Counting: Elements of Combinatorics

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Basic Counting Principles

- **THE PRODUCT RULE.** Suppose that a procedure can be broken down into a sequence of two tasks. If there are n ways to do the first task and for each of these ways of doing the first task, there are k ways to do the second task, then there are

$$n \cdot k$$

- ways to do the procedure.

Examples

- A new company with just two employees, Asan and Aman, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?
- *Solution:* The procedure of assigning offices to these two employees consists of assigning an office to Asan, which can be done in 12 ways, then assigning an office to Aman different from the office assigned to Asan, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

Examples

- The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
- *Solution:* The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled.

Examples

- There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?
- *Solution:* The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are $32 \cdot 24 = 768$ ports.

Examples

- How many different bit strings of length seven are there?
- *Solution:* Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Examples

- How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?
- *Solution:* There are 26 choices for each of the three uppercase English letters and ten choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates.

Examples

- **Counting Functions.** How many functions are there from a set with m elements to a set with n elements?
- *Solution:* A function corresponds to a choice of one of the n elements in the codomain for each of the*m* elements in the domain. Hence, by the product rule there are $n \cdot n \cdot \ \ldots \ \cdot \ n = n^m$ functions from a set with m elements to one with n elements.

Examples

- **Counting One-to-One Functions.** How many one-to-one functions are there from a set with m elements to one with n elements?
- *Solution:* First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.
- **Remark**. Counting the number of onto functions is harder.

Examples

- **Counting One-to-One Functions.** How many one-to-one functions are there from a set with m elements to one with n elements?
- Now let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, \dots, a_m$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways (because the value used for a_1 cannot be used again).

Examples

- In general, the value of the function at a_k can be chosen in $n-k+1$ ways. By the product rule, there are

$$n(n-1)(n-2)\ldots(n-m+1)$$

- one-to-one functions from a set with m elements to one with n elements.

Examples

- **Counting Subsets of a Finite Set.** Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.
- *Solution:* Let S be a finite set. List the elements of S in arbitrary order. Recall that there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$.

Examples

- Namely, a subset of S is associated with the bit string with a 1 in the ith position if the ith element in the list is in the subset, and a 0 in this position otherwise. By the product rule, there are $2^{|S|}$ bit strings of length $|S|$. Hence,

$$|P(S)| = 2^{|S|}.$$

The Basics of Counting

- **THE SUM RULE.** If a task can be done either in one of n ways or in one of k ways, where none of the set of n ways is the same as any of the set of k ways, then there are

$$n + k$$

- ways to do the task.

Examples

- Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Examples

- *Solution:* There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

The Basics of Counting

- We can extend the sum rule to more than two tasks. Suppose that a task can be done in one of n_1 ways, in one of n_2 ways, . . . , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \le i < j \le m$. Then the number of ways to do the task is

$$n_1 + n_2 + \ldots + n_m.$$

Example

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
- *Solution:* The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are 23 + 15 + 19 = 57 ways to choose a project.

More Complex Counting Problems

- Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- *Solution:* Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$. We will now find P_6, P_7, and P_8.

More Complex Counting Problems

- Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$$P_6 = 36^6 - 26^6 =$$
$$2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$$

More Complex Counting Problems

Similarly, we have

$$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920$$

and

$$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$$

Consequently,

$$P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$$

The Subtraction Rule (Inclusion-Exclusion for Two Sets)

- **THE SUBTRACTION RULE.** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

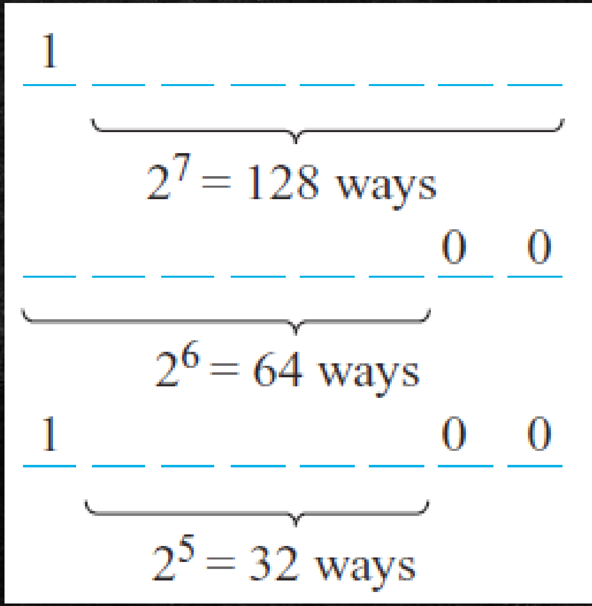
- **Remark**. The subtraction rule is also known as the **principle of inclusion-exclusion**.

Example

- How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
- *Solution:* We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00. We can construct a bit string of length eight that begins with a 1 in 2^7=128 ways.

Example

- Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6 = 64$ ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way.



Example

- Some of the ways to construct a bit string of length eight starting with a 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00. There are $2^5=32$ ways to construct such a string. This follows by the product rule, because the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in one way.

Example

- Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, which equals the number of ways to construct a bit string of length eight that begins with a 1 or that ends with 00, equals

$$128 + 64 - 32 = 160.$$

The Division Rule

- **THE DIVISION RULE.** There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.
- We can restate the division rule in terms of sets: "If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$."

Example

- How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?
- *Solution:* We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table.

Example

- Note that are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and one way to select the person for seat 4. Thus, there are $4! = 24$ ways to order the given four people for these seats.

Example

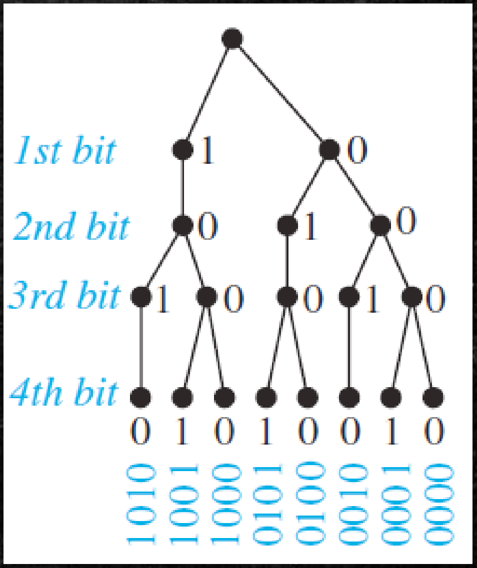
- However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table.

Tree Diagrams

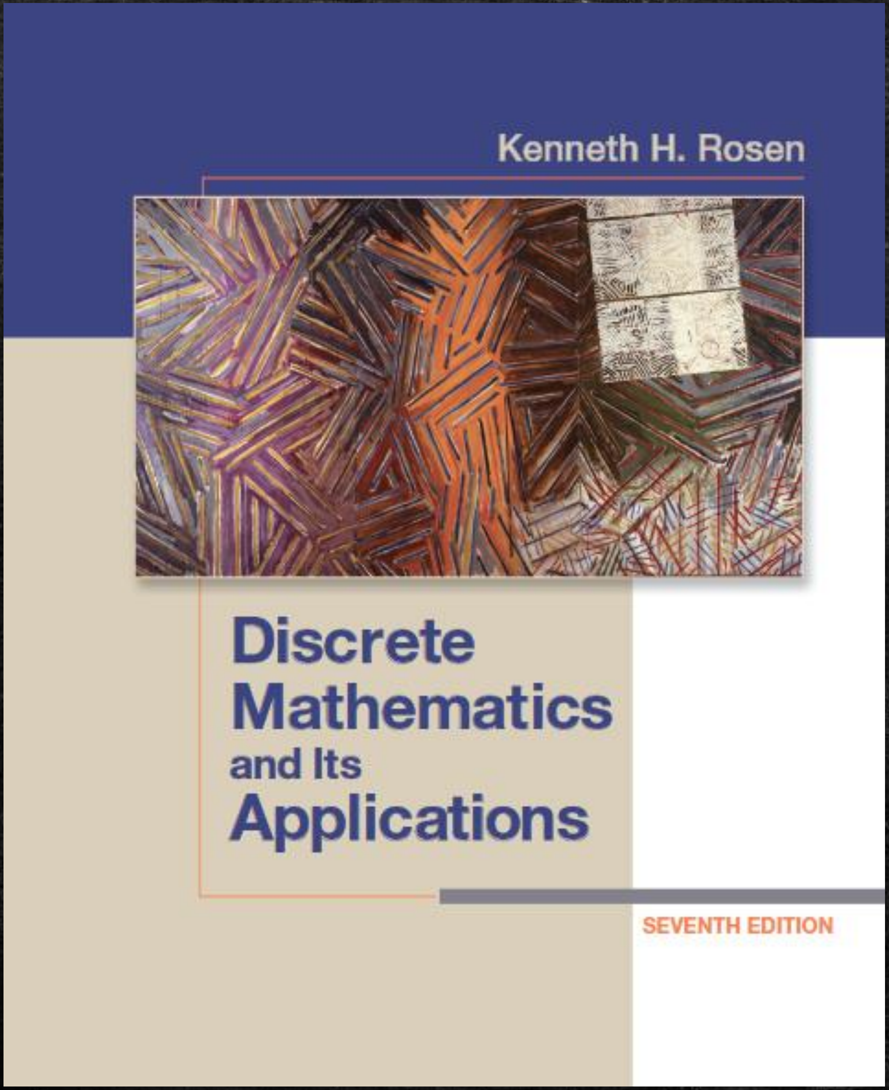
- Counting problems can be solved using **tree diagrams**. A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches. (We will study trees in detail in future.) To use trees in counting, we use a branch to represent each possible choice. We represent the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them.

Example

- How many bit strings of length four do not have two consecutive 1s?
- *Solution:* The tree diagram in Figure displays all bit strings of length four without two consecutive 1s.We see that there are eight bit strings of length four without two consecutive 1s.



HOMEWORK: Exercises 2, 4, 6, 14, 16, 22, 32, 36, 40, 62 on pp. 396-398;



The Pigeonhole Principle

- **THE PIGEONHOLE PRINCIPLE.** If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

The Pigeonhole Principle

- The pigeonhole principle is also called the **Dirichlet (drawer) principle**, after the nineteenth century German mathematician G. Lejeune Dirichlet, who often used this principle in his work. Dirichlet was not the first person to use this principle; a demonstration that there were at least two Parisians with the same number of hairs on their heads dates back to the 17th century

The Pigeonhole Principle

- **COROLLARY 1.** A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

Example

- How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
- *Solution:* There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Example

- Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.
- *Solution:* Let n be a positive integer. Consider the $n + 1$ integers $1, 11, 111, \ldots, 11\ldots1$ (where the last integer in this list is the integer with $n + 1$ 1s in its decimal expansion). Note that there are n possible remainders when an integer is divided by n.

Example

- Because there are $n + 1$ integers in this list, by the pigeonhole principle there must be two with the same remainder when divided by n. The larger of these integers less the smaller one is a multiple of n, which has a decimal expansion consisting entirely of 0s and 1s.

The Generalized Pigeonhole Principle

- The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes. However, even more can be said when the number of objects exceeds a multiple of the number of boxes. For instance, among any set of 21 decimal digits there must be 3 that are the same. This follows because when 21 objects are distributed into 10 boxes, one box must have more than 2 objects.

The Generalized Pigeonhole Principle

- **THEOREM 2. (THE GENERALIZED PIGEONHOLE PRINCIPLE)** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example

- What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Example

- *Solution:* The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

Example

- **a)** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
- **b)** How many must be selected to guarantee that at least three hearts are selected?

Example

- *Solution:* **a)** Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $\lceil N/4 \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$, so nine cards suffice.

Example

- *Solution:* **b)** We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.

Some Elegant Applications of the Pigeonhole Principle

- **Example**. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Some Elegant Applications of the Pigeonhole Principle

- *Solution:* Let a_j be the number of games played on or before the jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Moreover, $a_1 + 14, a_2 + 14,\ \ldots,\ a_{30} + 14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j + 14 \leq 59$.

Some Elegant Applications of the Pigeonhole Principle

- *Solution (cont.):* The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Because the integers a_j, $j = 1, 2, \ldots, 30$ are all distinct and the integers $a_j + 14$, $j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i.

Some Elegant Applications of the Pigeonhole Principle

- **Example**. Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

- *Solution:* Write each of the $n + 1$ integers $a_1, a_2, \ldots, a_n + 1$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$ for $j = 1, 2, \ldots, n + 1$, where k_j is a nonnegative integer and q_j is odd.

Some Elegant Applications of the Pigeonhole Principle

- *Solution (cont.):* The integers $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then, $a_i = 2^{k_i}q$ and $a_j = 2^{k_j}q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i.

Some Elegant Applications of the Pigeonhole Principle

- **THEOREM 3.** Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.
- **Example**. The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains 10 terms. Note that $10 = 3^2 + 1$. There are four strictly increasing subsequences of length four, namely, 1, 4, 6, 12; 1, 4, 6, 7; 1, 4, 6, 10; and 1, 4, 5, 7. There is also a strictly decreasing subsequence of length four, namely, 11, 9, 6, 5.

Some Elegant Applications of the Pigeonhole Principle

- The final example shows how the generalized pigeonhole principle can be applied to an important part of combinatorics called **Ramsey theory**, after the English mathematician F. P. Ramsey. In general, Ramsey theory deals with the distribution of subsets of elements of sets.
- **Ramsey’s problem**. Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Some Elegant Applications of the Pigeonhole Principle

- *Solution:* Let A be one of the six people. Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. This follows from the generalized pigeonhole principle, because when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements.

Some Elegant Applications of the Pigeonhole Principle

- *Solution (cont.):* In the former case, suppose that *B*, *C*, and *D* are friends of *A*. If any two of these three individuals are friends, then these two and *A* form a group of three mutual friends. Otherwise, *B*, *C*, and *D* form a set of three mutual enemies. The proof in the latter case, when there are three or more enemies of *A*, proceeds in a similar manner.

Some Elegant Applications of the Pigeonhole Principle

- The **Ramsey number** $R(m, n)$, where m and n are positive integers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies.

Some Elegant Applications of the Pigeonhole Principle

- It is possible to prove some useful properties about Ramsey numbers, but for the most part it is difficult to find their exact values. Note that by symmetry it can be shown that $R(m, n) = R(n, m)$. We also have $R(2, n) = n$ for every positive integer $n \geq 2$. The exact values of only nine Ramsey numbers $R(m, n)$ with $3 \leq m \leq n$ are known, including $R(4, 4) = 18$. Only bounds are known for many other Ramsey numbers, including $R(5, 5)$, which is known to satisfy $43 \leq R(5, 5) \leq 49$.

HOMEWORK: Exercises 4, 10, 14, 34, 40 on pp. 405-406;