Counting: Elements of Combinatorics (cont.)

Assylbek Issakhov,
Ph.D., professor

Permutations: Example

- In how many ways can we select three students from a group of five students to stand in line for a picture?
- In how many ways can we arrange all five of these students in a line for a picture?

Permutations: Example

- *Solution:* First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

Permutations: Example

- *Solution:* To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture.

Permutations

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an $\boldsymbol{r}$**-permutation**.
- The number of r-permutations of a set with n elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the product rule.

Permutations

- **THEOREM 1.** If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$

- r-permutations of a set with n distinct elements.

Permutations

- **COROLLARY 1.** If n and r are integers with $0 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Permutations: Example

- How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Permutations: Example

- *Solution:* Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200.$$

Permutations: Example

- Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Permutations: Example

- *Solution: The number of different ways to award the medals is the number of 3-permutations* of a set with eight elements. Hence, there are

$$P(8,3) = 8 \cdot 7 \cdot 6 = 336$$

- *possible ways to award* the medals.

Permutations: Example

- How many permutations of the letters $ABCDEFGH$ *contain the string* ABC *?*
- *Solution: Because the letters* ABC *must occur as a block, we can find the answer by finding the* number of permutations of six objects, namely, the block ABC *and the individual letters* $D, E, F, G,$ *and* H. *Because these six objects can occur in any order, there are* $6! = 720$ *permutations* of the letters $ABCDEFGH$ *in which* ABC *occurs as a block.*

Combinations: Example

- How many different committees of three students can be formed from a group of four students?
- *Solution: To answer this question, we need only find the number of subsets with three elements* from the set containing the four students.

Combinations: Example

- *Solution (cont.):* We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter.

Combinations

- Previous Example illustrates that many counting problems can be solved by finding the number of subsets of a particular size of a set with n *elements, where* n *is a positive integer.*
- An ***r-combination of elements of a set is an unordered selection of r elements from the set.***
- Thus, an r*-combination is simply a subset of the set with* r *elements.*

Combinations

- The number of r*-combinations of a set with* n *distinct elements is denoted by* $C(n, r)$. *Note* that $C(n, r)$ *is also denoted by* $\binom{n}{r}$ *and is called a* ***binomial coefficient.***

- **THEOREM 2.** The number of r*-combinations of a set with* n *elements, where* n *is a nonnegative integer and* r *is an integer with* $0 \leq r \leq n$, *equals*

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Combinations: Example

- How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?
- *Solution*: Because the order in which the five cards are dealt from a deck of 52 cards does not matter, there are $C(52,5) = \frac{52!}{5!47!}$ different hands of five cards that can be dealt.

$$C(52,5) = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960.$$

Combinations: Example

- *Solution (cont.)*: Note that there are $C(52, 47) = \frac{52!}{47!5!}$ different ways to select 47 cards from a standard deck of 52 cards. We do not need to compute this value because

$$C(52, 47) = C(52, 5).$$

- Only the order of the factors $5!$ and $47!$ is different in the denominators in the formulae for these quantities.

Combinations

- **COROLLARY 2.** Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.
- **DEFINITION 1.** A *combinatorial proof* of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called *double counting proofs* and *bijective proofs*, respectively.

Combinations: Example

- How many bit strings of length n contain exactly r 1s?
- *Solution:* The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s.

Combinations: Example

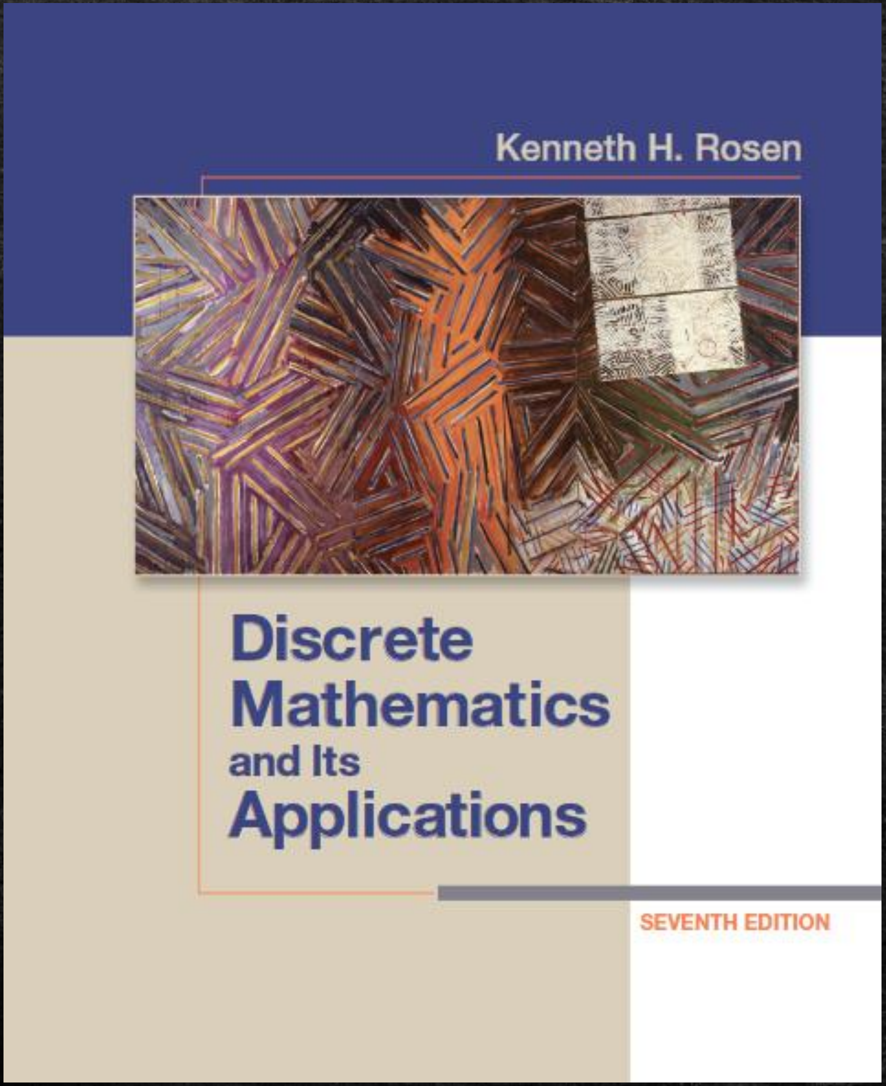
- Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Combinations: Example

- *Solution:* By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$C(9,3) \cdot C(11,4) = \frac{9!}{3!\,6!} \cdot \frac{11!}{4!\,7!} =$$
$$= 84 \cdot 330 = 27{,}720$$

HOMEWORK: Exercises 2, 4, 8, 12, 14, 16, 30, 34 on pp. 413-414



The Binomial Theorem

- The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A **binomial** expression is simply the sum of two terms, such as $x + y$.

The Binomial Theorem

- **THEOREM 1. (THE BINOMIAL THEOREM)** Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j =$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

The Binomial Theorem: Example

- What is the expansion of $(x+y)^4$?
- *Solution:* From the binomial theorem it follows that

$$(x+y)^4$$
$$= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j = \binom{4}{0} x^4 + \binom{4}{1} x^3 y$$
$$+ \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 =$$
$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

The Binomial Theorem: Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?
- *Solution:* From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300$$

The Binomial Theorem: Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?
- *Solution:* First, note that this expression equals $(2x - 3y)^{25}$. By the binomial theorem, we have

$$\left(2x + (-3y)\right)^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^{j}$$

- Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely,

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!\,12!} 2^{12}3^{13}$$

The Binomial Theorem

- **COROLLARY 1.** Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- **COROLLARY 2.** Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Pascal's Identity and Triangle

- **THEOREM 2. (PASCAL'S IDENTITY)** Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal's Triangle

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\dots
\end{array}
$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\dots
\end{array}
$$

HOMEWORK: Exercises 4, 6, 8, 12 on p. 421