Elements of Graph Theory (cont.)

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Connectivity

- Many problems can be modeled with paths formed by traveling along the edges of graphs. For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model. Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on can be solved using models that involve paths in graphs.

Paths

- Informally, a **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these edges.
- A formal definition of paths and related terminology is given in Definition 1.

Paths

- **DEFINITION 1.** Let n be a nonnegative integer and G an undirected graph. A *path* of *length* n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \ldots, n$, the endpoints x_{i-1} and x_i. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$ (because listing these vertices uniquely determines the path).

Paths

- **DEFINITION 1 (cont.).** The path is a *circuit* if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to *pass through* the vertices $x_1, x_2, \ldots, x_{n-1}$ or *traverse* the edges $e_1, e_2, \ldots, e_n$. A path or circuit is *simple* if it does not contain the same edge more than once.

Paths

- When it is not necessary to distinguish between multiple edges, we will denote a path $e_1, e_2, \ldots, e_n$, where e_i is associated with $\{x_{i-1}, x_i\}$ for $i = 1, 2, \ldots, n$ by its vertex sequence $x_0, x_1, \ldots, x_n$. This notation identifies a path only as far as which vertices it passes through. Consequently, it does not specify a unique path when there is more than one path that passes through this sequence of vertices, which will happen if and only if there are multiple edges between some successive vertices in the list. Note that a path of length zero consists of a single vertex.

Paths

- ***Remark:*** There is considerable variation of terminology concerning the concepts defined in Definition 1. For instance, in some books, the term **walk** is used instead of *path*, where a walk is defined to be an alternating sequence of vertices and edges of a graph, $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$, where v_{i-1} and v_i are the endpoints of e_i for $i = 1, 2, \ldots, n$.

Paths

- ***Remark (cont.):*** When this terminology is used, **closed walk** is used instead of *circuit* to indicate a walk that begins and ends at the same vertex, and **trail** is used to denote a walk that has no repeated edge (replacing the term *simple path*). When this terminology is used, the terminology **path** is often used for a trail with no repeated vertices, conflicting with the terminology in Definition 1. Because of this variation in terminology, you will need to make sure which set of definitions are used in a particular book or article when you read about traversing edges of a graph.

Paths: Example

- In the simple graph shown in Figure 1, a, d, c, f, e is a simple path of length 4, because $\{a, d\}, \{d, c\}, \{c, f\}$, and $\{f, e\}$ are all edges. However, d, e, c, a is not a path, because $\{e, c\}$ is not an edge.

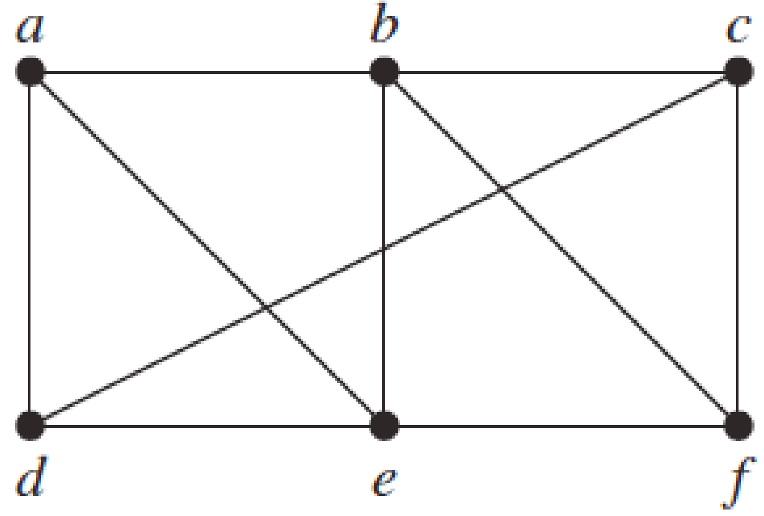


FIGURE 1 A Simple Graph.

Paths: Example

- Note that b, c, f, e, b is a circuit of length 4 because $\{b, c\}, \{c, f\}, \{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice.

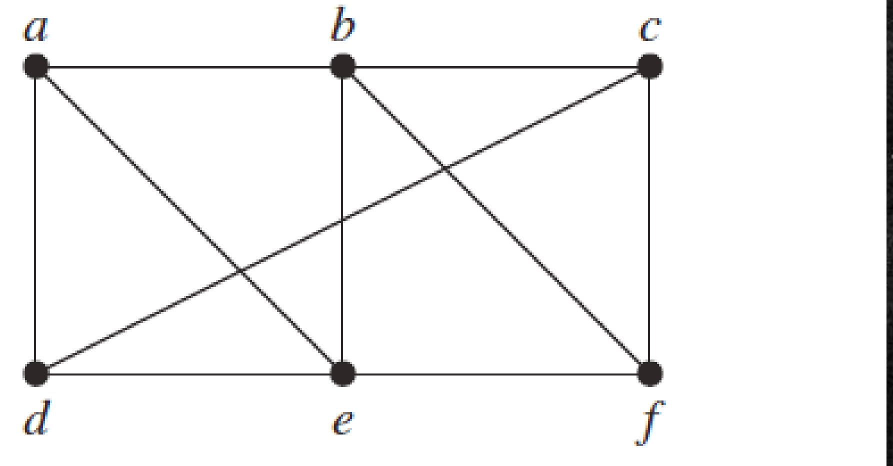


FIGURE 1 A Simple Graph.

Paths

- **DEFINITION 2.** Let n be a nonnegative integer and G a directed graph. A *path* of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$.

Paths

- **DEFINITION 2. (cont.)** When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \ldots, x_n$. A path of length greater than zero that begins and ends at the same vertex is called a *circuit* or *cycle*. A path or circuit is called *simple* if it does not contain the same edge more than once.

Paths

- ***Remark:*** Terminology other than that given in Definition 2 is often used for the concepts defined there. In particular, the alternative terminology that uses *walk, closed walk, trail*, and *path* (described in the remarks following Definition 1) may be used for directed graphs.

Paths: Example of Paths in Collaboration Graphs

- In a collaboration graph, two people a and b are connected by a path when there is a sequence of people starting with a and ending with b such that the endpoints of each edge in the path are people who have collaborated. We will consider two particular collaboration graphs here. First, in the academic collaboration graph of people who have written papers in mathematics, the **Erdo's number** of a person m is the length of the shortest path between m and the extremely prolific mathematician Paul Erdo's.

Paths: Paths in Collaboration Graphs

- That is, the Erdo's number of a mathematician is the length of the shortest chain of mathematicians that begins with Paul Erdo's and ends with this mathematician, where each adjacent pair of mathematicians have written a joint paper. The number of mathematicians with each Erdo's number as of early 2006, according to the Erdo's Number Project, is shown in Table 1.

TABLE 1 The Number of Mathematicians with a Given Erdős Number (as of early 2006).

Erdős Number	*Number of People*
0	1
1	504
2	6,593
3	33,605
4	83,642
5	87,760
6	40,014
7	11,591
8	3,146
9	819
10	244
11	68
12	23
13	5

Connectedness in Undirected Graphs: Example

- The graph G_1 in Figure 2 is connected. However, the graph G_2 in Figure 2 is not connected.

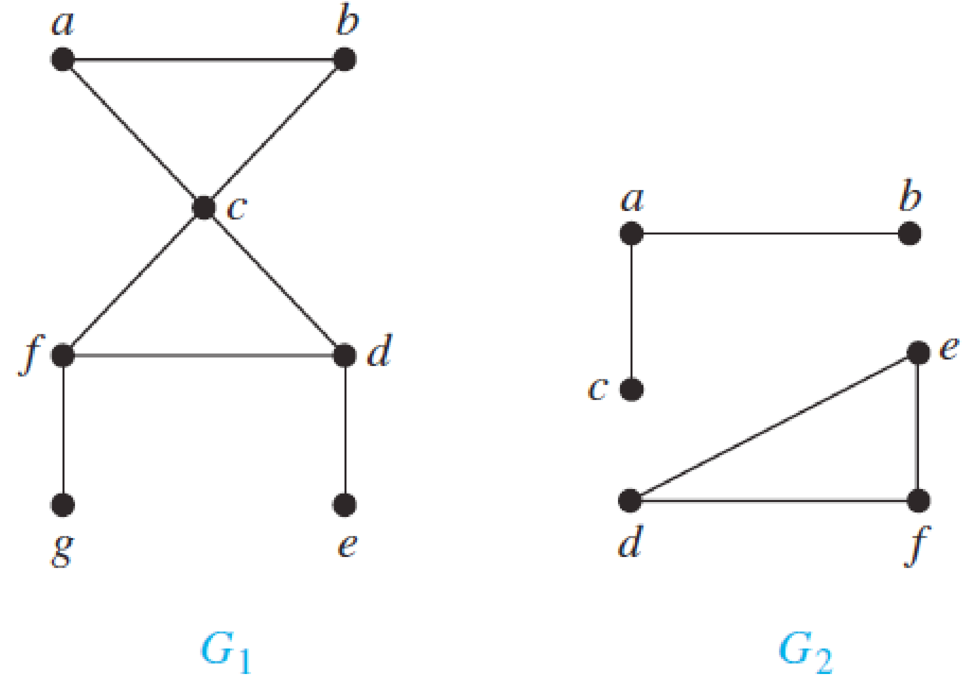


FIGURE 2 The Graphs G_1 and G_2.

Connectedness in Undirected Graphs

- **THEOREM 1.** There is a simple path between every pair of distinct vertices of a connected undirected graph.
- A **connected component** of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

Connectedness in Undirected Graphs: Example

- What are the connected components of the graph H shown in Figure 3?
- *Solution:* The graph H is the union of three disjoint connected subgraphs H_1, H_2, and H_3, which are the connected components of H.

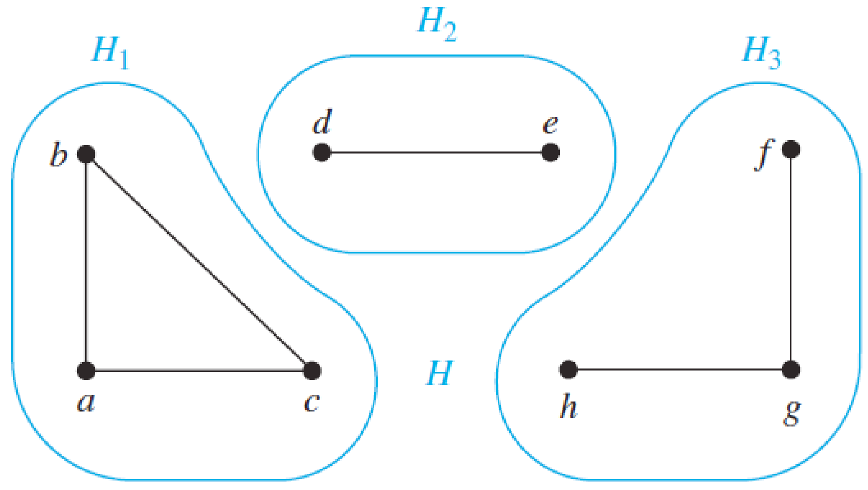


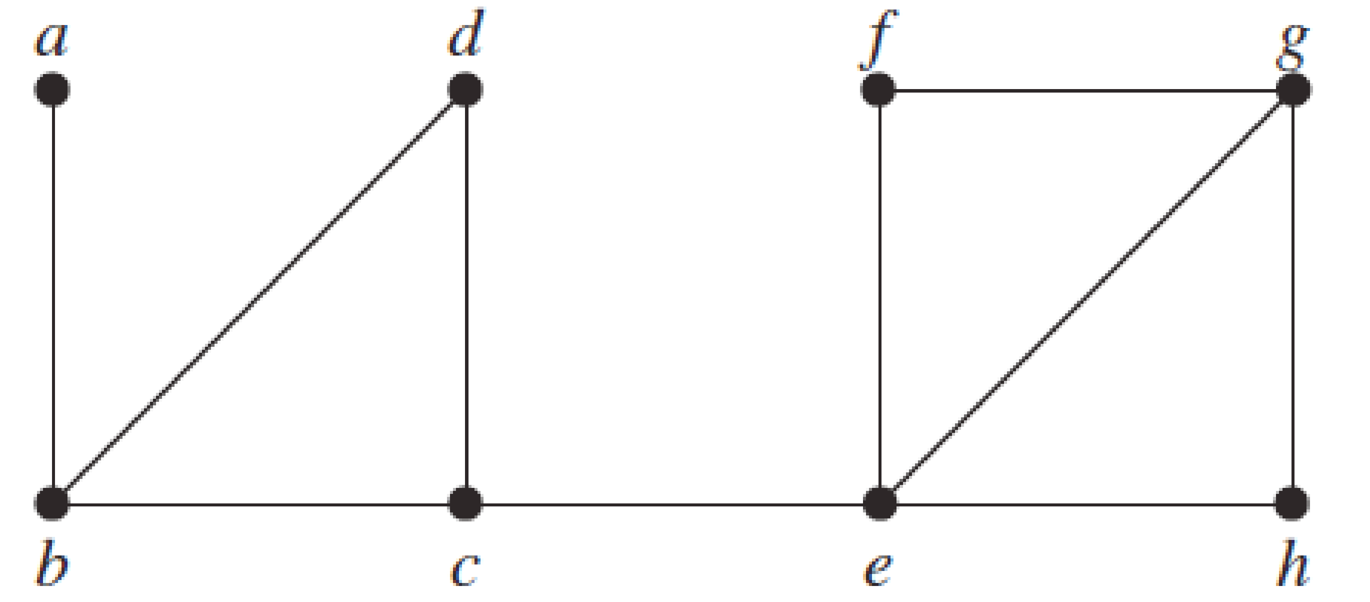
FIGURE 3 The Graph H and Its Connected Components H_1, H_2, and H_3.

How Connected is a Graph?

- Suppose that a graph represents a computer network. Knowing that this graph is connected tells us that any two computers on the network can communicate. However, we would also like to understand how reliable this network is. For instance, will it still be possible for all computers to communicate after a router or a communications link fails? To answer this and similar questions, we now develop some new concepts.

Connected Graph: Example

- Find the cut vertices and cut edges in the graph G_1.
- *Solution:* The cut vertices of G_1 are b, c, and e. The cut edges are $\{a, b\}$ and $\{c, e\}$.



G_1

Vertex Connectivity

- Not all graphs have cut vertices. For example, the complete graph K_n, where $n \geq 3$, has no cut vertices. When you remove a vertex from K_n and all edges incident to it, the resulting subgraph is the complete graph K_{n-1}, a connected graph. Connected graphs without cut vertices are called **nonseparable graphs**, and can be thought of as more connected than those with a cut vertex.

Vertex Connectivity

- A subset V' of the vertex set V of $G = (V, E)$ is a **vertex cut**, or **separating set**, if $G - V'$ is disconnected. Note that every connected graph, except a complete graph, has a vertex cut. We define the **vertex connectivity** of a noncomplete graph G, denoted by $\kappa(G)$, as the minimum number of vertices in a vertex cut.
- When G is a complete graph, it has no vertex cuts, because removing any subset of its vertices and all incident edges still leaves a complete graph.

Vertex Connectivity

- Consequently, we cannot define $\kappa(G)$ as the minimum number of vertices in a vertex cut when G is complete. Instead, we set $\kappa(K_n) = n - 1$, the number of vertices needed to be removed to produce a graph with a single vertex.
- Consequently, for every graph $G, \kappa(G)$ is minimum number of vertices that can be removed from G to either disconnect G or produce a graph with a single vertex.

Vertex Connectivity

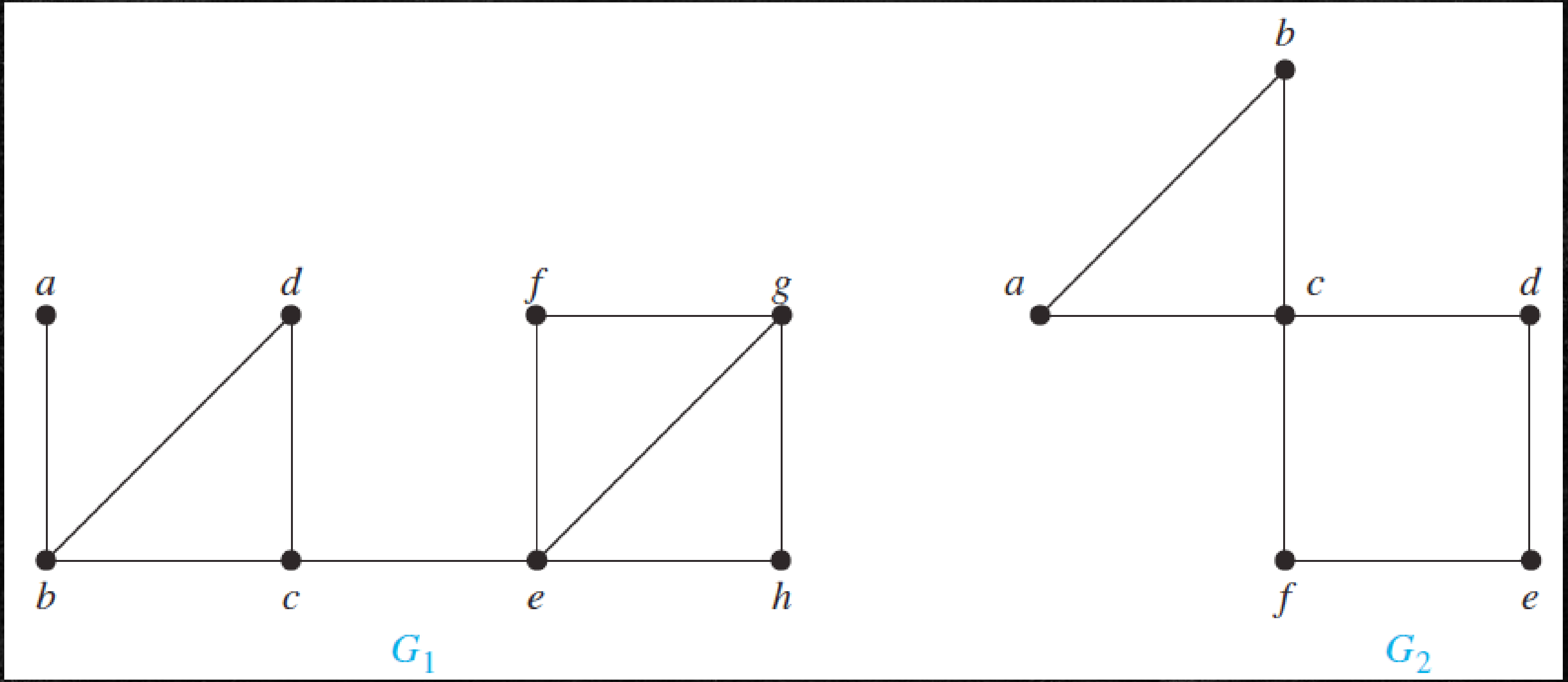
- We have $0 \leq \kappa(G) \leq n-1$ if G has n vertices, $\kappa(G)=0$ if and only if G is disconnected or $G=K_1$, and $\kappa(G)=n-1$ if and only if G is complete.
- The larger $\kappa(G)$ is, the more connected we consider G to be. Disconnected graphs and K_1 have $\kappa(G)=0$, connected graphs with cut vertices and K_2 have $\kappa(G)=1$, graphs without cut vertices that can be disconnected by removing two vertices and K_3 have $\kappa(G)=2$, and so on.

Vertex Connectivity: Example

- Find the vertex connectivity for each of the graphs in the next Figures.
- *Solution:* Each of the five graphs in the next Figures are connected and has more than vertex, so each of these graphs has positive vertex connectivity.

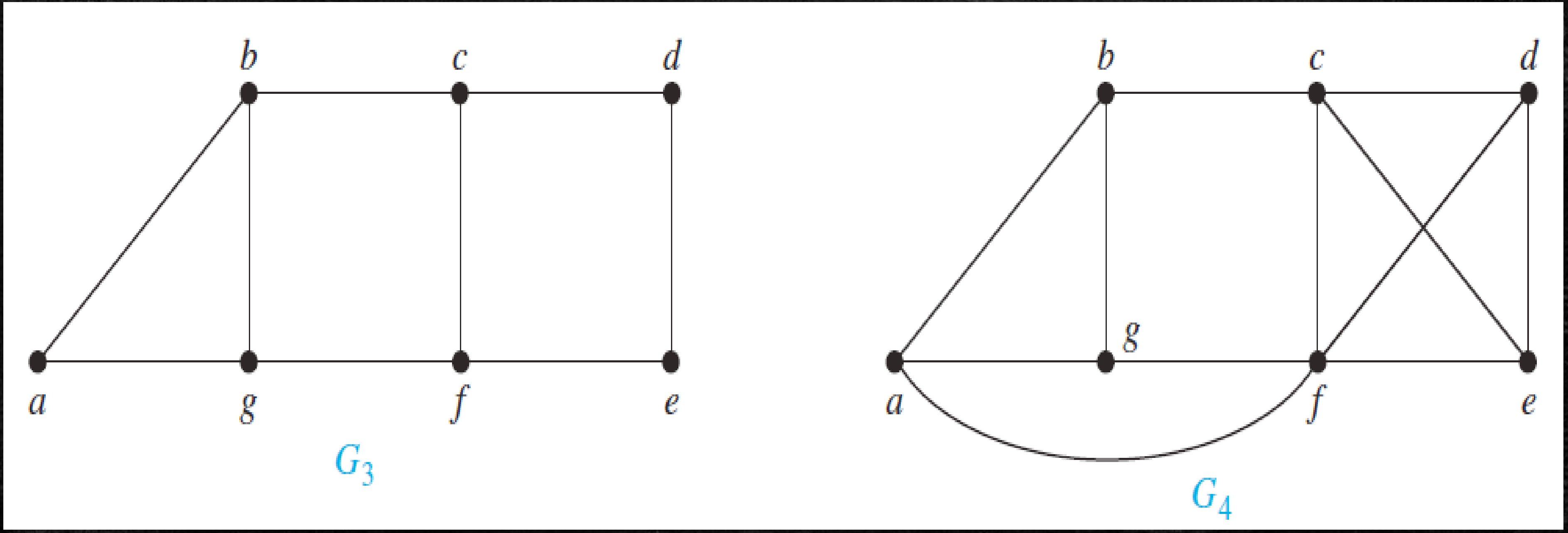
Vertex Connectivity: Example

- *Solution (cont.):* Because G_1 is a connected graph with a cut vertex, as shown in previous Example, we know that $\kappa(G_1) = 1$. Similarly, $\kappa(G_2) = 1$, because c is a cut vertex of G_2.



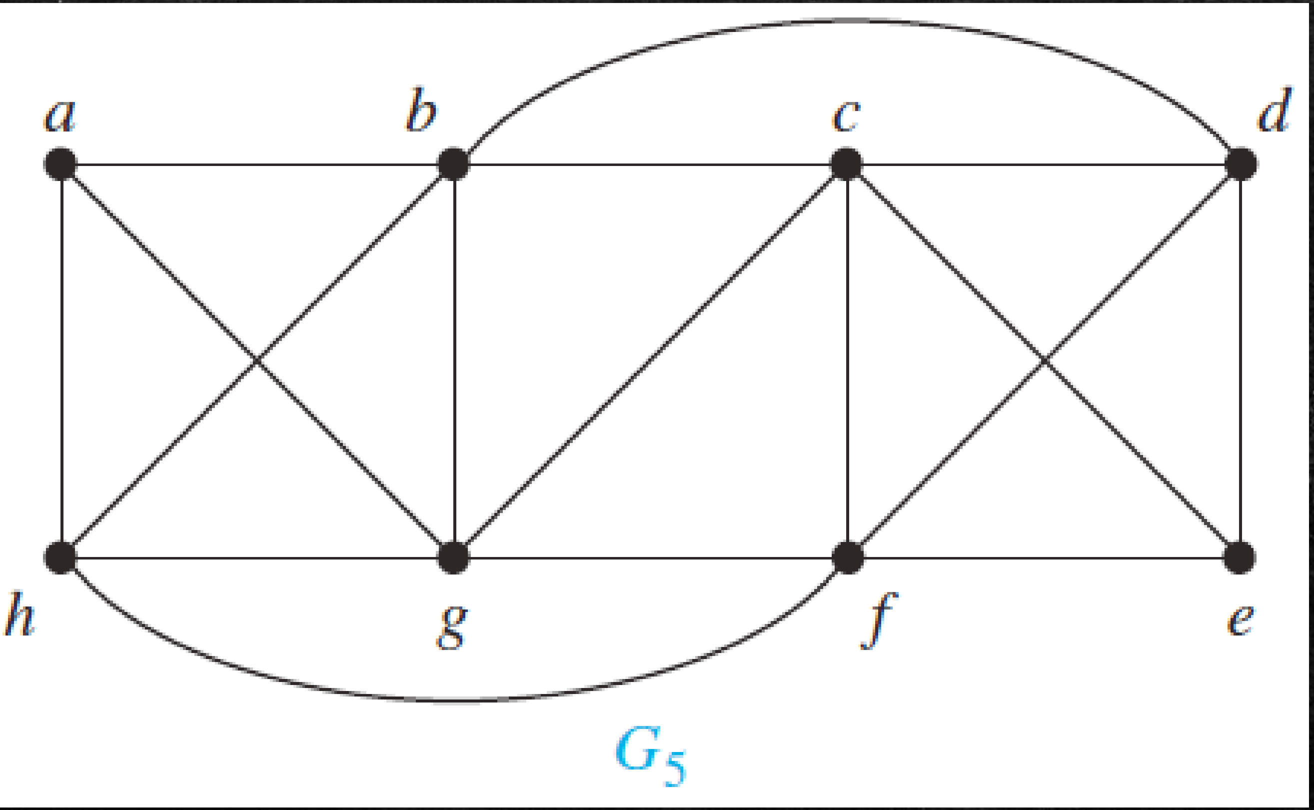
Vertex Connectivity: Example

- *Solution (cont.):* G_3 has no cut vertices, but that $\{b, g\}$ is a vertex cut. Hence, $\kappa(G_3) = 2$. Similarly, because G_4 has a vertex cut of size two, $\{c, f\}$, but no cut vertices. It follows that $\kappa(G_4) = 2$.



Vertex Connectivity: Example

- *Solution (cont.):* G_5 has no vertex cut of size two, but $\{b, c, f\}$ is a vertex cut of G_5. Hence, $\kappa(G_5) = 3$.



Edge Connectivity

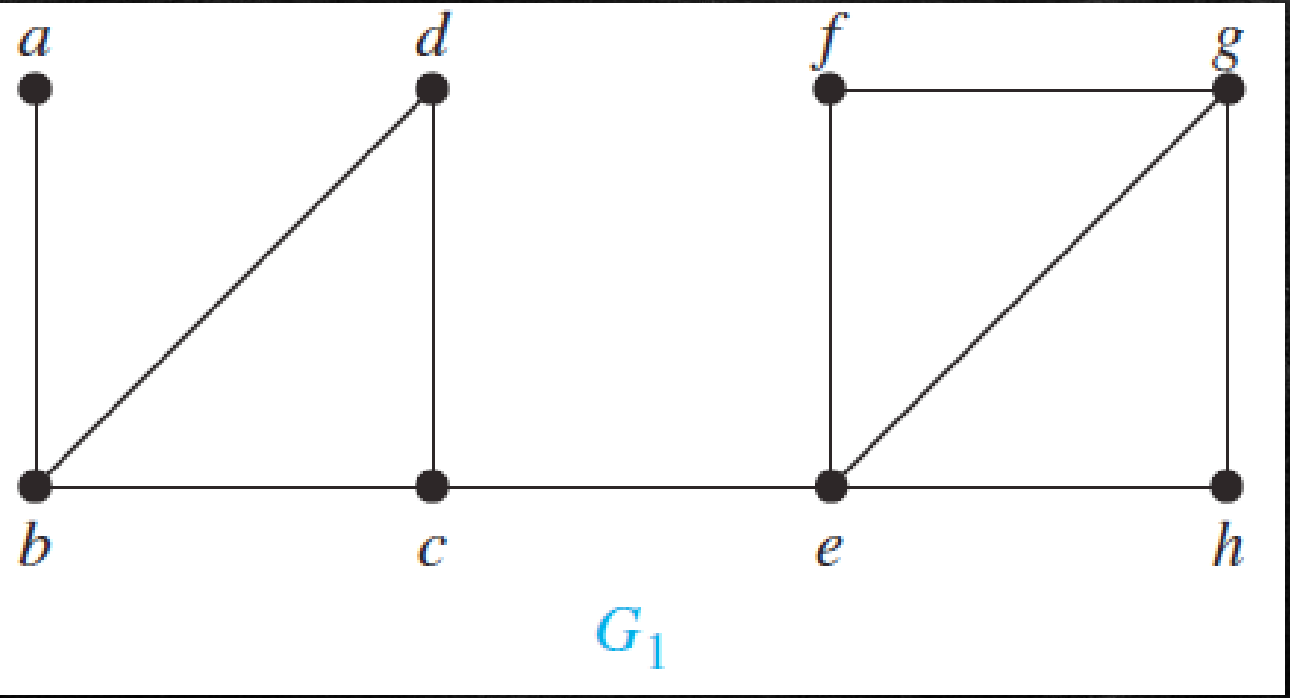
- We can also measure the connectivity of a connected graph $G = (V, E)$ in terms of the minimum number of edges that we can remove to disconnect it. If a graph has a cut edge, then we need only remove it to disconnect G. If G does not have a cut edge, we look for the smallest set of edges that can be removed to disconnect it. A set of edges E' is called an **edge cut** of G if the subgraph $G - E'$ is disconnected.

Edge Connectivity

- The **edge connectivity** of a graph G, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of G. This defines $\lambda(G)$ for all connected graphs with more than one vertex because it is always possible to disconnect such a graph by removing all edges incident to one of its vertices. Note that $\lambda(G) = 0$ if G is not connected. We also specify that $\lambda(G) = 0$ if G is a graph consisting of a single vertex. It follows that if G is a graph with n vertices, then $0 \leq \lambda(G) \leq n - 1$. Note that $\lambda(G) = n - 1$ where G is a graph with n vertices if and only if $G = K_n$.

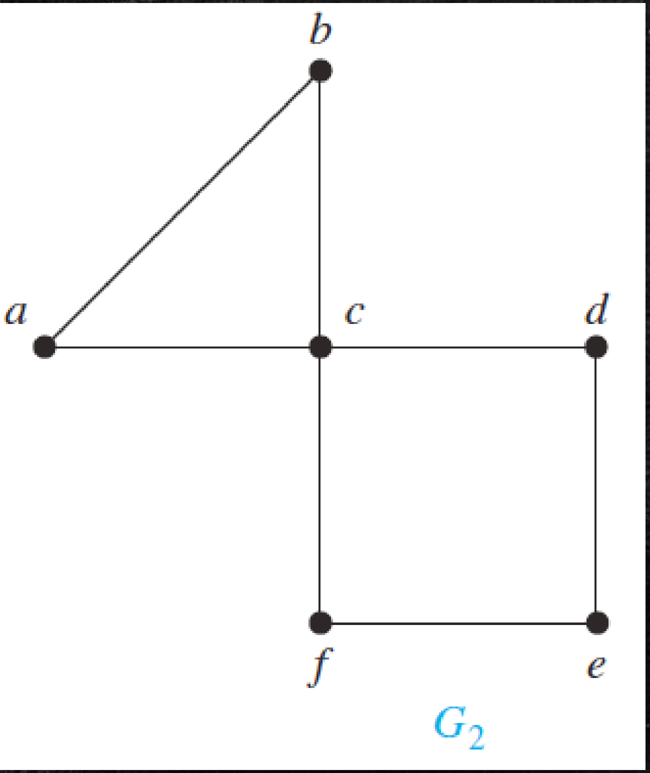
Edge Connectivity: Example

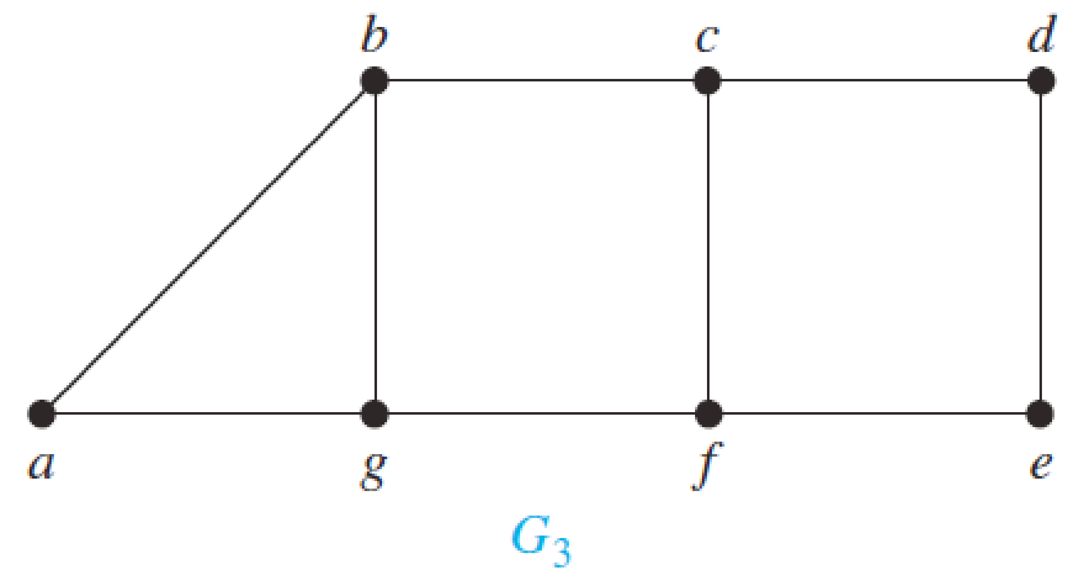
- Find the edge connectivity of each of the graphs in the next Figures.
- *Solution:* G_1 has a cut edge, so $\lambda(G_1) = 1$.



Edge Connectivity: Example

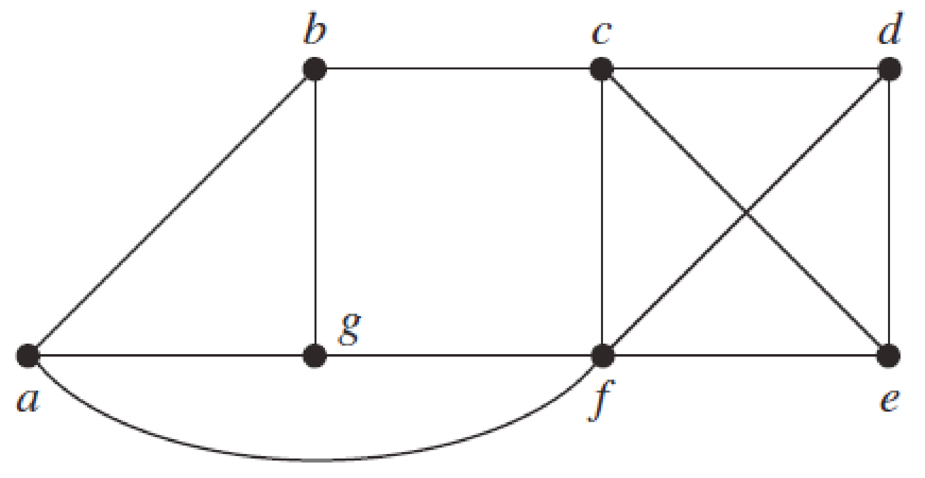
- *Solution (cont.):* The graph G_2 has no cut edges, but the removal of the two edges $\{a, b\}$ and $\{a, c\}$ disconnects it. Hence, $\lambda(G_2) = 2$. Similarly, $\lambda(G_3) = 2$, the removal of the two edges $\{b, c\}$ and $\{f, g\}$ disconnects it.



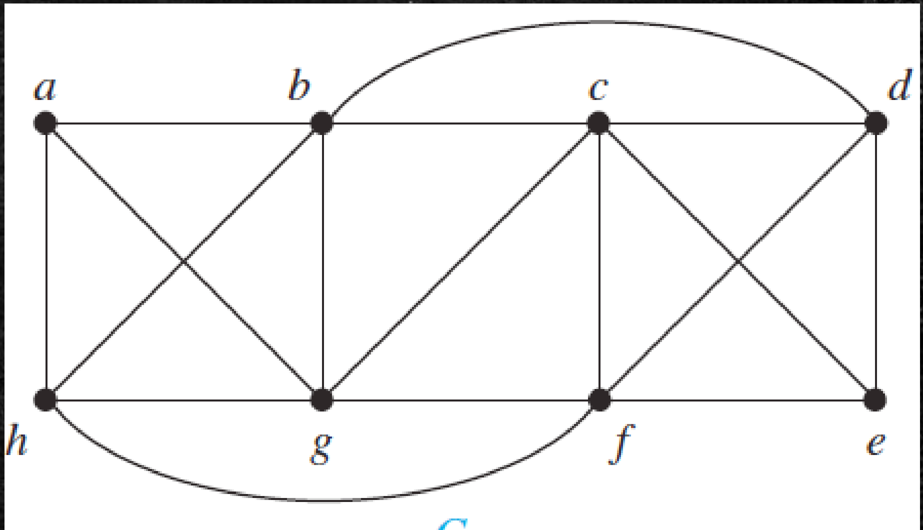


Edge Connectivity: Example

- *Solution (cont.):* The removal of the three edges $\{b, c\}, \{a, f\}$, and $\{f, g\}$ disconnects G_4. Hence, $\lambda(G_4) = 3$. $\lambda(G_5) = 3$, because the removal of $\{a, b\}, \{a, g\}$, and $\{a, h\}$ disconnect it.



G_4



G_5

An Inequality for Vertex connectivity and edge connectivity

- When $G = (V, E)$ is a noncomplete connected graph with at least three vertices, the minimum degree of a vertex of G is an upper bound for both the vertex connectivity of G and the edge connectivity of G. That is, $\kappa(G) \leq \min_{v \in V} \deg(v)$ and $\lambda(G) \leq \min_{v \in V} \deg(v)$. To see this, observe that deleting all the neighbors of a fixed vertex of minimum degree disconnects G, and deleting all the edges that have a fixed vertex of minimum degree as an endpoint disconnects G.

An Inequality for Vertex connectivity and edge connectivity

- Note that $\kappa(G) \leq \lambda(G)$ when G is a connected noncomplete graph. Note also that $\kappa(K_n) = \lambda(K_n) = \min_{v \in V} \deg(v) = n - 1$ when n is a positive integer and that $\kappa(G) = \lambda(G) = 0$ when G is a disconnected graph. Putting these facts together, establishes that for all graphs G,

$$\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v).$$

Connectedness in Directed Graphs

- **DEFINITION 4.** A directed graph is *strongly connected* if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
- For a directed graph to be strongly connected there must be a sequence of directed edges from any vertex in the graph to any other vertex. A directed graph can fail to be strongly connected but still be in “one piece.” Definition 5 makes this notion precise.

Connectedness in Directed Graphs

- **DEFINITION 5.** A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph.
- That is, a directed graph is weakly connected if and only if there is always a path between two vertices when the directions of the edges are disregarded. Clearly, any strongly connected directed graph is also weakly connected.

Connectedness in Directed Graphs: Example

- Are the directed graphs G and H shown in Figure 5 strongly connected? Are they weakly connected?

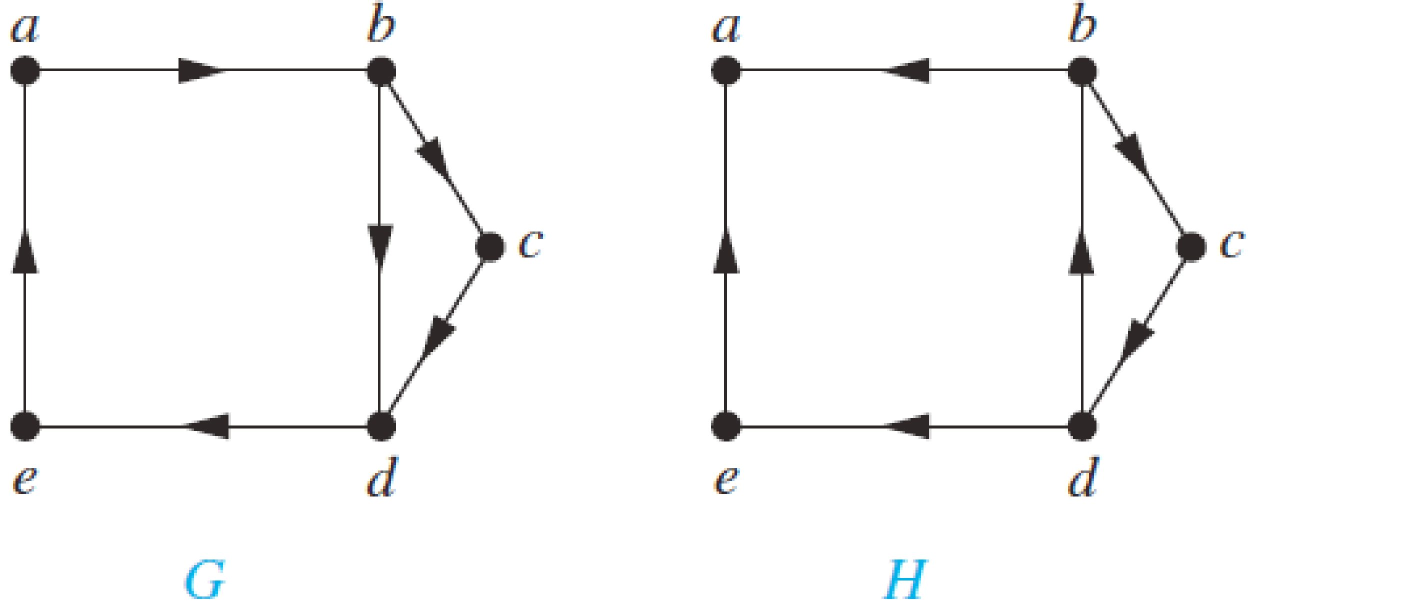


FIGURE 5 The Directed Graphs G and H.

Connectedness in Directed Graphs: Example

- *Solution:* G is strongly connected because there is a path between any two vertices in this directed graph. Hence, G is also weakly connected. The graph H is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, because there is a path between any two vertices in the underlying undirected graph of H.

Paths and Isomorphism

- There are several ways that paths and circuits can help determine whether two graphs are isomorphic. For example, the existence of a simple circuit of a particular length is a useful invariant that can be used to show that two graphs are not isomorphic. In addition, paths can be used to construct mappings that may be isomorphisms.
- As we mentioned, a useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer ≥ 2.

Paths and Isomorphism: Example

- Determine whether the graphs G and H shown in Figure 6 are isomorphic.

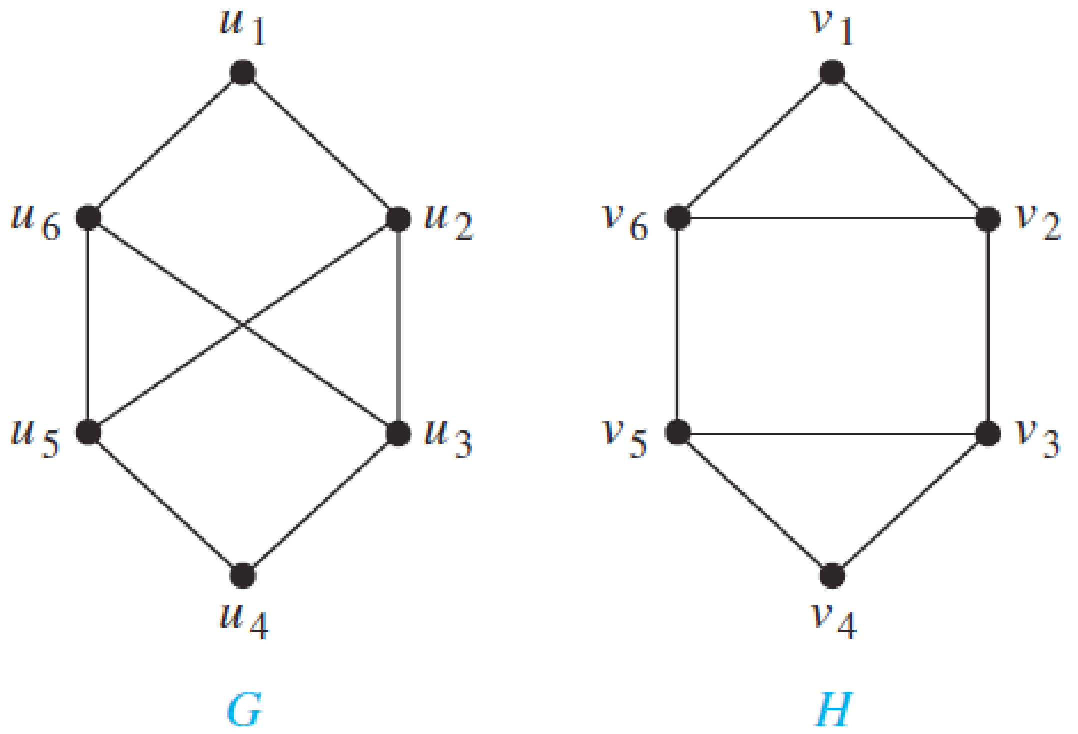


FIGURE 6 The Graphs G and H.

Paths and Isomorphism: Example

- *Solution:* Both G and H have six vertices and eight edges. Each has four vertices of degree three, and two vertices of degree two. So, the three invariants – number of vertices, number of edges, and degrees of vertices – all agree for the two graphs. However, H has a simple circuit of length three, namely, v_1, v_2, v_6, v_1, whereas G has no simple circuit of length three, as can be determined by inspection. Because the existence of a simple circuit of length three is an isomorphic invariant, G and H are not isomorphic.

Paths and Isomorphism: Example

- Determine whether the graphs G and H shown in Figure 7 are isomorphic.

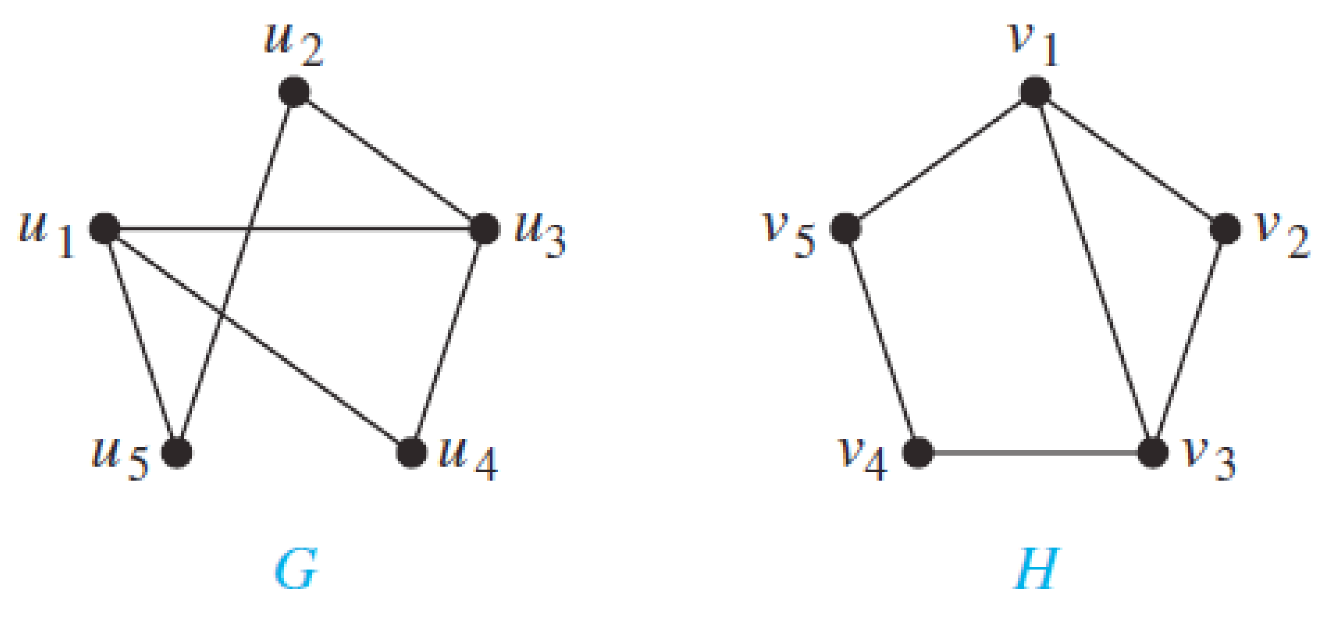


FIGURE 7 The Graphs G and H.

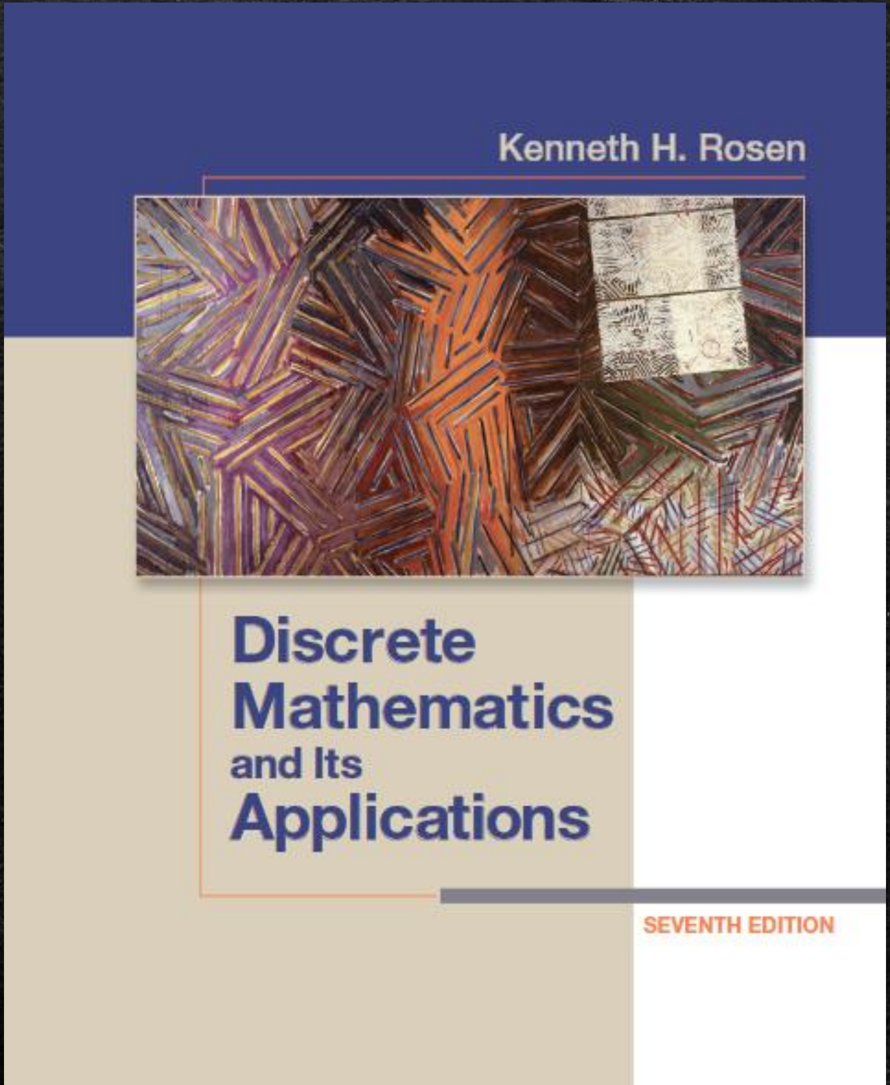
Paths and Isomorphism: Example

- *Solution:* Both G and H have five vertices and six edges, both have two vertices of degree three and three vertices of degree two, and both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five. Because all these isomorphic invariants agree, G and H may be isomorphic.
- To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree.

Paths and Isomorphism: Example

- *Solution (cont.):* For example, the paths u_1, u_4, u_3, u_2, u_5 in G and v_3, v_2, v_1, v_5, v_4 in H both go through every vertex in the graph; start at a vertex of degree three; go through vertices of degrees two, three, and two, respectively; and end at a vertex of degree two. By following these paths through the graphs, we define the mapping f with $f(u_1) = v_3, f(u_4) = v_2, f(u_3) = v_1, f(u_2) = v_5$, and $f(u_5) = v_4$. Note that f is an isomorphism, so G and H are isomorphic.

HOMEWORK: Exercises 2, 4, 12, 14, 20, 22, 50 on pp. 689-692



Euler and Hamilton Paths

- Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly once? Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly once? Although these questions seem to be similar, the first question, which asks whether a graph has an *Euler circuit*, can be easily answered simply by examining the degrees of the vertices of the graph, while the second question, which asks whether a graph has a *Hamilton circuit*, is quite difficult to solve for most graphs.

Euler Paths and Circuits

- The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian Federation), was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure 1 depicts these regions and bridges.

Euler Paths and Circuits

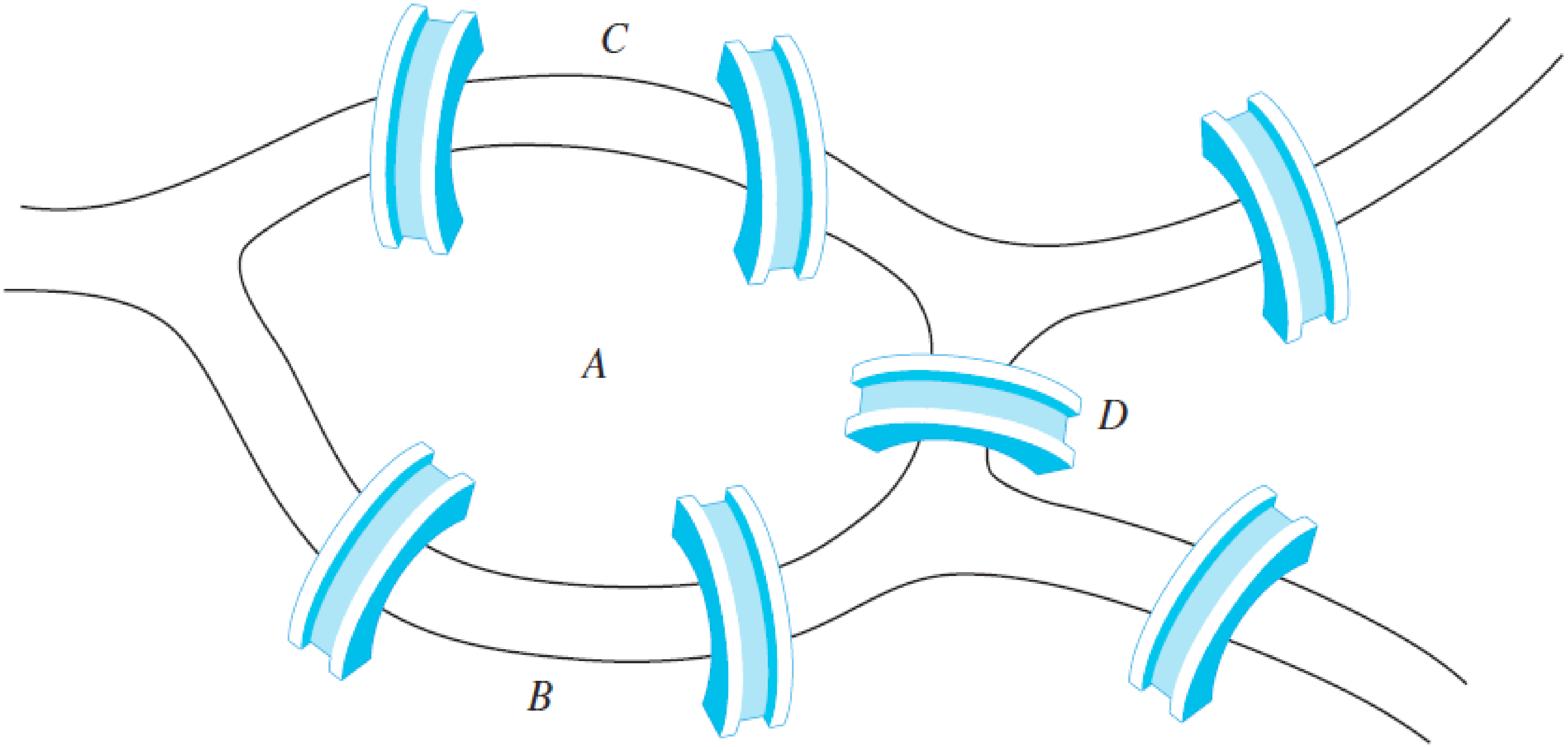


FIGURE 1 The Seven Bridges of Königsberg.

Euler Paths and Circuits

- The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.
- The Swiss mathematician Leonhard Euler solved this problem. His solution, published in 1736, may be the first use of graph theory. Euler studied this problem using the multigraph obtained when four regions are represented by vertices and the bridges by edges. This multigraph is shown in Figure 2.

Euler Paths and Circuits

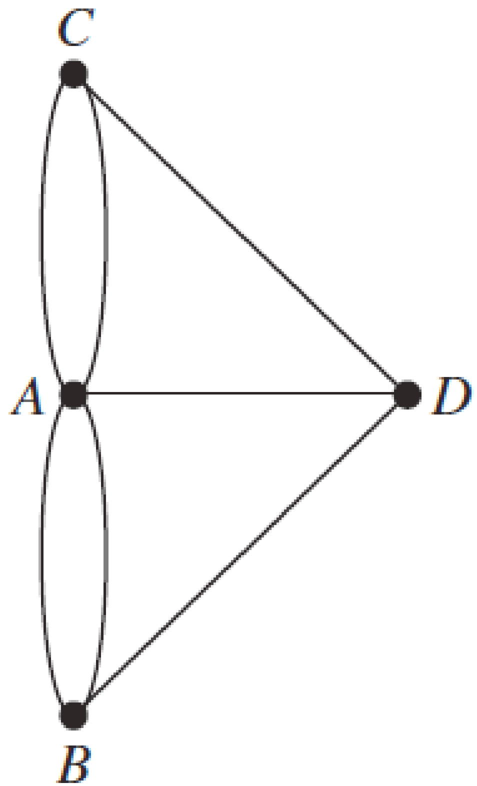


FIGURE 2 Multigraph Model of the Town of Königsberg.

Euler Paths and Circuits

- The problem of traveling across every bridge without crossing any bridge more than once can be rephrased in terms of this model. The question becomes: Is there a simple circuit in this multigraph that contains every edge?
- **DEFINITION 1.** An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

Euler Paths and Circuits: Example

- Which of the undirected graphs in Figure 3 have an Euler circuit? Of those that do not, which have an Euler path?

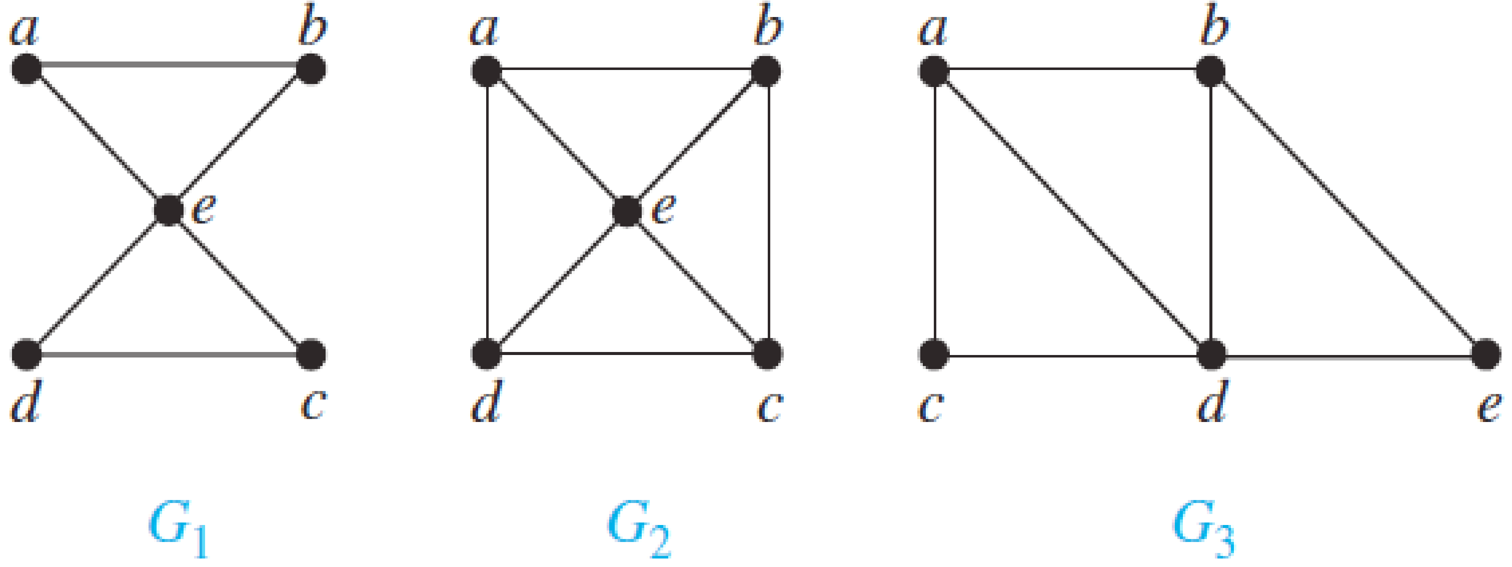


FIGURE 3 The Undirected Graphs G_1, G_2, and G_3.

Euler Paths and Circuits: Example

- *Solution:* The graph G_1 has an Euler circuit, for example, a, e, c, d, e, b, a. Neither of the graphs G_2 or G_3 has an Euler circuit. However, G_3 has an Euler path, namely, a, c, d, e, b, d, a, b. G_2 does not have an Euler path.

Euler Paths and Circuits: Example

- Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?

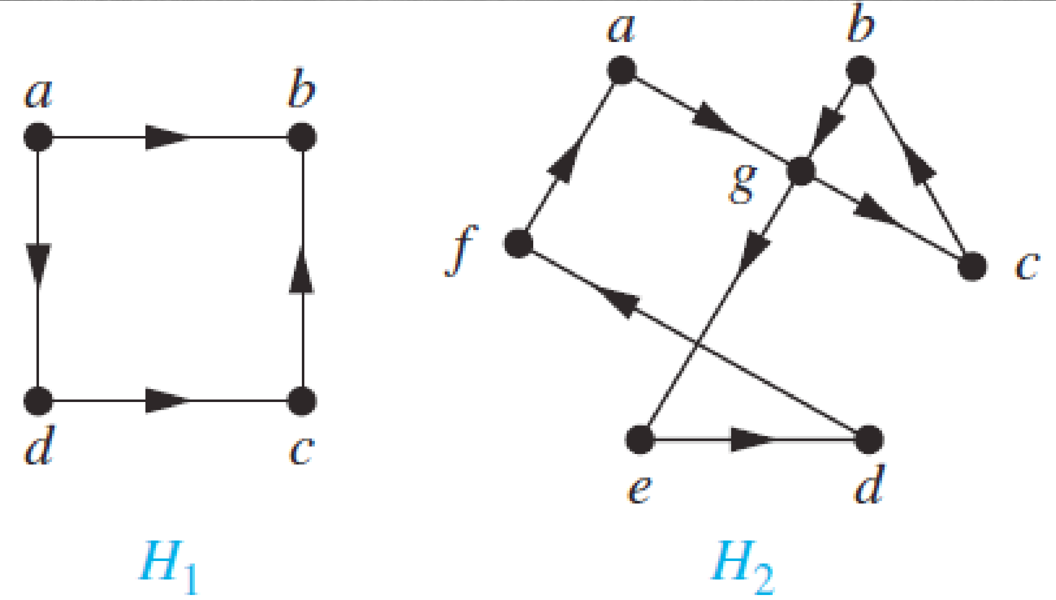


FIGURE 4 The Directed Graphs H_1, H_2, and H_3.

Euler Paths and Circuits: Example

- *Solution:* The graph H_2 has an Euler circuit, for example, $a, g, c, b, g, e, d, f, a$. Neither H_1 nor H_3 has an Euler circuit. H_3 has an Euler path, namely, c, a, b, c, d, b, but H_1 does not.

Necessary and sufficient conditions for Euler circuits and paths

- **THEOREM 1.** A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Euler circuits and paths: Example

- Many puzzles ask you to draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced. We can solve such puzzles using Euler circuits and paths. For example, can *Mohammed's scimitars*, shown in Figure 6, be drawn in this way, where the drawing begins and ends at the same point?

Euler circuits and paths: Example

- We can solve this problem because the graph G shown in Figure 6 has an Euler circuit. It has such a circuit because all its vertices have even degree. We will construct an Euler circuit. First, we form the circuit $a, b, d, c, b, e, i, f, e, a$.

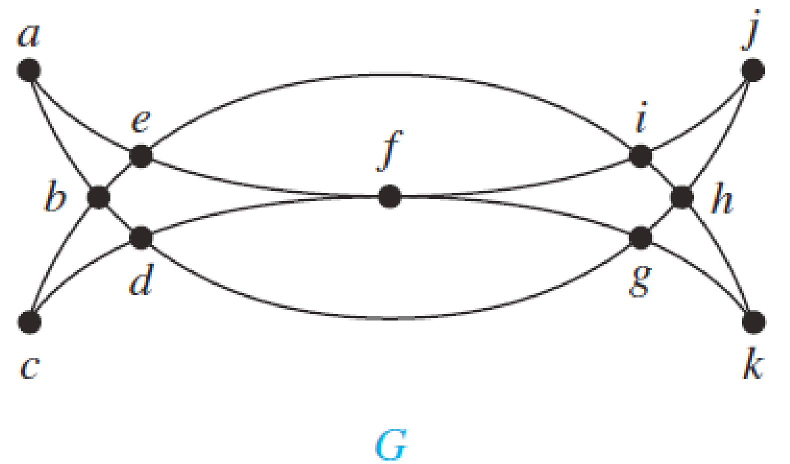


FIGURE 6 Mohammed's Scimitars.

Euler circuits and paths: Example

- We obtain the subgraph H by deleting the edges in this circuit and all vertices that become isolated when these edges are removed. Then we form the circuit $d, g, h, j, i, \; h, k, g, f, d$ in H. After forming this circuit we have used all edges in G. Splicing this new circuit into the first circuit at the appropriate place produces the Euler circuit
$a, b, d, g, h, j, i, \; h, k, g, f, d, c, b, e, i, f, e, a$. This circuit gives a way to draw the scimitars without lifting the pencil or retracing part of the picture.

Euler circuits and paths

- **THEOREM 2.** A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Euler circuits and paths: Example

- Which graphs shown in Figure 7 have an Euler path?

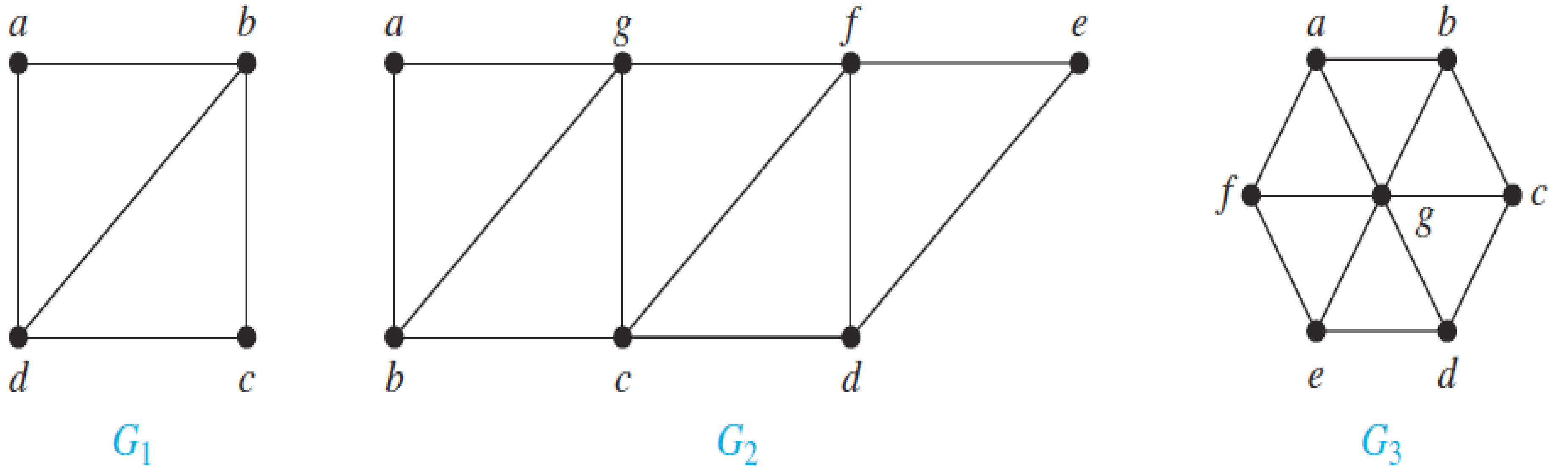


FIGURE 7 Three Undirected Graphs.

Euler circuits and paths: Example

- *Solution:* G_1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints. One such Euler path is d, a, b, c, d, b. Similarly, G_2 has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints. One such Euler path is $b, a, g, f, e, d, c, g, b, c, f, d$. G_3 has no Euler path because it has six vertices of odd degree.

Hamilton Paths and Circuits

- We have developed necessary and sufficient conditions for the existence of paths and circuits that contain every edge of a multigraph exactly once. Can we do the same for simple paths and circuits that contain every vertex of the graph exactly once?
- **DEFINITION 2.** A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*.

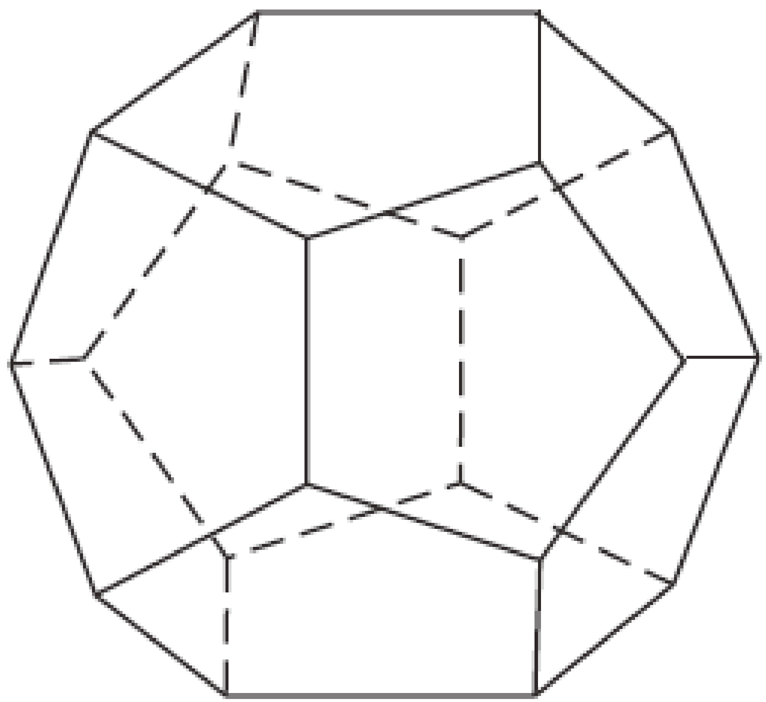
Hamilton Paths and Circuits

- **DEFINITION 2 (cont.).** That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i = x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

Hamilton Paths and Circuits

- This terminology comes from a game, called the *Icosian puzzle*, invented in 1857 by the Irish mathematician Sir William Rowan Hamilton. It consisted of a wooden dodecahedron [a polyhedron with 12 regular pentagons as faces, as shown in the next Figure (a)], with a peg at each vertex of the dodecahedron, and string. The 20 vertices of the dodecahedron were labeled with different cities in the world.

Hamilton Paths and Circuits



(a)

- The object of the puzzle was to start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and end back at the first city. The circuit traveled was marked off using the string and pegs.

Hamilton Paths and Circuits

- We will consider the equivalent question: Is there a circuit in the graph shown in Figure 8(b) that passes through each vertex exactly once?

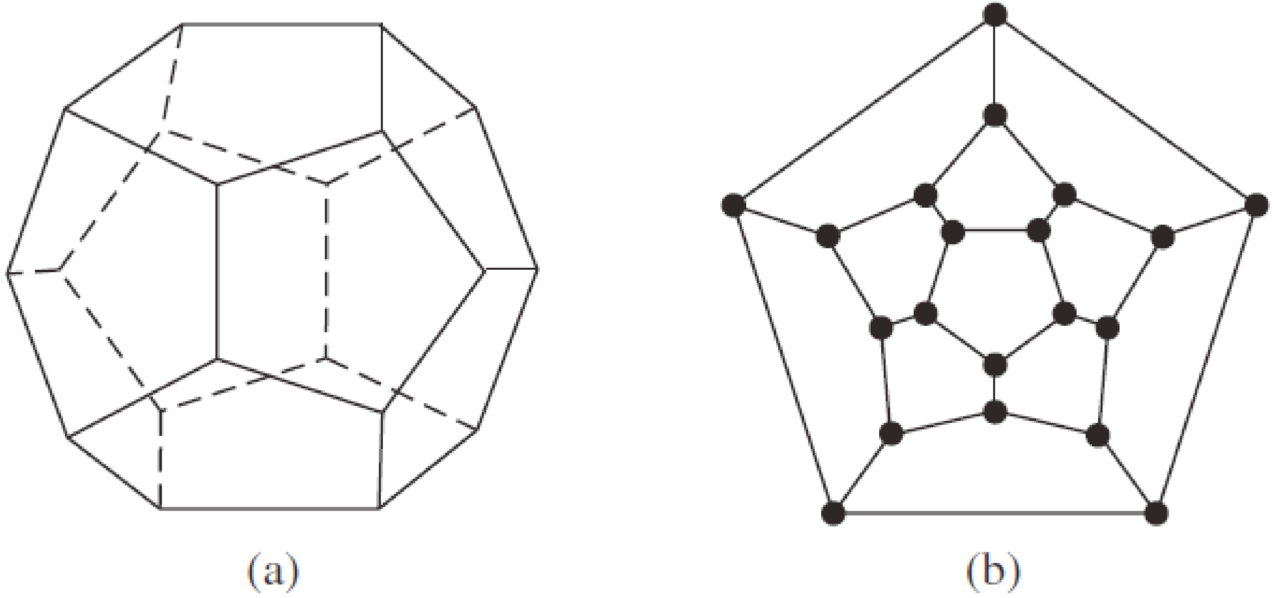


FIGURE 8 Hamilton's "A Voyage Round the World" Puzzle.

Hamilton Paths and Circuits

- This solves the puzzle because this graph is isomorphic to the graph consisting of the vertices and edges of the dodecahedron. A solution of Hamilton’s puzzle is shown in Figure 9.

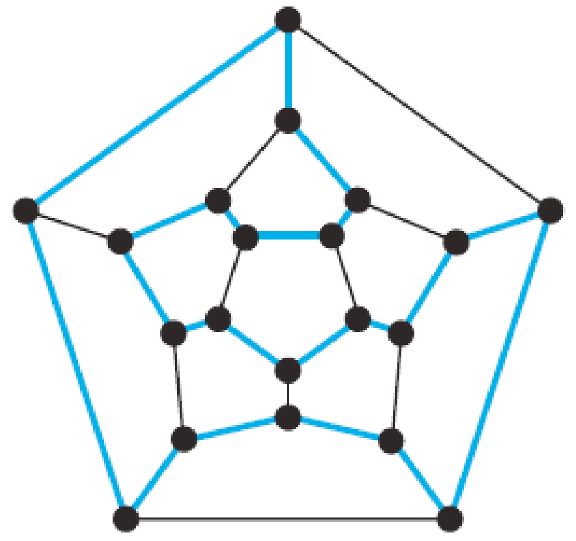


FIGURE 9 **A Solution to the “A Voyage Round the World” Puzzle.**

Hamilton Paths and Circuits: Example

- Which of the simple graphs in Figure 10 have a Hamilton circuit or, if not, a Hamilton path?

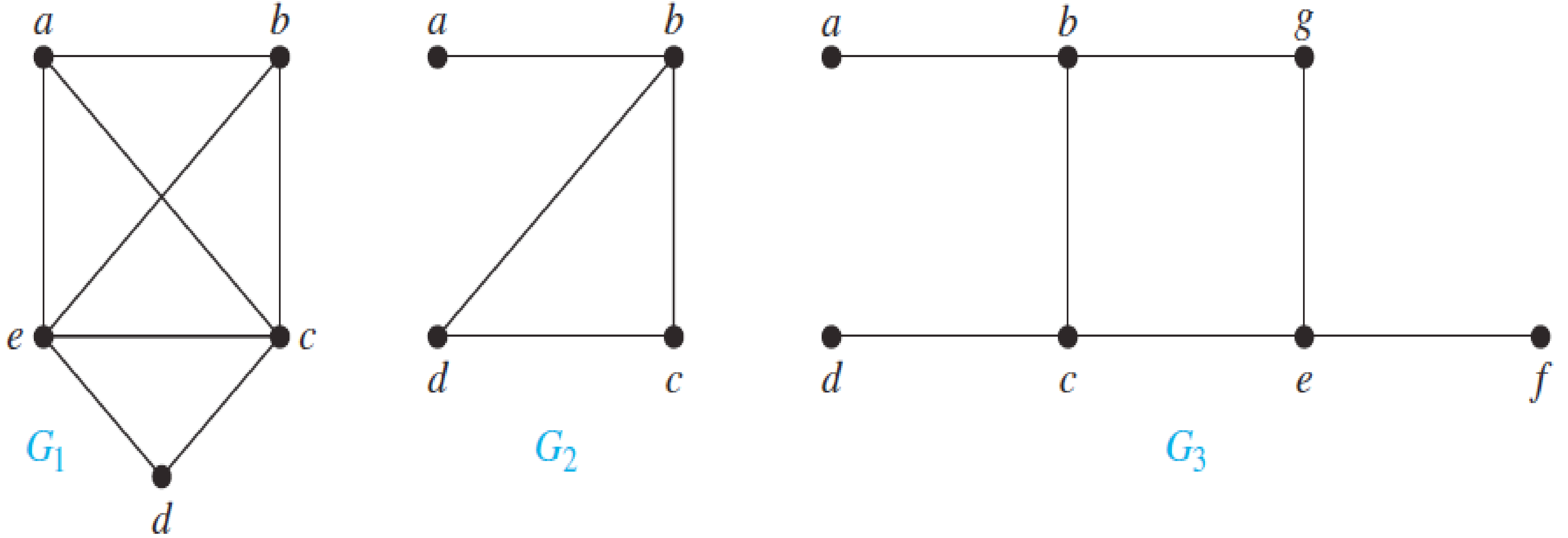


FIGURE 10 Three Simple Graphs.

Hamilton Paths and Circuits: Example

- *Solution:* G_1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G_2 (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice), but G_2 does have a Hamilton path, namely, a, b, c, d. G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}, \{e, f\}$, and $\{c, d\}$ more than once.

Conditions for the existence of Hamilton circuits

- Is there a simple way to determine whether a graph has a Hamilton circuit or path? At first, it might seem that there should be an easy way to determine this, because there is a simple way to answer the similar question of whether a graph has an Euler circuit. Surprisingly, there are no known simple necessary and sufficient criteria for the existence of Hamilton circuits. However, many theorems are known that give sufficient conditions for the existence of Hamilton circuits.

Conditions for the existence of Hamilton circuits

- Also, certain properties can be used to show that a graph has no Hamilton circuit. For instance, a graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in the circuit. Moreover, if a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.

Conditions for the existence of Hamilton circuits

- Also, note that when a Hamilton circuit is being constructed and this circuit has passed through a vertex, then all remaining edges incident with this vertex, other than the two used in the circuit, can be removed from consideration. Furthermore, a Hamilton circuit cannot contain a smaller circuit within it.

Conditions for the existence of Hamilton circuits: Example

- Show that neither graph displayed in Figure 11 has a Hamilton circuit.

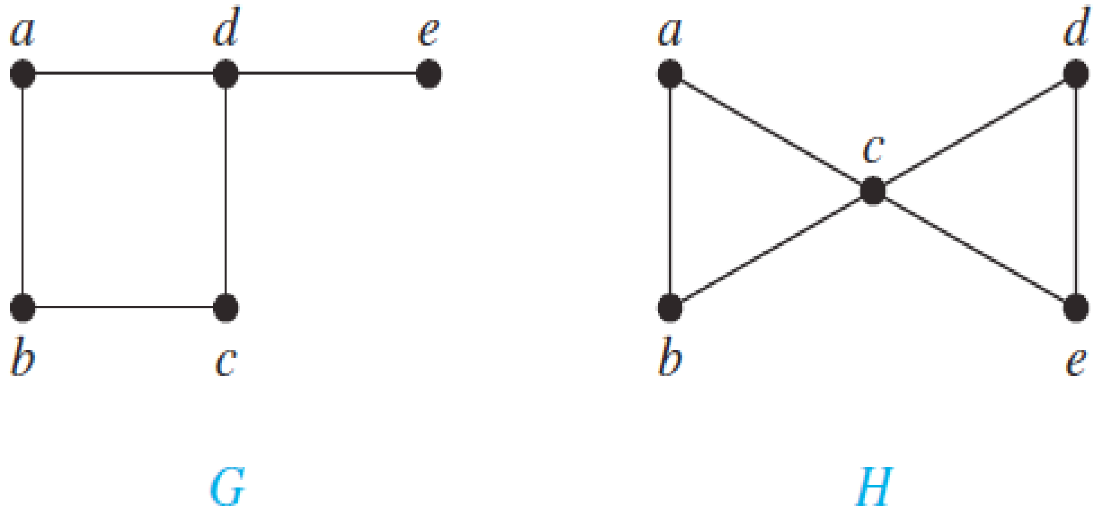


FIGURE 11 Two Graphs That Do Not Have a Hamilton Circuit.

Conditions for the existence of Hamilton circuits: Example

- *Solution:* There is no Hamilton circuit in G because G has a vertex of degree one, namely, e. Now consider H. Because the degrees of the vertices $a, b, d,$ and e are all two, every edge incident with these vertices must be part of any Hamilton circuit. It is now easy to see that no Hamilton circuit can exist in H, for any Hamilton circuit would have to contain four edges incident with *c*, which is impossible.

Conditions for the existence of Hamilton circuits: Example

- Show that K_n has a Hamilton circuit whenever $n \geq 3$.
- *Solution:* We can form a Hamilton circuit in K_n beginning at any vertex. Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once. This is possible because there are edges in K_n between any two vertices.

Conditions for the existence of Hamilton circuits

- Although no useful necessary and sufficient conditions for the existence of Hamilton circuits are known, quite a few sufficient conditions have been found. Note that the more edges a graph has, the more likely it is to have a Hamilton circuit. Furthermore, adding edges (but not vertices) to a graph with a Hamilton circuit produces a graph with the same Hamilton circuit. So as we add edges to a graph, especially when we make sure to add edges to each vertex, we make it increasingly likely that a Hamilton circuit exists in this graph.

Conditions for the existence of Hamilton circuits

- Consequently, we would expect there to be sufficient conditions for the existence of Hamilton circuits that depend on the degrees of vertices being sufficiently large. We state two of the most important sufficient conditions here. These conditions were found by Gabriel A. Dirac in 1952 and Øystein Ore in 1960.

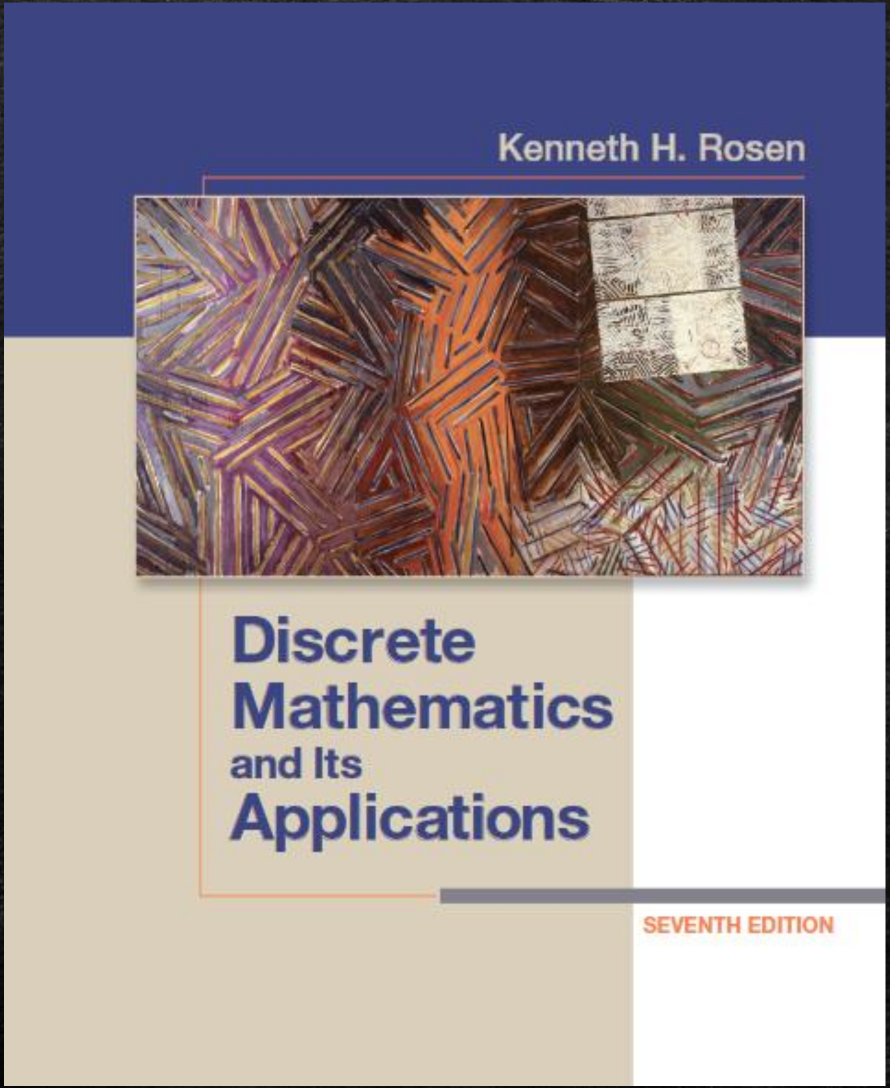
Conditions for the existence of Hamilton circuits

- **THEOREM 3. (DIRAC'S THEOREM)** If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.
- **THEOREM 4. (ORE'S THEOREM)** If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Conditions for the existence of Hamilton circuits

- Both Ore’s theorem and Dirac’s theorem provide sufficient conditions for a connected simple graph to have a Hamilton circuit. However, these theorems do not provide necessary conditions for the existence of a Hamilton circuit. For example, the graph C_5 has a Hamilton circuit but does not satisfy the hypotheses of either Ore’s theorem or Dirac’s theorem

HOMEWORK: Exercises 2, 4, 6, 8, 10, 14, 18, 20, 30, 32, 34, 36 on pp. 703-705



Planar Graphs

- Consider the problem of joining three houses to each of three separate utilities, as shown in Figure 1. Is it possible to join these houses and utilities so that none of the connections cross? This problem can be modeled using the complete bipartite graph $K_{3,3}$. The original question can be rephrased as: Can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?

Planar Graphs

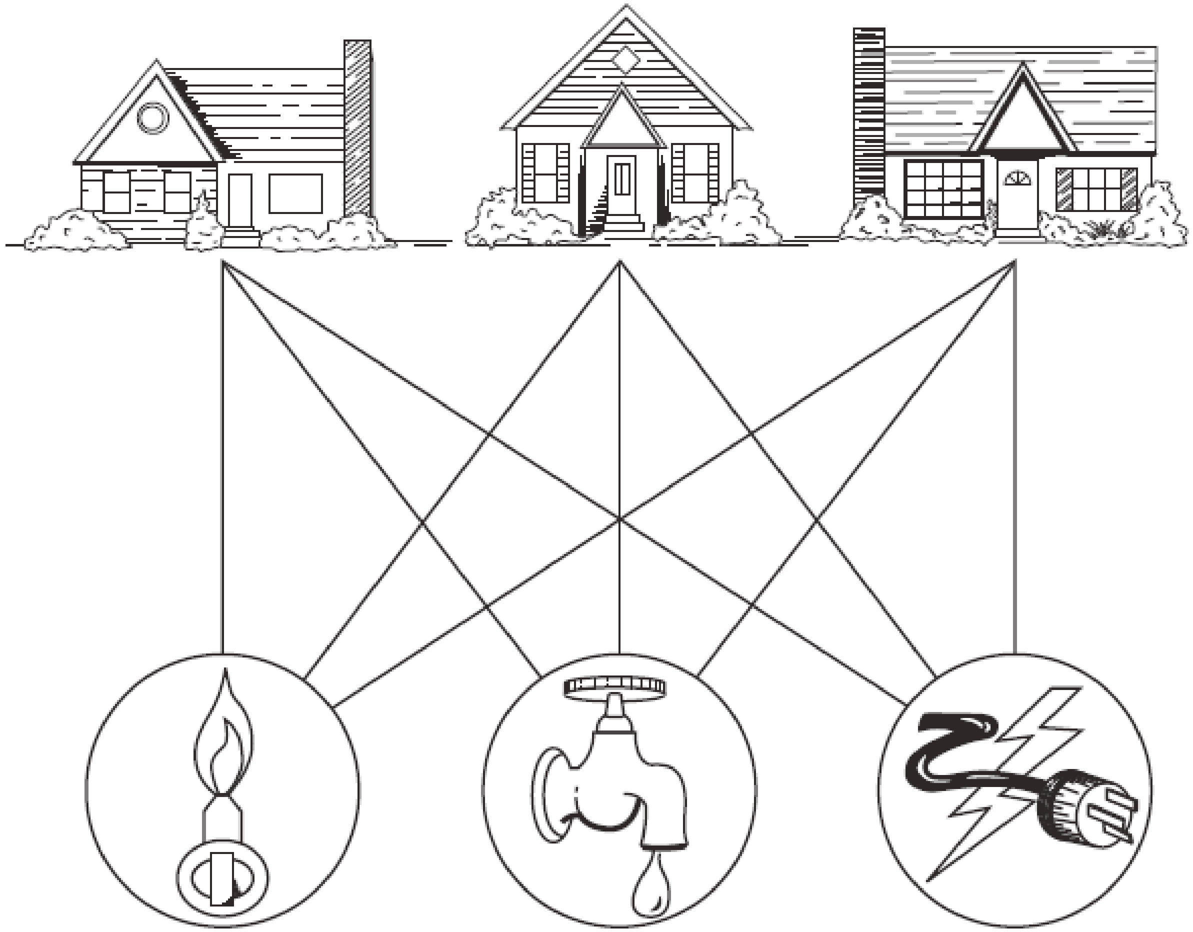


FIGURE 1 Three Houses and Three Utilities.

Planar Graphs

- **DEFINITION 1.** A graph is called *planar* if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a *planar representation* of the graph.
- A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

Planar Graphs: Example

- Is K_4 planar?
- *Solution:* K_4 is planar because it can be drawn without crossings, as shown in Figure 3.

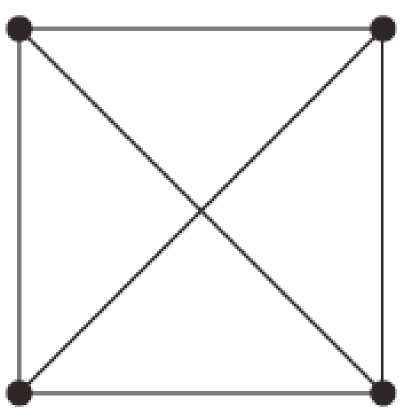


FIGURE 2 The Graph K_4.

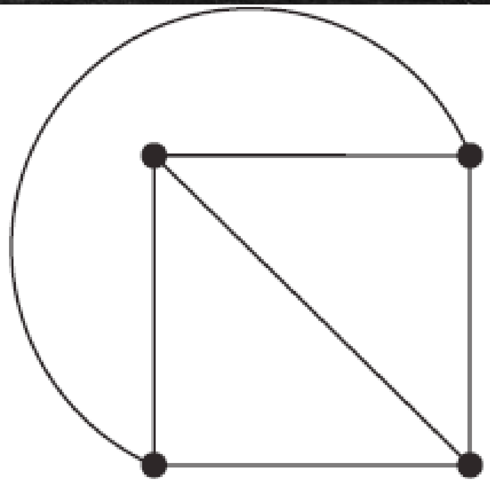


FIGURE 3 K_4 Drawn with No Crossings.

Planar Graphs: Example

- Is Q_3, shown in Figure 4, planar?
- *Solution:* Q_3 is planar, because it can be drawn without any edges crossing, as shown in Figure 5.

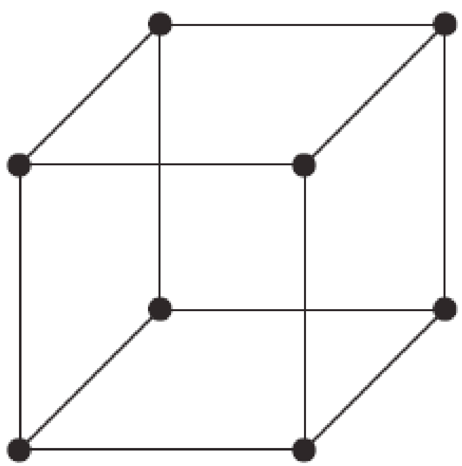


FIGURE 4 The Graph Q_3.

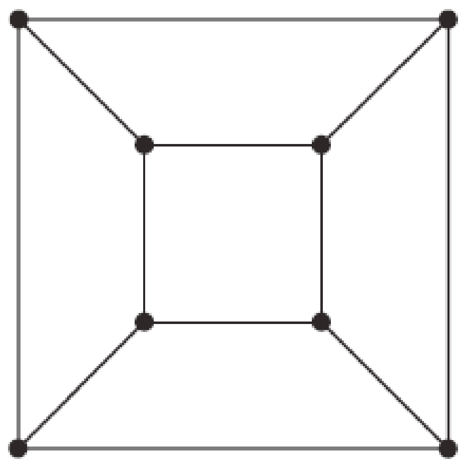


FIGURE 5 A Planar Representation of Q_3.

Planar Graphs: Example

- Is $K_{3,3}$, shown in Figure 6, planar?

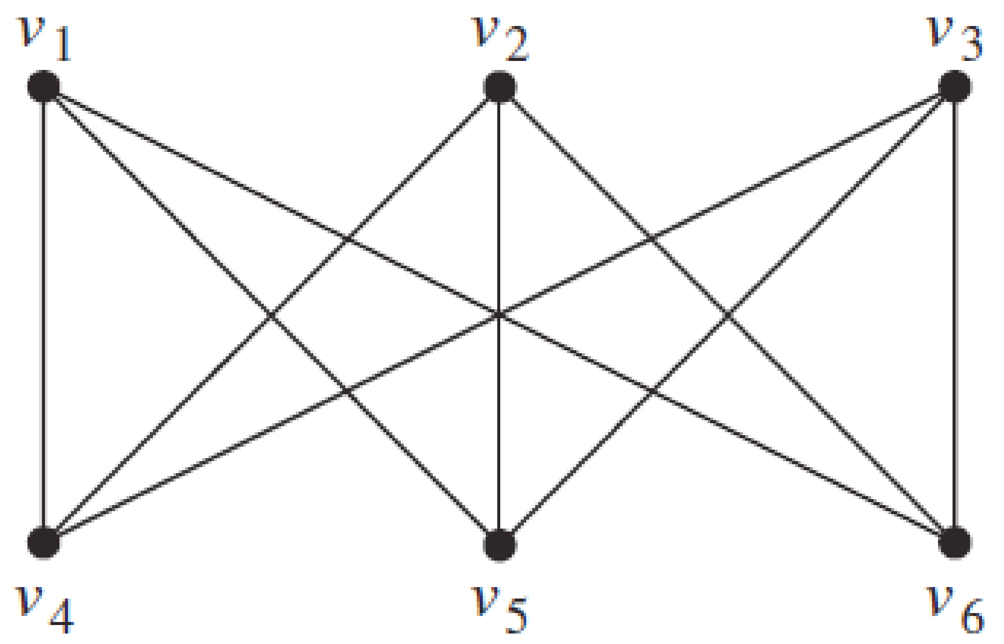


FIGURE 6 The Graph $K_{3,3}$.

Planar Graphs: Example

- *Solution:* Any attempt to draw $K_{3,3}$ in the plane with no edges crossing is doomed. We now show why. In any planar representation of $K_{3,3}$, the vertices v_1 and v_2 must be connected to both v_4 and v_5. These four edges form a closed curve that splits the plane into two regions, R_1 and R_2, as shown in Figure 7(a). The vertex v_3 is in either R_1 or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between v_3 and v_4 and between v_3 and v_5 separate R_2 into two subregions, R_{21} and R_{22}, as shown in Figure 7(b).

Planar Graphs: Example

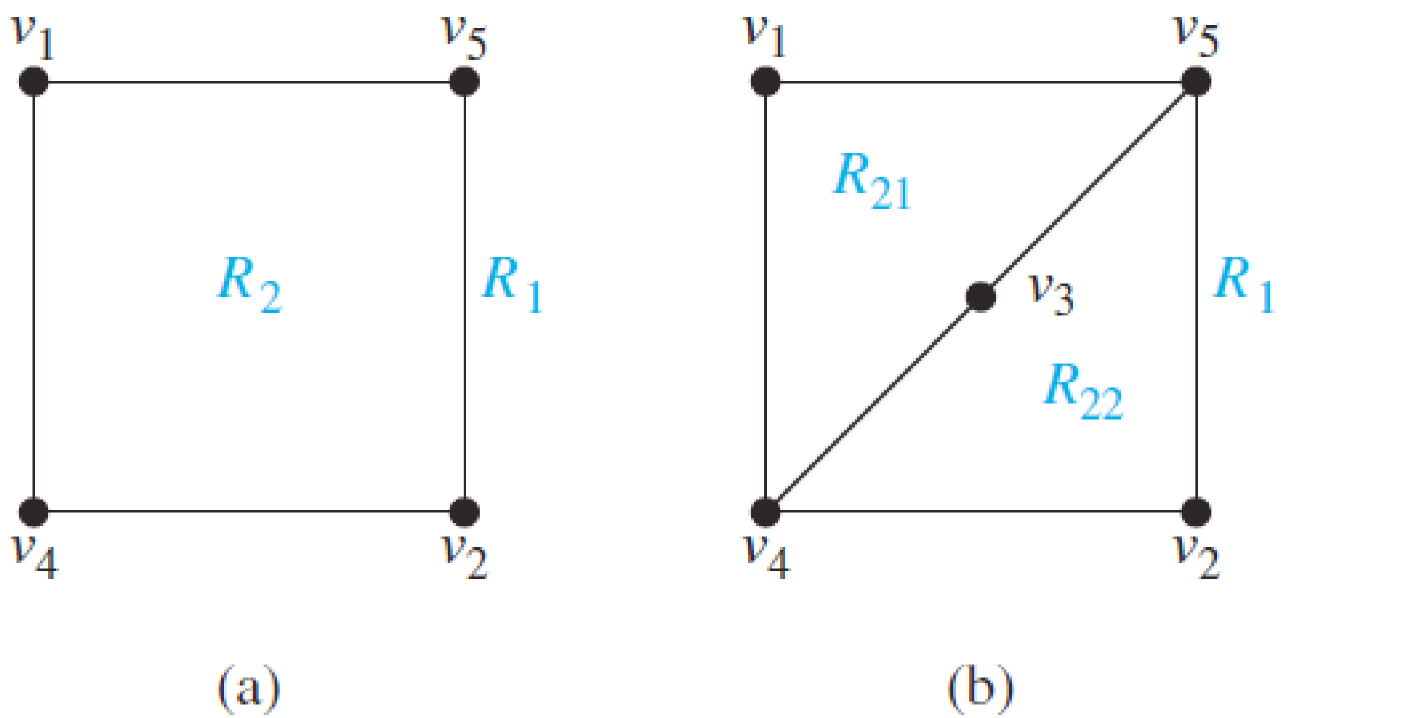


FIGURE 7 **Showing that $K_{3,3}$ Is Nonplanar.**

Planar Graphs: Example

- Next, note that there is no way to place the final vertex v_6 without forcing a crossing. For if v_6 is in R_1, then the edge between v_6 and v_3 cannot be drawn without a crossing. If v_6 is in R_{21}, then the edge between v_2 and v_6 cannot be drawn without a crossing. If v_6 is in R_{22}, then the edge between v_1 and v_6 cannot be drawn without a crossing.
- A similar argument can be used when v_3 is in R_1. It follows that $K_{3,3}$ is not planar.

Euler's Formula

- A planar representation of a graph splits the plane into **regions**, including an unbounded region. For instance, the planar representation of the graph shown in Figure 8 splits the plane into six regions. These are labeled in the figure. Euler showed that all planar representations of a graph split the plane into the same number of regions. He accomplished this by finding a relationship among the number of regions, the number of vertices, and the number of edges of a planar graph.

Euler’s Formula

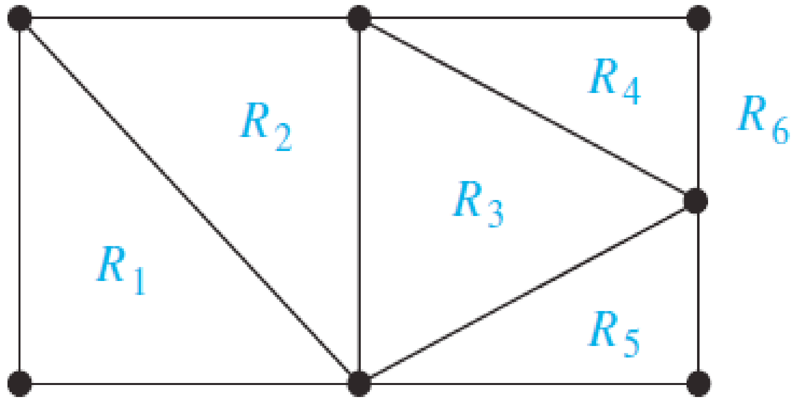


FIGURE 8 The Regions of the Planar Representation of a Graph.

- **THEOREM 1. (EULER’S FORMULA)** Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Euler’s Formula: Example

- Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?
- *Solution:* This graph has 20 vertices, each of degree 3, so $v = 20$. Because the sum of the degrees of the vertices, $3v = 3 \cdot 20 = 60$, is equal to twice the number of edges, $2e$, we have $2e = 60$, or $e = 30$. Consequently, from Euler’s formula, the number of regions is

$$r = e - v + 2 = 30 - 20 + 2 = 12.$$

Euler's Formula: Example

- **COROLLARY 1.** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.
- **COROLLARY 2.** If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Euler’s Formula: Example

- Show that K_5 is nonplanar using Corollary 1.
- *Solution:* The graph K_5 has five vertices and 10 edges. However, the inequality $e \leq 3v - 6$ is not satisfied for this graph because $e = 10$ and $3v - 6 = 9$. Therefore, K_5 is not planar.

Euler’s Formula

- It was previously shown that $K_{3,3}$ is not planar. Note, however, that this graph has six vertices and nine edges. This means that the inequality $e = 9 \leq 12 = 3 \cdot 6 - 6$ is satisfied. Consequently, the fact that the inequality $e \leq 3v - 6$ is satisfied does *not* imply that a graph is planar. However, the following corollary of Theorem 1 can be used to show that $K_{3,3}$ is nonplanar.

Euler’s Formula

- **COROLLARY 3.** If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Euler's Formula: Example

- Use Corollary 3 to show that $K_{3,3}$ is nonplanar.
- *Solution:* Because $K_{3,3}$ has no circuits of length three (this is easy to see because it is bipartite), Corollary 3 can be used. $K_{3,3}$ has six vertices and nine edges. Because $e = 9$ and $2v - 4 = 8$, Corollary 3 shows that $K_{3,3}$ is nonplanar.

Kuratowski’s Theorem

- We have seen that $K_{3,3}$ and K_5 are not planar. Clearly, a graph is not planar if it contains either of these two graphs as a subgraph. Surprisingly, all nonplanar graphs must contain a subgraph that can be obtained from $K_{3,3}$ or K_5 using certain permitted operations.

Kuratowski's Theorem

- If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an **elementary subdivision**. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.

Kuratowski’s Theorem: Example

- Show that the graphs G_1, G_2, and G_3 displayed in Figure 12 are all homeomorphic.

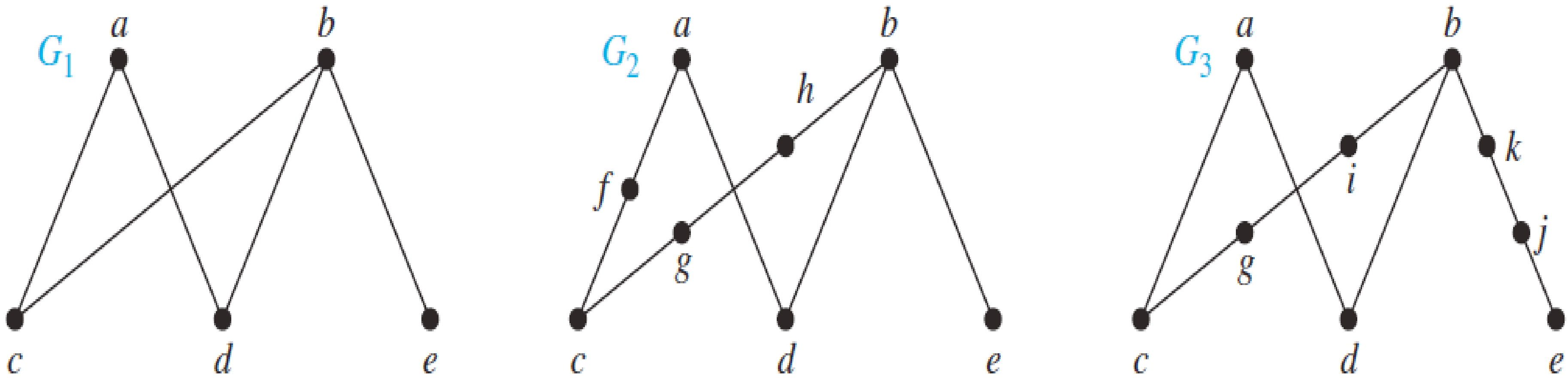


FIGURE 12 Homeomorphic Graphs.

Kuratowski's Theorem: Example

- *Solution:* These three graphs are homeomorphic because all three can be obtained from G_1 by elementary subdivisions. G_1 can be obtained from itself by an empty sequence of elementary subdivisions. To obtain G_2 from G_1 we can use this sequence of elementary subdivisions: (i) remove the edge $\{a, c\}$, add the vertex f, and add the edges $\{a, f\}$ and $\{f, c\}$; (ii) remove the edge $\{b, c\}$, add the vertex g, and add the edges $\{b, g\}$ and $\{g, c\}$; and (iii) remove the edge $\{b, g\}$, add the vertex h, and add the edges $\{g, h\}$ and $\{b, h\}$. Similarly we obtain G_3 from G_1.

Kuratowski's Theorem: Example

- The Polish mathematician Kazimierz Kuratowski established Theorem 2 in 1930, which characterizes planar graphs using the concept of graph homeomorphism.
- **THEOREM 2.** A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

HOMEWORK: Exercises 2, 4, 6, 8, 20, 22, 24 on pp. 725-726