \cdot R_2) = R_1^{-1}\big(range(R_1) \cap dom(R_2)\big)$;
   (b) $range(R_1 \cdot R_2) = R_2\big(range(R_1) \cap dom(R_2)\big)$.
8) What are the binary relations R satisfying $R^{-1} = \bar{R}$?
9) Let A and B be finite sets of m and n elements respectively.
   (a) How many relations between the elements of A and B are there?
   (b) How many functions from A to B are there?
10) Prove that for any binary relations: $(R_1 \cdot R_2)^{-1} = R_2^{-1} \cdot R_1^{-1}$.
11) Let A, B, A_1, B_1 be sets such that A is in 1-1 correspondence with A_1, and B with B_1. Show that there exists a 1-1 correspondence:
   (a) between $A \times B$ and $A_1 \times B_1$;
   (b) between A^B and $A_1^{B_1}$.
12) Prove that f satisfies the condition $f(A \cap B) = f(A) \cap f(B)$ for any A and B if and only if f is 1-1 function.
13) Prove that the function $\chi_A^U(x) = \begin{cases} 1, if\ x \in A, \\ 0, if\ x \in U \backslash A, \end{cases}$ where A is a subset of a non-empty set U, satisfies the following conditions:
   (a) $\chi_{A \cap B}^U(x) = \chi_A^U(x) \cdot \chi_B^U(x)$;
   (b) $\chi_{A \backslash B}^U(x) = \chi_A^U(x) \cdot \left(1 - \chi_B^U(x)\right)$.

HOME WORK 2

Relations and functions

1) Find a geometric interpretation of the following sets:
   (a) $[a;b]^2$;
   (b) $[a;b]^3$.
2) Prove that:
   (a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
   (b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times C) \cup (A \times D) \cup (B \times D)$;
   (c) $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$.
3) Find $dom(R), range(R), R^{-1}, R \cdot R, R \cdot R^{-1}, R^{-1} \cdot R$ for the following relations:
   (a) $R = \{(x,y) | x, y \in \mathbb{N}\ and\ y\ divides\ x\}$;
   (b) $R = \{(x,y) | x, y \in \mathbb{R}\ and\ 2x \geq 3y\}$;
   (c) $R = \{(x,y) | x, y \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]\ and\ y \geq sinx\}$.
4) Let A and B be finite sets of m and n elements respectively.
   (a) How many 1-1 functions from A to B are there?
   (b) For which m and n is there a 1-1 correspondence between A and B?
5) Prove that $f(A) \backslash f(B) \subseteq f(A \backslash B)$ for any function f.
6) Prove that the function $\chi_A^U(x) = \begin{cases} 1, if\ x \in A, \\ 0, if\ x \in U \backslash A, \end{cases}$ where A is a subset of a non-empty set U, satisfies the following conditions:
   (a) if $A = \cup_{i \in I} A_i$, then $\chi_A^U(x) = \max_{i \in I} \chi_{A_i}^U(x)$;
   (b) if $A = \cap_{i \in I} A_i$, then $\chi_A^U(x) = \min_{i \in I} \chi_{A_i}^U(x)$.