Mathematical Induction, Counting Principles.

Birzhan Kalmurzayev

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Proof Techniques

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- Checking each case individually is impossible
- We need a systematic proof method

Mathematical Induction

Principle of Mathematical Induction

Let $P(n)$ be a statement depending on $n \in \mathbb{N}$. If:

- $P(1)$ is true (base case),
- $P(n) \Rightarrow P(n+1)$ for all n (inductive step),

then $P(n)$ is true for all $n \in \mathbb{N}$.

Induction: Intuition

- The base case starts the process
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Like falling dominoes: once the first falls, all the others follow.

Example of Induction

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Prove that
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for all $n \in \mathbb{N}$.

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for all $n \in \mathbb{N}$.

- Base case: $n = 1$
- Inductive hypothesis: assume true for n
- Prove for $n + 1$

Strong Induction

Principle of Strong Induction

To prove $P(n)$ for all $n \in \mathbb{N}$, assume that

$$P(1), P(2), \dots, P(n)$$

are all true, and prove $P(n+1)$.

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Remark

Strong induction is logically equivalent to ordinary induction.

Typical Mistakes

- Forgetting the base case
- Using the statement to be proved inside the proof
- Confusing $P(n)$ with $P(n+1)$

One more example: Prove that for any n

$$\sum_{n^2 < i \leq (n+1)^2} i = (n^2+1) + (n^2+2) + \cdots (n+1)^2 = n^3 + (n+1)^3$$

Basic Counting Principles

Addition Principle

If a task can be performed either as task T_1 or as task T_2, and these two tasks are mutually exclusive, where T_1 can be done in n_1 ways and T_2 in n_2 ways, then the task can be done in

$$(n_1 + n_2) \text{ ways.}$$

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Multiplication Principle

If a task consists of two successive steps T_1 and T_2, where T_1 can be done in n_1 ways and for each of them T_2 can be done in n_2 ways, then the task can be done in

$$(n_1 \cdot n_2) \text{ ways.}$$

Examples of Counting Principles

Example (Addition Principle)

A student can choose *either*:

- one of 3 mathematics courses, or
- one of 5 computer science courses.

Since the choices are mutually exclusive, the total number of choices is $3 + 5 = 8$.

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Example (Multiplication Principle)

A password consists of:

- one letter (26 choices),
- followed by one digit (10 choices).

The total number of passwords is $26 \cdot 10 = 260$.

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General Form

If n objects are placed into m boxes, then at least one box contains at least

$$\left\lceil \frac{n}{m} \right\rceil$$

objects.

Inclusion–Exclusion Principle

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For finite sets A, B, and C,

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Idea

We first add all elements, subtract those counted twice, and add back those counted three times.

Inclusion-Exclusion Principle

General Case

For finite sets $A_1, A_2, \ldots A_n$

$$|\bigcup_{i=1}^{n} A_i| = \sum_{s=1}^{n} (-1)^{s+1} \sum_{1 \le i_1 < i_2 < \cdots < i_s \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_s}|$$

Proof by induction:

Example: Inclusion–Exclusion

Problem

In a group of students:

- 30 students study Mathematics,
- 25 students study Physics,
- 20 students study Computer Science,
- 10 students study Math and Physics,
- 8 students study Math and Computer Science,
- 7 students study Physics and Computer Science, and
- 5 students study all three courses.

How many students study at least one subject?

Example: Euler's Totient Function

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For a positive integer n, $\varphi(n)$ is the number of integers $1 \leq k \leq n$ that are coprime to n (i.e., $\gcd(k, n) = 1$).

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Compute $\varphi(12)$.

- Prime factors: $12 = 2^2 \cdot 3$
- Numbers divisible by 2: $\{2, 4, 6, 8, 10, 12\}$ — 6 numbers
- Numbers divisible by 3: $\{3, 6, 9, 12\}$ — 4 numbers
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By inclusion-exclusion: $\varphi(12) = 12 - 6 - 4 + 2 = 4$.

Euler's Totient Function

Euler Function

For any $n \in \mathbb{N}$ the Euler Function is

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

where $p_1, p_2, \ldots, p_k$ are all prime devisors of n.

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$$\varphi(12) = 12\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)$$

Thank you for your attention!