# *Binomial Coefficients and Combinatorial Counting*

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## *Example*

*Find a recurrence relation*

What is the number of bit strings of length $n$ that do not have two consecutive 0's?

Lets count: $n = 0$ : empty-string

$n = 1$ : 0, 1;

$n = 2$ : 01, 11, 10;

$n = 3$ : 011, 111, 101, 010, 110;

$n = 4$ : 0111, 1111, 1011, 0101, 1101, 0110, 1110, 1010;

$n = 5$ : 01111, 11111, 10111, 01011, 11011, 01101, 11101, 10101, 01110, 11110, 10110, 01010, 11010

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# Fibonacci Numbers

*Definition*

The Fibonacci sequence $\{F_n\}$ is defined by

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$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 \ldots$$

# *One more Example*

### *Codeword Enumeration*

A computer system considers a string of decimal digits a valid codeword if it contains an even number of "0" digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let $a_n$ be the number of valid $n$-digit codewords. Find a recurrence relation for $a_n$.

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Answer: $a_n = 8a_{n-1} + 10^{n-1}$

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so a direct computation requires finding all previous Fibonacci numbers

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**Question**

Is it possible to compute $F_n$ without computing all previous values?

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$$1, 3, 9, 27, \ldots$$

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- Fibonacci recurrence has order 2
- $a_n = 2a_{n-1} + 1$ has order 1

# *SolvingLinear Recurrence Relations*

### *Definition*

A linear homogeneous recurrence relation of degree $k$ with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_{n-1}, c_{n-2}, \ldots, c_{n-k}$ are real numbers, and $c_k \neq 0$.

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- The recurrence relation $a_n = 3a_{n-1}$ is linear homogeneous recurrence relation of degree 1;
- The recurrence relation $a_n = a_{n-1} + a_{n-2}$ is linear homogeneous recurrence relation of degree 2;
- The recurrence relation $a_n = a_{n-5}$ is linear homogeneous recurrence relation of degree 5;

*Theorem*

*Assume* $a_{n+2} = c_1 a_{n-1} + c_2 a_n$ *be a recurrence relation and corresponding equation* $x^2 = c_1 x + c_2$ *has two distinct roots* $r_1$ *and* $r_2$. *Then for any* $n$

$$a_n = d_1 r_1^n + d_2 r_2^n$$

*for some constants* $d_1$ *and* $d_2$.

Proof by Induction.

*Theorem*

*Assume $a_{n+2} = c_1 a_{n+1} + c_2 a_n$ be a recurrence relation and corresponding equation $x^2 = c_1 x + c_2$ has unique root $r$. Then for any $n$*

$$a_n = d_1 r^n + d_2 n r^n$$

*for some constants $d_1$ and $d_2$.*

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$$a_{n+2} = a_{n+1} + 2a_n$$

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## *Examples*

- Find an explicit formula for the Fibonacci numbers.
- What is the solution of the recurrence relation

$$a_{n+2} = a_{n+1} + 2a_n$$

  with $a_0 = 2$ and $a_1 = 7$.
- What is the solution of the recurrence relation

$$a_{n+2} = 6a_{n+1} - 9a_n$$

  with $a_0 = 1$ and $a_1 = 6$.

## THE GENERAL CASE

**Theorem**

*Assume*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k}$$

*be a recurrence relation and corresponding equation*

$$x^k = c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k$$

*has $k$ distinct roots $r_1, r_2, \ldots, r_k$. Then for any $n$*

$$a_n = d_1 r_1^n + d_2 r_2^n + \cdots + d_k r_k^n$$

*for some constants $d_1, d_2, \ldots, r_k$.*

# Example

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

### Theorem

*Assume*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k}$$

*be a recurrence relation and corresponding equation*

$$x^k = c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k$$

*has $t$ distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively and $m_1 + m_2 + \cdots + m_t = k$. Then for any $n$*

$$a_n = (d_{1,0} + d_{1,1} n + \cdots d_{1,m_1} n^{m_1}) r_1^n +$$

$$+(d_{2,0} + d_{2,1} n + \cdots d_{2,m_2} n^{m_2}) r_2^n + \cdots$$

$$+(d_{t,0} + d_{t,1} n + \cdots + d_{t,m_t} n^{m_t}) r_t^n$$

*for some constants $d_1, d_2, \ldots, r_k$.*

## *Example*

Find the solution to the recurrence relation

$$an = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1, a_1 = -2$, and $a_2 = -1$

# *Linear Non-homogeneous Recurrence Relations with Constant Coefficient*

*Definition*

A linear non-homogeneous recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on $n$.

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where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on $n$.
The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation.

# Solving non-homogeneous Recurrence relation

*Theorem*

*If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

*then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

# *Example*

Find solution of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$

with initial condition $a_0 = 1$ and $a_1 = 17$