Class work 3

Task 1. Compute the followings

(a) How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

(b) How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other?

(c) How many ways are there for four men and five women to stand in a line so that
  i) all men stand together?
  ii) all women stand together?

(d) How many ways are there for three penguins and six puffins to stand in a line so that
  i) all puffins stand together?
  ii) all penguins stand together?

(e) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if there are no restrictions?

(f) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if the person holding ticket 47 wins the grand prize?

(g) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if the person holding ticket 47 wins one of the prizes?

(h) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if the person holding ticket 47 does not win a prize?

(i) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if the people holding tickets 19 and 47 both win prizes?

(j) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if the people holding tickets 19, 47, and 73 all win prizes?

(k) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if the people holding tickets 19, 47, 73, and 97 all win prizes?

(l) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if none of the people holding tickets 19, 47, 73, and 97 wins a prize?

(m) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if the grand prize winner is a person holding ticket 19, 47, 73, or 97?

(n) One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

(o) Thirteen people on a softball team show up for a game. How many ways are there to choose 10 players to take the field?

(p) Thirteen people on a softball team show up for a game. How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?

(q) Thirteen people on a softball team show up for a game. Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

(r) A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

(s) The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain
  i) exactly one vowel?
  ii) exactly two vowels?
  iii) at least one vowel?
  iv) at least two vowels?

(t) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

(u) How many bit strings contain exactly eight 0's and 10 1's if every 0 must be immediately followed by a 1?

(v) How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from the remaining 69 countries?

(w) How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

(x) How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

(y) How many strings of six lowercase letters from the English alphabet contain the letters a and b in consecutive positions with a preceding b, with all the letters distinct?.

(z) How many strings of six lowercase letters from the English alphabet contain the letters a and b, where a is somewhere to the left of b in the string, with all the letters distinct?

Task 2. Count the followings:

(a) Find the expansion of $(x+y)^{13}$.

(b) What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

(c) Use the binomial theorem to expand $(3x-y^2)^4$ into a sum of terms of the form cx^ay^b, where c is a real number and a and b are non-negative integers.

(d) Use the binomial theorem to expand $(3x^4-2y^3)^5$ into a sum of terms of the form cx^ay^b, where c is a real number and a and b are non-negative integers.

(e) Use the binomial theorem to find the coefficient of x^8y^9 in the expansion of $(5x^2+2y^3)^6$.

(f) Use the binomial theorem to find the coefficient of x^9y^8 in the expansion of $(2x^3-4y^2)^7$.

(g) Give a formula for the coefficient of x^k in the expansion of $(x+\frac{1}{x})^{100}$, where k is an integer.

(h) Give a formula for the coefficient of x^k in the expansion of $(x^2-\frac{1}{x})^{100}$, where k is an integer.

(i) Find the coefficient of x^{50} in the expansion of $\left(x^2+\frac{1}{x}\right)^{60}$.

(j) Find the coefficient of $x^{12}y^{18}$ in the expansion of $(3x^2-2y^3)^{10}$, or show that such a term does not appear.

(k) Determine all integers k for which the coefficient of x^k in $(x-\frac{1}{x})^{100}$ is non-zero.

(l) Find the coefficient of x^0 in the expansion of $(2x^3-\frac{1}{x^2})^{25}$.

(m) Find the coefficient of $x^{20}y^{15}$ in the expansion of $(x^4+y^3)^n$, where n is a positive integer.

(n) Show that the coefficient of x^{101} in the expansion of $(x+1)^{200}$ is equal to the coefficient of x^{99}.

(o) Find a expression for the coefficient of x^{3k} in $(1+x+x^2)^n$.

(p) Find the coefficient of x^{10} in the expansion of $(1+x)^{20}(1-x)^{10}$.

(q) Find the coefficient of x^{15} in the expansion of $(x^2-1)^8(x+1)^5$.

(r) Prove that the sum of all coefficients in the expansion of $(x-2y)^n$ is equal to $(-y)^n$.

(s) Find the coefficient of x^8 in the expansion of $(x^3+x^{-1})^{10}$.

(t) Find the coefficient of x^{20} in the expansion of $(x+x^{-1}+1)^{30}$.

(u) Determine the number of distinct powers of x appearing in the expansion of $(x^2+\frac{1}{x})^n$.

(v) Find the coefficient of $x^{12}y^9$ in the expansion of $(2x^2-y)^6(x+y)^3$.

(w) Show that the coefficient of x^{2k+1} in the expansion of $(x-1)^n+(x+1)^n$ is equal to zero.

(x) Find the coefficient of x^0 in the expansion of $(x+\frac{1}{x})^{2n}$.

(y) Find a formula for the coefficient of x^k in the expansion of $(x^m+\frac{1}{x})^n$, where m, n are positive integers.

(z) Find the smallest power of x appearing in the expansion of $(x^3-\frac{1}{x^2})^{40}$.

Task 2. Count the followings:

(a) Every day a student randomly chooses a sandwich for lunch from a pile of wrapped sandwiches. If there are six kinds of sandwiches, how many different ways are there for the student to choose sandwiches for the seven days of a week if the order in which the sandwiches are chosen matters?

(b) How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

(c) A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose a dozen bagels with at least one of each kind?

(d) A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose a dozen bagels with at least three egg bagels and no more than two salty bagels?

(e) How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

(f) How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

(g) A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouses if the copies of the book are indistinguishable?

(h) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where x_1, x_2, x_3, and x_4 are non-negative integers?

(i) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = \overline{1,5}$, is a non-negative integer such that $x_i \geq 2$ for $i = \overline{1,5}$?

(j) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = \overline{1,5}$, is a non-negative integer such that $0 \leq x_i \leq 10$ for $i = \overline{1,5}$?

(k) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = \overline{1,5}$, is a non-negative integer such that $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?

(l) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where x_i, $i = \overline{1,6}$ is a non-negative integer such that $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 > 5$, and $x_6 \geq 6$?

(m) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where x_i, $i = \overline{1,6}$ is a non-negative integer such that $x_i < 8$, $i = \overline{1,6}$

(n) How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0's, three 1's, and five 2's?

(o) How many strings of 20-decimal digits are there that contain two 0's, four 1's, three 2's, one 3, two 4's, three 5's, two 7's, and three 9's?

(p) Suppose that a large family has 14 children, including two sets of identical triplets, three sets of identical twins, and two individual children. How many ways are there to seat these children in a row of chairs if the identical triplets or twins cannot be distinguished from one another?

(q) How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are non-negative integers?

(r) A Swedish tour guide has devised a clever way for his clients to recognize him. He owns 13 pairs of shoes of the same style, customized so that each pair has a unique color. How many ways are there for him to choose a left shoe and a right shoe from these 13 pairs so that the colors of the left and right shoe are different but which color is on which foot does not matter?

(s) A Swedish tour guide has devised a clever way for his clients to recognize him. He owns 13 pairs of shoes of the same style, customized so that each pair has a unique color. How many ways are there for him to choose a left shoe and a right shoe from these 13 pairs without restrictions, but which color is on which foot does not matter?

(t) How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

(u) How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

(v) How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them, respectively.

(w) How many positive integers less than 1 000 000 have the sum of their digits equal to 19?

(x) How many positive integers less than 1 000 000 have exactly one digit equal to 9 and have a sum of digits equal to 13?

(y) There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

(z) How many different strings can be made from the letters in "`MISSISSIPPI`", using all the letters?