A Posteriori Error Estimation in *hp*-FEM for the Stokes Equations

1 Introduction

We shall consider the steady state Stokes equations which describe slow motion of an incompressible fluid.

2 Preliminaries

In this section, we introduce some notations and state the basic assumptions on which we rely throughout this work. Further, we present our model problem and repeat some important theoretical results which will play an important role later in our work.

2.1 The model problem and basic assumptions

Let $\Omega \in \mathbb{R}^2$ be a polygonal domain with boundary Γ and $u(x)$ and $\varrho(x)$ be the velocity and the pressure of the fluid at some point $x \in \Omega$, respectively. Given $f \in L^2(\Omega)^2$ and $\nu \geq 1$, consider the Stokes equations as our model problem: Find $u : \overline{\Omega} \to \mathbb{R}^2$ and $\varrho : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nu\Delta u + \nabla\varrho &= f \qquad \text{in} \quad \Omega \\ \nabla \cdot u &= 0 \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \Gamma. \end{aligned} \tag{1}$$

In order to avoid some technical difficulties in the following sections, we restrict the model problem to homogeneous boundary conditions. The numerical results show that our methods can be applied equally-well to Stokes problems with non-homogeneous boundary conditions.
We denote the standard Sobolev spaces by $H^m(\Omega)$ for $m \in \mathbb{N}_0$. In particular, the norm and the scalar product of $L^2(\Omega) = H^0(\Omega)$ are denoted by $\|\cdot\|_\Omega$ and $(\cdot,\cdot)_\Omega$, respectively. To account for homogeneous Dirichlet boundary conditions, we set

$$H_0^1(\Omega) := \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma\}.$$

Further, we denote the space containing all functions from $L^2(\Omega)$ with zero mean-value by

$$L_0^2(\Omega) := \{\varphi \in L^2(\Omega) : (\varphi, 1)_\Omega = 0\}$$

and define

$$\mathcal{H} := H_0^1(\Omega)^2 \times L_0^2(\Omega).$$

Then, we introduce the bilinear form $\mathcal{L} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$\mathcal{L}([u, \varrho]; [v, q]) := (\nu \nabla u, \nabla v)_\Omega - (\varrho, \nabla \cdot v)_\Omega - (\nabla \cdot u, q)_\Omega \qquad \forall [u, \varrho], [v, q] \in \mathcal{H}. \tag{2}$$

The standard weak formulation of problem (1) is: Seek $[u, \varrho] \in \mathcal{H}$ such that

$$\mathcal{L}([u, \varrho]; [v, q]) = (f, v)_\Omega \qquad \forall [v, q] \in \mathcal{H}. \tag{3}$$

Due to the continuous inf-sup condition

$$\inf_{[u,\varrho]\in\mathcal{H}} \sup_{[v,q]\in\mathcal{H}} \frac{\mathcal{L}([u, \varrho]; [v, q])}{(\|\nabla u\|_\Omega + \|\varrho\|_\Omega)\,(\|\nabla v\|_\Omega + \|q\|_\Omega)} \geq \kappa > 0,$$

where κ is the inf-sup constant depending only on Ω, the weak problem is well-posed and has a unique solution $[u, \varrho] \in \mathcal{H}$; see [**?**, **?**] for more details.
Now, assume that $\mathcal{T} = \{K\}$ is a triangulation of Ω. Considering the affine transformation $T_K : \hat{K} \to K$, we assume that each element $K \in \mathcal{T}$ is the image of reference element $\hat{K} := [0, 1]^2$. Further, we define the mesh size vector $h := (h_K)_{K\in\mathcal{T}}$, where $h_K := \mathrm{diam}(K)$. With each element $K \in \mathcal{T}$, we associate a polynomial degree $p_K \in \mathbb{N}$ and collect them in a polynomial degree vector $p := (p_K)_{K\in\mathcal{T}}$. Throughout this work, we assume that the discretization $(\mathcal{T}, p)$ of Ω is (γ_h, γ_p)-regular [**?**, **?**, **?**].

Definition 1 ((γ_h, γ_p)-Regularity)**.** *A discretization $(\mathcal{T}, p)$ is called (γ_h, γ_p)-regular if and only if there exist constants $\gamma_h, \gamma_p > 0$ such that for all $K, K' \in \mathcal{T}$ with $K \cap K' \neq \emptyset$ it holds*

$$\gamma_h^{-1} h_K \leq h_{K'} \leq \gamma_h h_K$$

and

$$\gamma_p^{-1} p_K \leq p'_K \leq \gamma_p p_K.$$

For $K \in \mathcal{T}$ arbitrary, let $\mathcal{E}(K)$ denote the set of all interior edges of cell K. Then, $h_e := \mathrm{diam}(e)$ is the diameter of edge $e \in \mathcal{E}(K)$ and its polynomial degree p_e is given by $p_e := \max\{p_K, p_{K'}\}$ for $K, K' \in \mathcal{T}$ with $e = K \cap K'$. Further, we define the finite element spaces $V_u^p(\mathcal{T})$ and $V_\varrho^p(\mathcal{T})$ by

$$V_u^p(\mathcal{T}) := \left\{ u \in H_0^1(\Omega) : \;\; u|_K \circ T_K \in \mathcal{Q}_{p_K}\left(\hat{K}\right) \text{ for all } K \in \mathcal{T} \right\}$$

and

$$V_\varrho^p(\mathcal{T}) := \left\{ \varrho \in L_0^2(\Omega) : \;\; \varrho|_K \circ T_K \in \mathcal{Q}_{p_K-1}\left(\hat{K}\right) \text{ for all } K \in \mathcal{T} \right\},$$

respectively. Here, $\mathcal{Q}_r(\hat{K})$ is the tensor-product polynomial space of degree at most $r \in \mathbb{N}_0$ defined on reference cell $\hat{K}$. To simplify notations a little bit, we set

$$\mathcal{V}^p(\mathcal{T}) := V_u^p(\mathcal{T})^2 \times V_\varrho^p(\mathcal{T}).$$

Then, the discrete approximation to 3 is obtained by finding $[u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}] \in \mathcal{V}^p(\mathcal{T})$ such that

$$\mathcal{L}\left([u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}]\,; [v_{\mathrm{FE}}, q_{\mathrm{FE}}]\right) = (f, v_{\mathrm{FE}})_\Omega \qquad \forall\, [v_{\mathrm{FE}}, q_{\mathrm{FE}}] \in \mathcal{V}^p(\mathcal{T}). \tag{4}$$

For this discretization of Stokes problem (1), it can be shown easily that the following Galerkin orthogonality holds:

Lemma 1 (Galerkin Orthogonality)**.** *Let $[u, \varrho] \in \mathcal{H}$ be the solution of* 3 *and $[u_{FE}, \varrho_{FE}] \in \mathcal{V}^p(\mathcal{T})$ be the solution of* 4*. Then, it holds*

$$\mathcal{L}\left([u - u_{FE}, \varrho - \varrho_{FE}]\,; [v_{FE}, q_{FE}]\right) = 0 \qquad \forall\, [v_{FE}, q_{FE}] \in \mathcal{V}^p(\mathcal{T}).$$

2.2 Auxiliary Results

Now, we present some auxiliary results which we use later in our work. This includes an H^1-conforming interpolation operator, which preserves homogeneous Dirichlet boundary conditions, and some polynomial smoothing estimates.

First, let us consider the H^1-conforming interpolation operator. It is a Clément-type interpolation which replaces the point evaluation of the interpolated function by some local average [?]. This procedure does not require the extra regularity of a point evaluation, but is also well-defined for functions from the space $H^1(\Omega)$. In [?], this interpolation operator was modified in such a way that it also preserves polynomial boundary conditions. In [?], Melenk extended this H^1-conforming interpolation to *hp*-adaptive finite element spaces.

In order to be able to state the interpolation error estimates for this Clément-type interpolation error, we have to introduce some additional notation. For some arbitrary cell $K \in \mathcal{T}$, let the patch ω_K be given by

$$\omega_K := \bigcup \{L \in \mathcal{T} : K \cap L \neq \emptyset\}. \tag{5}$$

In [?], the following result, which gives us an estimate for the interpolation error in terms of the gradient of the interpolated function, was derived.

Theorem 1 (H^1-Conforming Interpolation)**.** *Let $\mathcal{T}$ be (γ_h, γ_p)-regular and $K \in \mathcal{T}$ be arbitrary. Then, there exists a bounded linear operator $\Pi^{hp} : H_0^1(\Omega)^d \to V_u^p(\mathcal{T}_h)^d$, $d \in \{1, 2\}$, and some constant $C > 0$ independent of mesh size h and polynomial degree p such that*

$$\left\|u - \Pi^{hp} u\right\|_K \leq C \frac{h_K}{p_K} \|\nabla u\|_{\omega_K}$$

and

$$\left\|u - \Pi^{hp} u\right\|_e \leq C \sqrt{\frac{h_e}{p_e}} \|\nabla u\|_{\omega_K}$$

for all $u \in H_0^1(\Omega)$ and all $e \in \mathcal{E}(K)$.

Proof. See Theorem 3.3 in [?]. □

Now, we want to present some polynomial smoothing estimates which allow us to insert some smoothing function ϕ into the L^2-norm of a polynomial π_p. These smoothing estimates are widely used in the error analysis of many numerical methods for partial differential equations and integral equations [?, ?, ?, ?, ?, ?, ?, ?].

On reference cell $\hat{K} := [0, 1]^2$ and reference edge $\hat{e} := (0, 1) \times \{0\}$, we define the smoothing functions $\Phi_{\hat{K}} : \hat{K} \to \mathbb{R}_+$ and $\Phi_{\hat{e}} : \hat{e} \to \mathbb{R}_+$ by

$$\Phi_{\hat{K}}(x) := \operatorname{dist}\left(x, \partial \hat{K}\right) \tag{6}$$

and

$$\Phi_{\hat{e}}(x) := x(1 - x), \tag{7}$$

respectively. The corresponding smoothing estimates for the reference cell $\hat{K}$ have been derived in [?] for the one-dimensional case and in [?] for the two-dimensional case.

Lemma 2. *Let $a, b \in \mathbb{R}$ such that $-1 \leq a \leq b$ and consider $\Phi_{\hat{K}}$ as the smoothing function given in (6). Then, for any $\pi_p \in \mathcal{Q}_p\left(\hat{K}\right)$, there exists some constant $C > 0$ independent of $p \in \mathbb{N}$ such that*

$$\int_{\hat{K}} \pi_p^2 \left(\Phi_{\hat{K}}\right)^a \leq C p^{2(b-a)} \int_{\hat{K}} \pi_p^2 \left(\Phi_{\hat{K}}\right)^b$$

and

$$\int_{\hat{K}} |\nabla \pi_p|^2 \left(\Phi_{\hat{K}}\right)^{2a} \leq C p^{2(2-a)} \int_{\hat{K}} \pi_p^2 \left(\Phi_{\hat{K}}\right)^a .$$

Proof. See Theorem 2.5 in [**?**]. ☐

The next lemma gives some results for the extension of a polynomial from an edge to a domain.

Lemma 3. *Let $\Phi_{\hat{e}}$ be the smoothing function from (7) and $\alpha \in \left(\frac{1}{2}, 1\right)$. Then, for any $\pi_{p_{\hat{e}}} \in \mathcal{Q}_p\left(\hat{e}\right)$ and every $\delta \in (0,1]$, there exists some extension $v_{\hat{e}} \in H^1\left(\hat{K}\right)$ and some constant $C > 0$ independent of $p \in \mathbb{N}$ such that:*

1. $v_{\hat{e}}|_{\hat{e}} = \pi_p \Phi_{\hat{e}}^{\alpha}$ *and* $v_{\hat{e}}|_{\partial\hat{K}\setminus\hat{e}} = 0$*;*
2. $\|v_{\hat{e}}\|_{\hat{K}}^2 \leq C\delta \left\|\pi_p \Phi_{\hat{e}}^{\alpha/2}\right\|_{\hat{e}}^2$*;*
3. $\|\nabla v_{\hat{e}}\|_{\hat{K}}^2 \leq C\left(\delta p^{2(2-\alpha)} + \delta^{-1}\right) \left\|\pi_p \phi_{\hat{e}}^{\alpha/2}\right\|_{\hat{e}}^2$.

Proof. See Lemma 2.6 in [**?**]. ☐

Lemma 4. *(Polynomial Inverse Estimate). Let $K \in \mathcal{T}$ and $(u, \varrho) \in \mathcal{Q}_{p_K, p_K-1}$ denote some polynomials. Then there exists a constant C_{inv} independent of polynomial degree p_K and mesh diameter h_K such that*

$$\|\partial u\|_K \leq C_{inv} \frac{p_K^2}{h_K} \|u\|_K$$

Proof. The proof follows exactly of [ref 194 thesis/ ref 22, 1D, Dorfler-Thesis?] ☐

Lemma 5. *(Polynomial Trace Estimate). Let $K \in \mathcal{T}$ and $(u, \varrho) \in \mathcal{Q}_{p_K, p_K-1}$ denote some polynomials. hen there exists a constant C_{tr} independent of polynomial degree p_K and mesh diameter h_K such that*

$$\|u\|_{\partial K} \leq C_{tr} \frac{p_K}{\sqrt{h_K}} \|u\|_K$$

Proof. Lemma 3.9 [thesis], thm 4.76 [ref 194 thesis]

☐

3 A Posteriori Error Estimation

The idea behind a posteriori error estimation is to assess the error between the exact solution $[u, \varrho] \in \mathcal{H}$ and its finite element approximation $[u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}] \in \mathcal{V}^p(\mathcal{T})$ in terms of known quantities, such as problem data and the approximate solution, only [**?**]. Therefore, a posteriori error estimates differ from a priori estimates in such a way that the upper and lower bounds of the error do not depend on the unknown quantities $[u, \varrho] \in \mathcal{H}$.

Definition 2 (A Posteriori Error Estimator)**.** *A functional $\eta\left(u_{FE}, \varrho_{FE}, f\right)$ is called an a posteriori error estimator, if and only if there exists some constant $C > 0$ such that*

$$\|\nabla\left(u - u_{FE}\right)\|_{\Omega} + \|\varrho - \varrho_{FE}\|_{\Omega} \leq C\eta\left(u_{FE}, \varrho_{FE}, f\right). \tag{8}$$

Furthermore, if $\eta\left(u_{FE}, \varrho_{FE}, f\right)$ can be decomposed into localized quantities $\eta_K\left(u_{FE}, \varrho_{FE}, f\right)$, $K \in \mathcal{T}$, such that

$$\eta^2(u_{FE}, \varrho_{FE}, f) = \sum_{K\in\mathcal{T}} \eta_K^2\left(u_{FE}, \varrho_{FE}, f\right), \tag{9}$$

then $\eta_K\left(u_{FE}, \varrho_{FE}, f\right)$ is called local error indicator.

Note, estimate (8) is usually called reliability estimate, since it guarantees that the error of the finite element approximation $[u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}]$ in the natural stability norm is controlled by error estimator $\eta\left(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f\right)$. Further, the local error indicators $\eta_K\left(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f\right)$ given in identity (9) provide the most important tool for adaptive mesh refinement by identifying those cells $K \in \mathcal{T}$ where the error is large and, subsequently, the mesh has to be refined locally. This procedure can be repeated several times until the error estimator $\eta\left(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f\right)$ is smaller than a prescribed tolerance.
Obviously, computational efficiency requires that the local error estimators also satisfy some efficiency property guaranteeing that the lower bound (8) is sharp enough. To this end, we would like to derive a local upper bound

$$\eta_K\left(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f\right) \leq C\left(\left\|\nabla\left(u-u_{\mathrm{FE}}\right)\right\|_K + \left\|\varrho-\varrho_{\mathrm{FE}}\right\|_K\right) \qquad \forall K \in \mathcal{T} \tag{10}$$

which, however, is still an open task for the much simpler Poisson problem (see, e.g., [**?**, **?**]) and, thus, out of the scope of this paper.

3.1 Residual-based a posteriori error estimator and error analysis

In this section, we define the residual-based a posteriori error estimator for Stokes problem (1) and derive upper and lower bounds for this error estimator in terms of the energy error of the approximated solution. As given in identity (9), the a posteriori error estimator η_α shall be the sum of local error indicators $\eta_{\alpha,K}$:

$$\eta_\alpha^2 := \sum_{K\in\mathcal{T}} \eta_{\alpha;K}^2$$

for $\alpha \in [0,1]$. The local error indicator $\eta_{\alpha;K}$ can be decomposed into a cell contribution and an interface contribution:

$$\eta_{\alpha;K}^2 := \eta_{\alpha;K;R}^2 + \eta_{\alpha;K;B}^2, \tag{11}$$

where $\eta_{\alpha;K;R}$ denotes the residual-based term and $\eta_{\alpha;K;B}$ indicates the jump-based term. These terms are defined by

$$\eta_{\alpha;K;R}^2 := \frac{h_K^2}{p_K^2}\left\|\left(I_{p_K}^K f + \nu\Delta u_{\mathrm{FE}} - \nabla\varrho_{\mathrm{FE}}\right)\Phi_K^{\frac{\alpha}{2}}\right\|_K^2 + \left\|\left(\nabla\cdot u_{\mathrm{FE}}\right)\Phi_K^{\frac{\alpha}{2}}\right\|_K^2 \tag{12}$$

and

$$\eta_{\alpha;K;B}^2 := \sum_{e\in\mathcal{E}(K)} \frac{h_e}{2p_e}\left\|\left[\nu\frac{\partial u_{\mathrm{FE}}}{\partial n_K}\right]\Phi_e^{\frac{\alpha}{2}}\right\|_e^2. \tag{13}$$

Here, $I_{p_K}^K f$ denotes the local L^2-projection of f into the space of vector-valued polynomials of degree less or equal than p_K and $[\cdot]$ is the jump across the edge. Further, Φ_K and Φ_e are the transformed smoothing weight functions from (6) and (7) under the inverse affine transformation $T_K^{-1}: K \to \hat{K}$ and $T_e^{-1}: e \to \hat{e}$, respectively.
Now, let us begin with the error analysis of the a posteriori error estimator η_α. First, we derive a lower bound for the error estimator η_α in terms of the energy error $\left\|\nabla\left(u-u_{\mathrm{FE}}\right)\right\|_\Omega + \left\|\varrho-\varrho_{\mathrm{FE}}\right\|_\Omega$. This type of estimate is often called reliability estimate.

Theorem 2 (Reliability)**.** *Let $[u_{FE}, \varrho_{FE}] \in \mathcal{V}^p(\mathcal{T})$ be the solution of discrete problem (4) and $[u,\varrho] \in \mathcal{H}$ be solution of weak problem (3). Further, let $\alpha \in [0,1]$ and assume that triangulation $\mathcal{T}$ is (γ_h,γ_p)-regular. Then, there exists some constant $C_{rel} > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\left\|\nabla\left(u-u_{FE}\right)\right\|_\Omega^2 + \left\|\varrho-\varrho_{FE}\right\|_\Omega^2 \leq C_{rel}\sum_{K\in\mathcal{T}}\left(p_K^{2\alpha}\eta_{\alpha;K}^2 + \frac{h_K^2}{p_K^2}\left\|I_{p_K}^K f - f\right\|_K^2\right).$$

Proof. Set $e_{\mathrm{FE}} := u - u_{\mathrm{FE}}$ and $\epsilon_{\mathrm{FE}} := \varrho - \varrho_{\mathrm{FE}}$. From Lemma 1, we have

$$\begin{aligned}\mathcal{L}\left([e_{\mathrm{FE}}, \epsilon_{\mathrm{FE}}]; [v,q]\right) &= \left(\nu \nabla e_{\mathrm{FE}}, \nabla\left(v - \Pi^{hp} v\right)\right)_{\Omega} - \left(\epsilon_{\mathrm{FE}}, \nabla \cdot \left(v - \Pi^{hp} v\right)\right)_{\Omega} - \left(\nabla \cdot e_{\mathrm{FE}}, q\right)_{\Omega} \\ &= \sum_{K \in \mathcal{T}} \left(\left(\nu \nabla e_{\mathrm{FE}}, \nabla\left(v - \Pi^{hp} v\right)\right)_{K} - \left(\epsilon_{\mathrm{FE}}, \nabla \cdot \left(v - \Pi^{hp} v\right)\right)_{K} - \left(\nabla \cdot e_{\mathrm{FE}}, q\right)_{K} \right),\end{aligned}$$

where $\Pi^{hp} : H_0^1(\Omega)^2 \to V_u^p(\mathcal{T})^2$ denotes the H^1-conforming interpolation operator from Theorem 1. Using integration by parts and the incompressibility condition $\nabla \cdot u = 0$ yields

$$\begin{aligned}\mathcal{L}\left([e_{\mathrm{FE}}, \epsilon_{\mathrm{FE}}]; [v,q]\right) = \sum_{K \in \mathcal{T}} \Bigg(&\left(f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}}, v - \Pi^{hp} v\right)_K - \left(\nabla \cdot u_{\mathrm{FE}}, q\right)_K \\ &+ \sum_{e \in \mathcal{E}(K)} \left(\left[\nu \frac{\partial u_{\mathrm{FE}}}{\partial n}\right], v - \Pi^{hp} v \right)_e \Bigg)\end{aligned}$$

and, by applying the continuous Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathcal{L}\left([e_{\mathrm{FE}}, \epsilon_{\mathrm{FE}}]; [v,q]\right) \le \sum_{K \in \mathcal{T}} \Bigg(&\left\| I_{p_K}^K f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}} \right\|_K \left\| v - \Pi^{hp} v \right\|_K + \left\| \nabla \cdot u_{\mathrm{FE}} \right\|_K \|q\|_K \\ &+ \left\| f - I_{p_K}^K f \right\|_K \left\| v - \Pi^{hp} v \right\|_K + \sum_{e \in \mathcal{E}(K)} \left\| \left[\nu \frac{\partial u_{\mathrm{FE}}}{\partial n_K}\right] \right\|_e \left\| v - \Pi^{hp} v \right\|_e \Bigg).\end{aligned}$$

With Theorem 1, we obtain

$$\begin{aligned}\mathcal{L}\left([e_{\mathrm{FE}}, \epsilon_{\mathrm{FE}}]; [v,q]\right) \le C \sum_{K \in \mathcal{T}} \Bigg(&\frac{h_K}{p_K} \left\| I_{p_K}^K f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}} \right\|_K + \left\| \nabla \cdot u_{\mathrm{FE}} \right\|_K + \frac{h_K}{p_K} \left\| f - I_{p_K}^K f \right\|_K \\ &+ \sum_{e \in \mathcal{E}(K)} \sqrt{\frac{h_e}{p_e}} \left\| \left[\nu \frac{\partial u_{\mathrm{FE}}}{\partial n_K}\right] \right\|_e \Bigg) \left(\|\nabla v\|_{\omega_K} + \|q\|_K \right)\end{aligned}$$

and, with the discrete Cauchy-Schwarz inequality, this implies

$$\begin{aligned}\mathcal{L}\left([e_{\mathrm{FE}}, \epsilon_{\mathrm{FE}}]; [v,q]\right) &\le C \left(\sum_{K \in \mathcal{T}} \left(\eta_{0;K}^2 + \frac{h_K^2}{p_K^2} \left\| f - I_{p_K}^K f \right\|_K^2 \right) \right)^{\frac{1}{2}} \left(\|\nabla v\|_{\Omega}^2 + \|q\|_{\Omega}^2 \right)^{\frac{1}{2}} \\ &\le C \left(\sum_{K \in \mathcal{T}} \left(\eta_{0;K}^2 + \frac{h_K^2}{p_K^2} \left\| f - I_{p_K}^K f \right\|_K^2 \right) \right)^{\frac{1}{2}} \left(\|\nabla v\|_{\Omega} + \|q\|_{\Omega} \right)\end{aligned}$$

for some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p. By applying the formal definition of the energy norm

$$\|\nabla e_{\mathrm{FE}}\|_{\Omega} + \|\epsilon_{\mathrm{FE}}\|_{\Omega} = \sup_{[v,q] \in \mathcal{H}} \frac{\mathcal{L}\left([e_{\mathrm{FE}}, \epsilon_{\mathrm{FE}}]; [v,q]\right)}{\|\nabla v\|_{\Omega} + \|q\|_{\Omega}},$$

the result follows for $\alpha = 0$. With the smooting estimates given in Lemma 2, we can bound $\eta_{0;K}$ in terms of $\eta_{\alpha;K}$ for $\alpha \in (0,1]$ from above. Therefore, set $a := 0$ and $b := \alpha$ in Lemma 2 and we get

$$\|\nabla e_{\mathrm{FE}}\|_{\Omega} + \|\epsilon_{\mathrm{FE}}\|_{\Omega} \le C_{rel} \left(\sum_{K \in \mathcal{T}} \left(p_K^{2\alpha} \eta_{\alpha;K}^2 + \frac{h_K^2}{p_K^2} \left\| f - I_{p_K}^K f \right\|_K^2 \right) \right)^{\frac{1}{2}}$$

which concludes the proof. $\square$

Next, we derive an upper bound for the a posteriori error estimator $\eta_{\alpha;K}$ in terms of the energy error $\|\nabla(u-u_{\mathrm{FE}})\|_{\omega_K} + \|\varrho-\varrho_{\mathrm{FE}}\|_{\omega_K}$. Therefore, we consider the residual-based term $\eta_{\alpha;K;R}$ and the jump-based term $\eta_{\alpha;K;B}$ separately and combine the derived efficiency estimates later to obtain an upper bound for the residual-based a posteriori error estimator from definition (11).

Lemma 6. *Let $[u,\varrho]\in\mathcal{H}$ be the solution of weak problem (3) and $[u_{FE},\varrho_{FE}]\in\mathcal{V}^p(\mathcal{T})$ be the solution of discrete problem (4). Further, we assume that triangulation $\mathcal{T}$ is (γ_h,γ_p)-regular and let $\alpha\in[0,1]$ be arbitrary. Then, there exists some constant $C>0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\eta^2_{\alpha;K;R} \le C\left(p_K^{2(1-\alpha)}\left(\nu^2\|\nabla(u-u_{FE})\|_K^2 + \|\varrho-\varrho_{FE}\|_K^2\right) + \frac{h_K^{2+\frac{\alpha}{2}}}{p_K^{1+\alpha}}\left\|f-I^K_{p_K}f\right\|_K^2\right).$$

Proof. For simplicity, we can write the residual-based term $\eta_{\alpha;K;R}$ as

$$\eta^2_{\alpha;K;R} = \eta^2_{\alpha;K;R_1} + \eta^2_{\alpha;K;R_2},$$

where $\eta_{\alpha;K;R;1}$ and $\eta_{\alpha;K;R;2}$ are defined as follows:

$$\begin{aligned}\eta^2_{\alpha;K;R_1} &:= \frac{h_K^2}{p_K^2}\left\|\left(I^K_{p_K}f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}}\right)\Phi_K^{\frac{\alpha}{2}}\right\|_K^2,\\ \eta^2_{\alpha;K;R_2} &:= \left\|\nabla\cdot u_{\mathrm{FE}}\Phi_K^{\frac{\alpha}{2}}\right\|_K^2.\end{aligned}\tag{14}$$

Using the idea in [?], for $0<\alpha\le1$, we define $w_K := \left(I^K_{p_K}f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}}\right)\Phi_K^\alpha \in H_0^1(K)$ and obtain

$$\left\|w_K\Phi_K^{-\frac{\alpha}{2}}\right\|_K^2 = \left(f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}},w_K\right)_K + \left(I^K_{p_K}f-f,w_K\right)_K.\tag{15}$$

With equation (3) and applying integration by parts, the first term reads

$$\left(f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}},w_K\right)_K = \left(\nu\nabla(u-u_{\mathrm{FE}}),\nabla w_K\right)_K - \left(\varrho-\varrho_{\mathrm{FE}},\nabla\cdot w_K\right)_K - \left(\nabla\cdot u,q\right)_K$$

and inserting into (15) and using the incompressibility condition $\nabla\cdot u=0$ implies

$$\left\|w_K\Phi_K^{-\frac{\alpha}{2}}\right\|_K^2 = \left(\nu\nabla(u-u_{\mathrm{FE}}),\nabla w_K\right)_K - \left(\varrho-\varrho_{\mathrm{FE}},\nabla\cdot w_K\right)_K + \left(I^K_{p_K}f-f,w_K\right)_K.$$

Then, by using the Cauchy-Schwarz inequality, we get

$$\left\|w_K\Phi_K^{-\frac{\alpha}{2}}\right\|_K^2 \le \left(\nu\|\nabla(u-u_{\mathrm{FE}})\|_K + \|\varrho-\varrho_{\mathrm{FE}}\|_K\right)\|\nabla w_K\|_K + \left\|\left(I^K_{p_K}f-f\right)\Phi_K^{\frac{\alpha}{2}}\right\|_K\left\|w_K\Phi_K^{-\frac{\alpha}{2}}\right\|_K.\tag{16}$$

Now, let us derive an upper bound for the H^1-semi norm of w_K, first. We see easily

$$\begin{aligned}\|\nabla w_K\|_K &= \left\|\nabla\left(\left(I^K_{p_K}f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}}\right)\Phi_K^\alpha\right)\right\|_K\\ &\le \left\|\nabla\left(I^K_{p_K}f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}}\right)\Phi_K^\alpha\right\|_K + \alpha\left\|\left(I^K_{p_K}f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}}\right)\Phi_K^{\alpha-1}\nabla\Phi_K\right\|_K\end{aligned}$$

and, from the fact that $\nabla\phi_K\le C$ for some constant $C>0$ independent of h_K, we obtain

$$\begin{aligned}\|\nabla w_K\|_K &\le \left\|\nabla\left(I^K_{p_K}f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}}\right)\Phi_K^\alpha\right\|_K + C\alpha\left\|\left(I^K_{p_K}f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}}\right)\Phi_K^{\alpha-1}\right\|_K\\ &\le C\left(\frac{p_K^{2-\alpha}}{h_K}\left\|w_K\Phi_K^{-\frac{\alpha}{2}}\right\|_K + \alpha\left\|\left(I^K_{p_K}f+\nu\Delta u_{\mathrm{FE}}-\nabla\varrho_{\mathrm{FE}}\right)\Phi_K^{\alpha-1}\right\|_K\right)\end{aligned}$$

with Lemma 2, where $C > 0$ denotes some constant independent of mesh size vector h and polynomial degree vector p. For the second term, we have to distinguish between two cases. Assuming $\alpha > \frac{1}{2}$, we set $a := 2(\alpha - 1)$ and $b := \alpha$ in Lemma 2 to get

$$\left\| \left(I^K_{p_K} f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}} \right) \Phi_K^{\alpha - 1} \right\|_K \leq C p_K^{1 - \frac{\alpha}{2}} \left\| w_K \Phi_K^{-\frac{\alpha}{2}} \right\|_K$$

and inserting into the estimate above yields

$$\left\| \nabla w_K \right\|_K \leq C \frac{p_K^{2-\alpha}}{h_K} \left\| w_K \Phi_K^{-\frac{\alpha}{2}} \right\|_K \tag{17}$$

for some constant $C > 0$ independent of h_K and p_K. Since

$$\phi_K \leq h_K, \tag{18}$$

inequality (16) reads

$$\left\| w_K \Phi_K^{-\frac{\alpha}{2}} \right\|_K \leq C \frac{p_K^{2-\alpha}}{h_K} \left(\nu \left\| \nabla \left(u - u_{\mathrm{FE}} \right) \right\|_K + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_K \right) + h_K^{\frac{\alpha}{2}} \left\| I^K_{p_K} f - f \right\|_K$$

and, after multiplying both sides by $\frac{h_K}{p_K}$ and using definition (14), we have

$$\eta_{\alpha;K;R_1} \leq C p_K^{1-\alpha} \left(\nu \left\| \nabla \left(u - u_{\mathrm{FE}} \right) \right\|_K + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_K \right) + \frac{h_K^{1+\frac{\alpha}{2}}}{p_K} \left\| I^K_{p_K} f - f \right\|_K . \tag{19}$$

Now, let us consider the case $0 \leq \alpha \leq \frac{1}{2}$. Therefore, let $\beta := \frac{1+\alpha}{2}$. Again, using the smoothing estimates given in Lemma 2 and considering the fact that $\beta > \alpha$, we find

$$\begin{aligned} \left\| w_K \Phi_K^{-\frac{\alpha}{2}} \right\|_K &\leq C p_K^{\beta - \alpha} \left\| \left(I^K_p f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}} \right) \Phi_K^{\frac{\beta}{2}} \right\|_K \\ &= C \frac{p_K^{1+\beta-\alpha}}{h_K} \eta_{\beta;K;R_1} \end{aligned}$$

and estimate (19) implies

$$\left\| w_K \Phi_K^{-\frac{\alpha}{2}} \right\|_K \leq C \left(\frac{p_K^{2-\alpha}}{h_K} \left(\nu \left\| \nabla \left(u - u_{\mathrm{FE}} \right) \right\|_K + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_K \right) + \frac{h_K^{\frac{\beta}{2}}}{p_K^{\alpha - \beta}} \left\| I^K_{p_K} f - f \right\|_K \right)$$

Then, the definition of β yields

$$\eta_{\alpha;K;R_1} \leq C \left(p_K^{1-\alpha} \left(\nu \left\| \nabla \left(u - u_{\mathrm{FE}} \right) \right\|_K + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_K \right) + \frac{h_K^{\frac{5+\alpha}{4}}}{p_K^{\frac{1+\alpha}{2}}} \left\| I^K_{p_K} f - f \right\|_K \right). \tag{20}$$

To obtain the upper bound for $\eta^2_{\alpha;K;R_2}$, we observe

$$\begin{aligned} \eta_{\alpha;K;R_2} &= \left\| \nabla \cdot u_{\mathrm{FE}} \Phi_K^{\frac{\alpha}{2}} \right\|_K \\ &\leq h_K^{\frac{\alpha}{2}} \left\| \nabla \cdot u_{\mathrm{FE}} \right\|_K \end{aligned}$$

by (18). Since $\nabla \cdot u = 0$, we have $\nabla \cdot u_{\mathrm{FE}} = \nabla \cdot (u - u_{\mathrm{FE}})$ and, hence,

$$\eta_{\alpha;K;R_2} \leq h_K^{\frac{\alpha}{2}} \left\| \nabla \left(u - u_{\mathrm{FE}} \right) \right\|_K . \tag{21}$$

Finally, combining estimates (19)-(21) gives the desired result. $\square$

Now, let us consider the jump-based term $\eta_{\alpha;K;B}$ from equation (13). In order to derive an upper bound for this term, we use the same ideas as in Lemma 6.

Lemma 7. *Let $[u,\varrho] \in \mathcal{H}$ be the solution of weak problem (3) and $[u_{FE},\varrho_{FE}] \in \mathcal{V}^p(\mathcal{T})$ be the solution of discrete problem (4). Further, we assume that triangulation $\mathcal{T}$ is (γ_h,γ_p)-regular. Then, there exists some constant $C>0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\eta_{\alpha;K;B}^2 \leq C\left(p_K^{\frac{3-\alpha}{2}}\left(\nu^2\left\|\nabla\left(u-u_{FE}\right)\right\|_{\omega_K}^2+\left\|\varrho-\varrho_{FE}\right\|_{\omega_K}^2\right)+\frac{h_K^2}{p_K^{\frac{3+\alpha}{2}}}\left\|I_{p_K}^K f-f\right\|_{\omega_K}^2\right)$$

for all $\alpha \in [0,1]$.

Proof. For given element $K \in \mathcal{T}$ and $e \in \mathcal{E}(K)$, there exists some $K_1 \in \mathcal{T}$ such that $e = \partial K \cap \partial K_1$ and $K_e := K \cup K_1$. Moreover, by Lemma 2, there exists some extension function $v_e \in H_0^1(K_e)$ such that $v_e|_e = \left[\nu \frac{\partial u_{\mathrm{FE}}}{\partial n}\right]\Phi_e^{\alpha}$. By construction, v_e is continuous on K and vanishes on ∂K_e. We can extend v_e by zero to $\Omega\backslash K_e$ which gives us $v_e \in H_0^1(\Omega)$. Now, to derive an upper bound for the jump-based term $\eta_{\alpha;K;B}^2$, we use the integration by parts formula to get

$$\left\|v_e\Phi_e^{-\frac{\alpha}{2}}\right\|_e^2 = (\nu\Delta u_{\mathrm{FE}}, v_e)_{K_e} + (\nu\nabla u_{\mathrm{FE}}, \nabla v_e)_{K_e}$$

and, from weak formulation (3), we have

$$\begin{aligned}\left\|v_e\Phi_e^{-\frac{\alpha}{2}}\right\|_e^2 &= (\nu\Delta u_{\mathrm{FE}}, v_e)_{K_e} - \left(\nu\nabla\left(u-u_{\mathrm{FE}}\right),\nabla v_e\right)_{K_e} + (f,v_e)_{K_e} + (\varrho,\nabla\cdot v_e)_{K_e} + (\nabla\cdot u, v_e)_{K_e}\\ &= (\nu\Delta u_{\mathrm{FE}}, v_e)_{K_e} - \left(\nu\nabla\left(u-u_{\mathrm{FE}}\right),\nabla v_e\right)_{K_e} + (f,v_e)_{K_e} + (\varrho_{\mathrm{FE}},\nabla\cdot v_e)_{K_e} + (\varrho-\varrho_{\mathrm{FE}},\nabla\cdot v_e)_{K_e}\end{aligned}$$

by incompressibility condition $\nabla\cdot u = 0$. Then, performing integration by parts gives

$$\begin{aligned}\left\|v_e\Phi_e^{-\frac{\alpha}{2}}\right\|_e^2 = &\left(I_{p_K}^K f + \nu\Delta u_{\mathrm{FE}} - \nabla\varrho_{\mathrm{FE}}, v_e\right)_{K_e} - (\nu\nabla(u-u_{\mathrm{FE}}),\nabla v_e)_{K_e} + (\varrho-\varrho_{\mathrm{FE}},\nabla\cdot v_e)_{K_e}\\ &+ \left(f - I_{p_K}^K f, v_e\right)_{K_e}\end{aligned}$$

and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}\left\|v_e\Phi_e^{-\frac{\alpha}{2}}\right\|_e^2 \leq &\left(\left\|I_{p_K}^K f + \nu\Delta u_{\mathrm{FE}} - \nabla\varrho_{\mathrm{FE}}\right\|_{K_e} + \left\|f - I_{p_K}^K f\right\|_{K_e}\right)\|v_e\|_{K_e} + \nu\|\nabla(u-u_{\mathrm{FE}})\|_{K_e}\|\nabla v_e\|_{K_e}\\ &+ \|\varrho-\varrho_{\mathrm{FE}}\|_{K_e}\|\nabla\cdot v_e\|_{K_e}.\end{aligned} \tag{22}$$

Now, we have to distinguish between two cases. First, let us assume $\alpha > \frac{1}{2}$. By using an affine transformation to get back to reference element $\hat{K}$ and, then, using Lemma 3, we obtain the following upper bounds for $\|v_e\|_{K_e}$ and $\|\nabla v_e\|_{K_e}$ on edge e:

$$\begin{aligned}\|\nabla v_e\|_{K_e}^2 &\leq C\frac{\delta p_K^{(2(2-\alpha))}+\delta^{-1}}{h_K}\left\|\left[\nu\frac{\partial u_{\mathrm{FE}}}{\partial n}\right]\Phi_e^{\frac{\alpha}{2}}\right\|_e^2,\\ \|v_e\|_{K_e}^2 &\leq C\delta h_K\left\|\left[\nu\frac{\partial u_{\mathrm{FE}}}{\partial n}\right]\Phi_e^{\frac{\alpha}{2}}\right\|_e^2.\end{aligned}$$

Knowing that $\|\nabla\cdot v_e\|_{K_e} \leq \|\nabla v_e\|_{K_e}$, estimate (22) yields

$$\begin{aligned}\left\|\left[\nu\frac{\partial u_{\mathrm{FE}}}{\partial n}\right]\Phi_e^{\frac{\alpha}{2}}\right\|_e \leq C\Bigg(&(\delta h_K)^{\frac{1}{2}}\left(\left\|I_{p_K}^K f + \nu\Delta u_{\mathrm{FE}} - \nabla\varrho_{\mathrm{FE}}\right\|_{K_e} + \left\|f - I_{p_K}^K f\right\|_{K_e}\right)\\ &+\sqrt{\frac{\delta p_K^{2(2-\alpha)}+\delta^{-1}}{h_K}}\left(\nu\left\|\nabla\left(u-u_{\mathrm{FE}}\right)\right\|_{K_e} + \|\varrho-\varrho_{\mathrm{FE}}\|_{K_e}\right)\Bigg)\end{aligned}$$

and it follows

$$\left\| \left[\nu \frac{\partial u_{\mathrm{FE}}}{\partial n} \right] \Phi_e^{\frac{\alpha}{2}} \right\|_e \leq C\Bigg((\delta h_K)^{\frac{1}{2}} \left(\frac{p_K^2}{h_K} \left(\nu \left\| \nabla (u - u_{\mathrm{FE}}) \right\|_{K_e} + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_{K_e} \right) + p_K^{\frac{1}{2}} \left\| f - I_{p_K}^K f \right\|_{K_e} \right) + \sqrt{\frac{\delta p_K^{2(2-\alpha)} + \delta^{-1}}{h_K}} \left(\nu \left\| \nabla (u - u_{\mathrm{FE}}) \right\|_{K_e} + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_{K_e} \right) \Bigg)$$

with Lemma 6. By squaring both sides and summing over all edges $e \in \mathcal{E}(K)$, we get

$$\eta_{\alpha;K;B}^2 \leq C\Bigg(\delta \left(p_K^3 \left(\nu^2 \left\| \nabla (u - u_{\mathrm{FE}}) \right\|_{\omega_e}^2 + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_{\omega_e}^2 \right) + h_K^2 \left\| f - I_{p_K}^K f \right\|_{\omega_e}^2 \right) + \frac{\delta p_K^{2(2-\alpha)} + \delta^{-1}}{p_K} \left(\nu^2 \left\| \nabla (u - u_{\mathrm{FE}}) \right\|_{\omega_e}^2 + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_{\omega_e}^2 \right) \Bigg) \tag{23}$$

and setting $\delta := p_K^{-2}$ gives the desired result. Now, let $0 \leq \alpha \leq \frac{1}{2}$. Similar to the proof of Lemma 6, we set $\beta := \frac{1+\alpha}{2}$ and apply Lemma 2 to get $\eta_{\alpha;K;B} \leq p_K^{\beta-\alpha} \eta_{\beta;K;B}$. Then, using estimate (23) gives

$$\eta_{\alpha;K;B}^2 \leq C\Bigg(\delta \left(p_K^{\frac{7-\alpha}{2}} \left(\nu^2 \left\| \nabla (u - u_{\mathrm{FE}}) \right\|_{\omega_e}^2 + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_{\omega_e}^2 \right) + \frac{h_K^2}{p_K^{\frac{\alpha-1}{2}}} \left\| f - I_{p_K}^K f \right\|_{\omega_e}^2 \right) + \frac{\delta p_K^{2(2-\alpha)} + \delta^{-1}}{p_K^{\frac{1+\alpha}{2}}} \left(\nu^2 \left\| \nabla (u - u_{\mathrm{FE}}) \right\|_{\omega_e}^2 + \left\| \varrho - \varrho_{\mathrm{FE}} \right\|_{\omega_e}^2 \right) \Bigg)$$

and setting $\delta := p_K^{-2}$ concludes the proof. □

By combining the results from Lemmas 6 and 7, we can derive an upper bound for the residual-based a posteriori error estimator η in terms of the quasi-local energy error.

Theorem 3. *Let $[u_{FE}, \varrho_{FE}] \in \mathcal{V}^p(\mathcal{T})$ be the solution of discrete problem (4) and $[u, \varrho] \in \mathcal{H}$ be solution of weak problem (3). Further, we assume that triangulation $\mathcal{T}$ is (γ_h, γ_p)-regular and let $\alpha \in [0,1]$ be arbitrary. Then, there exists some constant $C_{eff} > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\eta_{\alpha;K;B}^2 \leq C_{eff} \left(p_K^k \left(\nu^2 \left\| \nabla (u - u_{FE}) \right\|_{\omega_K}^2 + \left\| \varrho - \varrho_{FE} \right\|_{\omega_K}^2 \right) + \frac{h_K^2}{p_K^{1+\alpha}} \left\| I_{p_K}^K f - f \right\|_{\omega_K}^2 \right)$$

for all $K \in \mathcal{T}$, where $k := \max\left\{2(1-\alpha), \frac{3-\alpha}{2}\right\}$.

Proof. The result follows from Definition 1 and Lemmas 6 and 7. □

4 *hp*-Adaptive Refinement

As in [Dorfler 1D, Markus high D] the fully automatic hp-adaptive refinement strategy is based on the standard Adaptive Finite Element Method (AFEM).

$$SOLVE \longrightarrow ESTIMATE \longrightarrow MARK \longrightarrow REFINE. \tag{24}$$

The procedures SOLVE and REFINE are almost the same in all AFEM algorithms. Basically, the distinction between different adaptive approaches comes from the procedures ESTIMATE and MARK. Here we follow exactly the same hp-adaptive strategy as presented in [Markus high , Markus Maxwell,]. Some convergence results corresponding to this adaptive algorithm will be presented in the next subsections.

4.1 The refinement strategy

In order to explain our hp-refinement strategy, we will go through procedures ESTIMATE and MARK and will explain the way those procedures work in our refinement algorithm.
Procedure **ESTIMATE**, shows the accuracy of the finite element solution, obtained from procedure SOLVE. A reliable and efficient residual based a posteriori error estimator[...schwab], which gives an estimation of the exact energy error in terms of the given data and the computed solution from module SOLVE have been developed in Section 3. Procedure **MARK**, determines which cells are the best candidates for h- or p- refinement. Despite the pure h- or p- refinement, in hp- refinement just the information given from procedure ESTIMATE is not sufficient to choose the cells with the biggest error contribution to be refined. The reason comes from this fact that in hp-refinement, one needs also to determine which refinement pattern should be applied on the selected cells. Therefore, besides error estimator given from procedure ESTIMATE, some extra indicators should be defined here to help us in determining the best refinement strategy on candidate refinement cell.

Remark 1

- In application we just use the two common h- and p- refinement patterns, namely equal-sized bisection in every coordinate-direction and increase the polynomial degree p_K by one, for h- and p- refinement respectively. There are of course much more refinement algorithms which we could use. Therefore, generally in our theoretical part we assume there is $n \in \mathbb{N}$, $n > 2$ different refinement patterns to choose from.

- in h-refinement we have to reduce the error contribution of the boundary terms as much as we can. Therefore in order to make sure that no new hanging-nodes are produced at the edges of the cell, we also have to h-refine the neighboring cells, which we called them patch cells. The figure [...?] shows our h-refinement pattern on patch ω_K corresponding to cell K.

- in p-refinement we again need to ensure that by increasing polynomial degree on cell K , no new constraint degrees of freedom are created. Thus, the increment of polynomial degrees will be applied on neighboring cell, as well. Fig. [...2] shows how the p-refinement is applied on all cells included in patch ω_K.

4.1.1 Convergence Estimator

let $j \in \{1, 2, \cdots, n\}$, which n indicates the number of different refinement patterns, and consider $K \in \mathcal{T}_N$ be arbitrary an cell in N-th cycle of refinement. Then we show the local finite element space consisting of functions from $\mathcal{V}^p(\mathcal{T})$ compactly supported in the local patch ω_K with refinement pattern j applied to cell K, with $\mathcal{V}^p_{K,j}(\mathcal{T}|_{\omega_K})$. Following [Dorfler 1D], we define the convergence estimator $k_{K,j} \in \mathbb{R}^+$ as the solution of the following optimization problem.

$$k_{K,j} = \frac{1}{\eta_K(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}})} \sup_{(v,q)\in\mathcal{V}^p_{K,j}(\mathcal{T}_N|_{\omega_K})} \frac{\int_{\omega_K} v\left(I^{\omega_K}_{p_K} f + \nu\Delta u_{\mathrm{FE}} - \nabla\varrho_{\mathrm{FE}}\right) - \int_{\omega_K} q\,(\nabla\cdot u_{\mathrm{FE}})}{\|v\|_{(\omega_K)^d} + \|q\|_{(\omega_K)}} \tag{25}$$

let $(w_u^{N,j}, w_\varrho^{N,j}) \in \mathcal{V}_{K,j}^p(\mathcal{T}_N|_{\omega_K})$ be the solution of the following variational problem

$$\begin{aligned}\int_{\omega_K} (\nabla v)^T \nabla w_u^{N,j} - \int_{\omega_K} (\nabla \cdot v) w_\varrho^{N,j} - \int_{\omega_K} q(\nabla \cdot w_u^{N,j}) \\ = \int_{\omega_K} v\left(I_{p_K}^{\omega_K} f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}}\right) - \int_{\omega_K} q\left(\nabla \cdot u_{\mathrm{FE}}\right) \qquad \forall (v,q) \in \mathcal{V}_{K,j}^p(\mathcal{T}_N|_{\omega_K}, \omega_K)\end{aligned} \tag{26}$$

It can be easily shown that for all $(v,q) \in \mathcal{V}_{K,j}^p(\mathcal{T}_N|_{\omega_K})$,

$$\begin{aligned}\sup_{(v,q)\in \mathcal{V}_{K,j}^p(\mathcal{T}_N|_{\omega_K})} \frac{\int_{\omega_K} v\left(I_{p_K}^{\omega_K} f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}}\right) - \int_{\omega_K} q\left(\nabla \cdot u_{\mathrm{FE}}\right)}{\|v\|_{(\omega_K)^d} + \|q\|_{(\omega_K)}} \leqslant \\ \frac{\int_{\omega_K} w_u^{N,j}\left(I_{p_K}^{\omega_K} f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}}\right) - \int_{\omega_K} w_\varrho^{N,j}\left(\nabla \cdot u_{\mathrm{FE}}\right)}{\left\|w_u^{N,j}\right\|_{(\omega_K)^d} + \left\|w_\varrho^{N,j}\right\|_{(\omega_K)}} = \left\|\nabla w_u^{N,j}\right\|_{(\omega_K)^d} + \left\|w_p^{N,j}\right\|_{(\omega_K)}\end{aligned} \tag{27}$$

Therefore, the solution of optimization problem (25), obtained for $(w_u^{N,j}, w_\varrho^{N,j}) \in \mathcal{V}_{K,j}^p(\mathcal{T}_N|_{\omega_K})$.
It is remarkable that the solution of optimization problem (25), indicates which refinement pattern provides the biggest error reduction on every cell. Besides convergence estimator, In order to get the efficient refinement pattern we would like to define another parameter namely *workload number*[...MArkus thesis]. in fact these numbers, $\varpi_{K,j} \in \mathbb{R}^+$, show the required work of refinement pattern j on cell K. Here in this work, for $\varpi_{K,j}$ we set number of degrees of freedom in the local finite element space $\mathcal{V}_{K,j}^p(\mathcal{T}_N|_{\omega_K})$. There are more choices to set for the work load number,see [Thesis Markus]

4.1.2 Marking strategy

By seeting up all the required materials in order to mark cells for refinement, we start as follows. Basically the marking strategy comes from the solution of a nonlinear optimization problem, namely NP-hard [ref 87 thesis and paper Markus on high D]. Since soliving this problem is compplicated, we try a quite simple and computationally cheap algorithm to solve the maximization problem. The substituet approach works like this:
for every cell K we assign an integer $j_K \in \{1, 2, \cdots, n\}$ such that

$$\frac{k_{K,j_K}}{\varpi_{K,j_K}} = \max_{j\in\{1,2,\cdots,n\}} \frac{k_{K,j}}{\varpi_{K,j}} \tag{28}$$

under the constraint

$$\sum_{K\in\mathcal{A}} k_{K,j_k}^2 \eta_K^2 \geq \theta^2 \eta^2 \tag{29}$$

The solution of this maximization problem is $(\mathcal{A}, (j_K)_{K\in\mathcal{A}})$, where $\mathcal{A} \in \mathcal{T}$ is a set with minimal calrdinality. set $\mathcal{A}$ containing all cells satisfying the constraint condition (29) defines the candidate cells for refinement.

4.1.3 Adaptive hp-refinement Algorithm

- Initialization: Set $N = 0$, a coarse mesh $\mathcal{T}_0$, $\theta \in [0, 1]$ and also tolerance TOL.
- SOLVE: Fine the solution $(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}})$ of discrete problem (4).

- ESTIMATE: Compute aposteriori error estimation given by equation (11), if $\eta_K < TOL$ then STOP the slgorithm.
- MARK cells: For all cells $K \in \mathcal{T}_N$ and all refinement patterns $j \in \{1, 2, \cdots, n\}$, compute the convergence estimator $k_{K,j}$ and the work-load number $\varpi_{K,j}$. Then approximate the solution of constraint maximization problem given in equations (28) and (29)
- REFINE: Given $(\mathcal{A}_N, (j_K)_{K \in \mathcal{A}_N})$, we refine the cells contained in $\mathcal{A}_N$ with refinement patterns j_K corresponding to each cell. Then set $N = N + 1$ and go to step SOLVE.

5 Convergence of adaptive hp-refinement

In this section we want to show that the energy norm of our hp-adaptive FEM solution is reduced at each refinement step of our algorithm. The assumptions and the expected results will be shown in Theorem (4) and then in the next theorem (**??**) we try to proof that the weighted sum of exact energy error and our aposteriori error estimation will also be reduced at each refinement step.

Theorem 4 (Contraction Convergence)**.** *Let $N \in \mathbb{N}$ be an arbitrary refinement step, and $(u, \varrho) \in \mathcal{H}$ be solution of (3). Assume that for some $\theta \in (0, 1]$ solution of the constraint problem (28) and (29) exists. Moreover assume for a sufficiently small $\tau \in (0, 1]$ depending only on polynomial vector p, the data saturation assumption (48) holds. Consider $(u^N_{FE}, \varrho^N_{FE}) \in \mathcal{V}^p(\mathcal{T}_N)$ and $(u^{N+1}_{FE}, \varrho^{N+1}_{FE}) \in \mathcal{V}^p(\mathcal{T}_{N+1})$ as solutions of (4) two consecutive iterative steps. Then there is a constant $\mu \in (0, 1)$ independent of mesh size h and polynomial degree vector p such that*

$$\|\nabla(u - u^{N+1}_{FE})\|^2_{\Omega^d} + \|\varrho - \varrho^{N+1}_{FE}\|^2_{\Omega} \leq \mu \left(\|\nabla(u - u^N_{FE})\|^2_{\Omega^d} + \|\varrho - {\varrho^N}^2_{FE}\|_{\Omega} \right) \tag{30}$$

Where d is space dimension. In order to prove theorem (4), we need to state a couple of auxiliary results. The following lemmas would be our main tools in justifying the above convergence result.

Lemma 8 (Error Estimate Reduction)**.** *Let $N \in \mathbb{N}$ be an arbitrary refinement step, and $(u, \varrho) \in \mathcal{H}$ be solution of (3). Assume that for some $\theta \in (0, 1]$ solution of the constraint problem (28) and (29) exists. Also consider $(u^N_{FE}, \varrho^N_{FE}) \in \mathcal{V}^p(\mathcal{T}_N)$ and $(u^{N+1}_{FE}, \varrho^{N+1}_{FE}) \in \mathcal{V}^p(\mathcal{T}_{N+1})$ as solutions of (4) two consecutive iterative steps. Moreover, we assume for all those refinement patterns $j \in \{1, 2, \cdots, n\}$ there is a constant $\rho \in (0, 1)$ independent of mesh size h and polynomial vector p such that*

$$\frac{h_K}{p_K} \leq \rho \frac{h_{\tilde{K}}}{p_{\tilde{K}}} \tag{31}$$

for all refined cells $\tilde{K} \in \mathcal{T}_N$ and $K \in \mathcal{T}_{N+1}$ such tha $K \subseteq \tilde{K}$. Additionally assume the saturation assumption which says there is $\tau \in (0, 1]$ such that

$$\sum_{K \in \tau_N} \frac{h_K^2}{p_K^2} \|I^K_{p_K} f - f\|^2_K \leq \tau^2 \eta^2(u_{FE}, \varrho_{FE}, \tau_N) \tag{32}$$

holds. Then there exists a constant C_{red} independent of mesh size h and polynomial degree p, for all $\delta > 0$ such that

$$\eta^2(u^{N+1}_{FE}, \varrho^{N+1}_{FE}, \mathcal{T}_{N+1}) \leq (1 + \delta)^2 \left((1 + \frac{\tau^2 \rho^2}{\delta}) \eta^2(u^N_{FE}, \varrho^N_{FE}, \mathcal{T}_N) - (1 - \rho)\eta^2(u^N_{FE}, \varrho^N_{FE}, \mathcal{A}_N) \right)$$
$$+ C_{red}(1 + \frac{1}{\delta}) \max_{K \in \mathcal{T}_N} (p_K^2) \left(\|\nabla(u_{N+1} - u_N)\|^2_{K^d} + \|\varrho_{N+1} - \varrho_N\|^2_K \right)$$

Proof. Without loss of generality we assume $\nu = 1$, then by definition (11) we have

$$\eta^2(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) = \sum_{K \in \mathcal{T}_{N+1}} \left(\eta_{K;R}^2(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) + \eta_{K;B}^2(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) \right) \tag{33}$$

equation (12) for residual terms implies

$$\eta_{K;R}(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) := \frac{h_K}{p_K} \left\| (I_{p_K;N+1}^K f + \Delta u_{\mathrm{FE}}^{N+1} - \nabla \varrho_{\mathrm{FE}}^{N+1}) \right\|_K + \left\| (\nabla \cdot u_{\mathrm{FE}}^{N+1}) \right\|_K$$

Then Minkowski inequality implies:

$$\begin{aligned} \eta_{K;R}(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) \leq & \frac{h_K}{p_K} \left\| (I_{p_K;N}^K f + \Delta u_{\mathrm{FE}}^{N} - \nabla \varrho_{\mathrm{FE}}^{N}) \right\|_K + \left\| \nabla \cdot u_{\mathrm{FE}}^{N} \right\|_K \\ & + \frac{h_K}{p_K} \left\| I_{p_K;N+1}^K f - I_{p_K;N}^K f \right\|_K + \frac{h_K}{p_K} \left\| \Delta (u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N}) \right\|_K \\ & + \frac{h_K}{p_K} \left\| \nabla (\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^{N}) \right\|_K + \left\| \nabla \cdot (u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N}) \right\|_K \end{aligned} \tag{34}$$

we define the set

$$\mathcal{R}_N = \{K \in \mathcal{T}_N : K \text{ is refined}\}; \quad \text{where } \mathcal{A}_N \subseteq \mathcal{R}_N$$

Now consider two cases. Frist, if there exists some $\tilde{\mathrm{K}} \in \mathcal{R}_N$ such that $K \subseteq \tilde{\mathrm{K}}$, then by (31) we have

$$\frac{h_K}{p_K} \left\| (I_{p_K;N}^K f + \Delta u_{\mathrm{FE}}^{N} - \nabla \varrho_{\mathrm{FE}}^{N}) \right\|_K + \left\| \nabla \cdot u_{\mathrm{FE}}^{N} \right\|_K \leq \rho \frac{h_{\tilde{K}}}{p_{\tilde{K}}} \left\| (I_{p_K;N}^K f + \nu \Delta u_{\mathrm{FE}}^{N} - \nabla \varrho_{\mathrm{FE}}^{N}) \right\|_K + \left\| \nabla \cdot u_{\mathrm{FE}}^{N} \right\|_K \tag{35}$$

Similarly

$$\frac{h_K}{p_K} \left\| I_{p_K;N+1}^K f - I_{p_K;N}^K f \right\|_K \leq \rho \frac{h_{\tilde{K}}}{p_{\tilde{K}}} \left\| f - I_{p_K;N}^K f \right\|_K \tag{36}$$

Now, for the second case consider there is no such $\tilde{K} \in \mathcal{R}_N$, the equations (35) and (36) turns to be

$$\frac{h_K}{p_K} \left\| (I_{p_K;N}^K f + \nu \Delta u_{\mathrm{FE}}^{N} - \nabla \varrho_{\mathrm{FE}}^{N}) \right\|_K + \left\| \nabla \cdot u_{\mathrm{FE}}^{N} \right\|_K = \eta_{K;R}(u_{\mathrm{FE}}^{N}, \varrho_{\mathrm{FE}}^{K}, \mathcal{T}_N) \tag{37}$$

and

$$\frac{h_K}{p_K} \left\| I_{p_K;N+1}^K f - I_{p_K;N}^K f \right\|_K = 0. \tag{38}$$

In both cases, by lemma 4 we have

$$\frac{h_K}{p_K} \left\| \Delta (u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N}) \right\|_K \leq C_{inv}^1 p_K \left\| \nabla (u_{N+1} - u_N) \right\|_{K^d} \tag{39}$$

$$\frac{h_K}{p_K} \left\| \nabla (\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^{N}) \right\|_K \leq C_{inv}^2 p_k \left\| \varrho_{N+1} - \varrho_N \right\|_K \tag{40}$$

for the case if there exists some $\tilde{K} \in \mathcal{R}_N$ such that $K \subseteq \tilde{K}$, we insert equations (35), (36), (39) and (40) into equation (34)

$$\begin{aligned} \eta_{K;R}(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) \leq \rho \frac{h_{\tilde{K}}}{p_{\tilde{K}}} & \left(\| I_{p_K;N}^K f + \nu \Delta u_{\mathrm{FE}}^{N} - \nabla \varrho_{\mathrm{FE}}^{N} \|_K + \| f - I_{p_K;N}^K f \|_K \right) + \| \nabla \cdot u_{\mathrm{FE}}^{N} \|_K \\ & + p_K C_{inv}^{max} \left(\| \nabla (u_{N+1} - u_N) \|_{K^d} + \| \varrho_{N+1} - \varrho_N \|_K \right) \end{aligned} \tag{41}$$

where $C_{inv}^{max} = \max\{C_{inv}^1, C_{inv}^2\}$. And for the case that if there is no such $\tilde{K} \in \mathcal{R}_N$

$$\eta_{K;R}(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) \leq \eta_{K;R}(u_{\mathrm{FE}}^{N}, \varrho_{\mathrm{FE}}^{N}, \mathcal{T}_{N}) + p_K C_{inv}^{max}\Bigg(\|\nabla(u_{N+1} - u_N)\|_{K^d} + \|\varrho_{N+1} - \varrho_N\|_K \Bigg) \quad (42)$$

Now considering the boundary term given in equation (13)

$$\begin{aligned} \eta_{K;B}^2(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) = \frac{1}{2}\sum_{e\in\mathcal{E}(I)} \frac{h_e}{p_e}\left\| \left[\frac{\partial u_{\mathrm{FE}}^{N+1}}{\partial n_K}\right]\right\|_e^2 = \frac{1}{2}\sum_{e\in\mathcal{E}(I)} \frac{h_e}{p_e}\left\| \left[\frac{\partial u_{\mathrm{FE}}^{N+1}}{\partial n_K}\right]\right\|_e \\ \left(\left\| \left[\frac{\partial u_{\mathrm{FE}}^{N}}{\partial n_K}\right]\right\|_e + \left\| \left[\frac{\partial u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N}}{\partial n_K}\right]\right\|_e \right) \end{aligned} \quad (43)$$

Using Minkowski and the Cauchy-Schwarz inequality, we will get

$$\eta_{K;B}^2(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) \leq \eta_{K;B}^2(u_{\mathrm{FE}}^{N}, \varrho_{\mathrm{FE}}^{N}, \mathcal{T}_{N})(L_1 + L_2) \quad (44)$$

such that

$$L_1^2 = \frac{1}{2}\sum_{e\in\mathcal{E}(K)} \frac{h_e}{p_e}\left\| \left[\frac{\partial u_{\mathrm{FE}}^{N}}{\partial n_K}\right]\right\|_e^2$$

$$L_2^2 = \frac{1}{2}\sum_{e\in\mathcal{E}(K)} \frac{h_e}{p_e}\left\| \left[\frac{\partial u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N}}{\partial n_K}\right]\right\|_e^2$$

For the case that there is $\tilde{K} \in \mathcal{R}_N$ such that $K \subseteq \tilde{K}$ then

$$L_1^2 \leq \frac{\rho}{2}\sum_{e\in\mathcal{E}_I(\tilde{K})} \frac{h_e}{p_e}\left\| \left[\frac{\partial u_{\mathrm{FE}}^{N}}{\partial n_K}\right]\right\|_{e\cap\partial K}^2$$

But again for the case that there is no such $\tilde{K} \in \mathcal{R}_N$, we have

$$L_1^2 \leq \eta_{K;B}^2(u_{\mathrm{FE}}^{N}, \varrho_{\mathrm{FE}}^{N}, \mathcal{T}_{N})$$

for L_2 in both cases using lemma (5) we have

$$L_2^2 \leq d C_{tr}^2 p_K \|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N})\|_K^2$$

Inserting the above results in (**??**), for the first case we get

$$\eta_{K;B}(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) \leq \left(\frac{\rho}{2}\sum_{e\in\mathcal{E}_I(\tilde{K})} \frac{h_e}{p_e}\left\| \left[\frac{\partial u_{\mathrm{FE}}^{N}}{\partial n_K}\right]\right\|_{e\cap\partial K}^2 \right)^{\frac{1}{2}} + C_{tr}\sqrt{p_k d}\|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N})\|_K \quad (45)$$

and if there is no such $\tilde{K} \in \mathcal{R}_N$,

$$\eta_{K;B}(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) \leq \eta_{K;B}(u_{\mathrm{FE}}^{N}, \varrho_{\mathrm{FE}}^{N}, \mathcal{T}_{N}) + C_{tr}\sqrt{p_k d}\|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N})\|_K \quad (46)$$

now we gather all approximations given in (41), (42), (45) and (46) and will insert them into (33). Using Young inequality and also saturation assumption we have (48)

$$\begin{aligned} \eta^2(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}, \mathcal{T}_{N+1}) \leq (1+\delta)^2\Bigg((1 + \frac{\tau^2\rho^2}{\delta})\eta^2(u_{\mathrm{FE}}^{N}, \varrho_{\mathrm{FE}}^{N}, \mathcal{T}_{N}) - (1-\rho)\eta^2(u_{\mathrm{FE}}^{N}, \varrho_{\mathrm{FE}}^{N}, \mathcal{A}_{N}) \Bigg) \\ + C_{red}(1 + \frac{1}{\delta}) \max_{K\in\mathcal{T}_N}(p_K^2)\Bigg(\|\nabla(u_{N+1} - u_N)\|_{K^d}^2 + \|\varrho_{N+1} - \varrho_N\|_K^2 \Bigg) \end{aligned} \quad (47)$$

□

Lemma 9. *Let $N \in \mathbb{N}$ be an arbitrary refinement step, and $(u, \varrho) \in \mathcal{H}$ be solution of (3). Assume that for some $\theta \in (0,1]$ solution of the constraint problem (28) and (29) exists. Also consider $(u_{FE}^N, \varrho_{FE}^N) \in \mathcal{V}^p(\mathcal{T}_N)$ and $(u_{FE}^{N+1}, \varrho_{FE}^{N+1}) \in \mathcal{V}^p(\mathcal{T}_{N+1})$ as solutions of (4) two consecutive iterative steps. Additionally assume the saturation assumption which says there is $\tau \in (0,1]$ such that*

$$\sum_{K \in \tau_N} \frac{h_K^2}{p_K^2} \|I_{p_K}^K f - f\|_K^2 \leq \tau^2 \eta^2(u_{FE}, \varrho_{FE}, \tau_N) \tag{48}$$

holds. Then there exists constants C_1 and C_2 independent of mesh size h and vector of polynomial degree p and for all $\delta > 0$ such that

$$\left\|\nabla\left(u_{FE}^{N+1} - u_{FE}^N\right)\right\|_\Omega^2 + \left\|\varrho_{FE}^{N+1} - \varrho_{FE}^N\right\|_\Omega^2 \geqslant \delta\left(\frac{\theta^2}{C_1(1+\delta)}\left(\left\|\nabla\left(u - u_{FE}^N\right)\right\|_\Omega^2 + \left\|\varrho - \varrho_{FE}^N\right\|_\Omega^2\right) - C_2\tau^2\eta^2(u_{FE}^N, \varrho_{FE}^N, \mathcal{T}_N)\right)$$

Proof. Consider $K \in \mathcal{T}_N$ and $(\psi_u^{N+1}, \psi_\varrho^{N+1}) \in \mathcal{V}^p(\mathcal{T}_{N+1})$ which their support set in a subset of ω_K. Since $(u_{\mathrm{FE}}^{N+1}, \varrho_{\mathrm{FE}}^{N+1}) \in \mathcal{V}^p(\mathcal{T}_{N+1})$ solve equation (4), we have

$$\begin{aligned}&\int_{\omega_K} (\nabla\psi_u^{N+1})^T \nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N+1}) - \int_{\omega_K} \nabla\cdot\psi_u^{N+1}(\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^N) - \int_{\omega_K} \psi_{\varrho\mathrm{FE}}^{N+1}\nabla\cdot(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^N)\\ &= \int_{\omega_K} \psi_u^{N+1} I_{p_K}^{\omega_K} f + \int_{\omega_K} \psi_u^{N+1}(f - I_{p_K}^{\omega_K} f) - \int_{\omega_K} \nabla\psi_u^{N+1}\nabla u_{\mathrm{FE}}^N + \int_{\omega_K} \nabla\cdot\psi_u^{N+1}\varrho_{\mathrm{FE}}^N + \int_{\omega_K} \psi_\varrho^{N+1}\nabla\cdot u_{\mathrm{FE}}^N\end{aligned} \tag{49}$$

we abbreviate the left hand side of equation (49) by L.H.S, integrating by parts and the L^2 interpolation property implies

$$L.H.S = \sum_{K\in\omega_K}\left(\int_K \psi_u^{N+1}(I_{p_K}^K f + \Delta u_{\mathrm{FE}}^N - \nabla\varrho_{\mathrm{FE}}^N) + \int_K \psi_\varrho^{N+1}\nabla\cdot u_{\mathrm{FE}}^N\right) + \int_{\omega_K}(\psi_u^{N+1} - \psi_u^N)(f - I_{p_K}^{\omega_K} f)$$

$$L.H.S \geqslant \left|\sum_{K\in\omega_K}\left(\int_K \psi_u^{N+1}(I_{p_K}^K f + \Delta u_{\mathrm{FE}}^N - \nabla\varrho_{\mathrm{FE}}^N) + \int_K \psi_\varrho^{N+1}\nabla\cdot u_{\mathrm{FE}}^N\right)\right| - \left|\int_{\omega_K}(\psi_u^{N+1} - \psi_u^N)(f - I_{p_K}^{\omega_K} f)\right|$$

With Holder inequality we have

$$\begin{aligned}&\left|\sum_{K\in\omega_K}\left(\int_K \psi_u^{N+1}(I_{p_K}^K f + \Delta u_{\mathrm{FE}}^N - \nabla\varrho_{\mathrm{FE}}^N) + \int_K \psi_\varrho^{N+1}\nabla\cdot u_{\mathrm{FE}}^N\right)\right| \leq \left\|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^N)\right\|_{\omega_K^d}\left\|\nabla\psi_u^{N+1}\right\|_{\omega_K^d}\\ &+ \left\|\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^N\right\|_{\omega_k}\left\|\nabla\psi_u^{N+1}\right\|_{\omega_K^d} + \left\|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^N)\right\|_{\omega_K^d}\left\|\psi_\varrho^{N+1}\right\|_{\omega_K} + \left\|\psi_u^{N+1} - \psi_u^N\right\|_{\omega_K}\left\|f - I_{p_{\omega_K}}^{\omega_K} f\right\|_{\omega_K}\end{aligned} \tag{50}$$

Now consider H^1 conforming interpolation operator $\Pi^{hp} : H_0^1(\Omega)^d \to V_u^p(\mathcal{T}_h)^d$, we choose $\psi_u^N = \Pi^{hp}\psi_u^{N+1}$. Using (1) implies

$$\|\psi_u^{N+1} - \psi_u^N\|_{\omega_K} = \|\psi_u^{N+1} - \Pi^{hp}\psi_u^{N+1}\|_{\omega_K} \leq C_{clem}\frac{h_K}{p_K}\|\nabla\psi_u^{N+1}\|_{\omega_K^d}$$

Inserting into equation (50)

$$\begin{aligned}&\left|\sum_{K\in\omega_K}\int_K \psi_u^{N+1}(I_{p_K}^K f + \Delta u_{\mathrm{FE}}^N - \nabla\varrho_{\mathrm{FE}}^N) + \int_K \psi_\varrho^{N+1}\nabla\cdot u_{\mathrm{FE}}^N\right| \leq \Bigg(\|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^N)\|_{\omega_K^d}\\ &+ \|\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^N\|_{\omega_k} + C_{clem}\frac{h_K}{p_K}\left\|f - I_{p_{\omega_K}}^{\omega_K} f\right\|_{\omega_K}\Bigg)\left(\|\nabla\psi_u^{N+1}\|_{\omega_K^d} + \|\psi_\varrho^{N+1}\|_{\omega_K}\right)\end{aligned}$$

equation (25) gives

$$k_{K,j}\eta_K \leq \|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N})\|_{\omega_K^d} + \|\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^{N}\|_{\omega_k} + C_{clem}\frac{h_K}{p_K}\|f - I_{p_{\omega_K}}^{\omega_K} f\|_{\omega_K}$$

Taking summation over all $K \in \mathcal{T}_N$, knowing that $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{T}_{\mathcal{N}}$, and then using Young's inequality yields

$$\sum_{K\in\mathcal{A}_N} k_{K,j}^2\eta_K^2 \leq (1+\frac{1}{\delta})\left(\sum_{K\in\mathcal{T}_N}\|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N})\|_{\omega_K^d}^2 + \|\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^{N}\|_{\omega_k}^2\right)$$
$$+ C_{clem}^2(1+\delta)\sum_{K\in\mathcal{T}_N}\frac{h_K^2}{p_K^2}\|f - I_{p_{\omega_K}}^{\omega_K} f\|_{\omega_K}^2$$

for some $\delta > 0$. Applying (48) and let $\hat{C} = \max_{K\in\mathcal{T}_N} |\{K \in \mathcal{T}_{\mathcal{N}} : K \in \omega_K\}|$

$$\sum_{K\in\mathcal{A}_N} k_{K,j}^2\eta_K^2 \leq \hat{C}\left((1+\frac{1}{\delta})(\|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^{N})\|_{\Omega^d}^2 + \|\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^{N}\|_{\Omega}^2)\right) + C_{clem}^2(1+\delta)\tau^2\eta^2(u_{\mathrm{FE}}^N, \varrho_{\mathrm{FE}}^N, \mathcal{T}_N) \tag{51}$$

by theorem (2) and (48)

$$\begin{aligned}\left\|\nabla\left(u - u_{\mathrm{FE}}^N\right)\right\|_{\Omega^d}^2 + \left\|\varrho - \varrho_{\mathrm{FE}}^N\right\|_{\Omega}^2 &\leq C_{rel}\left(\eta^2(u_{\mathrm{FE}}^N, \varrho_{\mathrm{FE}}^N, \mathcal{T}_N) + \sum_{K\in\mathcal{T}}\frac{h_K^2}{p_K^2}\left\|I_{p_K}^K f - f\right\|_K^2\right)\\ &\leq C_{rel}\left((1+\tau^2)\tau^2\eta^2\right)\\ &\leq 2C_{rel}\,\eta^2\end{aligned}$$

Applying the constraint condition given in (29)

$$\theta^2\left(\left\|\nabla\left(u - u_{\mathrm{FE}}^N\right)\right\|_{\Omega^d}^2 + \left\|\varrho - \varrho_{\mathrm{FE}}^N\right\|_{\Omega}^2\right) \leq 2C_{rel}\sum_{K\in\mathcal{A}_N} k_{K,j_k}^2\eta_K^2 \tag{52}$$

imposing equation () into equation (), and assuming $C_1 = 2\hat{C}C_{rel}$ and $C_2 = \frac{C_{clem}^2}{\hat{C}}$ the result comes for free. ☐

Lemma 10. *Let $N \in \mathbb{N}$ be an arbitrary refinement step, and $(u, \varrho) \in \mathcal{H}$ be solution of (3). Assume that for some $\theta \in (0, 1]$ solution of the constraint problem (28) and (29) exists. Also consider $(u_{FE}^N, \varrho_{FE}^N) \in \mathcal{V}^p(\mathcal{T}_N)$ and $(u_{FE}^{N+1}, \varrho_{FE}^{N+1}) \in \mathcal{V}^p(\mathcal{T}_{N+1})$ as solutions of (4) two consecutive iterative steps. Additionally assume the saturation assumption given in equation (**??**) for some $\tau \in (0, 1]$ holds. Then there exists constants C_1 and C_2 independent of mesh size h and vector of polynomial degree p and for all $\delta > 0$ such that*

$$\left\|\nabla\left(u - u_{FE}^{N+1}\right)\right\|_{\Omega^d}^2 + \left\|\varrho - \varrho_{FE}^{N+1}\right\|_{\Omega}^2 \leq \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right)\left(\|\nabla(u - u_{FE}^N)\|_{\Omega^d}^2 + \|\varrho - \varrho_{FE}^{N+1}\|_{\Omega}^2\right)$$
$$+ \frac{C_2\delta\tau^2}{2}\eta(u_{FE}^N, \varrho_{FE}^N, \mathcal{T}_N) - \frac{1}{2}\left(\|\nabla(u_{FE}^{N+1} - u_{FE}^N)\|_{\Omega^d}^2 + \|\varrho_{FE}^{N+1} - \varrho_{FE}^N\|_{\Omega}^2\right)$$

Proof. Galerkin orthogonality gives

$$\mathcal{L}\left(\left[u - u_{\mathrm{FE}}^{N+1}, \varrho - \varrho_{\mathrm{FE}}^N\right]; \left[u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^N, \varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^N\right]\right) = 0$$

$$
\begin{aligned}
\|\nabla(u-u^N_{\mathrm{FE}})\|^2_{\Omega^d}+\|\varrho-\varrho^N_{\mathrm{FE}}\|^2_{\Omega} &\geqslant \mathcal{L}\left(\left[u-u^N_{\mathrm{FE}},\varrho-\varrho^N_{\mathrm{FE}}\right];\left[u-u^N_{\mathrm{FE}},\varrho-\varrho^N_{\mathrm{FE}}\right]\right)\\
&\geqslant \mathcal{L}\left(\left[u-u^{N+1}_{\mathrm{FE}},\varrho-\varrho^{N+1}_{\mathrm{FE}}\right];\left[u-u^{N+1}_{\mathrm{FE}},\varrho-\varrho^{N+1}_{\mathrm{FE}}\right]\right)\\
&+\mathcal{L}\left(\left[u^{N+1}_{\mathrm{FE}}-u^N_{\mathrm{FE}},\varrho^{N+1}_{\mathrm{FE}}-\varrho^N_{\mathrm{FE}}\right];\left[u^{N+1}_{\mathrm{FE}}-u^N_{\mathrm{FE}},\varrho^{N+1}_{\mathrm{FE}}-\varrho^N_{\mathrm{FE}}\right]\right)
\end{aligned}
$$

Then using lemma (9), and letting $\hat{C}_\delta=\max\left\{\hat{C}(1+\frac{1}{\delta})\right\}$, we get

$$
\begin{aligned}
\|\nabla(u-u^N_{\mathrm{FE}})\|^2_{\Omega^d}+\|\varrho-\varrho^N_{\mathrm{FE}}\|^2_{\Omega} \geqslant{}& \|\nabla(u-u^{N+1}_{\mathrm{FE}})\|^2_{\Omega^d}+\|\varrho-\varrho^{N+1}_{\mathrm{FE}}\|^2_{\Omega}\\
&+\frac{1}{2}\left(\|\nabla(u^{N+1}_{\mathrm{FE}}-u^N_{\mathrm{FE}})\|^2_{\Omega^d}+\|\varrho^{N+1}_{\mathrm{FE}}-\varrho^N_{\mathrm{FE}}\|^2_{\Omega}\right)\\
&+\frac{1}{2}\left(\frac{\theta^2}{2\hat{C}_\delta C_{rel}}\left(\left\|\nabla\left(u-u^N_{\mathrm{FE}}\right)\right\|^2_{\Omega^d}+\left\|\varrho-\varrho^N_{\mathrm{FE}}\right\|^2_{\Omega}\right)\right.\\
&\left.-\frac{C^2_{clem}(1+\delta)\tau^2}{2\hat{C}_\delta}\eta^2(u^N_{\mathrm{FE}},\varrho^N_{\mathrm{FE}},\mathcal{T}_N)\right)
\end{aligned}
\tag{53}
$$

the result follows by assuming $C_1=2\hat{C}C_{rel}$ and $C_2=\frac{C^2_{clem}}{\hat{C}}$ into equation (53). ☐

now using the results given in lemmas (8), (9) and (10) proof for theorem (4) follows: Assume viscosity $\nu=1$, by theorem (3)

Proof. Assume viscosity $\nu=1$, by theorem (3) we have

$$
\eta^2(u^N_{\mathrm{FE}},\varrho^N_{\mathrm{FE}},\mathcal{T}_N)\leq C_{eff}\sum_{K\in\mathcal{T}_N}\left(p_K^2\left(\left\|\nabla\left(u-u^N_{\mathrm{FE}}\right)\right\|^2_{\omega_K^d}+\left\|\varrho-\varrho^N_{\mathrm{FE}}\right\|^2_{\omega_K}\right)+\frac{h_K^2}{p_K}\left\|I^K_{p_K}f-f\right\|^2_{\omega_K}\right)
$$

saturation assumption (48) and (γ_h,γ_p)-Regularity given in definition (1), implies

$$
\begin{aligned}
\eta^2(u^N_{\mathrm{FE}},\varrho^N_{\mathrm{FE}},\mathcal{T}_N)\leq{}& C_{eff}\hat{C}\max_{K\in\mathcal{T}_N}(p_K)\left(\max_{K\in\mathcal{T}_N}(p_K)\left(\left\|\nabla\left(u-u^N_{\mathrm{FE}}\right)\right\|^2_{\Omega^d}+\left\|\varrho-\varrho^N_{\mathrm{FE}}\right\|^2_{\Omega}\right)\right.\\
&\left.+C\tau^2\eta^2(u^N_{\mathrm{FE}},\varrho^N_{\mathrm{FE}},\mathcal{T}_N)\right)
\end{aligned}
$$

where $\hat{C}=\max_{K\in\mathcal{T}_N}|\{K\in\mathcal{T}_N:K\in\omega_K\}|$ and $C>2$. For $\tau<\frac{1}{\sqrt{CC_{eff}\hat{C}\max_{K\in\mathcal{T}_N}(p_K)^{1/2}}}$ we have

$$
\eta^2(u^N_{\mathrm{FE}},\varrho^N_{\mathrm{FE}},\mathcal{T}_N)\leq\frac{C_{eff}\hat{C}\max_{K\in\mathcal{T}_N}(p_K)^2}{1-CC_{eff}\hat{C}\tau^2\max_{K\in\mathcal{T}_N}(p_K)}\left(\left\|\nabla\left(u-u^N_{\mathrm{FE}}\right)\right\|^2_{\Omega^d}+\left\|\varrho-\varrho^N_{\mathrm{FE}}\right\|^2_{\Omega}\right)
$$

if we assume $\tau<\frac{1}{\sqrt{2CC_{eff}\hat{C}\max_{K\in\mathcal{T}_N}(p_K)^{1/2}}}$

$$
\eta^2(u^N_{\mathrm{FE}},\varrho^N_{\mathrm{FE}},\mathcal{T}_N)\leq 2\hat{C}C_{eff}\max_{K\in\mathcal{T}_N}(p_K)^2\left(\left\|\nabla\left(u-u^N_{\mathrm{FE}}\right)\right\|^2_{\Omega^d}+\left\|\varrho-\varrho^N_{\mathrm{FE}}\right\|^2_{\Omega}\right)
\tag{54}
$$

by lemma (10) we get

$$
\begin{aligned}
\left\|\nabla\left(u-u^{N+1}_{\mathrm{FE}}\right)\right\|^2_{\Omega^d}+\left\|\varrho-\varrho^{N+1}_{\mathrm{FE}}\right\|^2_{\Omega}\leq{}&\left(1-\frac{\delta\theta^2}{2C_1(1+\delta)}\right)\left(\|\nabla(u-u^N_{\mathrm{FE}})\|^2_{\Omega^d}+\|\varrho-\varrho^{N+1}_{\mathrm{FE}}\|^2_{\Omega}\right)\\
&+\frac{C_2\delta\tau^2}{2}\eta^(u^N_{\mathrm{FE}},\varrho^N_{\mathrm{FE}},\mathcal{T}_N)-\frac{1}{2}\left(\|\nabla(u^{N+1}_{\mathrm{FE}}-u^N_{\mathrm{FE}})\|^2_{\Omega^d}+\|\varrho^{N+1}_{\mathrm{FE}}-\varrho^N_{\mathrm{FE}}\|^2_{\Omega}\right)
\end{aligned}
$$

Using the results given by lemma (9),

$$\|\nabla(u - u_{\mathrm{FE}}^{N+1})\|_{\Omega^d}^2 + \|\varrho - \varrho_{\mathrm{FE}}^{N+1}\|_{\Omega}^2 \leq \left(1 + 2\delta\tau^2 C_2 \hat{C} C_{eff} \max_{K\in\mathcal{T}_\mathcal{N}} (p_K)^2 - \frac{\delta\theta^2}{C_1(1+\delta)}\right)\left(\left\|\nabla\left(u - u_{\mathrm{FE}}^N\right)\right\|_{\Omega^d}^2 + \left\|\varrho - \varrho_{\mathrm{FE}}^N\right\|_{\Omega}^2\right)$$

for $\tau < \frac{1}{\sqrt{2CC_1C_2\hat{C}C_{eff}}\max_{K\in\mathcal{T}_\mathcal{N}}(p_K)}$ and $\delta < C\theta^2 - 2$

$$\|\nabla(u - u_{\mathrm{FE}}^{N+1})\|_{\Omega^d}^2 + \|\varrho - \varrho_{\mathrm{FE}}^{N+1}\|_{\Omega}^2 \leq \left(1 + \frac{\delta}{CC_1} - \frac{\delta\theta^2}{C_1(1+\delta)}\right)\left(\left\|\nabla\left(u - u_{\mathrm{FE}}^N\right)\right\|_{\Omega^d}^2 + \left\|\varrho - \varrho_{\mathrm{FE}}^N\right\|_{\Omega}^2\right)$$

□

Theorem 5 (Quasi Convergence)**.** *Let $N \in \mathbb{N}$ be an arbitrary refinement step, and $(u, \varrho) \in \mathcal{H}$ be solution of (3). Assume that for some $\theta \in (0, 1]$ solution of the constraint problem (28) and (29) exists. Moreover assume for a sufficiently small $\tau \in (0, 1]$ depending only on polynomial vector p, the data saturation assumption (48) holds. Consider $(u_{FE}^N, \varrho_{FE}^N) \in \mathcal{V}^p(\mathcal{T}_N)$ and $(u_{FE}^{N+1}, \varrho_{FE}^{N+1}) \in \mathcal{V}^p(\mathcal{T}_{N+1})$ as solutions of (4) two consecutive iterative steps. Then for some $\vartheta > 0$ depends on polynomial degree vector p, there is a constant $\mu \in (0, 1)$ independent of mesh size h and polynomial degree vector p such that*

$$\|\nabla(u - u_{FE}^{N+1})\|_{\Omega^d}^2 + \|\varrho - \varrho_{FE}^{N+1}\|_{\Omega}^2 + \vartheta\eta^2(\mathcal{T}_{N+1}) \leq \mu\left(\|\nabla(u - u_{FE}^N)\|_{\Omega^d}^2 + \|\varrho - \varrho^{N}{}^2_{FE}\|_{\Omega} + \vartheta\eta^2(\mathcal{T}_\mathcal{N})\right) \quad (55)$$

Proof. By lemma (10)

$$\begin{aligned}\|\nabla(u - u_{\mathrm{FE}}^{N+1})\|_{\Omega^d}^2 + \|\varrho - \varrho_{\mathrm{FE}}^{N+1}\|_{\Omega}^2 + \vartheta\eta^2(\mathcal{T}_{N+1}) \leq{}& \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right)\left(\|\nabla(u - u_{\mathrm{FE}}^N)\|_{\Omega^d}^2 + \|\varrho - \varrho_{\mathrm{FE}}^N\|_{\Omega}^2\right)\\ &+ \frac{C_2\delta\tau^2}{2}\eta^2(\mathcal{T}_\mathcal{N}) - \frac{1}{2}\Bigg(\|\nabla(u_{\mathrm{FE}}^{N+1} - u_{\mathrm{FE}}^N)\|_{\Omega^d}^2\\ &+ \|\varrho_{\mathrm{FE}}^{N+1} - \varrho_{\mathrm{FE}}^N\|_{\Omega}^2\Bigg) + \vartheta\eta^2(\mathcal{T}_{N+1})\end{aligned}$$

Lemma (8) implies

$$\begin{aligned}\|\nabla(u - u_{\mathrm{FE}}^{N+1})\|_{\Omega^d}^2 + \|\varrho - \varrho_{\mathrm{FE}}^{N+1}\|_{\Omega}^2 + \vartheta\eta^2(\mathcal{T}_{N+1}) \leq{}& \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right)\left(\|\nabla(u - u_{\mathrm{FE}}^N)\|_{\Omega^d}^2 + \|\varrho - \varrho_{\mathrm{FE}}^N\|_{\Omega}^2\right)\\ &+ \left(\frac{C_2\delta\tau^2}{2} + \vartheta(1+\delta^2)(1 + \frac{\tau^2\rho^2}{\delta})\right)\eta^2(\mathcal{T}_\mathcal{N})\\ &- \vartheta(1-\rho)(1+\delta)^2\eta^2(\mathcal{A}_\mathcal{N})\end{aligned} \quad (56)$$

choosing $\vartheta \leq \frac{\delta}{2(1+\delta)C_{red}\max_{K\in\mathcal{T}_\mathcal{N}} p_K}$ and knowing that there exists a positive constan $C_k = \max\{\max_{K\in\mathcal{A}_\mathcal{N}}(k_{K,j_K}), 1\}$ such that

$$C_k^2\eta^2(\mathcal{A}_\mathcal{N}) \geq \sum_{K\in\mathcal{A}_\mathcal{N}} k_{K,j_K}^2 \eta_K^2(\mathcal{T}_\mathcal{N})$$

Therefore equation (56) reads as

$$\|\nabla(u-u_{\mathrm{FE}}^{N+1})\|_{\Omega^d}^2+\|\varrho-\varrho_{\mathrm{FE}}^{N+1}\|_{\Omega}^2+\vartheta\eta^2(\mathcal{T}_{N+1})\leq\left(1-\frac{\delta\theta^2}{2C_1(1+\delta)}\right)\left(\|\nabla(u-u_{\mathrm{FE}}^{N})\|_{\Omega^d}^2+\|\varrho-\varrho_{\mathrm{FE}}^{N}\|_{\Omega}^2\right)$$
$$+\hat{L}\eta^2(\mathcal{T}_N)$$

where $\hat{L}=\frac{C_2\delta\tau^2}{2}+\vartheta(1+\delta^2)(1+\frac{\tau^2\rho^2}{\delta})-\frac{\vartheta\theta^2(1-\rho)(1+\delta)^2}{C_k^2}$. For τ and δ sufficienty small the result holds. □

6 Numerical Results

In this section we present numerical experiments to demonstrate the practical performance of the fully automatic hp-adaptive refinement strategy described in Section 4. Here we consider some Stokes model problems in both two and three dimensional space. Our goal is to illustrate and test the efficiency and reliability of the proposed a posteriori error indicator η_α within the automatic hp-adaptive refinement procedure which was mentioned in the previous Section.

6.1 L-shaped domain

The fist example is a two-dimensional Stokes problem defined in a L-shaped domain $\Omega=(-1,1)^2\backslash[0,1)\times(-1,0]$; Further we assume $\nu=1,\ f=0$. The boundary condition is set such that the exact solution to equation 1 is given by [**?**, **?**, **?**]

$$\begin{pmatrix} u_1\\ u_2\\ \varrho\end{pmatrix}=\begin{pmatrix} -e^x(y\cos(y)+\sin(y))\\ e^x y\sin(y)\\ 2e^x\sin(y)+\frac{2(1-e)(1-\cos(1))}{3}\\ ,\end{pmatrix} \tag{57}$$

Explanation about
the domain hp refinement
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References