# Seminar 4

**THEOREM 12—The Ratio Test** Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or $\rho$ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

**EXAMPLE 1** Investigate the convergence of the following series.

$$\textbf{(b)}\quad \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

**(b)** If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n+2)!}{(n+1)!(n+1)!}$ and

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!}\\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \to 4.\end{aligned}$$

The series diverges because $\rho = 4$ is greater than 1.

**EXAMPLE 1** Investigate the convergence of the following series.

$$\textbf{(a)}\quad \sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$$

**Solution** We apply the Ratio Test to each series.

**(a)** For the series $\sum_{n=0}^{\infty} (2^n+5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3}\cdot\frac{2^{n+1}+5}{2^n+5} = \frac{1}{3}\cdot\left(\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}\right) \to \frac{1}{3}\cdot\frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty}\frac{5}{3^n} = \frac{1}{1-(2/3)} + \frac{5}{1-(1/3)} = \frac{21}{2}.$$

**THEOREM 13—The Root Test** Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n \to \infty} \sqrt[n]{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or $\rho$ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

**THEOREM 14—The Alternating Series Test (Leibniz's Test)** The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The $u_n$'s are all positive.
2. The positive $u_n$'s are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer $N$.
3. $u_n \to 0$.

**DEFINITION** A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

The geometric series in Example 3 converges absolutely because the corresponding series of absolute values

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

converges. The alternating harmonic series does not converge absolutely because the corresponding series of absolute values is the (divergent) harmonic series.

## Using the Ratio Test

In Exercises 1–8, use the Ratio Test to determine if each series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

2. $\sum_{n=1}^{\infty} \frac{n+2}{3^n}$

3. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$

4. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$

## Using the Root Test

In Exercises 9–16, use the Root Test to determine if each series converges or diverges.

9. $\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$

10. $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

11. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

12. $\sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$

13. $\sum_{n=1}^{\infty} \frac{8}{(3+(1/n))^{2n}}$

14. $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

## Determining Convergence or Divergence

In Exercises 17–44, use any method to determine if the series converges or diverges. Give reasons for your answer.

17. $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$

18. $\sum_{n=1}^{\infty} n^2 e^{-n}$

19. $\sum_{n=1}^{\infty} n! e^{-n}$

20. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

21. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

22. $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

## Determining Convergence or Divergence

In Exercises 1–14, determine if the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$

3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n3^n}$

4. $\sum_{n=2}^{\infty} (-1)^n \frac{4}{(\ln n)^2}$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$

# Seminar 5

**DEFINITIONS** A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** $a$ and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

**THEOREM 18—The Convergence Theorem for Power Series** If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all $x$ with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all $x$ with $|x| > |d|$.

**COROLLARY TO THEOREM 18** The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number $R$ such that the series diverges for $x$ with $|x - a| > R$ but converges absolutely for $x$ with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every $x$ $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

### How to Test a Power Series for Convergence

1. *Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely*. Ordinarily, this is an open interval

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. *If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$* (it does not even converge conditionally) because the $n$th term does not approach zero for those values of $x$.

$R$ is called the **radius of convergence** of the power series, and the interval of radius $R$ centered at $x = a$ is called the **interval of convergence**. The interval of convergence may be open, closed, or half-open, depending on the particular series. At points $x$ with $|x - a| < R$, the series converges absolutely. If the series converges for all values of $x$, we say its radius of convergence is infinite. If it converges only at $x = a$, we say its radius of convergence is zero.

## Operations on Power Series

**THEOREM 19—The Series Multiplication Theorem for Power Series** If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Finding the general coefficient $c_n$ in the product of two power series can be very tedious and the term may be unwieldy. The following computation provides an illustration of a product where we find the first few terms by multiplying the terms of the second series by each term of the first series:

$$\left(\sum_{n=0}^{\infty} x^n\right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right)$$

$$= (1 + x + x^2 + \cdots)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) \qquad \text{Multiply second series . . .}$$

$$= \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)}_{\text{by } 1} + \underbrace{\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right)}_{\text{by } x} + \underbrace{\left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \cdots\right)}_{\text{by } x^2} + \cdots$$

$$= x + \frac{x^2}{2} + \frac{5x^3}{6} - \frac{x^4}{6} \cdots. \qquad \text{and gather the first four powers.}$$

We can also substitute a function $f(x)$ for $x$ in a convergent power series.

**THEOREM 21—The Term-by-Term Differentiation Theorem** If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{on the interval} \qquad a - R < x < a + R.$$

This function $f$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

**EXAMPLE 4** Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1$$

**Solution** We differentiate the power series on the right term by term:

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1;$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1$$

**THEOREM 22—The Term-by-Term Integration Theorem** Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

converges for $a - R < x < a + R \;\; (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + C$$

for $a - R < x < a + R$.

## Intervals of Convergence

In Exercises 1–36, **(a)** find the series' radius and interval of convergence.

**1.** $\displaystyle\sum_{n=0}^{\infty} x^n$

**2.** $\displaystyle\sum_{n=0}^{\infty} (x + 5)^n$

**3.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n (4x + 1)^n$

**4.** $\displaystyle\sum_{n=1}^{\infty} \frac{(3x - 2)^n}{n}$

**5.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x - 2)^n}{10^n}$

**6.** $\displaystyle\sum_{n=0}^{\infty} (2x)^n$