Propositional Logic

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Propositional Logic: Propositions

- Our discussion begins with an introduction to the basic building blocks of logic propositions. A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Propositions: Example

- All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.

2. Toronto is the capital of Canada.

3. $1 + 1 = 2$.

4. $2 + 2 = 3$.

- Propositions 1 and 3 are true, whereas 2 and 4 are false.

Propositions: Example

- Consider the following sentences.

1. What time is it? 2. Read this carefully.

3. $x + 1 = 2$. 4. $x + y = z$.

- Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

Propositions

- We use letters to denote **propositional variables** (or **statement variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are $p, q, r, s, \ldots$. The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

Propositions

- The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**. It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

Propositions

- We now turn our attention to methods for producing new propositions from those that we already have. These methods were discussed by the English mathematician George Boole in 1854 in his book *The Laws of Thought.* Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

Propositions

- **DEFINITION 1.** Let *p* be a proposition. The *negation of p*, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement

  "It is not the case that p."

- The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Propositions: Example

- Find the negation of the proposition

“Asan’s PC runs Linux”

and express this in simple English.

- *Solution:* The negation is

“It is not the case that Asan’s PC runs Linux.”

This negation can be more simply expressed as

“Asan’s PC does not run Linux.”

Propositions

- Table below displays the **truth table** for the negation of a proposition p. This table has a row for each of the two possible truth values of a proposition p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

Propositions

- The negation of a proposition can also be considered the result of the operation of the **negation operator** on a proposition. The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives**.

Propositions

- **DEFINITION 2.** Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Propositions

- **DEFINITION 2.** Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.
- Note that in logic the word "but" sometimes is used instead of "and" in a conjunction. For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining."

Propositions: Example

- Find the conjunction of the propositions p and q where p is the proposition “Aizhan’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Aizhan’s PC runs faster than 1 GHz.”

Propositions: Example

- *Solution:* The conjunction of these propositions, $p \wedge q$, is the proposition "Aizhan's PC has more than 16 GB free hard disk space, and the processor in Aizhan's PC runs faster than 1 GHz." This conjunction can be expressed more simply as "Aizhan's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz." For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

Propositions

- **DEFINITION 3.** Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

TABLE 3 **The Truth Table for the Disjunction of Two Propositions.**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Propositions

- The use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, as an **inclusive or**. A disjunction is true when at least one of the two propositions is true. For instance, the inclusive or is being used in the statement

"Students who have taken calculus or computer science can take this class."

Propositions

- Here, we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects. On the other hand, we are using the **exclusive or** when we say

"Students who have taken calculus or computer science, but not both, can enroll in this class."

Propositions

- Here, we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.
- Similarly, when a menu at a restaurant states, “Soup or salad comes with an entrée,” the restaurant almost always means that customers can have either soup or salad, but not both. Hence, this is an exclusive, rather than an inclusive, or.

Propositions

- **DEFINITION 4.** Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Propositions

- **DEFINITION 5.** Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

Propositions

- The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Propositions

- Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \rightarrow q$. You will encounter most if not all of the following ways to express this conditional statement:

"if p, then q"; "p implies q"; "if p, q"; "p only if q"; "p is sufficient for q"; "a sufficient condition for q is p"; "q if p"; "q whenever p"; "q when p"; "q is necessary for p"; "a necessary condition for p is q"; "q follows from p"; "q unless $\neg p$".

Propositions: Example

- A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is

  "If I am elected, then I will lower taxes."

Propositions: Example

- If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in $p \to q$.

Propositions

- The if-then construction used in many programming languages is different from that used in logic. Most programming languages contain statements such as **if** p **then** S, where p is a proposition and S is a program segment (one or more statements to be executed).When execution of a program encounters such a statement, S is executed if p is true, but S is not executed if p is false.

Propositions

- We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \to p$ is called the **converse** of $p \to q$. The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$. We will see that of these three conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

Propositions: Example

- What are the contrapositive, the converse, and the inverse of the conditional statement

  "The home team wins whenever it is raining?"

- *Solution:* Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as

  "If it is raining, then the home team wins."

Propositions: Example

- *Solution (cont.):* Consequently, the contrapositive of this conditional statement is "If the home team does not win, then it is not raining." The converse is "If the home team wins, then it is raining." The inverse is "If it is not raining, then the home team does not win."
- Only the contrapositive is equivalent to the original statement.

Propositions: BICONDITIONALS

- **DEFINITION 6.** Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

Propositions: BICONDITIONALS

- Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise. That is why we use the words "if and only if" to express this logical connective and why it is symbolically written by combining the symbols $\rightarrow$ and $\leftarrow$. There are some other common ways to express $p \leftrightarrow q$:
- "p is necessary and sufficient for q"; "if p then q, and conversely"; "p iff q."

Propositions: BICONDITIONALS

- Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \land (q \rightarrow p)$.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Tables of Compound Propositions: Example

- Construct the truth table of the compound proposition: $(p \vee \neg q) \rightarrow (p \wedge q)$.
- *Solution:* Because this truth table involves two propositional variables *p* and *q*, there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF. The first two columns are used for the truth values of p and q, respectively. In the third column we find the truth value of $\neg q$, needed to find the truth value of $p \vee \neg q$, found in the fourth column.

Truth Tables of Compound Propositions: Example

- *Solution (cont.):* The fifth column gives the truth value of $p \land q$. Finally, the truth value of $(p \lor \neg q) \rightarrow (p \land q)$ is found in the last column.

TABLE 7 The Truth Table of $(p \lor \neg q) \rightarrow (p \land q)$.

p	q	$\neg q$	$p \lor \neg q$	$p \land q$	$(p \lor \neg q) \rightarrow (p \land q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Logic and Bit Operations

- Computers represent information using bits. A **bit** is a symbol with two possible values, namely, 0 (zero) and 1 (one). This meaning of the word bit comes from *b*inary dig*it*, because zeros and ones are the digits used in binary representations of numbers. The well-known statistician John Tukey introduced this terminology in 1946. A bit can be used to represent a truth value, because there are two truth values, namely, *true* and *false*.

Logic and Bit Operations

- As is customarily done, we will use a 1 bit to represent true and a 0 bit to represent false. That is, 1 represents T (true), 0 represents F (false). A variable is called a **Boolean variable** if its value is either true or false. Consequently, a Boolean variable can be represented using a bit.

Logic and Bit Operations

- Computer **bit operations** correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators $\wedge$, $\vee$, and $\oplus$, the tables shown in Table 9 for the corresponding bit operations are obtained. We will also use the notation *OR*, *AND*, and *XOR* for the operators $\vee$,$\wedge$, and $\oplus$, as is done in various programming languages.

Logic and Bit Operations

TABLE 9 Table for the Bit Operators *OR*, *AND*, and *XOR*.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

HOMEWORK: Exercises 2, 4, 8, 12, 16, 22, 28, 30, 32, 38 on pp. 12-15