

The Trapezoidal Rule

To approximate $\int_a^b f(x) \, dx$, use

$$T = \frac{\Delta x}{2} \left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n \right).$$

The y ’s are the values of f at the partition points

$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n - 1)\Delta x, x_n = b,$
where $\Delta x = (b - a)/n$.

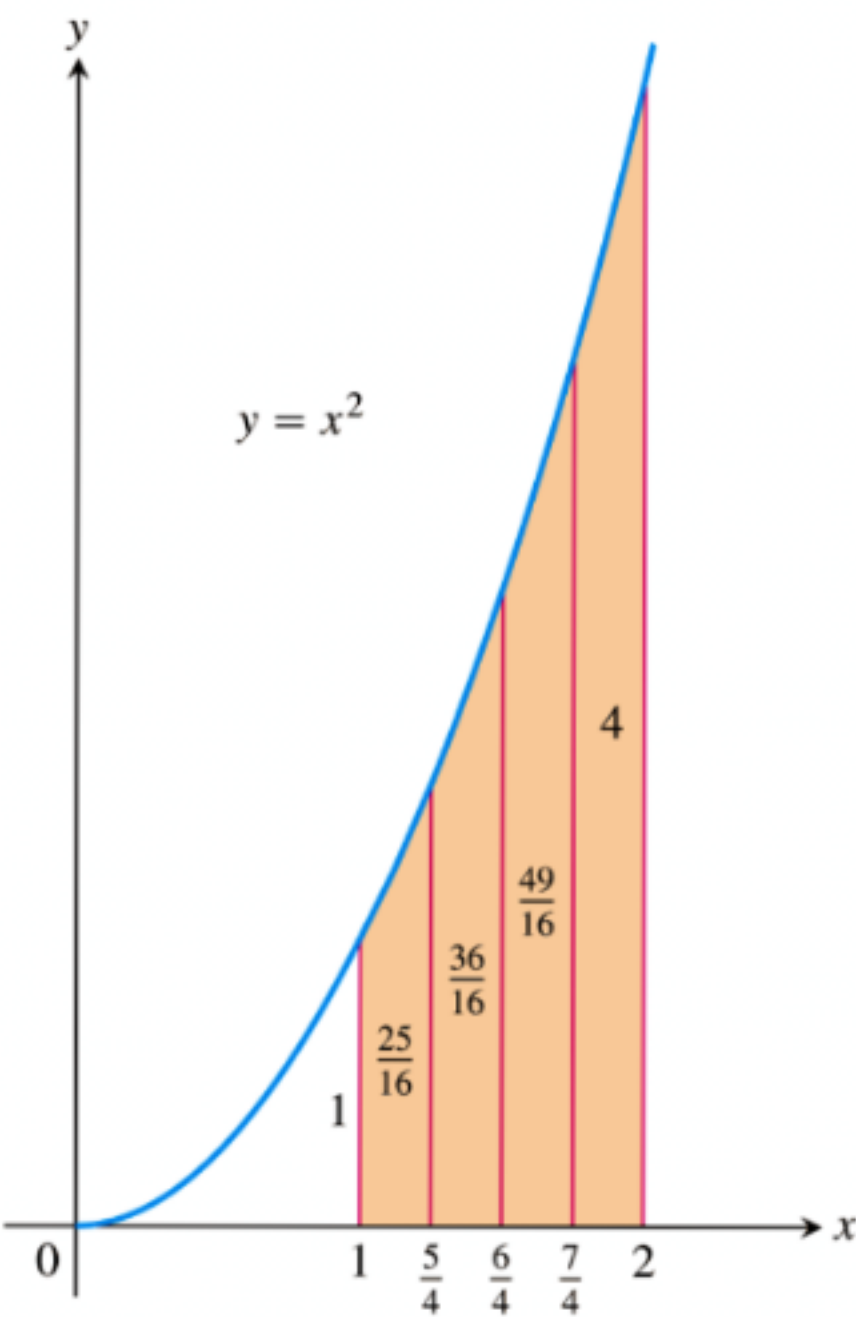


FIGURE 8.8 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight over-estimate (Example 1).

EXAMPLE 1 Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2 \, dx$. Compare the estimate with the exact value.

Solution Partition $[1, 2]$ into four subintervals of equal length (Figure 8.8). Then evaluate $y = x^2$ at each partition point (Table 8.2).
Using these y values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2} \left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4 \right) \\ &= \frac{1}{8} \left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4 \right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

Since the parabola is concave *up*, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. The exact value of the integral is

$$\int_1^2 x^2 \, dx = \left. \frac{x^3}{3} \right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

TABLE 8.2

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

Simpson’s Rule

To approximate $\int_a^b f(x) \, dx$, use

$$S = \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y ’s are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n - 1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

EXAMPLE 2 Use Simpson’s Rule with $n = 4$ to approximate $\int_0^2 5x^4 \, dx$.

Solution Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.3). Then apply Simpson’s Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned} S &= \frac{\Delta x}{3} \left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4 \right) \\ &= \frac{1}{6} \left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80 \right) \\ &= 32\frac{1}{12}. \end{aligned}$$

This estimate differs from the exact value (32) by only $1/12$, a percentage error of less than three-tenths of one percent, and this was with just four subintervals.

TABLE 8.3

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

I. Using the Trapezoidal Rule

- a. Estimate the integral with $n = 4$ steps and find an upper bound for the error.

II. Using Simpson's Rule

- a. Estimate the integral with $n = 4$ steps and find an upper bound for the error.

1. $\int_1^2 x \, dx$

3. $\int_{-1}^1 (x^2 + 1) \, dx$

5. $\int_0^2 (t^3 + t) \, dt$

7. $\int_1^2 \frac{1}{s^2} \, ds$

9. $\int_0^\pi \sin t \, dt$

2. $\int_1^3 (2x - 1) \, dx$

4. $\int_{-2}^0 (x^2 - 1) \, dx$

6. $\int_{-1}^1 (t^3 + 1) \, dt$

8. $\int_2^4 \frac{1}{(s - 1)^2} \, ds$

10. $\int_0^1 \sin \pi t \, dt$

I

II