# Boolean Algebra

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# Boolean Functions

- Boolean algebra provides the operations and the rules for working with the set $\{0, 1\}$. Electronic and optical switches can be studied using this set and the rules of Boolean algebra. The three operations in Boolean algebra that we will use most are complementation, the Boolean sum, and the Boolean product. The **complement** of an element, denoted with a bar, is defined by $\overline{0} = 1$ and $\overline{1} = 0$.

# Boolean Functions

- The Boolean sum, denoted by $+$ or by $OR$, has the following values:

$$1 + 1 = 1, 1 + 0 = 1, 0 + 1 = 1, 0 + 0 = 0.$$

- The Boolean product, denoted by $\cdot$ or by $AND$, has the following values:

$$1 \cdot 1 = 1, 1 \cdot 0 = 0, 0 \cdot 1 = 0, 0 \cdot 0 = 0.$$

# Boolean Functions

- When there is no danger of confusion, the symbol · can be deleted, just as in writing algebraic products. Unless parentheses are used, the rules of precedence for Boolean operators are: first, all complements are computed, followed by all Boolean products, followed by all Boolean sums.

# Boolean Functions: Example

- Find the value of $1 \cdot 0 + \overline{(0 + 1)}$.
- *Solution:* Using the definitions of complementation, the Boolean sum, and the Boolean product, it follows that

$$1 \cdot 0 + \overline{(0 + 1)} = 0 + \bar{1} = 0 + 0 = 0.$$

# Boolean Functions

- The complement, Boolean sum, and Boolean product correspond to the logical operators, $\neg, \vee$, and $\wedge$, respectively, where 0 corresponds to $\boldsymbol{F}$ (false) and 1 corresponds to $\boldsymbol{T}$ (true). Equalities in Boolean algebra can be directly translated into equivalences of compound propositions. Conversely, equivalences of compound propositions can be translated into equalities in Boolean algebra.

# Boolean Functions: Example

- Translate $1 \cdot 0 + \overline{(0 + 1)} = 0$, the equality found in previous Example, into a logical equivalence.
- *Solution:* We obtain a logical equivalence when we translate each 1 into a $\boldsymbol{T}$, each 0 into an $\boldsymbol{F}$, each Boolean sum into a disjunction, each Boolean product into a conjunction, and each complementation into a negation. We obtain

$$(\boldsymbol{T} \wedge \boldsymbol{F}) \vee \neg(\boldsymbol{T} \vee \boldsymbol{F}) \equiv \boldsymbol{F}.$$

# Boolean Functions: Example

- Translate the logical equivalence $(\boldsymbol{T} \wedge \boldsymbol{T}) \vee \neg \boldsymbol{F} \equiv \boldsymbol{T}$ into an identity in Boolean algebra.
- *Solution:* We obtain an identity in Boolean algebra when we translate each $\boldsymbol{T}$ into a 1, each $\boldsymbol{F}$ into a 0, each disjunction into a Boolean sum, each conjunction into a Boolean product, and each negation into a complementation. We obtain

$$(1 \cdot 1) + \bar{0} = 1.$$

# Boolean Expressions and Boolean Functions

- Let $B = \{0, 1\}$. Then

  $$B^n = \{(x_1, x_2, \ldots, x_n) | x_i \in B \; for \; 1 \le i \le n\}$$

- is the set of all possible $n$-tuples of 0s and 1s. The variable $x$ is called a **Boolean variable** if it assumes values only from $B$, that is, if its only possible values are 0 and 1. A function from $B^n$ to $B$ is called a **Boolean function of degree $n$**.

# Boolean Expressions and Boolean Functions: Example

- The function $F(x,y) = x\bar{y}$ from the set of ordered pairs of Boolean variables to the set $\{0,1\}$ is a Boolean function of degree 2 with $F(1,1) = 0, F(1,0) = 1, F(0,1) = 0,$ and $F(0,0) = 0$. We display these values of $F$ in Table 1.

**TABLE 1**

| $x$ | $y$ | $F(x,y)$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

# Boolean Expressions and Boolean Functions

- Boolean functions can be represented using expressions made up from variables and Boolean operations. The **Boolean expressions** in the variables $x_1, x_2, \ldots, x_n$ are defined recursively as

- 0, 1, $x_1, x_2, \ldots, x_n$ are Boolean expressions;
- if $E_1$ and $E_2$ are Boolean expressions, then $\overline{E_1}$, $(E_1 E_2)$, and $(E_1 + E_2)$ are Boolean expressions.

# Boolean Expressions and Boolean Functions: Example

- Each Boolean expression represents a Boolean function. The values of this function are obtained by substituting 0 and 1 for the variables in the expression.
- **Example**. Find the values of the Boolean function represented by $F(x, y, z) = xy + \bar{z}$.

# Boolean Expressions and Boolean Functions: Example

- *Solution:* The values of this function are displayed in Table 2.

**TABLE 2**

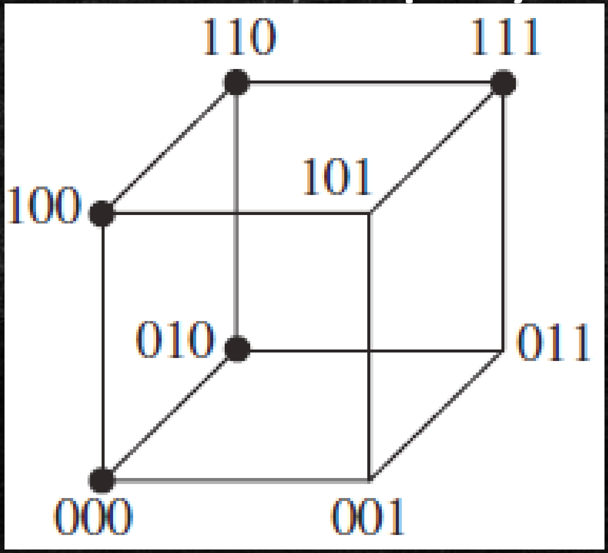
| $x$ | $y$ | $z$ | $xy$ | $\overline{z}$ | $F(x, y, z) = xy + \overline{z}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 |

# Boolean Expressions and Boolean Functions

- Note that we can represent a Boolean function graphically by distinguishing the vertices of the $n$-cube that correspond to the $n$-tuples of bits where the function has value 1.

# Boolean Expressions and Boolean Functions: Example

- The function $F(x, y, z) = xy + \bar{z}$ from $B^3$ to $B$ from previous Example can be represented by distinguishing the vertices that correspond to the five 3-tuples $(1, 1, 1), (1, 1, 0), (1, 0, 0), (0, 1, 0)$, and $(0, 0, 0)$, where $F(x, y, z) = 1$, as shown in Figure. These vertices are displayed using solid black circles.



# Boolean Expressions and Boolean Functions

- Boolean functions $F$ and $G$ of $n$ variables are equal if and only if
$$F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$$
- whenever $b_1, b_2, \ldots, b_n$ belong to $B$. Two different Boolean expressions that represent the same function are called **equivalent**. For instance, the Boolean expressions $xy$, $xy + 0$, and $xy \cdot 1$ are equivalent. The **complement** of the Boolean function $F$ is the function $F$, where
$$\bar{F}(x_1, \ldots, x_n) = \overline{F(x_1, \ldots, x_n)}.$$

# Boolean Expressions and Boolean Functions

- Let $F$ and $G$ be Boolean functions of degree $n$. The **Boolean sum** $F + G$ and the **Boolean product** $FG$ are defined by

$$(F + G)(x_1, \ldots, x_n) = F(x_1, \ldots, x_n) + G(x_1, \ldots, x_n),$$

$$(FG)(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)G(x_1, \ldots, x_n).$$

# Boolean Expressions and Boolean Functions

- A Boolean function of degree two is a function from a set with four elements, namely, pairs of elements from $B = \{0, 1\}$, to $B$, a set with two elements. Hence, there are 16 different Boolean functions of degree two. In Table 3 we display the values of the 16 different Boolean functions of degree two, labeled $F_1, F_2, \ldots, F_{16}$.

# Boolean Expressions and Boolean Functions

**TABLE 3** The 16 Boolean Functions of Degree Two.

| $x$ | $y$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ | $F_{12}$ | $F_{13}$ | $F_{14}$ | $F_{15}$ | $F_{16}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

# Boolean Expressions and Boolean Functions: Example

- How many different Boolean functions of degree $n$ are there?
- *Solution:* From the product rule for counting, it follows that there are $2^n$ different $n$-tuples of 0s and 1s. Because a Boolean function is an assignment of 0 or 1 to each of these $2^n$ different $n$-tuples, the product rule shows that there are $2^{2^n}$ different Boolean functions of degree $n$.

# Identities of Boolean Algebra

- There are many identities in Boolean algebra. The most important of these are displayed in Table 5. These identities are particularly useful in simplifying the design of circuits. Each of the identities in Table 5 can be proved using a table.

# Identities of Boolean Algebra

**TABLE 5** **Boolean Identities.**

| *Identity* | *Name* |
|---|---|
| $\overline{\overline{x}} = x$ | Law of the double complement |
| $x + x = x$<br>$x \cdot x = x$ | Idempotent laws |
| $x + 0 = x$<br>$x \cdot 1 = x$ | Identity laws |
| $x + 1 = 1$<br>$x \cdot 0 = 0$ | Domination laws |
| $x + y = y + x$<br>$xy = yx$ | Commutative laws |

# Identities of Boolean Algebra

| | |
|---|---|
| $x + (y + z) = (x + y) + z$<br>$x(yz) = (xy)z$ | Associative laws |
| $x + yz = (x + y)(x + z)$<br>$x(y + z) = xy + xz$ | Distributive laws |
| $\overline{(xy)} = \overline{x} + \overline{y}$<br>$\overline{(x + y)} = \overline{x}\,\overline{y}$ | De Morgan's laws |
| $x + xy = x$<br>$x(x + y) = x$ | Absorption laws |
| $x + \overline{x} = 1$ | Unit property |
| $x\overline{x} = 0$ | Zero property |

# Identities of Boolean Algebra: Example

- Show that the distributive law $x(y+z) = xy + xz$ is valid.

**TABLE 6** **Verifying One of the Distributive Laws.**

| $x$ | $y$ | $z$ | $y+z$ | $xy$ | $xz$ | $x(y+z)$ | $xy+xz$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Identities of Boolean Algebra: Example

- Prove the **absorption law** $x(x+y)=x$ using the other identities of Boolean algebra shown in Table 5.
- *Solution:*

| | | |
|---|---|---|
| $x(x+y)$ | $= (x+0)(x+y)$ | Identity law for the Boolean sum |
| | $= x + 0 \cdot y$ | Distributive law of the Boolean sum over the Boolean product |
| | $= x + y \cdot 0$ | Commutative law for the Boolean product |
| | $= x + 0$ | Domination law for the Boolean product |
| | $= x$ | Identity law for the Boolean sum. |

# Duality

- The identities in Table 5 come in pairs (except for the law of the double complement and the unit and zero properties). To explain the relationship between the two identities in each pair we use the concept of a dual. The **dual** of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

# Duality

- The dual of a Boolean function $F$ represented by a Boolean expression is the function represented by the dual of this expression. This dual function, denoted by $F^d$, does not depend on the particular Boolean expression used to represent $F$. An identity between functions represented by Boolean expressions remains valid when the duals of both sides of the identity are taken. This result, called the **duality principle**, is useful for obtaining new identities.

# Duality: Example

- Construct an identity from the absorption law $x(x+y)=x$ by taking duals.
- *Solution:* Taking the duals of both sides of this identity produces the identity $x+xy=x$, which is also called an absorption law and is shown in Table 5.

# The Abstract Definition of a Boolean Algebra

- Here we have focused on Boolean functions and expressions. However, the results we have established can be translated into results about propositions or results about sets. Because of this, it is useful to define Boolean algebras abstractly. Once it is shown that a particular structure is a Boolean algebra, then all results established about Boolean algebras in general apply to this particular structure.

# The Abstract Definition of a Boolean Algebra

- Boolean algebras can be defined in several ways. The most common way is to specify the properties that operations must satisfy, as is done in the next Definition.

# The Abstract Definition of a Boolean Algebra

A *Boolean algebra* is a set $B$ with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $\overline{\phantom{x}}$ such that these properties hold for all $x$, $y$, and $z$ in $B$:

$$\left.\begin{aligned} x \vee 0 &= x \\ x \wedge 1 &= x \end{aligned}\right\} \quad \text{Identity laws}$$

$$\left.\begin{aligned} x \vee \overline{x} &= 1 \\ x \wedge \overline{x} &= 0 \end{aligned}\right\} \quad \text{Complement laws}$$

$$\left.\begin{aligned} (x \vee y) \vee z &= x \vee (y \vee z) \\ (x \wedge y) \wedge z &= x \wedge (y \wedge z) \end{aligned}\right\} \quad \text{Associative laws}$$

$$\left.\begin{aligned} x \vee y &= y \vee x \\ x \wedge y &= y \wedge x \end{aligned}\right\} \quad \text{Commutative laws}$$

$$\left.\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \end{aligned}\right\} \quad \text{Distributive laws}$$

# The Abstract Definition of a Boolean Algebra

- Using the laws given in Definition, it is possible to prove many other laws that hold for every Boolean algebra, such as idempotent and domination laws.

- From our previous discussion, $B = \{0, 1\}$ with the *OR* and *AND* operations and the complement operator, satisfies all these properties. The set of propositions in $n$ variables, with the $\vee$ and $\wedge$ operators, $\boldsymbol{F}$ and $\boldsymbol{T}$, and the negation operator, also satisfies all the properties of a Boolean algebra.

# The Abstract Definition of a Boolean Algebra

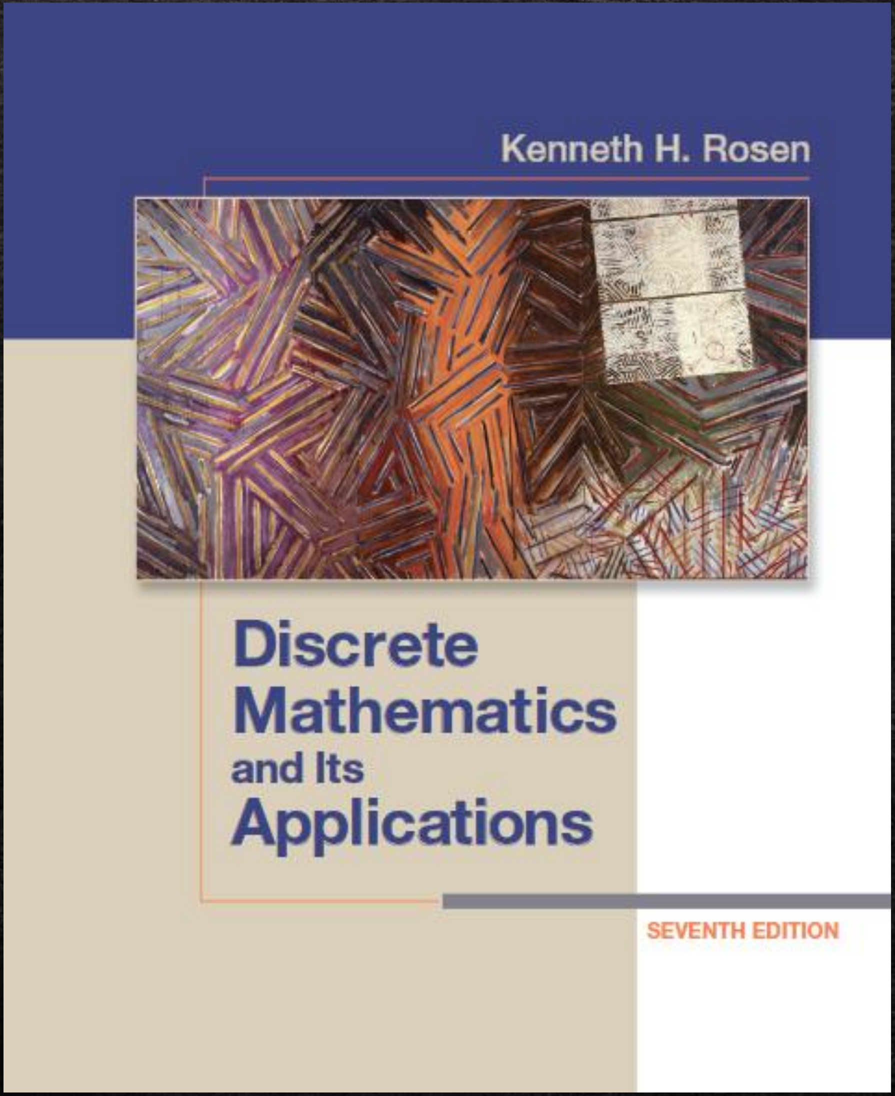
- Boolean algebras may also be defined using the notion of a lattice. A lattice $L$ is a partially ordered set in which every pair of elements $x, y$ has a least upper bound, denoted by $lub(x, y)$ and a greatest lower bound denoted by $glb(x, y)$. Given two elements $x$ and $y$ of $L$, we can define two operations ∨ and ∧ on pairs of elements of $L$ by

$$x \vee y = lub(x, y) \text{ and } x \wedge y = glb(x, y).$$

# The Abstract Definition of a Boolean Algebra

- For a lattice $L$ to be a Boolean algebra as specified in Definition, it must have two properties. First, it must be **complemented**. For a lattice to be complemented it must have a least element 0 and a greatest element 1 and for every element $x$ of the lattice there must exist an element $\bar{x}$ such that $x \vee \bar{x} = 1$ and $x \wedge \bar{x} = 0$. Second, it must be **distributive**. This means that for every $x$, $y$, and $z$ in $L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Showing that a complemented, distributive lattice is a Boolean algebra has been left as Supplementary Exercise.

# HOMEWORK: Exercises 2, 4, 6, 8, 12, 24, 28 on p. 818



# Representing Boolean Functions

- Two important problems of Boolean algebra will be considered here. The first problem is: Given the values of a Boolean function, how can a Boolean expression that represents this function be found? This problem will be solved by showing that any Boolean function can be represented by a Boolean sum of Boolean products of the variables and their complements. The solution of this problem shows that every Boolean function can be represented using the three Boolean operators $\cdot$, $+$, and $\bar{\ }$.

# Representing Boolean Functions

- The second problem is: Is there a smaller set of operators that can be used to represent all Boolean functions? We will answer this question by showing that all Boolean functions can be represented using only one operator. Both of these problems have practical importance in circuit design.

# Sum-of-Products Expansions: Example

- Find Boolean expressions that represent the functions $F(x, y, z)$ and $G(x, y, z)$, which are given in Table 1.

**TABLE 1**

| $x$ | $y$ | $z$ | $F$ | $G$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

# Sum-of-Products Expansions: Example

- *Solution:* An expression that has the value 1 when $x = z = 1$ and $y = 0$, and the value 0 otherwise, is needed to represent $F$. Such an expression can be formed by taking the Boolean product of $x$, $\bar{y}$, and $z$. This product, $x\bar{y}z$, has the value 1 if and only if $x = \bar{y} = z = 1$, which holds if and only if $x = z = 1$ and $y = 0$.

# Sum-of-Products Expansions: Example

- *Solution:* To represent $G$, we need an expression that equals 1 when $x = y = 1$ and $z = 0$, or $x = z = 0$ and $y = 1$. We can form an expression with these values by taking the Boolean sum of two different Boolean products. The Boolean product $xy\bar{z}$ has the value 1 if and only if $x = y = 1$ and $z = 0$. Similarly, the product $\bar{x}y\bar{z}$ has the value 1 if and only if $x = z = 0$ and $y = 1$. The Boolean sum of these two products, $xy\bar{z} + \bar{x}y\bar{z}$, represents $G$, because it has the value 1 if and only if $x = y = 1$ and $z = 0$, or $x = z = 0$ and $y = 1$.

# Sum-of-Products Expansions

- This Example illustrates a procedure for constructing a Boolean expression representing a function with given values. Each combination of values of the variables for which the function has the value 1 leads to a Boolean product of the variables or their complements.

# Sum-of-Products Expansions

- **DEFINITION 1.** A *literal* is a Boolean variable or its complement. A *minterm* of the Boolean variables $x_1, x_2, \ldots, x_n$ is a Boolean product $y_1 y_2 \ldots y_n$, where $y_i = x_i$ or $y_i = \overline{x_i}$. Hence, a minterm is a product of $n$ literals, with one literal for each variable.

# Sum-of-Products Expansions

- A minterm has the value 1 for one and only one combination of values of its variables. More precisely, the minterm $y_1 y_2 \ldots y_n$ is 1 if and only if each $y_i$ is 1, and this occurs if and only if $x_i = 1$ when $y_i = x_i$ and $x_i = 0$ when $y_i = \overline{x_i}$.

# Sum-of-Products Expansions: Example

- Find a minterm that equals 1 if $x_1 = x_3 = 0$ and $x_2 = x_4 = x_5 = 1$, and equals 0 otherwise.
- *Solution:* The minterm $\overline{x_1}x_2\overline{x_3}x_4x_5$ has the correct set of values.

# Sum-of-Products Expansions

- The minterms in this Boolean sum correspond to those combinations of values for which the function has the value 1. The sum of minterms that represents the function is called the **sum-of-products expansion** or the **disjunctive normal form (DNF)** of the Boolean function.

# Sum-of-Products Expansions: Example

- Find the sum-of-products expansion for the function $F(x, y, z) = (x + y)\bar{z}$. <u>*Solution:*</u> We will find the sum-of-products expansion of $F(x, y, z)$ in two ways. First, we will use Boolean identities to expand the product and simplify. We find that

$$
\begin{aligned}
F(x, y, z) &= (x + y)\overline{z} && \\
&= x\overline{z} + y\overline{z} && \text{Distributive law} \\
&= x1\overline{z} + 1y\overline{z} && \text{Identity law} \\
&= x(y + \overline{y})\overline{z} + (x + \overline{x})y\overline{z} && \text{Unit property} \\
&= xy\overline{z} + x\overline{y}\,\overline{z} + xy\overline{z} + \overline{x}y\overline{z} && \text{Distributive law} \\
&= xy\overline{z} + x\overline{y}\,\overline{z} + \overline{x}y\,\overline{z}. && \text{Idempotent law}
\end{aligned}
$$

# Sum-of-Products Expansions: Example

- Second, we can construct the sum-of-products expansion by determining the values of $F$ for all possible values of the variables $x$, $y$, and $z$. These values are found in Table 2.

**TABLE 2**

| $x$ | $y$ | $z$ | $x + y$ | $\overline{z}$ | $(x + y)\overline{z}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 0 |

# Sum-of-Products Expansions: Example

- The sum-of products expansion of $F$ is the Boolean sum of three minterms corresponding to the three rows of this table that give the value 1 for the function. This gives

$$F(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}.$$

# Sum-of-Products Expansions

- It is also possible to find a Boolean expression that represents a Boolean function by taking a Boolean product of Boolean sums. The resulting expansion is called the **conjunctive normal form (CNF)** or **product-of-sums expansion** of the function. These expansions can be found from sum-of-products expansions by taking duals. To find such expansions directly we use the same method as for disjunctive normal form.

# Sum-of-Products Expansions

- The Boolean sum $y_1 + y_2 + \cdots + y_n$, where $y_i = x_i$ or $y_i = \overline{x_i}$, has the value 0 for exactly one combination of the values of the variables, namely, when $x_i = 0$ if $y_i = x_i$ and $x_i = 1$ if $y_i = \overline{x_i}$. This Boolean sum is called a **maxterm.** A Boolean function can be represented as a Boolean product of maxterms. This representation is called the **product-of-sums expansion** or **conjunctive normal form (CNF)** of the function.

# Functional Completeness

- Every Boolean function can be expressed as a Boolean sum of minterms. Each minterm is the Boolean product of Boolean variables or their complements. This shows that every Boolean function can be represented using the Boolean operators $\cdot$, $+$, and $\bar{\ }$. Because every Boolean function can be represented using these operators we say that the set $\{\cdot, +, \bar{\ }\}$ is **functionally complete**.

# Functional Completeness

- Can we find a smaller set of functionally complete operators? We can do so if one of the three operators of this set can be expressed in terms of the other two. This can be done using one of De Morgan's laws. We can eliminate all Boolean sums using the identity

$$x + y = \overline{\bar{x}\bar{y}},$$

which is obtained by taking complements of both sides in the second De Morgan law.

# Functional Completeness

- This means that the set $\{\cdot, \bar{\ }\}$ is functionally complete.
- Similarly, we could eliminate all Boolean products using the identity

$$xy = \overline{\bar{x} + \bar{y}},$$

which is obtained by taking complements of both sides in the first De Morgan law. Consequently $\{+, \bar{\ }\}$ is functionally complete. Note that the set $\{+, \cdot\}$ is not functionally complete.

# Functional Completeness

- We have found sets containing two operators that are functionally complete. Can we find a smaller set of functionally complete operators, namely, a set containing just one operator? Such sets exist. Define two operators, the | or ***NAND*** operator (or Sheffer stroke), defined by

$$1|1 = 0 \text{ and } 1|0 = 0|1 = 0|0 = 1;$$

- and the ↓ or ***NOR*** operator (Peirce's arrow),

$$1 \downarrow 1 = 1 \downarrow 0 = 0 \downarrow 1 = 0 \text{ and } 0 \downarrow 0 = 1.$$

# Functional Completeness

- Both of the sets $\{|\}$ and $\{\downarrow\}$ are functionally complete. To see that $\{|\}$ is functionally complete, because $\{\cdot, \bar{}\}$ is functionally complete, all that we have to do is show that both of the operators $\cdot$ and $\bar{}$ can be expressed using just the $|$ operator. This can be done as

$$\bar{x} = x|x,$$

$$xy = (x|y)|(x|y).$$

# HOMEWORK: Exercises 2, 4, 6, 12, 14, 20 on p. 822