DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the **nth term** of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

is the **sequence of partial sums** of the series, the number s_n being the **nth partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots, \qquad r = 1/2, a = 1$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots. \qquad r = -1/3, a = 1$$

If $r = 1$, the nth partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of a. If $r = -1$, the series diverges because the nth partial sums alternate between a and 0. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n \qquad \text{Multiply } s_n \text{ by } r.$$

$$s_n - rs_n = a - ar^n \qquad \text{Subtract } rs_n \text{ from } s_n. \text{ Most of the terms on the right cancel.}$$

$$s_n(1 - r) = a(1 - r^n) \qquad \text{Factor.}$$

$$s_n = \frac{a(1 - r^n)}{1 - r}, \qquad (r \neq 1). \qquad \text{We can solve for } s_n \text{ if } r \neq 1.$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ (as in Section 10.1) and $s_n \to a/(1 - r)$. If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1-r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

EXAMPLE 1   The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1-(1/3)} = \frac{1}{6}.$$

EXAMPLE 2   The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

$S = \frac{a}{1-r}$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1-r} = \frac{5}{1+(1/4)} = 4.$$
■

THEOREM 7   If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

The nth-Term Test for Divergence

$\displaystyle\sum_{n=1}^{\infty} a_n$ diverges if $\displaystyle\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

EXAMPLE 7   The following are all examples of divergent series.

(a) $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\dfrac{n+1}{n} \to 1$. $\lim_{n\to\infty} a_n \neq 0$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \dfrac{-n}{2n+5} = -\dfrac{1}{2} \neq 0$.

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1. Every nonzero constant multiple of a divergent series diverges.

2. If Σa_n converges and Σb_n diverges, then $\Sigma(a_n + b_n)$ and $\Sigma(a_n - b_n)$ both diverge.

Caution   Remember that $\Sigma(a_n + b_n)$ can converge when Σa_n and Σb_n both diverge. For example, $\Sigma a_n = 1 + 1 + 1 + \cdots$ and $\Sigma b_n = (-1) + (-1) + (-1) + \cdots$ diverge, whereas $\Sigma(a_n + b_n) = 0 + 0 + 0 + \cdots$ converges to 0.

EXAMPLE 9   Find the sums of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right)$

$\displaystyle = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$ Difference Rule

$\displaystyle = \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)}$ Geometric series with $a = 1$ and $r = 1/2, 1/6$

$\displaystyle = 2 - \frac{6}{5} = \frac{4}{5}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n}$ Constant Multiple Rule

$\displaystyle = 4\left(\frac{1}{1 - (1/2)} \right)$ Geometric series with $a = 1$, $r = 1/2$

$= 8$

The Integral Test

We now introduce the Integral Test with a series that is related to the harmonic series, but whose nth term is $1/n^2$ instead of $1/n$.

EXAMPLE 2 Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

FIGURE 10.10 The sum of the areas of the rectangles under the graph of $f(x) = 1/x^2$ is less than the area under the graph (Example 2).

Solution We determine the convergence of $\sum_{n=1}^{\infty}(1/n^2)$ by comparing it with $\int_1^{\infty}(1/x^2)\,dx$. To carry out the comparison, we think of the terms of the series as values of the function $f(x) = 1/x^2$ and interpret these values as the areas of rectangles under the curve $y = 1/x^2$.

As Figure 10.10 shows,

$$
\begin{aligned}
s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\
&= f(1) + f(2) + f(3) + \cdots + f(n) \\
&< f(1) + \int_1^n \frac{1}{x^2}\,dx \qquad \text{Rectangle areas sum to less than area under graph.} \\
&< 1 + \int_1^{\infty} \frac{1}{x^2}\,dx \qquad \int_1^n (1/x^2)\,dx < \int_1^{\infty}(1/x^2)\,dx \\
&< 1 + 1 = 2. \qquad \text{As in Section 8.7, Example 3,}\ \int_1^{\infty}(1/x^2)\,dx = 1.
\end{aligned}
$$

Thus the partial sums of $\sum_{n=1}^{\infty}(1/n^2)$ are bounded from above (by 2) and the series converges. The sum of the series is known to be $\pi^2/6 \approx 1.64493$. ∎
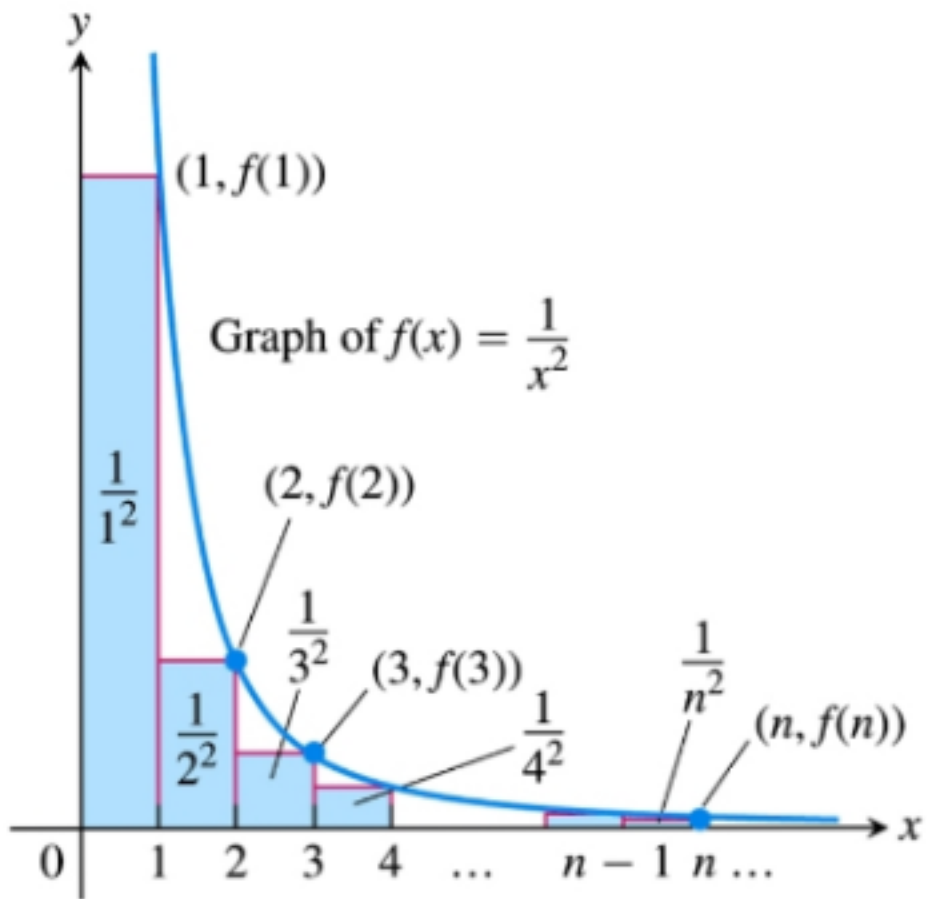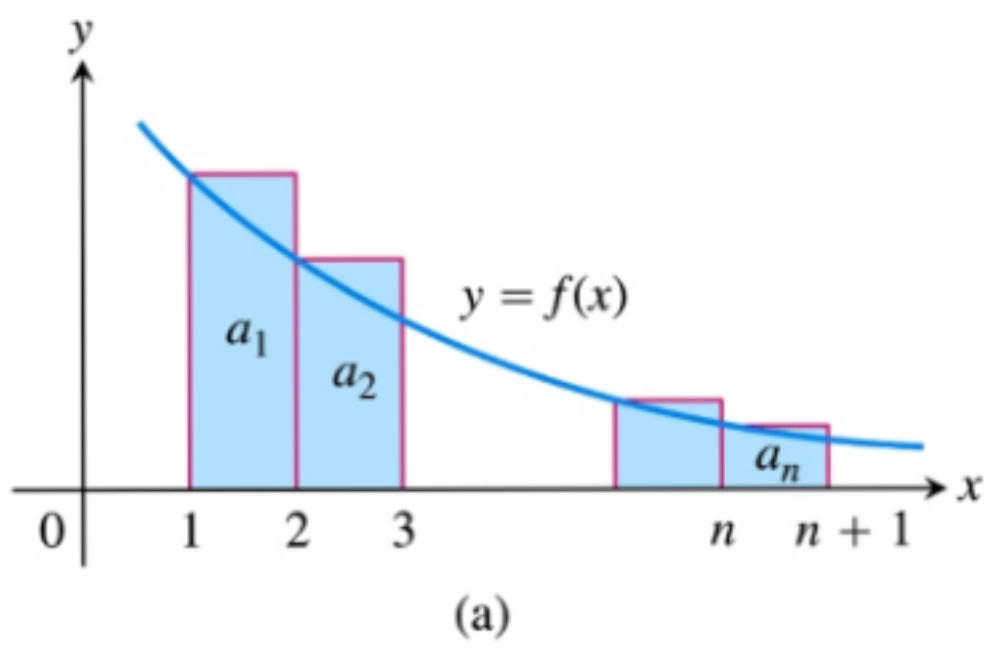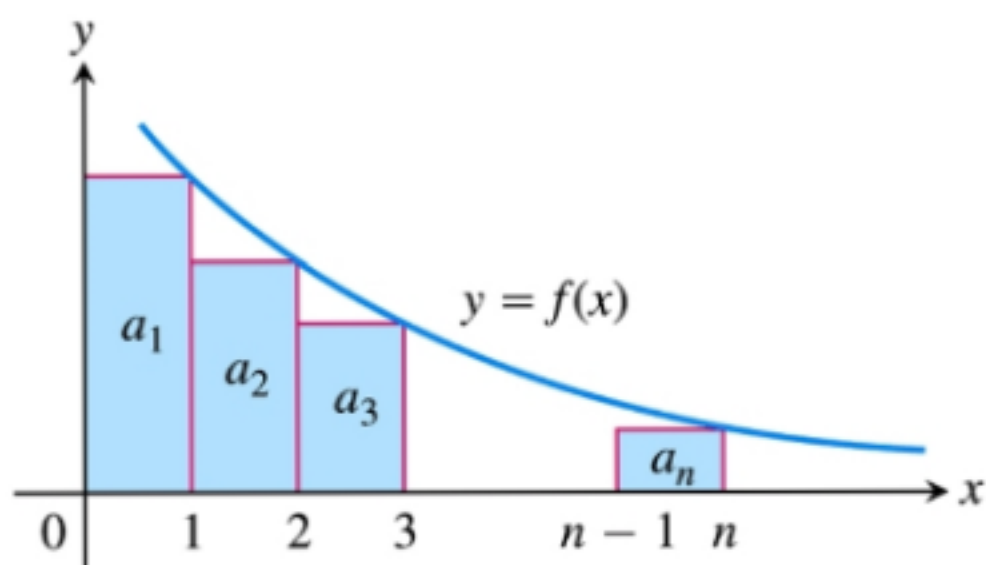

THEOREM 9—The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

FIGURE 10.11 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty}(x)\,dx$ both converge or both diverge.

Proof We establish the test for the case $N = 1$. The proof for general N is similar.

We start with the assumption that f is a decreasing function with $f(n) = a_n$ for every n. This leads us to observe that the rectangles in Figure 10.11a, which have areas $a_1, a_2, \ldots, a_n$, collectively enclose more area than that under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$. That is,

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n.$$

In Figure 10.11b the rectangles have been faced to the left instead of to the right. If we momentarily disregard the first rectangle of area a_1, we see that

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx.$$

If we include a_1, we have

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

Combining these results gives

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

These inequalities hold for each n, and continue to hold as $n \to \infty$.

If $\int_1^{\infty} f(x)\,dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite. If $\int_1^{\infty} f(x)\,dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite. Hence the series and the integral are both finite or both infinite. ∎

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that Σa_n converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx. \tag{1}$$

If we add the partial sum s_n to each side of the inequalities in (1), we get

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq s_n + \int_{n}^{\infty} f(x)\,dx \tag{2}$$

since $s_n + R_n = S$. The inequalities in (2) are useful for estimating the error in approximating the sum of a convergent series. The error can be no larger than the length of the interval containing S, as given by (2).

THEOREM 10—The Comparison Test   Let Σa_n, Σc_n, and Σd_n be series with nonnegative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \qquad \text{for all} \qquad n > N.$$

(a) If Σc_n converges, then Σa_n also converges.
(b) If Σd_n diverges, then Σa_n also diverges.

THEOREM 11—Limit Comparison Test   Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then Σa_n and Σb_n both converge or both diverge.

2. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and Σb_n converges, then Σa_n converges.

3. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and Σb_n diverges, then Σa_n diverges.

Finding nth Partial Sums

In Exercises 1–6, find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \dfrac{2}{3} + \dfrac{2}{9} + \dfrac{2}{27} + \cdots + \dfrac{2}{3^{n-1}} + \cdots$

2. $\dfrac{9}{100} + \dfrac{9}{100^2} + \dfrac{9}{100^3} + \cdots + \dfrac{9}{100^n} + \cdots$

3. $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \cdots + (-1)^{n-1}\dfrac{1}{2^{n-1}} + \cdots$

4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1} 2^{n-1} + \cdots$

5. $\dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \cdots + \dfrac{1}{(n+1)(n+2)} + \cdots$

6. $\dfrac{5}{1\cdot 2} + \dfrac{5}{2\cdot 3} + \dfrac{5}{3\cdot 4} + \cdots + \dfrac{5}{n(n+1)} + \cdots$

Series with Geometric Terms

In Exercises 7–14, write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

7. $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{4^n}$

8. $\displaystyle\sum_{n=2}^{\infty} \dfrac{1}{4^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \left(1 - \dfrac{7}{4^n}\right)$

10. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{5}{4^n}$

11. $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{5}{2^n} + \dfrac{1}{3^n}\right)$

12. $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{5}{2^n} - \dfrac{1}{3^n}\right)$

13. $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{1}{2^n} + \dfrac{(-1)^n}{5^n}\right)$

14. $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{2^{n+1}}{5^n}\right)$

In Exercises 15–18, determine if the geometric series converges or diverges. If a series converges, find its sum.

15. $1 + \left(\dfrac{2}{5}\right) + \left(\dfrac{2}{5}\right)^2 + \left(\dfrac{2}{5}\right)^3 + \left(\dfrac{2}{5}\right)^4 + \cdots$

16. $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$

17. $\left(\dfrac{1}{8}\right) + \left(\dfrac{1}{8}\right)^2 + \left(\dfrac{1}{8}\right)^3 + \left(\dfrac{1}{8}\right)^4 + \left(\dfrac{1}{8}\right)^5 + \cdots$

18. $\left(\dfrac{-2}{3}\right)^2 + \left(\dfrac{-2}{3}\right)^3 + \left(\dfrac{-2}{3}\right)^4 + \left(\dfrac{-2}{3}\right)^5 + \left(\dfrac{-2}{3}\right)^6 + \cdots$

Applying the Integral Test

Use the Integral Test to determine if the series in Exercises 1–10 converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1. $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2}$

2. $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^{0.2}}$

3. $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2 + 4}$

4. $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n + 4}$

5. $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$

7. $\displaystyle\sum_{n=1}^{\infty} \dfrac{n}{n^2 + 4}$

Using the nth-Term Test

In Exercises 27–34, use the nth-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

27. $\displaystyle\sum_{n=1}^{\infty} \dfrac{n}{n + 10}$

28. $\displaystyle\sum_{n=1}^{\infty} \dfrac{n(n + 1)}{(n + 2)(n + 3)}$

29. $\displaystyle\sum_{n=0}^{\infty} \dfrac{1}{n + 4}$

30. $\displaystyle\sum_{n=1}^{\infty} \dfrac{n}{n^2 + 3}$

Find the sum of each series in Exercises 41–48.

41. $\displaystyle\sum_{n=1}^{\infty} \dfrac{4}{(4n - 3)(4n + 1)}$

42. $\displaystyle\sum_{n=1}^{\infty} \dfrac{6}{(2n - 1)(2n + 1)}$

43. $\displaystyle\sum_{n=1}^{\infty} \dfrac{40n}{(2n - 1)^2(2n + 1)^2}$

44. $\displaystyle\sum_{n=1}^{\infty} \dfrac{2n + 1}{n^2(n + 1)^2}$

45. $\displaystyle\sum_{n=1}^{\infty} \left(\dfrac{1}{\sqrt{n}} - \dfrac{1}{\sqrt{n + 1}}\right)$

46. $\displaystyle\sum_{n=1}^{\infty} \left(\dfrac{1}{2^{1/n}} - \dfrac{1}{2^{1/(n+1)}}\right)$