Generating Functions and Applications in Combinatorics

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Definition

Assume Γ is the set of infinite sequences $a_0, a_1, a_2, \ldots$, where $a_i \in \mathbb{R}$. Lets define the following operation in Γ: Assume a and b are infinite sequences with n-th terms a_n and b_n, respectively. Then

- $(a + b)_n = a_n + b_n$;
- $(\lambda a)_n = \lambda a_n$ for scalar λ;
- $(a \star b)_n = a_0 b_n + a_1 b_{n-1} + \cdots a_n b_0 =$

$$= \sum_{i=0}^{n} a_i b_{n-i}$$

By $\mathbb{O} \in \Gamma$ lets denote the sequence $0, 0, 0, 0, \ldots, 0, \ldots$
and by $\mathbb{I} \in \Gamma$ lets denote the sequence $1, 0, 0, 0, \ldots, 0, \ldots$
Then $(\Gamma, \mathbb{O}, \mathbb{I}, +, \star)$ is an algebra satisfying the following axioms:

- $\mathbb{O} + a = a$;
- $\mathbb{I} \star a = a$;
- $a + b = b + a$;
- $a \star b = b \star a$;
- $\lambda(a + b) = \lambda a + \lambda b$;
- $a + (b + c) = (a + b) + c$;
- $(a + b) \star c = (a \star c) + (b \star c)$;
- $(a \star b) \star c = a \star (b \star c)$.

Definition

The generating function for the sequence a

$$a_0, a_1, \dots, a_k, \dots$$

of real numbers is the infinite series

$$G[a](x) = a_0 + a_1 x + \dots + a_k x^k + \dots = \sum_{k=1}^{\infty} a_k x^k.$$

Theorem

A mapping $\phi : \Gamma \to C(x)$ *which defined* $a \to G[a]$ *is homomorphism: i.e. satisfy the following*

- $G[\mathbb{O}] = 0$;
- $G[\mathbb{I}] = 1$;
- $G[\lambda a] = \lambda G[a]$;
- $G[a + b] = G[a] + G[b]$;
- $G[a \star b] = G[a] \cdot G[b]$.

Examples

Find Generating functions for the following sequences:

- $1, 1, 1, 1, 1, \ldots$

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- $1, 3, 5, 7, 9, 11, , \ldots$

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Lemma

Let b be a sequence which obtained from a sequence a with shifting k times to the Right, i.e.

$$b_i = 0 \text{ for } i < k \text{ and } b_{k+i} = a_i \text{ for all } i \in \mathbb{N}$$

Then $G[b] = x^k \cdot G[a]$.

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Example. Find Generating function for the sequence

$$0, 0, 0, 0, 1, 3, 5, 7, 9, 11, 13, 15, \dots$$

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Let b be a sequence which obtained from a sequence a with shifting k times to the Left, i.e.

$$b_i = a_{i+k} \text{ for all } i \in \mathbb{N}$$

Then

$$G[b] = \frac{G[a] - a_0 - a_1 x - a_2 x^2 - \cdots - a_{k-1} x^{k-1}}{x^k}$$

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Example. Find Generating function for the sequence

$$11, 13, 15, 17, 19, 21, 23, 25, \ldots$$

Lemma

Let α be a number and a sequence b obtained from a sequence a by the way

$$b_n = \alpha^n \cdot a_n$$

Then $G[b]$ is obtained from $G[a]$ by replacing x with $\alpha \times x$.

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Example. Find Generating function for the sequence

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$$

Lemma

Let b be a sequence obtained from a sequence a by the way

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$$b_n = n \cdot a_n$$

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Let b be a sequence which obtained from a by the following

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- $a_n = n2^n$

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- Find coefficient for x^5 in $(1-2x)^{-7}$