Binomial Coefficients and Combinatorial Counting

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Motivation

- How many ways can we arrange objects?
- How many subsets of a given size exist?
- How many functions are there between finite sets?

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Goal

Develop systematic counting tools to answer these questions.

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- The first position can be filled in 5 ways
- The second position can be filled in 4 ways
- The third position can be filled in 3 ways
- The fourth position can be filled in 2 ways
- The last position can be filled in 1 way

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Answer

There are 120 different arrangements.

Permutations

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The number of permutations of an n-element set is

$$n!.$$

Permutations of a Set

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Permutations

$$(1,2,3),\ (1,3,2),\ (2,1,3),\ (2,3,1),\ (3,1,2),\ (3,2,1)$$

here (x, y, z) mean $f(1) = x,\ f(2) = y,\ f(3) = z$.

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k-Permutations

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In how many ways can we select three students from a group of five students to stand in line for a picture?

- First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line.
- Once this student has been selected, there are four ways to select the second student in the line.
- After the first and second students have been selected, there are three ways to select the third student in the line.

By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

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Example

Number of ways to choose and order 3 students from 10:

$$P_3^{10} = 10 \cdot 9 \cdot 8$$

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- Suppose that there are eight runners in a race. The winner receives a gold medal, the secondplace finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

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- Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Combinations

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To answer this question, we need only find the number of subsets with three elements from the set containing the four students. We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter.

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Example

Let S be the set $\{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from S. (Note that $\{4, 1, 3\}$ is the same 3-combination as $\{1, 3, 4\}$, because the order in which the elements of a set are listed does not matter.)

Theorem

The number of k-combinations of a set with n elements, where n is a non-negative integer and k is an integer with $0 \le k \le n$, equals

$$C_n^k = \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

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- A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?
- How many bit strings of length n contain exactly k 1s?

Properties

- Symmetry:

$$\binom{n}{k} = \binom{n}{n-k}$$

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- Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Theorem

Binomial Theorem Let x and y be variables, and let n be a non-negative integer. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k =$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Exercises

- What is the expansion of $(x + y)^7$?

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- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?
- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Permutations with Repetition

Example

How many strings of length r can be formed from the uppercase letters of the English alphabet?

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Theorem

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Combinations with Repetition

Example

How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Combinations with Repetition

Example

How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

- 4 apples
- 3 apples, 1 orange
- 3 oranges, 1 pear
- 2 apples, 2 oranges
- 2 apples, 1 orange, 1 pear
- 4 oranges
- 3 apples, 1 pear
- 3 pears, 1 apple
- 2 apples, 2 pears
- 2 oranges, 1 apple, 1 pear
- 4 pears
- 3 oranges, 1 apple
- 3 pears, 1 orange
- 2 oranges, 2 pears
- 2 pears, 1 apple, 1 orange

r-Combinations with Repetition

Theorem

There are $C^{r}_{n+r-1} = C^{n-1}_{n+r-1}$ r-combinations from a set with n elements when repetition of elements is allowed.

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Each r-combination of a set with n elements when repetition is allowed can be represented by a list of $n-1$ bars and r stars. The $n-1$ bars are used to mark off n different cells, with the i-th cell containing a star for each time the i-th element of the set occurs in the combination. For instance, a 6-combination of a set with four elements is represented with three bars and six stars. Here

$$**\,|\,*\,|\,|\,***$$

represents the combination containing exactly two of the first element, one of the second element, none of the third element, and three of the fourth element of the set.

Examples

- Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

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- Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.
- How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are non-negative integers?

Functions on Finite Sets

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Counting Functions

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Counting Functions

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Example

Number of functions from a 3-element set to a 2-element set:

$$2^3 = 8.$$

Special Types of Functions

- Injective:

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Remark

Counting surjections naturally leads back to inclusion–exclusion.

Summary

- Factorials count orderings
- Permutations and combinations describe selections
- Binomial coefficients count subsets
- Functions on finite sets unify many counting problems