Elements of Graph Theory

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Graphs and Graph Models

- **DEFINITION 1.** A *graph* $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*) and E, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.
- ***Remark:*** The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an **infinite graph**, and in comparison, a graph with a finite vertex set and a finite edge set is called a **finite graph**. In this lecture we will usually consider only finite graphs.

Graphs and Graph Models

- Graphs that may have **multiple edges** connecting the same vertices are called **multigraphs**. When there are m different edges associated to the same unordered pair of vertices $\{u, v\}$, we also say that $\{u, v\}$ is an edge of multiplicity m.
- That is, we can think of this set of edges as m different copies of an edge $\{u, v\}$.

Graphs and Graph Models

- Sometimes we need to include edges that connect a vertex to itself. Such edges are called **loops**, and sometimes we may even have more than one loop at a vertex. Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called **pseudographs**.
- So far the graphs we have introduced are **undirected graphs**. Their edges are also said to be **undirected**.

Graphs and Graph Models

- **DEFINITION 2.** A *directed graph* (or *digraph*) (V, E) consists of a nonempty set of vertices V and a set of *directed edges* (or *arcs*) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.
- When we depict a directed graph with a line drawing, we use an arrow pointing from u to v to indicate the direction of an edge that starts at u and ends at v.

Graphs and Graph Models

- When a directed graph has no loops and has no multiple directed edges, it is called a **simple directed graph**. Because a simple directed graph has at most one edge associated to each ordered pair of vertices (u, v), we call (u, v) an edge if there is an edge associated to it in the graph. In some computer networks, multiple communication links between two data centers may be present, as illustrated in Figure 5. Directed graphs that may have **multiple directed edges** from a vertex to a second (possibly the same) vertex are used to model such networks.

Graphs and Graph Models

- For some models we may need a graph where some edges are undirected, while others are directed. A graph with both directed and undirected edges is called a **mixed graph**. For example, a mixed graph might be used to model a computer network containing links that operate in both directions and other links that operate only in one direction.

Graphs and Graph Models

- This terminology for the various types of graphs is summarized in Table 1.

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Basic Terminology

- **DEFINITION 1.** Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.
- **DEFINITION 2.** The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So,
$$N(A) = \cup_{v \in A} N(v).$$

Basic Terminology

- **DEFINITION 3.** The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Basic Terminology: Example

- What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure 1?

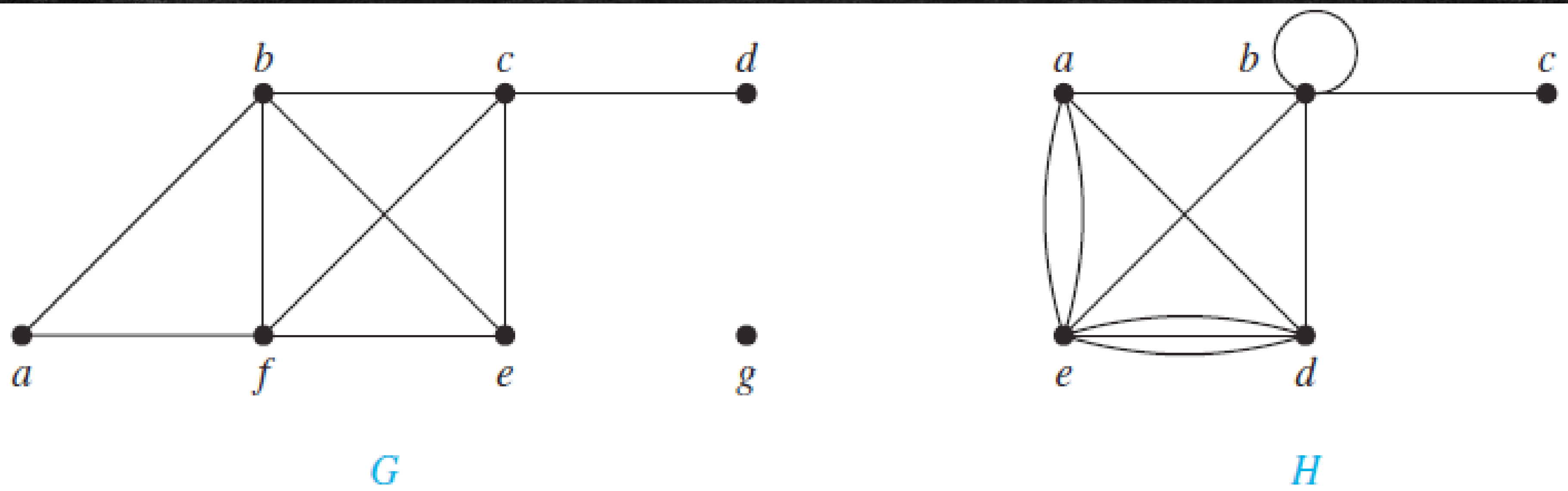


FIGURE 1 The Undirected Graphs G and H.

Basic Terminology: Example

- *Solution:* In G, $\deg(a) = 2, \deg(b) = \deg(c) = \deg(f) = 4, \deg(d) = 1, \deg(e) = 3$, and $\deg(g) = 0$. The neighborhoods of these vertices are $N(a) = \{b, f\}, N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}, N(d) = \{c\}, N(e) = \{b, c, f\}, N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$.
- In H, $\deg(a) = 4, \deg(b) = \deg(e) = 6, \deg(c) = 1$, and $\deg(d) = 5$. The neighborhoods of these vertices are $N(a) = \{b, d, e\}, N(b) = \{a, b, c, d, e\}, N(c) = \{b\}, N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.

Basic Terminology

- A vertex of degree zero is called **isolated**. It follows that an isolated vertex is not adjacent to any vertex. Vertex g in graph G in Example 1 is isolated. A vertex is **pendant** if and only if it has degree one. Consequently, a pendant vertex is adjacent to exactly one other vertex. Vertex d in graph G in previous Example is pendant.

Basic Terminology

- **THEOREM 1. (THE HANDSHAKING THEOREM)** Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

- (Note that this applies even if multiple edges and loops are present.)

Basic Terminology: Example

- How many edges are there in a graph with 10 vertices each of degree six?
- *Solution:* Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$.

Basic Terminology

- **THEOREM 2.** An undirected graph has an even number of vertices of odd degree.
- **DEFINITION 4.** When (u, v) is an edge of the graph G with directed edges, u is said to be *adjacent to* v and v is said to be *adjacent from* u. The vertex u is called the *initial vertex* of (u, v), and v is called the *terminal* or *end vertex* of (u, v). The initial vertex and terminal vertex of a loop are the same.

Basic Terminology

- **DEFINITION 5.** In a graph with directed edges the *in-degree of a vertex* v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree of* v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Basic Terminology: Example

- Find the in-degree and out-degree of each vertex in the graph *G* with directed edges shown in Figure 2.

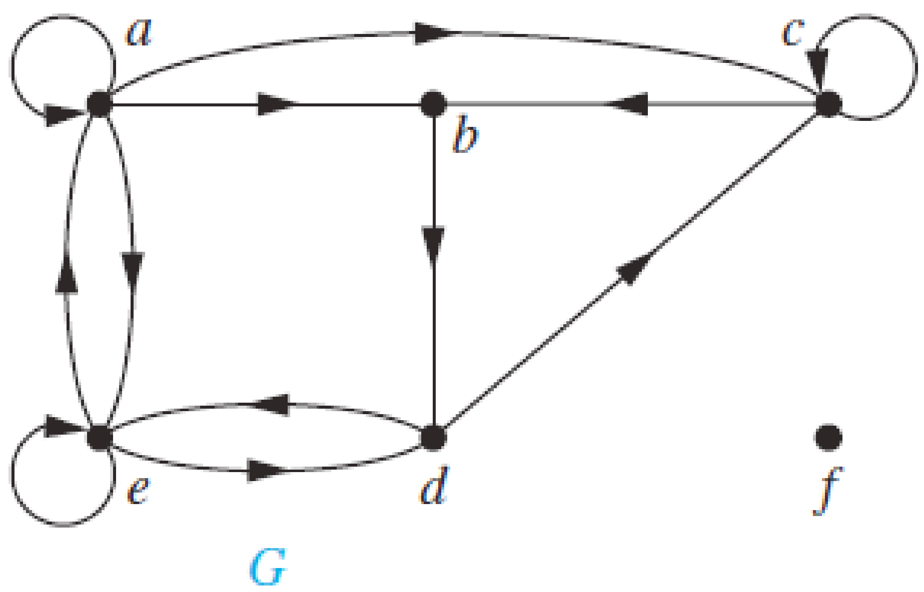


FIGURE 2 The Directed Graph *G*.

Basic Terminology: Example

- *Solution:* The in-degrees in G are $\deg^-(a) = 2, \deg^-(b) = 2, \deg^-(c) = 3, \deg^-(d) = 2, \deg^-(e) = 3$, and $\deg^-(f) = 0$.
- The out-degrees are $\deg^+(a) = 4, \deg^+(b) = 1, \deg^+(c) = 2, \deg^+(d) = 2, \deg^+(e) = 3$, and $\deg^+(f) = 0$.

Basic Terminology

- **THEOREM 3.** Let $G = (V\ , E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|.$$

Some Special Simple Graphs

- **Complete Graphs.** A **complete graph on n vertices**, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs K_n, for $n = 1, 2, 3, 4, 5, 6$, are displayed in Figure 3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete**.

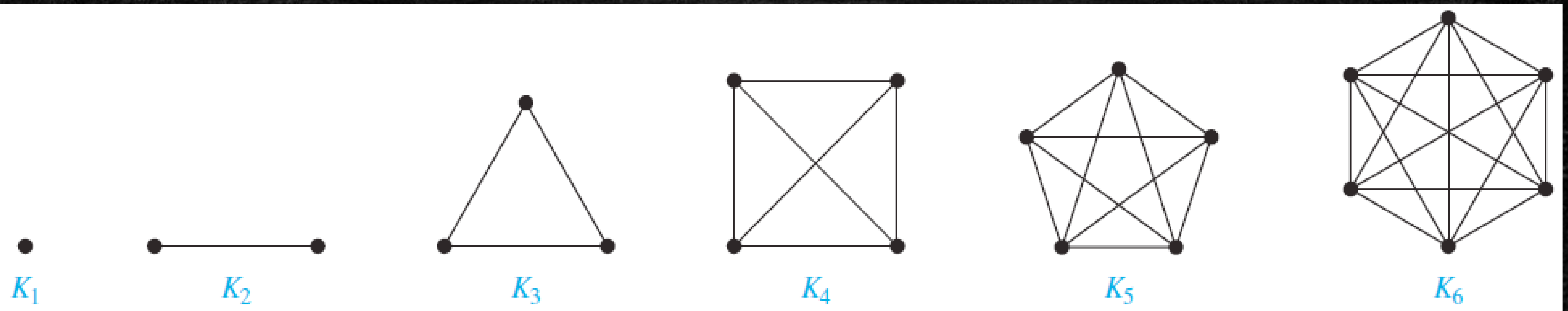


FIGURE 3 The Graphs K_n for $1 \leq n \leq 6$.

Bipartite Graphs

- **DEFINITION 6.** A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

Bipartite Graphs: Example

- C_6 is bipartite, as shown in Figure 7, because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.

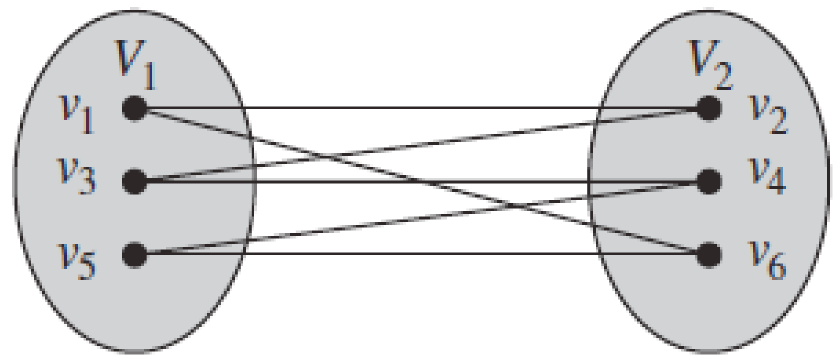


FIGURE 7 Showing That C_6 Is Bipartite.

Bipartite Graphs

- **THEOREM 4.** A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Bipartite Graphs: Example

- Are the graphs *G* and *H* displayed in Figure 8 bipartite?

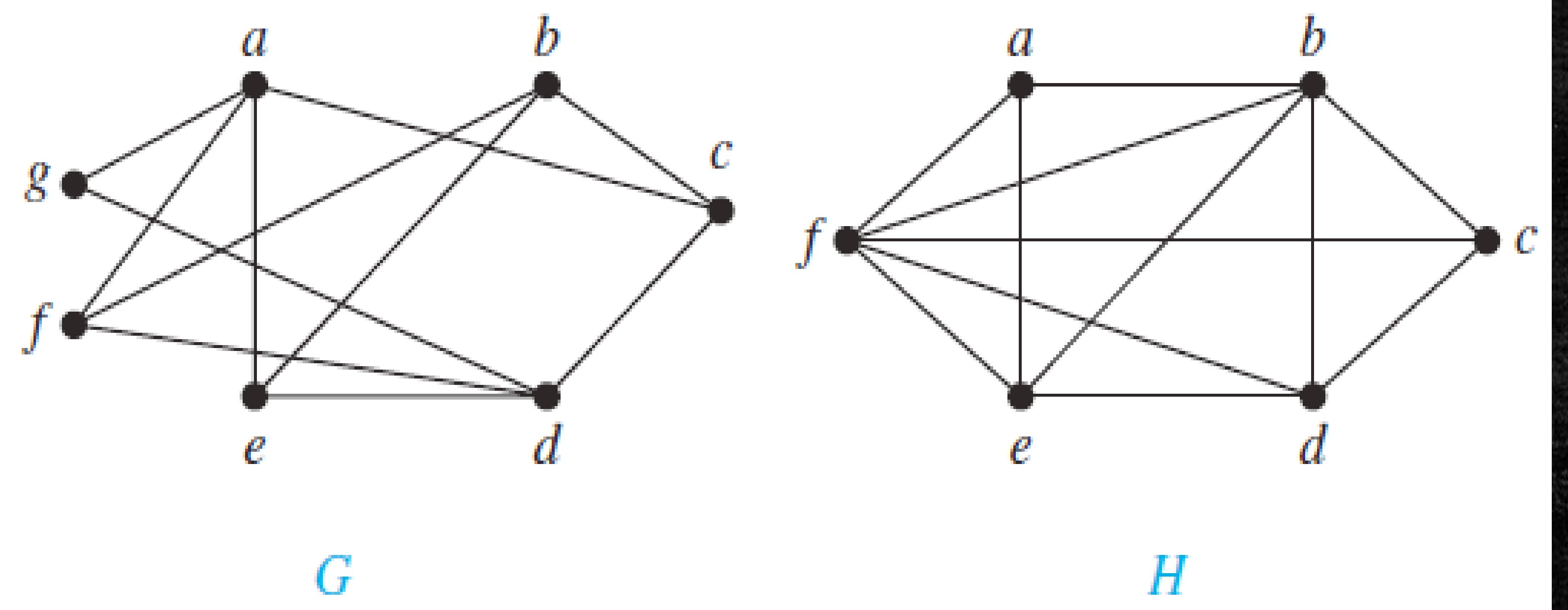


FIGURE 8 The Undirected Graphs *G* and *H*.

Bipartite Graphs

- **Complete Bipartite Graphs.** A **complete bipartite graph $K_{m,n}$** is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset. The complete bipartite graphs $K_{2,3}, K_{3,3}, K_{3,5}$, and $K_{2,6}$ are displayed in Figure 9.

Bipartite Graphs

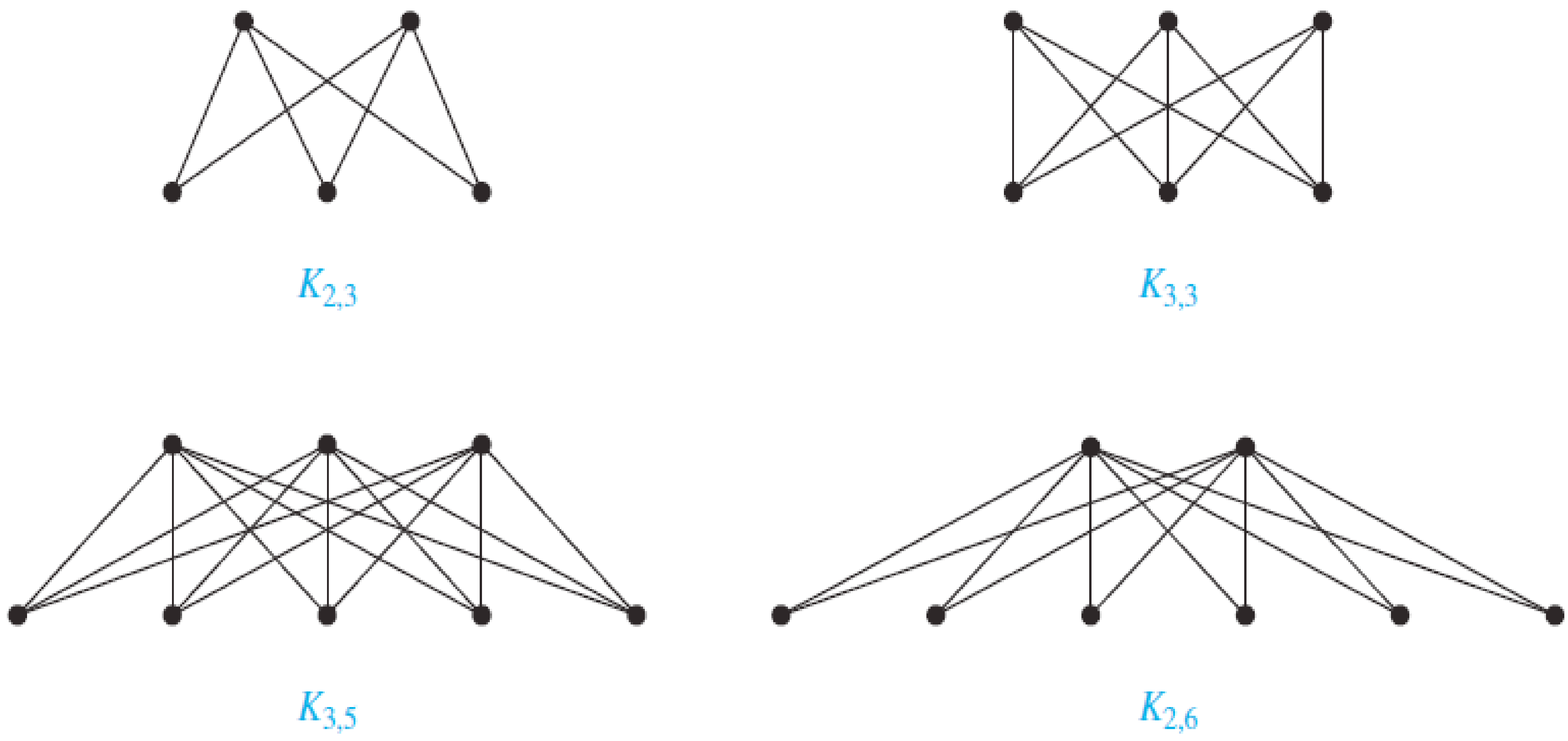


FIGURE 9 Some Complete Bipartite Graphs.

New Graphs from Old

- When edges and vertices are removed from a graph, without removing endpoints of any remaining edges, a smaller graph is obtained. Such a graph is called a **subgraph** of the original graph.
- **DEFINITION 7.** A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

New Graphs from Old

- **DEFINITION 8.** Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.

New Graphs from Old: Example

- The graph G shown in Figure 15 is a subgraph of K_5. If we add the edge connecting c and e to G, we obtain the subgraph induced by $W = \{a, b, c, e\}$.

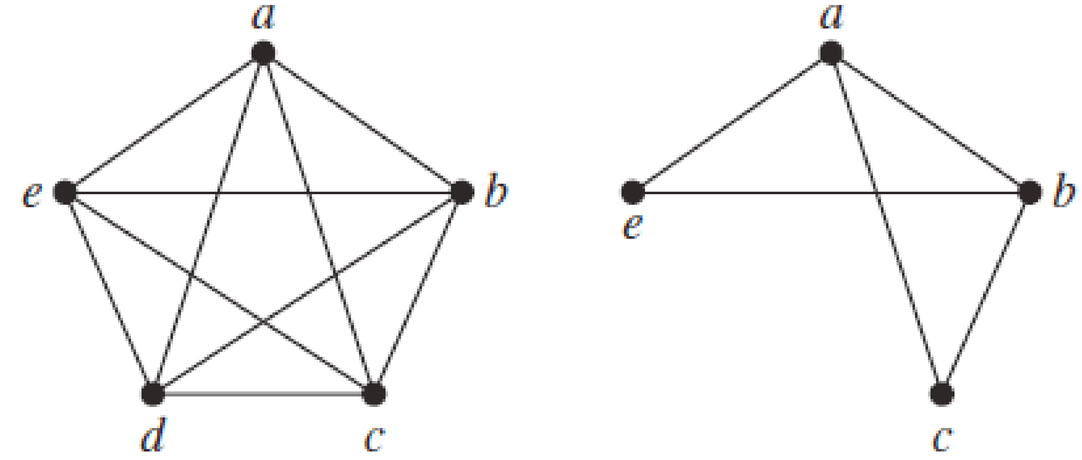


FIGURE 15 A Subgraph of K_5.

New Graphs from Old

- **GRAPH UNIONS.** Two or more graphs can be combined in various ways. The new graph that contains all the vertices and edges of these graphs is called the **union** of the graphs. We will give a more formal definition for the union of two simple graphs.
- **DEFINITION 9.** The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

New Graphs from Old: Example

- Find the union of the graphs G_1 and G_2 shown in Figure 16(a).
- *Solution:* The union is displayed in Figure 16(b).

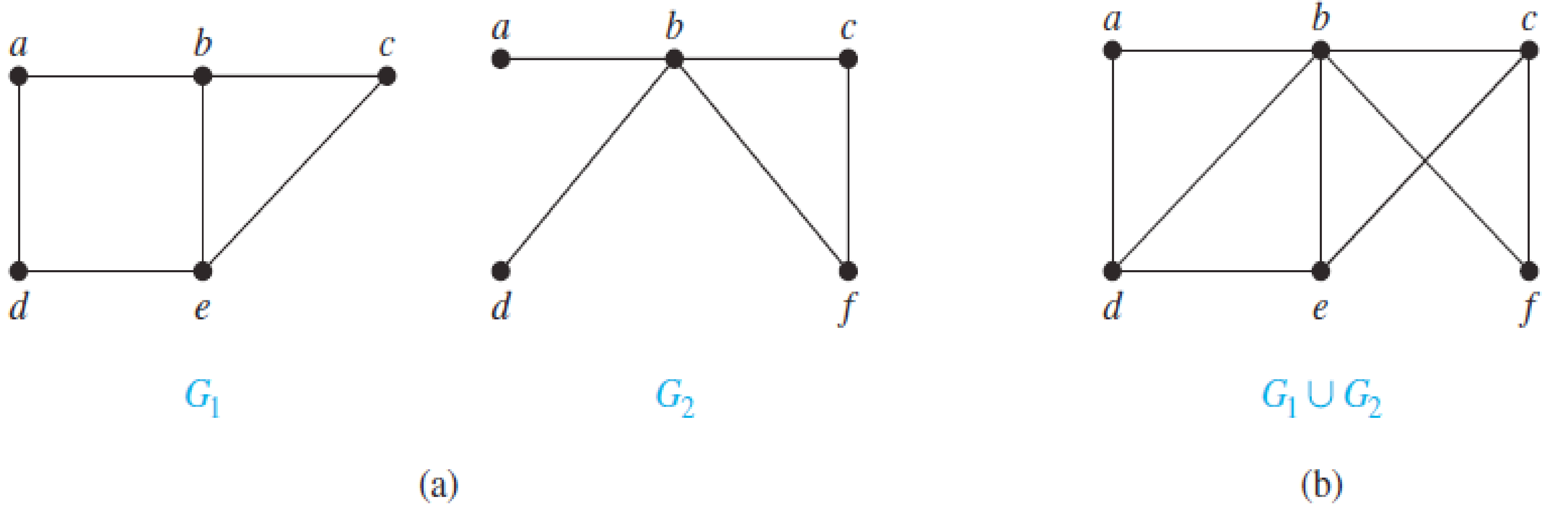
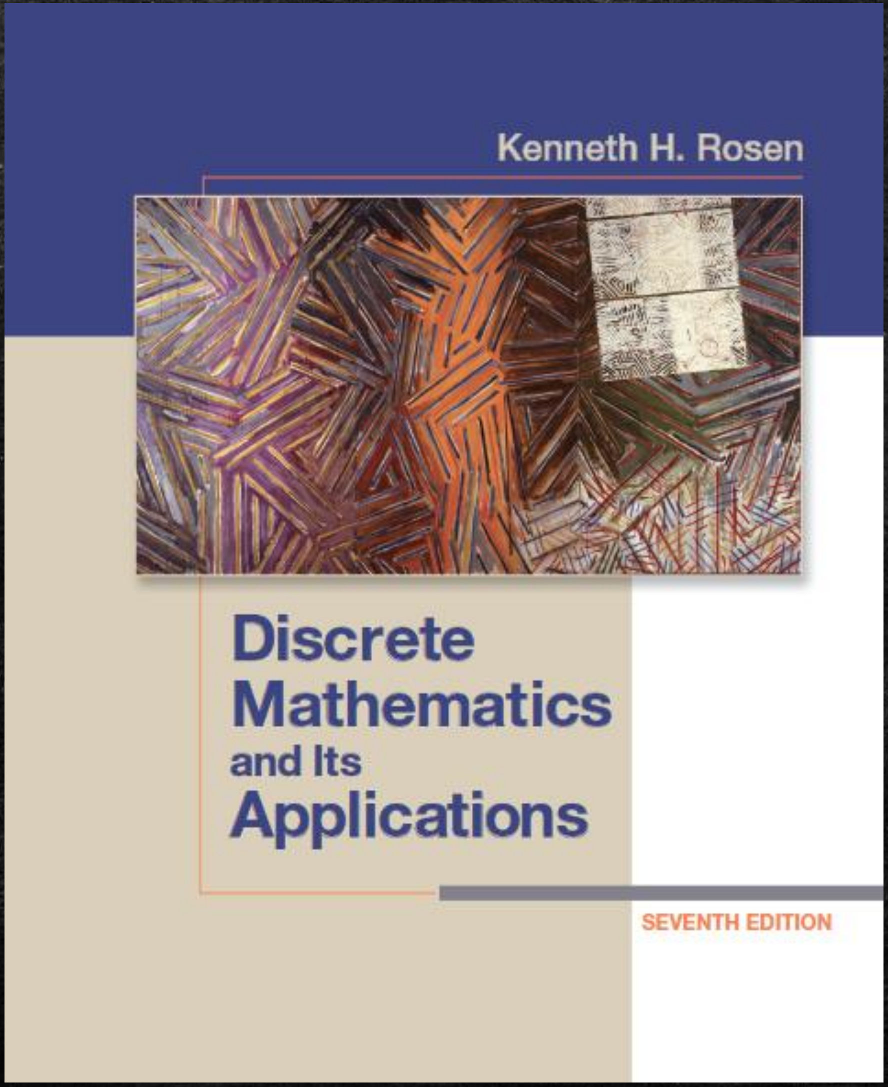


FIGURE 16 **(a) The Simple Graphs G_1 and G_2; (b) Their Union $G_1 \cup G_2$.**

HOMEWORK: Exercises 2, 8, 22, 24, 34, 56, 58, on pp. 665-667



Isomorphism of Graphs

- **DEFINITION 1.** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism*.
- Two simple graphs that are not isomorphic are called *nonisomorphic*.

Isomorphism of Graphs: Example

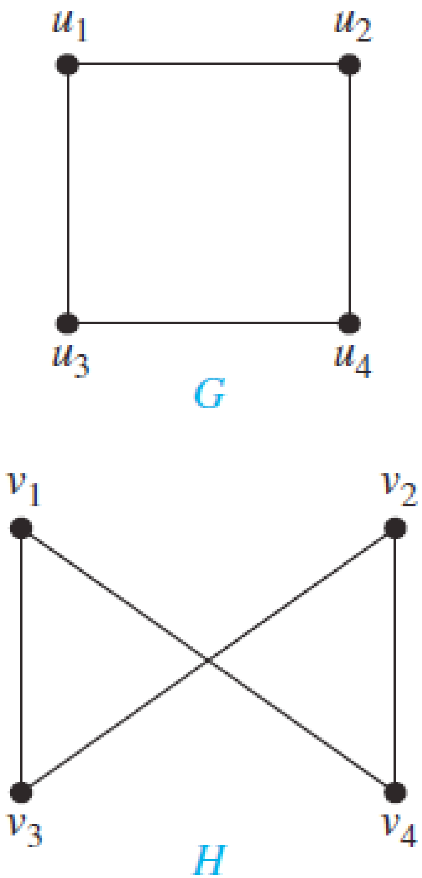


FIGURE 8 The Graphs G and H.

- Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure 8, are isomorphic.

Isomorphism of Graphs: Example

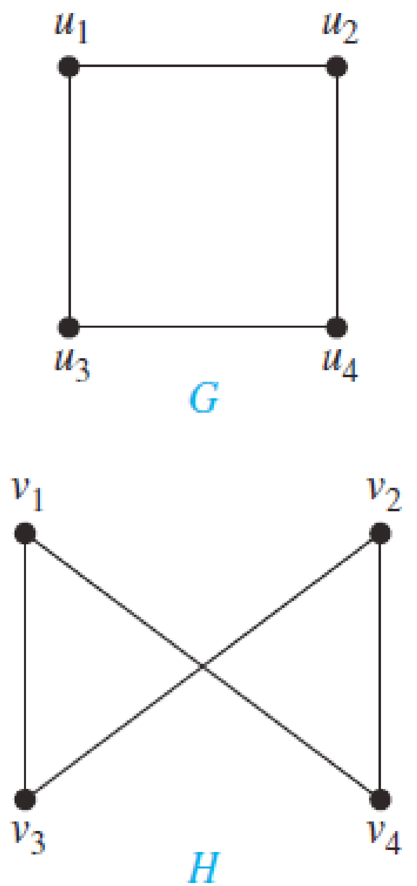


FIGURE 8 The Graphs G and H.

- *Solution*: The function f with $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in H.

Isomorphism of Graphs: Example

- Show that the graphs displayed in Figure 9 are not isomorphic.

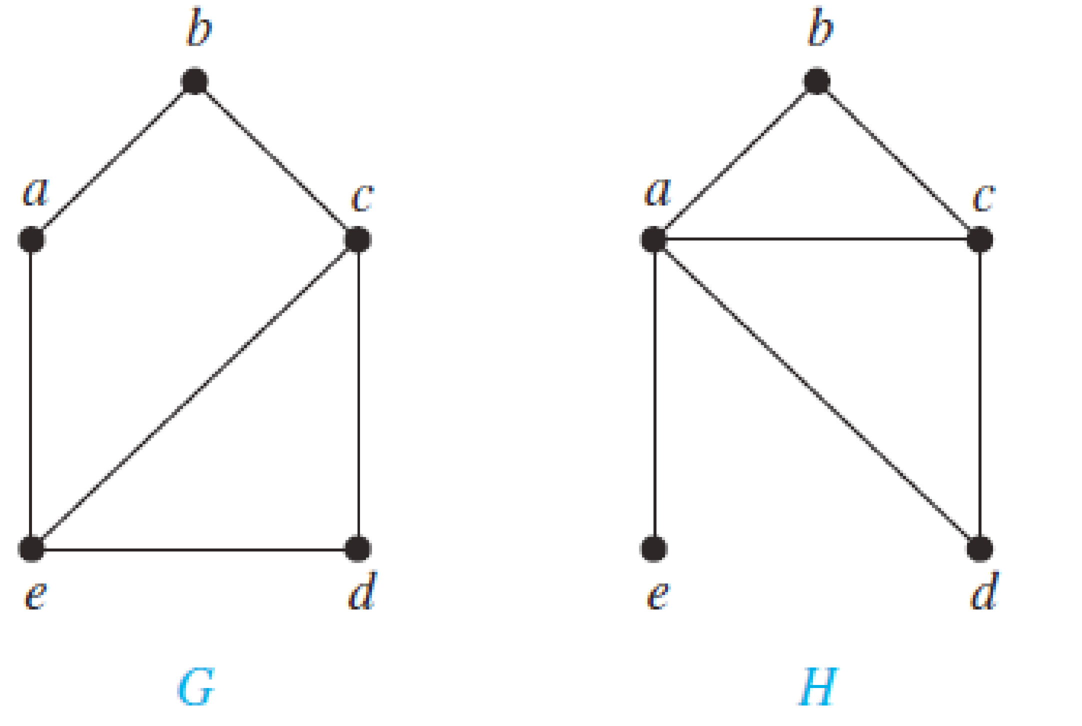


FIGURE 9 The Graphs G and H.

Isomorphism of Graphs: Example

- Determine whether the graphs shown in Figure 10 are isomorphic.

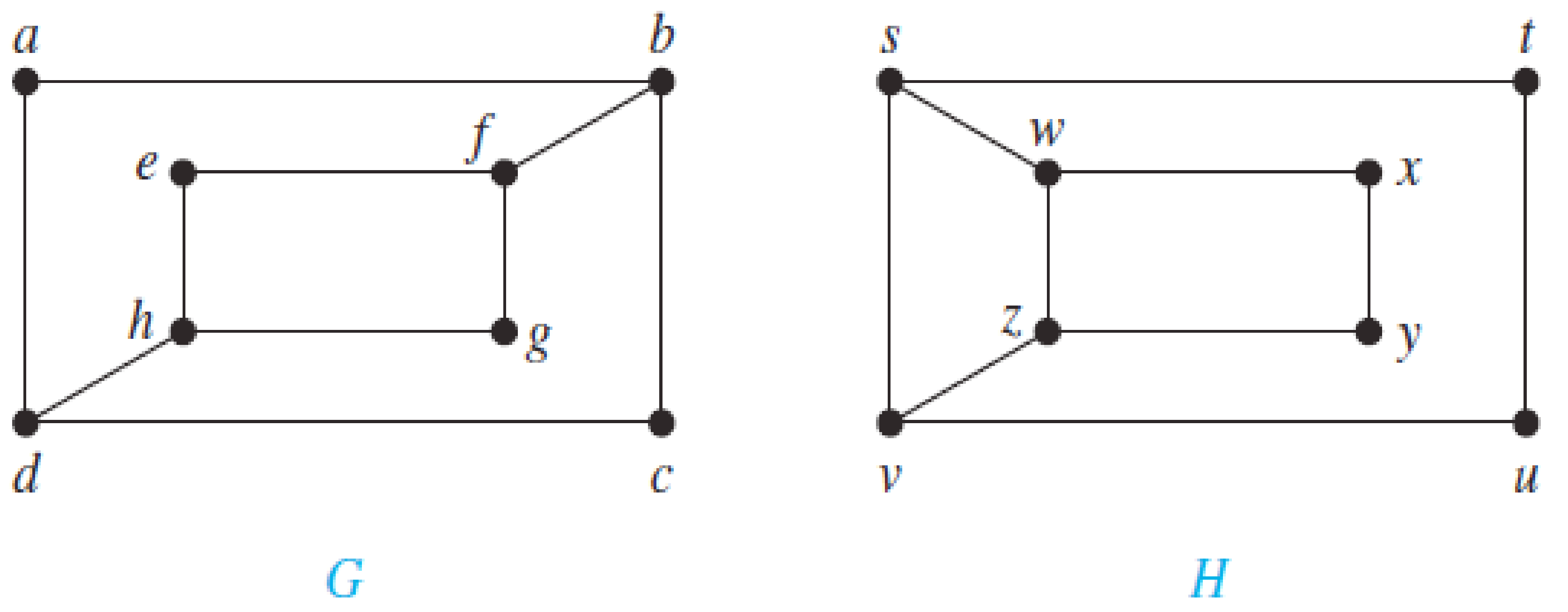


FIGURE 10 The Graphs G and H.

Isomorphism of Graphs: Example

- *Solution:* The graphs G and H both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three. Because these invariants all agree, it is still conceivable that these graphs are isomorphic.
- However, G and H are not isomorphic. To see this, note that because $\deg(a) = 2$ in G, a must correspond to either t, u, x, or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

HOMEWORK: Exercises 34, 36, 38, 40, 42, 44 on pp. 676-677