Sets

Assylbek Issakhov,
Ph.D., professor

SETS AND THEIR ELEMENTS

- A *set* may be viewed as any well-defined collection of objects, called the *elements* or *members* of the set.
- One usually uses capital letters, $A, B, X, Y, \ldots$, to denote sets, and lowercase letters, $a, b, x, y, \ldots$, to denote elements of sets.
- Synonyms for "set" are

"class", "collection", and "family".

SETS AND THEIR ELEMENTS

Membership in a set is denoted as follows:

- $a \in S$ denotes that $\boldsymbol{a}$ belongs to a set S
- $a, b \in S$ denotes that $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to a set S

Here $\in$ is the symbol meaning "is an element of."

- We use $\notin$ to mean "is not an element of."

Specifying Sets

- There are essentially two ways to specify a particular set:
- One way, if possible, is to list its members separated by commas and contained in braces { }.
- A second way is to state those properties which characterized the elements in the set.
- Example.

$A = \{1, 3, 5, 7, 9\}$, and

$B = \{x \mid x \textit{ is an even integer}, x > 0\}$

Specifying Sets

- Example.

  $A = \{1, 3, 5, 7, 9\}$, and

  $B = \{x \mid x\ is\ an\ even\ integer, x > 0\}$

That is, A consists of the numbers 1, 3, 5, 7, 9. The second set, which reads: B is the set of x such that x is an even integer and x is greater than 0, denotes the set B whose elements are the positive integers. Note that a letter, usually x, is used to denote a typical member of the set; and the vertical line | is read as “such that” and the comma as “and.”

Subsets

- Suppose every element in a set A is also an element of a set B, that is, suppose $\boldsymbol{a} \in A$ implies $\boldsymbol{a} \in B$. Then A is called a *subset* of B. We also say that A is *contained* in B or that B *contains* A. This relationship is written:

$$A \subseteq B \text{ or } B \supseteq A$$

- Two sets are equal if they both have the same elements or, equivalently, if each is contained in the other. That is:

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

- If A is not a subset of B, that is, if at least one element of A does not belong to B, we write $A \nsubseteq B$.

Subsets

- **Theorem 1.** Let A, B, C be any sets. Then:

(i) $A \subseteq A$

(ii) If $A \subseteq B$ and $B \subseteq A$, then $A = B$

(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Special symbols

- $\mathbb{N}$ – the set of *natural numbers* or positive integers: 1, 2, 3, . . .
- $\mathbb{Z}$ – the set of all integers: . . . , −2, −1, 0, 1, 2, . . .
- $\mathbb{Q}$ – the set of rational numbers
- $\mathbb{R}$ – the set of real numbers
- $\mathbb{C}$ – the set of complex numbers
- Observe that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Universal Set, Empty Set

- All sets under investigation in any application of set theory are assumed to belong to some fixed large set called the *universal set* which we denote by U, unless otherwise stated or implied.

- Given a universal set U and a property P, there may not be any elements of U which have property P. For example, the following set has no elements:

$$S = \{x \mid x \text{ is a positive integer}, x^2 = 3\}$$

Such a set with no elements is called the *empty set* or *null set* and is denoted by $\emptyset$.

Universal Set, Empty Set

- **Theorem 2.** For any set A, we have

$$\emptyset \subseteq A \subseteq U.$$

Disjoint Sets

- Two sets A and B are said to be *disjoint* if they have no elements in common. For example, suppose

$$A = \{1, 2\}, B = \{4, 5, 6\}, \text{ and } C = \{5, 6, 7, 8\}$$

Then A and B are disjoint, and A and C are disjoint. But B and C are not disjoint since B and C have elements in common, e.g., 5 and 6.

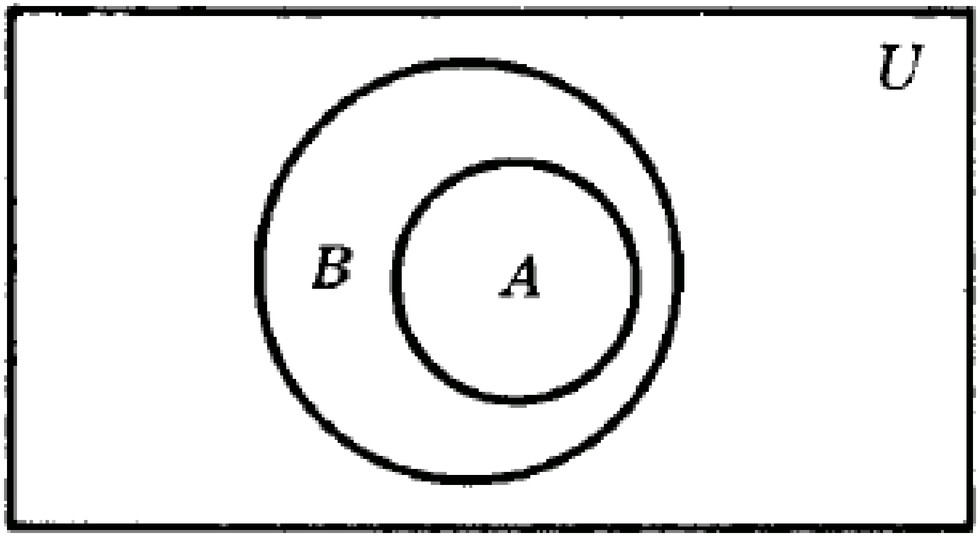
- We note that if A and B are disjoint, then neither is a subset of the other (unless one is the empty set).

VENN DIAGRAMS

- A Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane.
- The universal set U is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle.

VENN DIAGRAMS

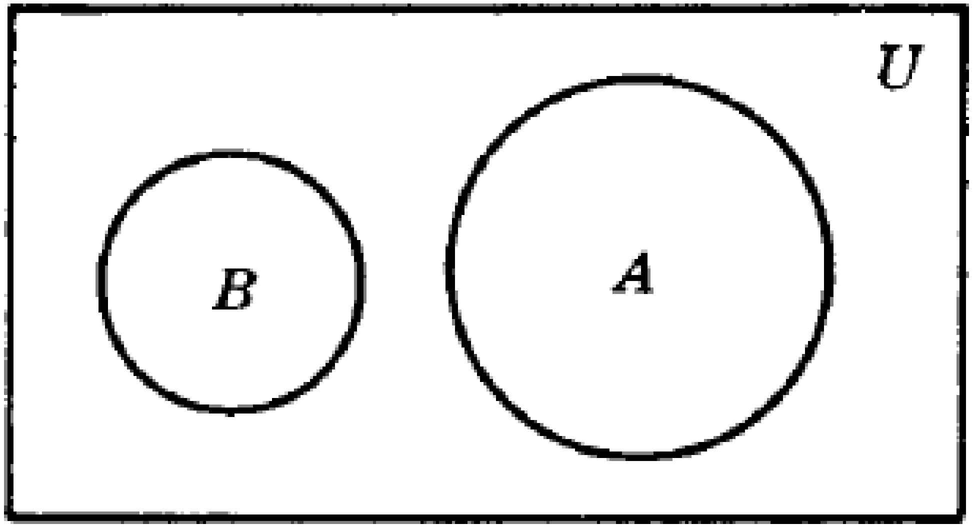
- If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B as in Fig. *(a)*



(a) $A \subseteq B$

VENN DIAGRAMS

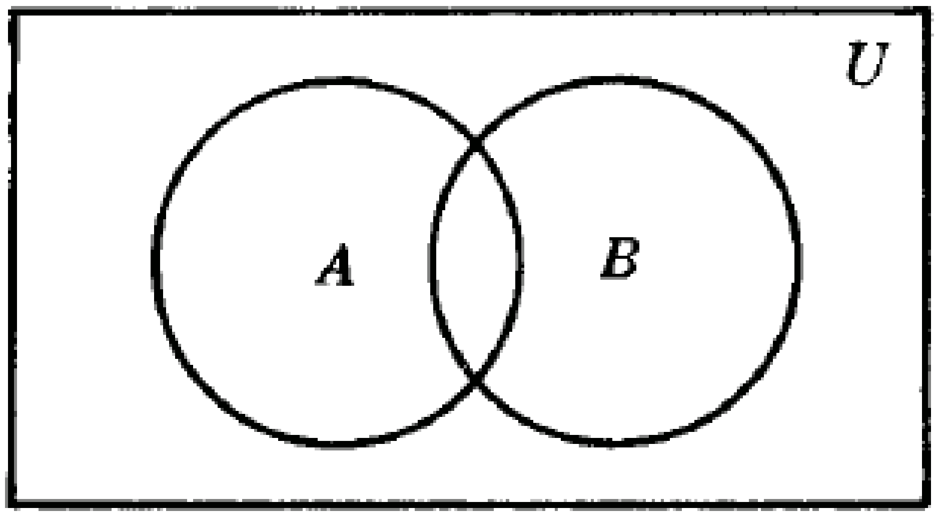
- If A and B are disjoint, then the disk representing A will be separated from the disk representing B as in Fig. *(b)*



(b) A and B are disjoint

VENN DIAGRAMS

- However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Fig. *(c)*.



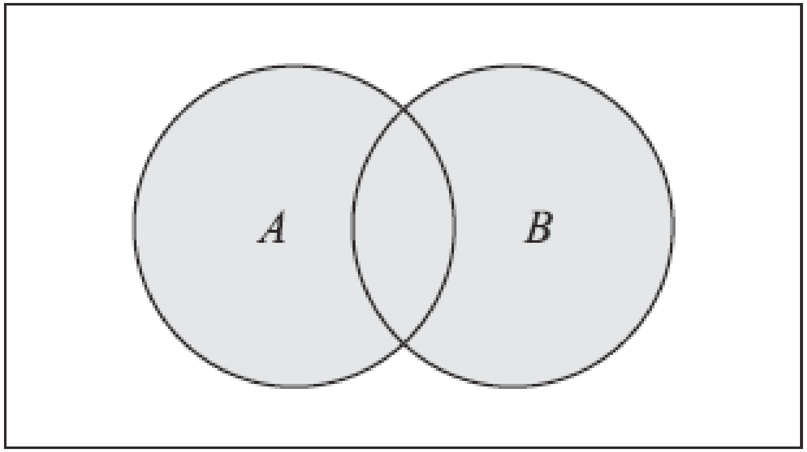
(c)

SET OPERATIONS

- The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is,

$$A \cup B = \{x \mid x \in A \text{ } or \text{ } x \in B\}$$

Here "or" is used in the sense of and/or. Figure *(a)* is a Venn diagram in which $A \cup B$ is shaded.



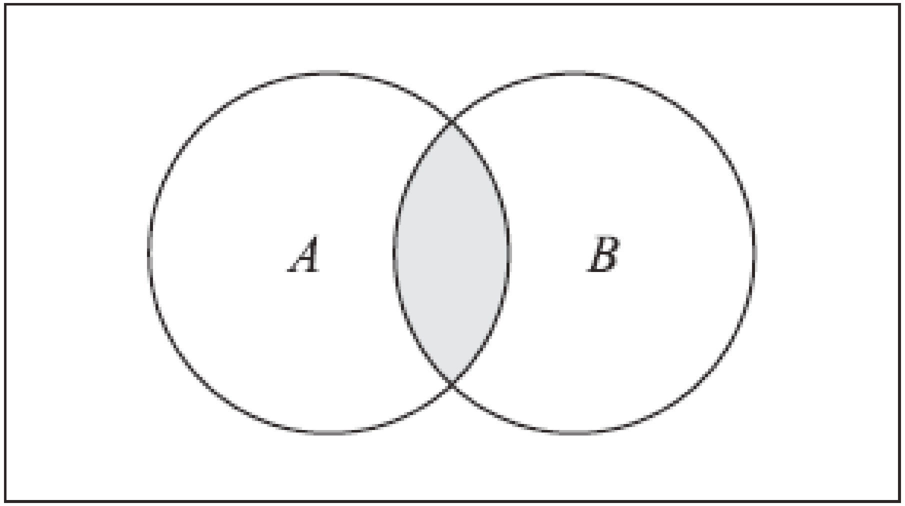
(*a*) $A \cup B$ is shaded

SET OPERATIONS

- The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B:

$$A \cap B = \{x \mid x \in A \; and \; x \in B\}.$$

- Figure *(b)* is a Venn diagram in which $A \cap B$ is shaded.



(*b*) $A \cap B$ is shaded

SET OPERATIONS

- Recall that sets A and B are said to be *disjoint* or *nonintersecting* if they have no elements in common or, using the definition of intersection, if $A \cap B = \emptyset$, the empty set. Suppose

$$S = A \cup B \text{ and } A \cap B = \emptyset$$

Then S is called the *disjoint union* of A and B.

SET OPERATIONS

- **Theorem 3.** For any sets A and B, we have:

(i) $A \cap B \subseteq A \subseteq A \cup B$ and

(ii) $A \cap B \subseteq B \subseteq A \cup B$.

- **Theorem 4.** The following are equivalent:

(i) $A \subseteq B$,

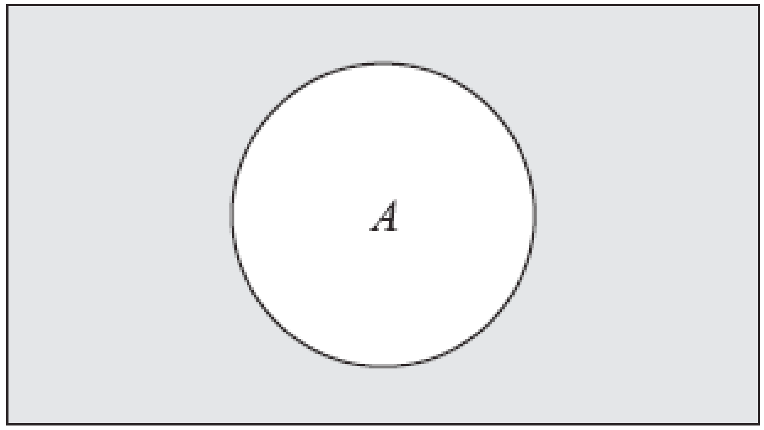
(ii) $A \cap B = A$,

(iii) $A \cup B = B$.

Complements, Differences, Symmetric Differences

- Recall that all sets under consideration at a particular time are subsets of a fixed universal set U. The *absolute complement* or, simply, *complement* of a set A, denoted by A^C (or $\bar{A}$), is the set of elements which belong to U but which do not belong to A. That is,

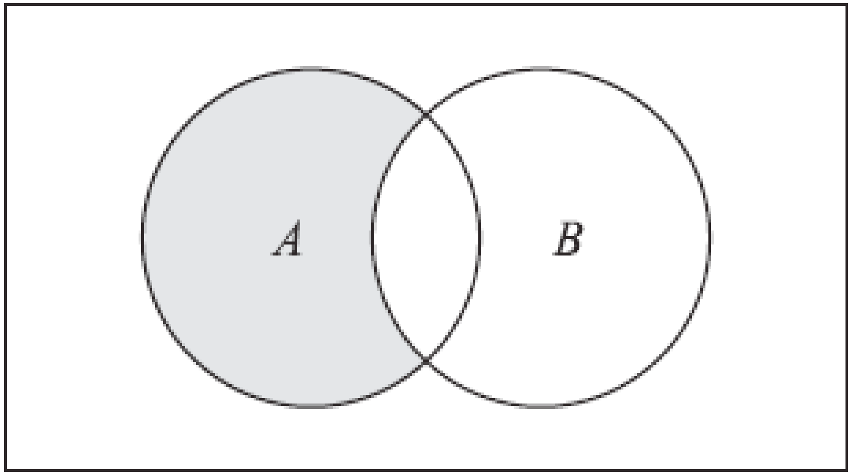
$$A^C = \{x \mid x \in U, x \notin A\}.$$



(a) A^C is shaded

Complements, Differences, Symmetric Differences

- The *relative complement* of a set B with respect to a set A or, simply, the *difference* of A and B, denoted by $A\backslash B$, is the set of elements which belong to A but which do not belong to B; that is $A\backslash B = \{x \mid x \in A, x \notin B\}$.
- The set $A\backslash B$ is read "A minus B." Sometimes $A\backslash B$ is denoted by $A - B$.

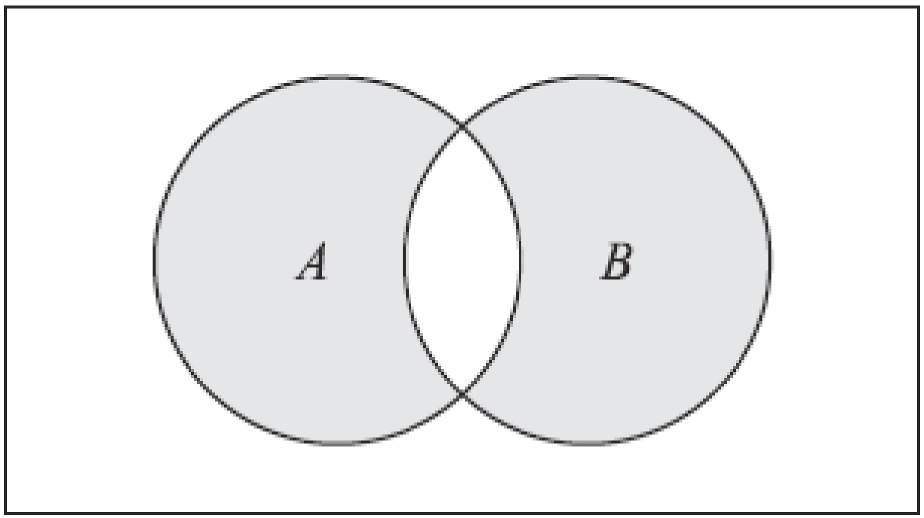


(b) $A\backslash B$ is shaded

Complements, Differences, Symmetric Differences

- The *symmetric difference* of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both. That is,

$$A \oplus B = (A \cup B) \backslash (A \cap B) \; or \; A \oplus B = (A \backslash \mathrm{B}) \cup (B \backslash \mathrm{A}).$$



(c) $A \oplus B$ is shaded

Fundamental Products

- Consider n distinct sets $A_1, A_2, \ldots, A_n$. A *fundamental product* of the sets is a set of the form
  $$A_1^* \cap A_2^* \cap \cdots \cap A_n^*, where\ A_i^* = A_i\ or\ A_i^* = A_i^C$$
- We note that:

(i) There are 2^n such fundamental products,

(ii) Any two such fundamental products are disjoint,

(iii) The universal set U is the union of all fundamental products.

ALGEBRA OF SETS, DUALITY

- Sets under the operations of union, intersection, and complement satisfy various laws (identities) which are listed in the Table:

Idempotent laws:	(1a) $A \cup A = A$	(1b) $A \cap A = A$
Associative laws:	(2a) $(A \cup B) \cup C = A \cup (B \cup C)$	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws:	(3a) $A \cup B = B \cup A$	(3b) $A \cap B = B \cap A$
Distributive laws:	(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws:	(5a) $A \cup \emptyset = A$	(5b) $A \cap \mathbf{U} = A$
	(6a) $A \cup \mathbf{U} = \mathbf{U}$	(6b) $A \cap \emptyset = \emptyset$
Involution laws:	(7) $(A^C)^C = A$	
Complement laws:	(8a) $A \cup A^C = \mathbf{U}$	(8b) $A \cap A^C = \emptyset$
	(9a) $\mathbf{U}^C = \emptyset$	(9b) $\emptyset^C = \mathbf{U}$
DeMorgan's laws:	(10a) $(A \cup B)^C = A^C \cap B^C$	(10b) $(A \cap B)^C = A^C \cup B^C$

FINITE SETS, COUNTING PRINCIPLE

- Sets can be finite or infinite. A set S is said to be *finite* if S is empty or if S contains exactly m elements where m is a positive integer; otherwise S is *infinite.*
- A set S is *countable* if S is finite or if the elements of S can be arranged as a sequence, in which case S is said to be *countably infinite*; otherwise S is said to be *uncountable*. The set E of even integers is countably infinite, whereas one can prove that the unit interval $H = [0, 1]$ of real numbers is uncountable.

Counting Elements in Finite Sets

- The notation $n(S)$ or $|S|$ will denote the number of elements in a set S. (Sometimes we use $card(S)$ instead of $n(S)$.)
- Thus $n(A) = 26$, where A is the set of letters in the English alphabet, and $n(D) = 7$, where D is the set of days of the week.
- Also $n(\emptyset) = 0$ since the empty set has no elements.

Counting Elements in Finite Sets

- **Lemma 6:** Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

- **Corollary 7:** Let A and B be finite sets. Then

$$n(A \backslash \mathrm{B}) = n(A) - n(A \cap B)$$

- **Corollary 8:** Let A be a subset of a finite universal set U. Then

$$n(A^C) = n(U) - n(A)$$

Inclusion-Exclusion Principle

- **Theorem (Inclusion-Exclusion Principle) 9:** Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
- **Corollary 10:** Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and
$$n(A \cup B \cup C) =$$
$$= n(A) + n(B) + n(C) - n(A \cap B) -$$
$$-n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

CLASSES OF SETS

- Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a *set of sets*.
- Whenever such a situation occurs, to avoid confusion, we will speak of a *class* of sets or *collection* of sets rather than a *set* of sets. If we wish to consider some of the sets in a given class of sets, then we speak of *subclass* or *subcollection*.

Power Sets

- For a given set S , we may speak of the class of all subsets of S. This class is called the *power set* of S, and will be denoted by $P(S)$.
- If S is finite, then so is $P(S)$. In fact, the number of elements in $P(S)$ is 2 raised to the power $n(S) = k$. That is,

$$n\big(P(S)\big) = 2^k$$

Partitions

- Let S be a nonempty set. A *partition* of S is a subdivision of S into non-overlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i.

(ii) The sets of $\{A_i\}$ are mutually disjoint; that is,

$$if\ A_j \neq A_k, then\ A_j \cap A_k = \emptyset$$

(iii) $S = \cup_i A_i$

MATHEMATICAL INDUCTION

- **Principle of Mathematical Induction I:**

Let P be a proposition defined on the positive integers $\mathbb{N}$; that is, $P(n)$ is either true or false for each $n \in \mathbb{N}$. Suppose P has the following two properties:

(i) $P(1)$ is true.

(ii) $P(k+1)$ is true whenever $P(k)$ is true.

Then P is true for every positive integer $n \in \mathbb{N}$.

MATHEMATICAL INDUCTION

- **Principle of Mathematical Induction II:**

Let P be a proposition defined on the positive integers $\mathbb{N}$ such that:

(i) $P(1)$ is true.

(ii) $P(k+1)$ is true whenever $P(i)$ is true

for all $1 \leq i \leq k$.

Then P is true for every positive integer $n \in \mathbb{N}$.

HOMEWORK: Problems 1.1 – 1.25 on pp. 13-18